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Research article

A note on inference for the mixed fractional Ornstein-Uhlenbeck process with drift

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Abstract: This paper is devoted to the controlled drift estimation of the mixed fractional Ornstein-Uhlenbeck process. We will consider two models: one is the optimal input where we will find the controlled function which maximize the Fisher information for the unknown parameter and the other one with a constant as the controlled function. Large sample asymptotical properties of the Maximum Likelihood Estimator (MLE) is deduced using the Laplace transform computations or the Cameron-Martin formula with extra part from [12]. As a a supplement of [12] we will also prove that the MLE is strongly consistent.

Keywords: mixed fractional Brownian motion; fundamental martingale; Laplace transform; optimal input
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1. Introduction

The drift parametric estimation of the Ornstein-Uhlenbeck process has been paid more and more attention in the past decades. These years, researchers not only considered the process with standard Brownian motion or Levy process but also the fractional case (see *e.g.,* [15, 17]). The MLE and its large deviation of the mixed fractional case has been studied by Chigansky *et al.* [12, 19]. In this paper we will consider still the MLE of the drift parameter but with an extra part in some space which can maximizer the Fisher information which is called experiment design.

Let us define $X = (X_t, 0 \le t \le T)$ a real-valued process, representing the observation, which is governed by:

$$dX_t = -\vartheta X_t dt + u(t)dt + d\xi_t,\ t \in [0, T],\ X_0 = 0 \tag{1.1}$$

where $\xi = (\xi_t,\ 0 \le t \le T)$ is a mixed fractional fractional Brownian motion (mfBm for short) which is defined by $\xi_t = W_t + B^H_t$, here $W = (W_t,\ 0 \le t \le T)$ and $B^H = (B^H_t,\ 0 \le t \le T$ are independent standard

Brownian motion and fractional Brownian motion with $H \in (0, 1)$, $H \neq 1/2$.

In the statistical aspect, the classical approach for experiment design consists on a two-step procedure: maximize the Fisher information under energy constraint of the input and find an adaptive estimation procedure. Ovseevich *et al.* [20] has first consider this type problem for the diffusion equation with continuous observation. When the kernel in [20] is not with explicit formula in the fractional diffusion case, Brouste *et al.* [3, 4] deduce the lower bound and upper bound with the method of spectral gap and solve the same problem. Base on this method, Brouste and Cai [1] have extended the result to the partially observed fractional Ornestein-Uhlenbeck process, in this work the asymptotical normality has been demonstrated with linear filtering of Gaussian processes and Laplace transform presented in [2, 14–16, 18]. These previous work, the common point is that: the optimal input does not depend on the unknown parameter and maximum likelihood estimator can be found directly from the likelihood equation. The one-step estimator will be used following the Newton-Raphson method and this work was introduced by Cai and LV [8].

For a fixed value of parameter ϑ, let $\mathbf{P}_\vartheta^T$ denote the probability measure, induced by X^T on the function space $C_{[0,T]}$ and let $\mathcal{F}_t^X$ be the nature filtration of X, $\mathcal{F}_t^X = \sigma(X_s, 0 \leq s \leq t)$. Let $\mathcal{L}(\vartheta, X^T)$ be the likelihood, *i.e.,* the Radon-Nikodym derivative of $\mathbf{P}_\vartheta^T$, restricted to $\mathcal{F}_T^Y$ with respect to some reference measure on $C_{[0,T]}$. In this setting, Fisher information stands for

$$\mathcal{I}_T(\vartheta, u) = -\mathbf{E}_\vartheta \frac{\partial^2}{\partial \vartheta^2} \ln \mathcal{L}(\vartheta, X^T).$$

Let us denote $\mathcal{U}_T$ some functional space of controls, that is defined by Eqs (2.7) and (2.6). Let us therefore note

$$\mathcal{J}_T(\vartheta) = \sup_{u \in \mathcal{U}_T} \mathcal{I}_T(\vartheta, u). \tag{1.2}$$

our main goal is to find estimator $\overline{\vartheta}_T$ of the parameter ϑ which is asymptotically efficient in the sense that, for any compact $\mathbb{K} \in \mathbb{R}_*^+ = \{\vartheta \in \mathbb{R}, \vartheta > 0\}$,

$$\sup_{\vartheta \in \mathbb{K}} \mathcal{J}_T(\vartheta) \mathbf{E}_\vartheta \left(\overline{\vartheta}_T - \vartheta\right)^2 = 1 + o(1)\,, \tag{1.3}$$

as $T \to \infty$.

As the optimal input does not depend on ϑ (see Proposition 2.1), a possible candidate is the Maximum Likelihood Estimator (MLE) $\hat{\vartheta}_T$, defined as the maximizer of the likelihood:

$$\hat{\vartheta}_T = \arg\max_{\vartheta > 0} \mathcal{L}(\vartheta, X^T).$$

We want to find the asymptotical normality of the MLE of ϑ.

The interest to mixed fractional Brownian motion was triggered by Cheridito [9]. The resent works of Cai, Chigansky, Kleptsyna and Marushkevych ([6, 11, 12, 19]) present a great value for the purpose of this paper. The process ξ_t satisfies a number of curious properties with applications in mathematical finance, see [5]. In particular, as shown in [9, 10] , it is a semimartingale if and only if $H \in \{\frac{1}{2}\} \bigcup (\frac{3}{4}, 1]$ and the measure μ^ξ induced by ξ on the space of continuous functions on $[0, T]$, is equivalent to the standard Wiener measure μ^B for $H > \frac{3}{4}$. On the other hand, μ^ξ and μ^{B^H} are equivalent if and only if $H < \frac{1}{4}$.

The paper falls into five parts. In Section 2, we present some main results of this paper and the Section 3 will contribute to the proofs of the main results. Section 4 is devoted to another special constant case. Some Lemmas will be given in Appendix.

2. Main results

2.1. Transformation of the model

Even if the mixed fractional Brownian motion ξ is a semimartingale when $H > \frac{3}{4}$, it is hard to write the likelihood function directly. We will try to transform our model with the fundamental martingale in [6] and get the explicit representation of the likelihood function. In what follows, all random variables and processes are defined on a given stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ satisfying the usual conditions and processes are $(\mathcal{F}_t)-$ adapted. Moreover the *natural filtration* of a process is understood as the $\mathbf{P}$-completion of the filtration generated by this process.

From the canonical innovation representation in [6], the fundamental martingale is defined as $M_t = \mathbf{E}(B_t|\mathcal{F}_t^\xi)$, $t \in [0, T]$, then for $H \in (0,1)$ and $H \neq 1/2$ this martingale satisfies

$$M_t = \int_0^t g(s,t)d\xi_s, \qquad \langle M\rangle_t = \int_0^t g(s,t)ds \tag{2.1}$$

where $g(s,t)$ is the solution of the integro-differential equation

$$g(s,t) + H\frac{d}{ds}\int_0^t g(r,t)|r-s|^{2H-1}\mathrm{sign}(s-r)dr = 1,\ 0 < s \leq t \leq T \tag{2.2}$$

Following from [6], let us introduce a process $Z = (Z_t,\ 0 \leq t \leq T)$ the fundamental semimartingale associated to X, defined as

$$Z_t = \int_0^t g(s,t)dX_s.$$

Note that X can be represented as $X_t = \int_0^t \hat{g}(s,t)dZ_s$ where

$$\hat{g}(s,t) = 1 - \frac{d}{d\langle M\rangle_s}\int_0^t g(r,s)dr \tag{2.3}$$

for $0 \leq s \leq t$ and there for the nature filtration of X and Z coincide. Moreover, we have the following representations:

$$dZ_t = -\vartheta Q_t d\langle M\rangle_t + v(t)d\langle M\rangle_t + dM_t, \tag{2.4}$$

where

$$Q_t = \frac{d}{d\langle M\rangle_t}\int_0^t g(s,t)X_s ds,\ v(t) = \frac{d}{d\langle M\rangle_t}\int_0^t g(s,t)u(s)ds. \tag{2.5}$$

2.2. Fisher information and optimal input

First of all, let us define the space of control for $v(t)$:

$$\mathcal{V}_T = \left\{h\middle|\frac{1}{T}\int_0^T |v(t)|^2 d\langle M\rangle_t \leq 1\right\}. \tag{2.6}$$

Remark that with (2.5) the following relationship between control u and its transformation v holds:

$$u(t) = \frac{d}{dt}\int_0^t \hat{g}(t,s)v(s)d\langle M\rangle_s \tag{2.7}$$

we can set the admissible control as $\mathcal{U}_T = \{u|v \in \mathcal{V}_T\}$. Note that these set are non-empty.

From [12], we know $Q_t = \int_0^t \psi(s,t)dZ_s$ where

$$\psi(s,t) = \frac{1}{2}\left(\frac{dt}{d\langle M\rangle_t} + \frac{ds}{d\langle M\rangle_s}\right). \tag{2.8}$$

Moreover, $Q_t = \frac{1}{2}\ell(t)^*\zeta_t$, where $\ell(t) = \begin{pmatrix} \psi(t,t) \\ 1 \end{pmatrix}$, $*$ standing for the transposition and $\zeta = (\zeta_t,\ t \geq 0)$ is the solution of the stochastic differential equation

$$d\zeta_t = -\frac{\vartheta}{2}A(t)\zeta_t d\langle M\rangle_t + b(t)v(t)d\langle M\rangle_t + b(t)dM_t, \zeta_0 = \mathbf{0}_{2\times 1}, \tag{2.9}$$

with

$$A(t) = \begin{pmatrix} \psi(t,t) & 1 \\ \psi^2(t,t) & \psi(t,t) \end{pmatrix}, \ b(t) = \begin{pmatrix} 1 \\ \psi(t,t) \end{pmatrix}. \tag{2.10}$$

The classical Girsanov theorem gives

$$\mathcal{L}(\vartheta, Z^T) = \mathbf{E}_\vartheta \exp\left\{-\int_0^T (-\vartheta Q_t + v(t))dZ_t - \frac{1}{2}\int_0^T (-\vartheta Q_t + v(t))^2 d\langle M\rangle_t\right\}. \tag{2.11}$$

Now from (2.11) the Fisher information can be easily obtained by

$$\begin{aligned} \mathcal{I}_T(\vartheta, v) &= -\mathbf{E}_\vartheta \frac{\partial^2}{\partial\vartheta^2} \ln \mathcal{L}(\vartheta, Z^T) \\ &= \frac{1}{4}\mathbf{E}_\vartheta \int_0^T (\ell(t)^*\zeta_t)^2\, d\langle M\rangle. \end{aligned}$$

Then we have the following results for the optimal input:

Theorem 2.1. *The asymptotic optimal input in the class of controls* $\mathcal{U}_T$ *is* $u_{opt}(t) = \frac{d}{dt}\int_0^t \hat{g}(s,t)\psi(s,s)d\langle M\rangle_s$ *where* $\hat{g}(s,t)$, $\psi(s,t)$, $\langle M\rangle_t$ *are defined in* (2.1), (2.3), (2.8). *Moreover,*

$$\lim_{T\to+\infty} \frac{\mathcal{J}_T(\vartheta)}{T} = \mathcal{I}(\vartheta),$$

where

$$\mathcal{I}(\vartheta) = \frac{1}{2\vartheta} + \frac{1}{\vartheta^2}. \tag{2.12}$$

The $\mathcal{J}_T(\vartheta)$ *is defined in* (1.2).

2.3. Asymptotical normality and strong consistency of the MLE

From the Theorem 2.1, we can see that the optimal input $u_{opt}(t)$ does not depend on the unknown parameter ϑ, we can easily obtain the estimator error of the MLE of the $\hat{\vartheta}_T$:

$$\hat{\vartheta}_T - \vartheta = \frac{\int_0^T Q_t dM_t}{\int_0^T Q_t^2 d\langle M\rangle_t}. \tag{2.13}$$

Then, the MLE reaches efficiency and we deduce its large sample asymptotic properties:

Theorem 2.2. *The MLE is uniformly consistent on compacts* $K \subset \mathbb{R}_*^+$, *i.e. for any* $\nu > 0$,

$$\lim_{T\to\infty} \sup_{\vartheta\in\mathbb{K}} \mathbf{P}_\vartheta^T \left\{ \left| \hat{\vartheta}_T - \vartheta \right| > \nu \right\} = 0\,,$$

uniformly on compacts asymptotically normal: as T tends to $+\infty$,

$$\lim_{T\to\infty} \sup_{\vartheta\in\mathbb{K}} \left| \mathbf{E}_\vartheta f\left(\sqrt{T}\left(\hat{\vartheta}_T - \vartheta \right) \right) - \mathbf{E} f(\eta) \right| = 0 \quad \forall f \in C_b$$

and ξ *is a zero mean Gaussian random variable of variance* $\mathcal{I}(\vartheta)^{-1}$ *(see* (2.12) *for the explicit value)* **which does not depend on** *H and we have the uniform on* $\vartheta \in \mathbb{K}$ *convergence of the moments: for any* $p > 0$,

$$\lim_{T\to\infty} \sup_{\vartheta\in\mathbb{K}} \left| \mathbf{E}_\vartheta \left| \sqrt{T}\left(\hat{\vartheta}_T - \vartheta \right) \right|^p - \mathbf{E}\left|\eta\right|^p \right| = 0.$$

Finally, the MLE is efficient in the sense of (1.3).

Theorem 2.3. *The MLE* $\hat{\vartheta}_T$ *is strong consistency that is*

$$\hat{\vartheta}_T \xrightarrow{a.s.} \vartheta, \quad T \to \infty.$$

3. Proofs of main results

3.1. Proof of Theorem 2.1

We will compute the Fisher information with the same method in [1], that is to separate the Fisher information into two parts, on into the control, the other without, we focus on the following decomposition:

$$\begin{aligned} \mathcal{I}_T(\vartheta, v) &= \frac{1}{4}\mathbf{E}_\vartheta \left\{ \int_0^T (\ell(t)^* \zeta_t - \mathbf{E}_\vartheta \ell(t)^* \zeta_t + \mathbf{E}_\vartheta \ell(t)^* \zeta_t)^2 \right\} \\ &= \mathcal{I}_{1,T}(\vartheta, v) + \mathcal{I}_{2,T}(\vartheta, v) \end{aligned} \tag{3.1}$$

where

$$\mathcal{I}_{1,T}(\vartheta, v) = \frac{1}{4} \int_0^T \mathbf{E}_\vartheta (\ell(t)^* \zeta_t - \mathbf{E}_\vartheta \ell(t)^* \zeta_t)^2 \langle M \rangle_t \tag{3.2}$$

and

$$\mathcal{I}_{2,T}(\vartheta, v) = \frac{1}{4} \int_0^T (\ell(t)^* \mathbf{E}_\vartheta \zeta_t)^2 d\langle M \rangle_t. \tag{3.3}$$

The deterministic function ($\mathcal{P}(t) = \mathbf{E}_\vartheta \zeta_t,\ t \geq 0$) satisfies the following equation:

$$\frac{d\mathcal{P}(t)}{d\langle M \rangle_t} = -\frac{1}{2}\vartheta A(t)\mathcal{P}(t) + b(t)v(t), \mathcal{P}(0) = \mathbf{0}_{2\times 1}, \tag{3.4}$$

at the same time the process $\overline{P} = (\overline{P}_t = \zeta_t - \mathbf{E}_\vartheta \zeta_t,\ t \geq 0)$ satisfies the following stochastic equation:

$$d\overline{P}_t = -\frac{1}{2}\vartheta A(t)\overline{P}_t d\langle M \rangle_t + b(t) dM_t,$$

which is just the ζ_t with $v(t) = 0$ which can be found in [12].

With the technical separation of (3.1) and the precedent remarks, we have

$$\mathcal{J}_T(\vartheta) = \mathcal{I}_{1,T}(\vartheta) + \mathcal{J}_{2,T}(\vartheta),$$

where

$$\mathcal{J}_{2,T}(\vartheta) = \sup_{v \in \mathcal{V}_T} \mathcal{I}_{2,T}(\vartheta,\ v).$$

From [12], we know

$$\lim_{T\to\infty} \frac{\mathcal{I}_{1,T}(\vartheta)}{T} = \frac{1}{2\vartheta},$$

so we just need to check that $\lim\limits_{T\to\infty} \frac{\mathcal{J}_{2,T}(\vartheta)}{T} = \frac{1}{\vartheta^2}$. From (3.4), we get

$$\mathcal{P}(t) = \varphi(t) \int_0^t \varphi^{-1}(s) b(s) v(s) d\langle M\rangle_s, \tag{3.5}$$

where $\varphi(t)$ is the matrix defined by

$$\frac{d\varphi(t)}{d\langle M\rangle_t} = -\frac{\vartheta}{2} A(t)\varphi(t),\ \varphi(0) = \mathbf{Id}_{2\times 2} \tag{3.6}$$

with $\mathbf{Id}_{2\times2}$ the 2×2 identity matrix. Substituting into (3.3), we get

$$\mathcal{I}_{1,T}(\vartheta,\ v) = \int_0^T \int_0^T \mathcal{K}_T(s,\ \sigma) \frac{1}{\sqrt{\psi(s,\ s)}} v(s) \frac{1}{\sqrt{\psi(\sigma,\ \sigma)}} v(\sigma) ds d\sigma, \tag{3.7}$$

where the operator

$$\mathcal{K}_T(s,\ \sigma) = \int_{\max(s,\sigma)}^T \mathcal{G}(t,\ s)\mathcal{G}(t,\ \sigma) dt \tag{3.8}$$

and

$$\mathcal{G}(t,\ \sigma) = \frac{1}{2}\left(\frac{1}{\sqrt{\psi(t,t)}} \ell(t)^* \varphi(t)\varphi^{-1}(\sigma) b(\sigma) \frac{1}{\sqrt{\psi(\sigma,\ \sigma)}} \right). \tag{3.9}$$

Then

$$\begin{aligned} \mathcal{J}_{2,T}(\vartheta) &= T \sup_{\widetilde{v}\in L^2[0,T], \|\widetilde{v}\|\le 1} \int_0^T \int_0^T \mathcal{K}_T(s,\sigma)\widetilde{v}(s)\widetilde{v}(\sigma) ds d\sigma, \\ &= T \sup_{\widetilde{v}\in L^2[0,T], \|\widetilde{v}\|\le 1} (\mathcal{K}_T\widetilde{v}, \widetilde{v}) \end{aligned} \tag{3.10}$$

where $\widetilde{v}(s) = \frac{v(s)}{\sqrt{T}} \frac{1}{\sqrt{\psi(t,t)}}$ and $\| \bullet \|$ stands for the usual norm in $L^2[0,\ T]$. Thus, Lemma 5.1 completes our proof.

3.2. Proof of Theorem 2.2

The proof of Theorem 2.2 is based on the Ibragimov-Khasminskii program of Theorem I.10.1 in [13]. Taking $v_{opt}(t) = \sqrt{\psi(t,t)}$ into the Eq (2.11), then the likelihood function is

$$\mathcal{L}(\vartheta, Z^T) = \mathbf{E}_\vartheta \exp\left\{-\int_0^T (-\vartheta Q_t + v_{opt}(t))dZ_t - \frac{1}{2}\int_0^T (-\vartheta Q_t + v_{opt}(t))^2 d\langle M\rangle_t\right\},$$

then the MLE will be

$$\hat{\vartheta}_T = \frac{\int_0^T v_{opt}(t)Q_t d\langle M\rangle_t - \int_0^T Q_t dZ_t}{\int_0^T Q_t^2 d\langle M\rangle_t} \tag{3.11}$$

and the estimation error has the form

$$\hat{\vartheta}_T - \vartheta = -\frac{\int_0^T Q_t dM_t}{\int_0^T Q_t^2 d\langle M\rangle_t}, \tag{3.12}$$

just take attention that here the Q_t will be with the relationship with $v_{opt}(t)$. Because $\int_0^t Q_s dMs$, $0 \le t \le T$ is a martingale and $\int_0^t Q_s^2 d\langle M\rangle_s$ is its quadratic variation, In order to prove the Theorem 2.2, we only need to check the Laplace transform of the quadratic variation and Lemma 5.2 achieves the proof.

3.3. Proof of theorem 2.3

With the law of large numbers, in order to obtain the strong consistency of ϑ, we only need to prove that

$$\lim_{T\to\infty} \int_0^T Q_t^2 d\langle M\rangle_t = +\infty \tag{3.13}$$

or there exists a positive constant μ such that the limit of the Laplace transform

$$\lim_{T\to\infty} \mathbf{E}\exp\left(-\mu\int_0^T Q_t^2 d\langle M\rangle_t\right) = 0.$$

In Lemma 5.4 if we take a big enough $\mu > 0$ such that the limit is negative (the μ can be easily found), then the Eq (3.13) is directly from this Lemma which implies the strong consistency.

4. The Laplace transform proof for the constance case

In fact, the previous method of the Laplace transform is also useful for the case $u(t)$ is a known constant. This problem has been considered in [7], here we use Cameron-Martin formula to reprove the result.

Let us consider $u(t) = \alpha$ a constant not 0. In this case we will denote the processes X, Z, Q by X^α, Z^α and Q^α and it is not hart to find that the MLE of the unknown parameter ϑ is

$$\hat{\vartheta}_T^\alpha = \frac{\int_0^T \alpha Q_t^\alpha d\langle M\rangle_t - \int_0^T Q_t^\alpha dZ_t^\alpha}{\int_0^T (Q_t^\alpha)^2 d\langle M\rangle_t} \tag{4.1}$$

where

$$dZ_t^\alpha = (\alpha - \vartheta Q_t^\alpha)d\langle M\rangle_t + dM_t,\ t \in [0,T]. \tag{4.2}$$

From [7] the estimation error can be presented by

$$\hat{\vartheta}_T^\alpha - \vartheta = \frac{\int_0^T Q_t^\alpha dM_t}{\int_0^T (Q_t^\alpha)^2 d\langle M\rangle_t}. \tag{4.3}$$

In order to obtain the result in [7]

$$\sqrt{T}(\hat{\vartheta}_T^\alpha - \vartheta) \xrightarrow{d} \mathcal{N}(0, 2\vartheta)$$

for $H > 1/2$ and

$$\sqrt{T}(\hat{\vartheta}_T^\alpha - \vartheta) \xrightarrow{d} \mathcal{N}\left(0, \frac{2\vartheta^2}{2\alpha^2 + \vartheta}\right)$$

for $H < 1/2$, we prove the stronger result of the Laplace transform:

Lemma 4.1. *For $H > 1/2$. the limit of the Laplace transform is*

$$\lim_{T\to\infty} \mathcal{L}_T^\alpha(\mu) = \lim_{T\to\infty} \mathbf{E}_\vartheta \exp\left(-\frac{\mu}{T}\int_0^T (Q_t^\alpha)^2 d\langle M\rangle_t\right) = \exp\left(-\frac{\mu}{2\vartheta}\right),\ \forall\mu > 0$$

and for $H < 1/2$,

$$\lim_{T\to\infty} \mathcal{L}_T^\alpha(\mu) = \lim_{T\to\infty} \mathbf{E}_\vartheta \exp\left(-\frac{\mu}{T}\int_0^T (Q_t^\alpha)^2 d\langle M\rangle_t\right) = \exp\left(-\mu\left(\frac{1}{2\vartheta} + \left(\frac{\alpha}{\vartheta}\right)^2\right)\right),\ \forall\mu > 0.$$

The proof will be presented in the Appendix.

Remark 4.2. The strong consistency of $\hat{\vartheta}_T^\alpha$ can also be obtained with the same proof of Theorem 2.3.

Remark 4.3. If we only consider this special case with $u(t) = \alpha$, the Laplace transform has no advantage because we can find a very kind solution of X^α with respect to the classical O-U process and every term of the estimator error can be easily computed as presented in [7]. But from the optimal input case, even we can find the explicit solution but the components of the estimator error are complicated, so the Laplace transform will be more efficient.

Remark 4.4. We only consider the MLE of ϑ when $u(t)$ is known, but the Laplace transform will be more useful for the case of O-U process with periodic drift of the form

$$dX_t = \left(\sum_{i=1}^p \mu_i \varphi_i(t) - \vartheta X_t\right)dt + d\xi_t,\ X_0 = 0$$

where μ_i and ϑ are all unknown and to be estimated. We will use the Cameron-Martin formula for the quadratic variation of the martingale of $p+1$ dimension especially when $H < 1/2$ and this will be our future work.

5. Appendix

Lemma 5.1. *For the kernel* $\mathcal{K}_T(s,\sigma)$ *defined in Eq* (3.10)

$$\lim_{T\to\infty}\sup_{\widetilde{v}\in L^2[0,T],\|\widetilde{v}\|\le 1}(\mathcal{K}_T\widetilde{v},\widetilde{v})=\frac{1}{\vartheta^2}\tag{5.1}$$

with an optimal input $v_{opt}(t)=\sqrt{\psi(t,t)}$

Proof. When we take $v(t)=v_{opt}(t)=\sqrt{\psi(t,t)}$, then

$$\frac{d\mathcal{P}(t)}{d\langle M\rangle_t}=-\frac{1}{2}\vartheta A(t)\mathcal{P}(t)+b(t)v_{opt}(t),\mathcal{P}(0)=\mathbf{0}_{2\times 1}.$$

Because for $H>1/2$, $\frac{d\langle M\rangle_t}{dt}=g^2(t,t)$. From [12]

$$\langle M\rangle_T\sim T^{2-2H}\lambda_H^{-1},\ T\to\infty,\ \lambda_H=\frac{2H\Gamma(3-2H)\Gamma(H+1/2)}{\Gamma(3/2-H)}.$$

then with the calculus of [4] we can easily obtain

$$\lim_{T\to\infty}\frac{1}{4T}\int_0^T(\ell(t)^*\mathcal{P}(t))^2d\langle M\rangle_t=\frac{1}{\vartheta^2}.\tag{5.2}$$

On the other hand for $H<1/2$ we have

$$\lim_{T\to\infty}\frac{\langle M\rangle_T}{T}=1$$

and we can also easily obtain the result of (5.2), that is to say the lower bound at least will be $\frac{1}{\vartheta^2}$.

Now we will try to find the upper bound. Let us introduce the Gaussian process $(\xi_t,\ 0\le t\le T)$

$$\xi_t=\left(\frac{1}{\sqrt{\psi(\sigma,\sigma)}}\ell(\sigma)^*\varphi(\sigma)\odot dW_\sigma\right)\varphi^{-1}(t),\ \xi_T=0$$

where $(W_\sigma,\sigma\ge 0)$ is a Wiener process and $\odot$ denotes the Itô backward integral (see [21]). It is worth emphasizing that

$$\mathcal{K}_T(s,\sigma)=\frac{1}{4}\mathbf{E}\left(\xi_s b(s)\frac{1}{\sqrt{\psi(s,s)}}\xi_\sigma b(\sigma)\frac{1}{\sqrt{\psi(\sigma,\sigma)}}\right)=\mathbf{E}(\mathcal{X}_\sigma\mathcal{X}_s).$$

where $\mathcal{X}$ is the centered Gaussian process defined by $\mathcal{X}_t=\frac{1}{2}\xi_t b(t)\frac{1}{\sqrt{\psi(s,s)}}$. The process $(\xi_t,0\le t\le T)$ satisfies the following dynamic

$$-d\xi_t=-\frac{\vartheta}{2}\xi_t A(t)d\langle M\rangle_t+\ell(t)^*\frac{1}{\sqrt{\psi(t,t)}}\odot dW_t,\ \xi_T=0.$$

Obviously, $\mathcal{K}_T(s,\sigma)$ is a compact symmetric operator for fixed T, so we should estimate the spectral gap (the first eigenvalue $\nu_1(T)$) of the operator. The estimation of the spectral gap is based on the

Laplace transform computation. Let us compute, for sufficiently small negative $a < 0$ the Laplace transform of $\int_0^T \mathcal{X}_t^2 dt$:

$$\begin{aligned} L_T(a) &= \mathbf{E}_\vartheta \exp\left(-a \int_0^T \mathcal{X}_t^2 dt\right) \\ &= \mathbf{E}_\vartheta \exp\left(-a \int_0^T \left(\frac{1}{2}\xi_t b(t) \frac{1}{\sqrt{\psi(t,t)}}\right)^2 dt\right) \end{aligned}$$

On one hand, for $a > -\frac{1}{\nu_1(T)}$, since $\mathcal{X}$ is a centered Gaussian process with covariance operator $\mathcal{K}_T$, using Mercer's theorem and Parseval's inequality, $L_T(a)$ can be represented as :

$$L_T(a) = \prod_{i\geq 1}(1 + 2a\nu_i(T))^{-\frac{1}{2}}, \tag{5.3}$$

where $\nu_i(T)$, $i \geq 1$ is the sequence of positive eigenvalues of the covariance operator. On the other hand,

$$\begin{aligned} L_T(a) &= \mathbf{E}_\vartheta \left(-\frac{a}{4} \int_0^T \xi_t b(t) b(t)^* \xi_t^* d\langle M\rangle_t\right) \\ &= \exp\left(\frac{1}{2} \int_0^T \text{trace}(\mathcal{H}(t)\mathcal{M}(t) d\langle M\rangle_t\right) \end{aligned}$$

where $\mathcal{M}(t) = \ell(t)^*\ell(t)$ and $\mathcal{H}(t)$ is the solution of Ricatti differential equation:

$$\frac{d\mathcal{H}(t)}{d\langle M\rangle_t} = \mathcal{H}(t)\mathcal{A}(t)^* + \mathcal{A}(t)\mathcal{H} + \mathcal{H}(t)\mathcal{M}(t)\mathcal{H}(t) - \frac{a}{2} b(t) b(t)^*,$$

with $\mathcal{A}(t) = -\frac{\vartheta}{2}A(t)$ and the initial condition $\mathcal{H}(0) = \mathbf{0}_{2\times 2}$, provided that the solution of this equation exists for any $0 \leq t \leq T$.

It is well know that if $\det \Psi_1(t) > 0$, for any $t \in [0, T]$, then $\mathcal{H}(t) = \Psi_1^{-1}(t)\Psi_2(t)$, where the pair of 2×2 matrices (Ψ_1, Ψ_2) satisfies the system of linear differential equations:

$$\begin{aligned} \frac{d\Psi_1(t)}{d\langle M\rangle_t} &= -\Psi_1(t)\mathcal{A}(t) - \Psi_2(t)\mathcal{M}(t), && \Psi_1(0) = \mathbf{Id}_{2\times 2}, \\ \frac{d\Psi_2(t)}{d\langle M\rangle_t} &= -\frac{a}{2}\Psi_1(t) b(t) b(t)^* + \Psi_2(t)\mathcal{A}(t)^*, && \Psi_2(0) = \mathbf{0}_{2\times 2} \end{aligned} \tag{5.4}$$

and

$$L_T(a) = \exp\left(-\frac{1}{2} \int_0^T \text{trace}\,(\mathcal{A}(t))\, d\langle N\rangle_t\right) (\det \Psi_1(T))^{-\frac{1}{2}}. \tag{5.5}$$

Rewriting the system (5.4) in the following form

$$\frac{d(\Psi_1(t),\ \Psi_2(t)\mathbf{J})}{d\langle M\rangle_t} = (\Psi_1(t),\ \Psi_2(t)\mathbf{J}) \cdot (\Upsilon \otimes A(t)), \tag{5.6}$$

where $\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\Upsilon = \begin{pmatrix} \frac{\vartheta}{2} & -\frac{a}{2} \\ -1 & -\frac{\vartheta}{2} \end{pmatrix}$

When $-\frac{\vartheta^2}{2} \le a \le 0$, we have two real eigenvalues of the matrix Υ, we denote them $(x_i)_{i=1,2}$. It can be checked that there exists a constant $C > 0$ such that

$$\det \Psi_1(T) = \exp((x_1)T)(C + \underset{T\to\infty}{O}(\frac{1}{T}))$$

where $x_1 = \sqrt{\frac{\vartheta^2}{4} + \frac{a}{2}}$. Therefore, due to the (5.5), we have $\prod_{i\ge1}(1 + 2a\nu_i(T)) > 0$ for any $a > -\frac{\vartheta^2}{2}$. It means that

$$\nu_1(T) \le \frac{1}{\vartheta^2}$$

□

Lemma 5.2. *For $v(t) = v_{opt}(t)$ defined in Lemma 5.1, the Laplace Transform*

$$\mathcal{L}_T(\mu) = \mathbf{E}_\vartheta \exp\left(-\frac{\mu}{T}\int_0^T Q_t^2 d\langle M\rangle_t\right) \xrightarrow[T\to\infty]{} \exp\left(-\mu\left(\frac{1}{2\vartheta} + \frac{1}{\vartheta^2}\right)\right) \tag{5.7}$$

for every $\mu > 0$.

Proof. First, we replace Q_t with ζ_t and rewrite the Laplace transform, that is

$$\mathcal{L}_T(\mu) = \mathbf{E}_\vartheta \exp\left\{-\frac{\mu}{T}\int_0^T \zeta_t R(t)\zeta_t^*\, d\langle M\rangle_t\right\}$$

where ζ_t is defined in (2.9) and $R(t) = \frac{1}{4}\begin{pmatrix} \psi^2(t,t) & \psi(t,t) \\ \psi(t,t) & 1 \end{pmatrix}$. Following from [14], we have

$$\mathcal{L}_T(\mu) = \exp\left\{-\frac{\mu}{T}\int_0^T [\mathrm{tr}(\Gamma(t)R(t)) + Z^*(t)R(t)Z(t)]\, d\langle M\rangle_t\right\}$$

where

$$\frac{d\Gamma(t)}{d\langle M\rangle_t} = -\frac{\vartheta}{2}A(t)\Gamma(t) - \frac{\vartheta}{2}\Gamma(t)A(t)^* + b(t)b(t)^* - \frac{2\mu}{T}\Gamma(t)R(t)\Gamma(t)$$

and

$$Z(t) = \mathbf{E}_\vartheta\zeta_t - \frac{\mu}{T}\int_0^t \varphi(t)\varphi^{-1}\Gamma(s)R(s)Z(s)d\langle M\rangle_s \tag{5.8}$$

with

$$\frac{d\varphi(t)}{d\langle M\rangle_t} = -\frac{\vartheta}{2}A(t)\varphi(t).$$

From [12] we know that

$$\lim_{T\to\infty} \exp\left(-\frac{\mu}{T}\int_0^T (\ \mathrm{tr}\ (\Gamma(t)R(t)))d\langle M\rangle_t = \exp\left(\frac{\mu}{2\vartheta}\right)\right)$$

On the other hand we know $\mathbf{E}\zeta_t = \mathcal{P}(t)$ defined in Lemma 5.1 with $v(t) = v_{opt}(t)$, thus

$$\lim_{T\to\infty} \exp\left(-\frac{\mu}{T}\mathbf{E}\zeta_t R(t)(\mathbf{E}\zeta_t)^*\right) = \lim_{T\to\infty} \exp\left(-\frac{\mu}{4T}\int_0^T (\ell^*(t)\mathcal{P}(t))^2 d\langle M\rangle_t\right) = \exp\left(-\frac{\mu}{\vartheta^2}\right)$$

Now, the conclusion is true provided that

$$\lim_{T\to\infty}\left(\frac{\mu}{T}\int_0^t \varphi(t)\varphi^{-1}\Gamma(s)R(s)Z(s)d\langle M\rangle_s\right)R(t)\left(\frac{\mu}{T}\int_0^t \varphi(t)\varphi^{-1}\Gamma(s)R(s)Z(s)d\langle M\rangle_s\right)^* = 0.$$

On one hand, from [4] and [12] when t is large enough

$$\int_0^t |F(t,s)|ds = \left|\frac{\mu}{T}\int_0^t \varphi(t)\varphi^{-1}\Gamma(s)R(s)\right| = O\left(\frac{1}{T}\right),\ T\to\infty \tag{5.9}$$

where

$$F(t,s) = \left|\frac{\mu}{T}\varphi(t)\varphi^{-1}\Gamma(s)R(s)\right|$$

and $|\cdot|$ denotes L^1 norm of the vector. On the other hand, If we define the operator S by

$$S(f)(t) = \int_0^t\int_0^t |F(t,s)|f(s)ds$$

then Eq (5.8) leads to

$$|Z(t)| \le |\mathcal{P}(t)| + S(|Z|)(t)$$

or we can say $(I-S)(|Z|)(t) \le |\mathcal{P}(t)| \le Const.$ From Eq (5.9) we have for t and T large enough

$$|Z(t)| \le (I-S)^{-1}(Const)(t) = \prod_{n=1}^{\infty} S^n(Const.)(t) \le Const. \tag{5.10}$$

The $Const.$ means some constant, but in different equation they may be different. Combining (5.9) and (5.10) we have for t large enough

$$\int_0^t |F(t,s)||Z(s)| = O\left(\frac{1}{T}\right),\ T\to\infty$$

which achieves the proof. □

In the following we will use the same method to prove the Lemma 4.1. When $u(t)=\alpha$, our two dimensional observed process $\zeta^\alpha = (\zeta_t^\alpha,\ 0\le t\le T)$ satisfies the following equation:

$$d\zeta_t^\alpha = \alpha b(t)d\langle M\rangle_t - \frac{\vartheta}{2}A(t)\zeta_t^\alpha d\langle M\rangle_t + b(t)dM_t \tag{5.11}$$

where $A(t)$, $b(t)$ are defined in (2.10). From the previous proof we know

$$\mathcal{L}_T^\alpha(\mu) = \exp\left\{-\frac{\mu}{T}\int_0^T [\ \mathrm{tr}\,(\Gamma(t)R(t)) + \mathcal{Z}^*(t)R(t)\mathcal{Z}(t)]\, d\langle M\rangle_t\right\} \tag{5.12}$$

where

$$\mathcal{Z}(t) = \mathbf{E}\zeta_t^\alpha - \frac{\mu}{T}\int_0^t \varphi(t)\varphi^{-1}\Gamma(s)R(s)\mathcal{Z}(s)d\langle M\rangle_s. \tag{5.13}$$

The functions $\Gamma(t)$, $\varphi(t)$ and the matrix $R(t)$ are defined in the previous Lemma. Let us recall that

$$\mathbf{E}Q_t^\alpha = \mathbf{E}\frac{d}{d\langle M\rangle_t}\int_0^t g(s,t)X_s^\alpha ds = \frac{d}{d\langle M\rangle_t}\int_0^t g(s,t)\mathbf{E}X_s^\alpha ds.$$

When

$$dX_t^\alpha = (\alpha - \vartheta X_t)dt + d\xi_t$$

we have

$$\mathbf{E}X_t^\alpha = \frac{\alpha}{\vartheta} - \frac{\alpha}{\vartheta}e^{-\vartheta t}.$$

It is obvious that when we calculate the limit of $\frac{1}{T}\int_0^T(\mathbf{E}Q_t^\alpha)^2 d\langle M\rangle_t$, the term $\frac{\alpha}{\vartheta}e^{-\vartheta t}$ has no contribution and will be 0. Now

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T(\mathbf{E}Q_t^\alpha)^2 d\langle M\rangle_t = \lim_{T\to\infty}\left(\frac{\alpha}{\vartheta}\right)^2\frac{1}{T}\langle M\rangle_T.$$

From [12], this limit will be 0 when $H > 1/2$ and $\left(\frac{\alpha}{\beta}\right)^2$ when $H < 1/2$. When

$$\int_0^T(\mathbf{E}\zeta_t^\alpha R(t))(\mathbf{E}\zeta_t^\alpha)^* d\langle M\rangle_t = \int_0^T\left(\frac{1}{2}\ell^*(t)\mathbf{E}\zeta_t^\alpha R(t)\right)^2 d\langle M\rangle_t = \int_0^T(\mathbf{E}Q_t^\alpha)^2 d\langle M\rangle_t,$$

the conclusion is true provided that

$$\lim_{T\to\infty}\left(\frac{\mu}{T}\int_0^t\varphi(t)\varphi^{-1}\Gamma(s)R(s)\mathcal{Z}(s)d\langle M\rangle_s\right)R(t)\left(\frac{\mu}{T}\int_0^t\varphi(t)\varphi^{-1}\Gamma(s)R(s)\mathcal{Z}(s)d\langle M\rangle_s\right)^* = 0$$

and this proof can be also found in the previous Lemma.

Remark 5.3. From the previous proof, we can see that the most difference compared with [12] is the extra part (Z(t) or $\mathcal{Z}(t)$) of the Laplace Transform coming from the function $u(t)$. Even in our two cases (optimal input and constant case) the extra part converges to 0 and does not have a decisive influence on the final result but we still can not ignore it. On the other hard, the limit of the main part is the sum of the uncontrolled mixed fractional O-U process and the additional part of $u(t)$.

Lemma 5.4. *For the controlled mixed fractional Ornstein-Uhlenbeck process with the drift parameter ϑ, we have the following limit:*

$$\mathcal{K}_T(\mu) = -\frac{\mu}{T}\log\mathbf{E}\exp\left(-\mu\int_0^T Q_t^2 d\langle M\rangle_t\right) \to \frac{\mu}{\vartheta^2} + \frac{\vartheta}{2} - \sqrt{\frac{\vartheta^2}{4} + \frac{\mu}{2}},\ T\to\infty.$$

for all $\mu > -\frac{\vartheta^2}{2}$.

Proof. This proof is directly from [19] and Lemma 5.2 or more specially, the term $\frac{\vartheta}{2} - \sqrt{\frac{\vartheta^2}{4} + \frac{\mu}{2}}$ comes from [19] and $\frac{1}{\vartheta^2}$ from Lemma 5.2. □

6. Conclusions

In this paper, we have considered the controlled drift parametric estimation of the mixed fractional Ornstein-Uhlenbeck process

$$dX_t = -\vartheta X_t dt + u(t)dt + dW_t + dB_t^H,\ \ H \in (0,1),\ H \neq 1/2.$$

First of all, we have found an explicit controlled function $u_{opt}(t)$ which maximize the Fisher information of the unknown drift parameter ϑ. Then under this special function we use the Laplace transform to compute the asymptotical normality and strong consistency of the maximum likelihood estimator. On the other hand, we use the same Laplace transform method to analyze the MLE of ϑ when $u(t)$ is a known constant.

Of course, when $u(t)$ is known we can find the solution of X_t and use relations between our controlled model and the uncontrolled one to study the MLE of ϑ such as presented in [7]. But there was no doubt that the direct Laplace transform is easier to understand and manipulate.

Remark 6.1. For the simulation of the MLE, even in the mixed fractional O-U process case with $u(t) = 0$ we do not have a proper method because the process $Q_t = \frac{d\int_0^t g(s,t)X_s ds}{d\langle M\rangle_t}$ is hard to simulate. However, we can use the one-step MLE to achieve this goal: that is with the initial Least Square Estimator (LSE) and local asymptotical property (LAN) of the process $X = (X_t, 0 \leq t \leq T)$. It will be our future work when the LAN property needs more tools such as the Malliavin calculus and Wick product.

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Conflict of interest

The authors declared that they have no conflicts of interest to this work.

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