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Research article

Darbo-Type $\mathcal{Z}_{\mathfrak{M}}$ and $\mathcal{L}_{\mathfrak{M}}$ contractions and its applications to Caputo fractional integro-differential equations

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Abstract: Darbo type $\mathcal{Z}_{\mathfrak{M}}$-contraction and Darbo type $\mathcal{L}_{\mathfrak{M}}$-contraction are introduced and some fixed point results are established for such contraction mappings. As an application, we prove the existence of solution of a Caputo fractional Volterra-Fredholm integro-differential equation via integral type boundary conditions and verify the validity of our application by an appropriate example.

Keywords: measure of noncompactness; fractional integro-differential equation; simulation function; $\mathcal{Z}_{\mathfrak{M}}$-contraction; $\mathcal{L}_{\mathfrak{M}}$-contraction; fixed point
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1. Introduction

Let $\mathbb{R}$ and $\mathbb{R}_+$ denote the sets of all real numbers and nonnegative real numbers respectively, $\mathbb{N}$ the set of all positive integers and $\overline{\mathbb{A}}$ the closure and $\overline{co}\mathbb{A}$ the convex hull closure of $\mathbb{A}$. Additionally, Ξ denotes a Banach space, $\Omega = \{\Lambda : \Lambda \neq \emptyset$, bounded, closed and convex subset of $\Xi\}$, $\mathfrak{B}(\Xi) = \{\Lambda \neq \emptyset : \Lambda$ is bounded subset of $\Xi\}$, $\ker \mathfrak{M} = \{\Lambda \in \mathfrak{B}(\Xi) : \mathfrak{M}(\Lambda) = 0\}$ be the kernel of function $\mathfrak{M} : \mathfrak{B}(\Xi) \to \mathbb{R}_+$.

Fixed point theory has been developed in two directions. One deals with contraction mappings on metric spaces, Banach contraction principle being the first important result in this direction. In the second direction, continuous operators are dealt with convex and compact subsets of a Banach space.

Brouwer's fixed point theorem and its infinite dimensional form, Schuader's fixed point theorems are the two important theorems in this second direction. In this paper, $Fix(\Upsilon)$ denotes a set of fixed points of a mapping Υ in Λ.

Theorem 1.1 (Brouwer's Fixed Point Theorem)**.** *[2] Every continuous mapping from the unit ball of $\mathbb{R}^n$ into itself has a fixed point.*

Theorem 1.2 (Schauder's Fixed Point Theorem)**.** *[15] Let $\Upsilon : \Lambda \to \Lambda$ a compact continuous operator, where $\Lambda \in \mathfrak{K}$. Then $Fix(\Upsilon) \neq \emptyset$.*

In Brouwer's and Schuader's fixed point theorems, compactness of the space under consideration is required as a whole or as a part. However, later the requirement of the compactness was relaxed by making use of the notion of a measure of noncompactness (in short MNC). Using the notion of MNC, the following theorem was proved by Darbo [5].

Theorem 1.3. *[5] Let $\Lambda \in \mathfrak{K}$ and $\Upsilon : \Lambda \to \Lambda$ be a continuous function. If there exists $k \in [0, 1)$ such that*

$$\mathfrak{M}(\Upsilon(\Lambda_0)) \leqslant k\mathfrak{M}(\Lambda_0),$$

where $\Lambda_0 \subset \Lambda$ and $\mathfrak{M}$ is MNC defined on Ξ. Then $Fix(\Upsilon) \neq \emptyset$.

It generalizes the renowned Schuader fixed point result and includes the existence portion of Banach contraction principle. In the sequel many extensions and generalizations of Darbo's theorem came into existence.

The Banach principle has been improved and extended by several researchers (see [7, 13, 14, 16]). Jleli and Samet [7] introduced the notion of θ-contractions and gave a generalization of the Banach contraction principle in generalized metric spaces, where $\theta : (0, \infty) \to (1, \infty)$ is such that:

(θ_1) θ is non-decreasing;
(θ_2) for every sequence $\{\kappa_j\} \subset (0, \infty)$, we have

$$\lim_{j\to\infty} \theta(\kappa_j) = 1 \Longleftrightarrow \lim_{j\to\infty} \kappa_j = 0^+;$$

(θ_3) there exists $L \in (0, \infty)$ and $\ell \in (0, 1)$ such that

$$\lim_{\kappa\to 0^+} \frac{\theta(\kappa) - 1}{\kappa^\ell} = L.$$

Khojasteh *et al.* [9] introduced the concept of $\mathcal{Z}$-contraction using simulation functions and established fixed point results for such contractions. Isik *et al.* [6] defined almost $\mathcal{Z}$-contractions and presented fixed point theorems for such contractions. Cho [3] introduced the notion of $\mathcal{L}$-contractions, and proved fixed point results under such contraction in generalized metric spaces. Using specific form of $\mathscr{Z}$ and $\mathscr{L}$, we can deduce other known existing contractions. For some results concerning $\mathcal{Z}$-contractions and its generalizations we refer the reader to [3] and the references cited therein. In particular, Chen and Tang [4] generalized $\mathcal{Z}$-contraction with $\mathcal{Z}_{\mathfrak{M}}$-contraction and established Darbo type fixed point results.

The aim of the present work is two fold. First we prove fixed point theorems under generalized $\mathcal{Z}_{\mathfrak{M}}$-contraction and then we prove fixed point results under Darbo type $\mathcal{L}_{\mathfrak{M}}$-contraction in Banach spaces.

It is interesting to see that several existing results in fixed point theory can be concluded from our main results. Furthermore, as an application of our results, we have proved the existence of solution to the Caputo fractional Volterra–Fredholm integro-differential equation

$$^{c}D^{\wp}\mu(\varkappa) = \mathfrak{g}(\varkappa) + \lambda_1 \int_0^{\varkappa} \mathfrak{I}_1(\varkappa, t)\, \xi_1(t, \mu(t))\, dt + \lambda_2 \int_0^{\mathfrak{T}} \mathfrak{I}_2(\varkappa, t)\, \xi_2(t, \mu(t))\, dt,$$

under boundary conditions:

$$a\mu(0) + b\mu(\mathfrak{T}) = \frac{1}{\Gamma(\wp)} \int_0^{\mathfrak{T}} (\mathfrak{T} - t)^{\wp - 1} \mathfrak{I}_3(\varkappa, t) dt,$$

where $\wp \in (0, 1]$, ^{c}D is the Caputo fractional derivative, λ_1, λ_2 are parameters, and $a, b > 0$ are real constants, $\mu, \mathfrak{g} : [0, \mathfrak{T}] \to \mathbb{R}$, $\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_3 : [0, \mathfrak{T}] \times [0, \mathfrak{T}] \to \mathbb{R}$ and $\xi_1, \xi_2 : [0, \mathfrak{T}] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. For the validity of existence result we construct an example.

2. Preliminaries

In this section, we recall some definitions and results which are further considered in the next sections, allowing us to present the results. The concept of MNC was introduced in [1] as follows:

Definition 2.1. [1] A map $\mathfrak{M}: \mathfrak{B}(\Xi) \to \mathbb{R}_+$ is MNC in Ξ if for all $\Lambda_1, \Lambda_2 \in \mathfrak{B}(\Xi)$ it satisfies the following conditions:

(i) $\ker \mathfrak{M} \neq \emptyset$ and relatively compact in Ξ;
(ii) $\Lambda_1 \subset \Lambda_2 \;\Rightarrow\; \mathfrak{M}(\Lambda_1) \leqslant \mathfrak{M}(\Lambda_2)$;
(iii) $\mathfrak{M}(\overline{\Lambda_1}) = \mathfrak{M}(\Lambda_1)$;
(iv) $\mathfrak{M}(\overline{co}\Lambda_1) = \mathfrak{M}(\Lambda_1)$;
(v) $\mathfrak{M}(\eta\Lambda_1 + (1 - \eta)\Lambda_2) \leqslant \eta\mathfrak{M}(\Lambda_1) + (1 - \eta)\mathfrak{M}(\Lambda_2)\ \forall \eta \in [0, 1]$;
(vi) if $\{\Lambda_n\}$ is a sequence of closed sets in $\mathfrak{B}(\Xi)$ with $\Lambda_{n+1} \subset \Lambda_n$, $\forall n \in \mathbb{N}$ and $\lim_{n\to+\infty} \mathfrak{M}(\Lambda_n) = 0$, then $\Lambda_\infty = \bigcap_{n=1}^{+\infty} \Lambda_n \neq \emptyset$.

The Kuratowski MNC [11] is the function $\mathfrak{M}: \mathfrak{B}(\Xi) \to \mathbb{R}_+$ defined by

$$\mathfrak{M}(\mathcal{K}) = \inf \left\{ \varepsilon > 0 : \mathcal{K} \subset \bigcup_{i=1}^{n} \mathcal{S}_i, \mathcal{S}_i \subset \Xi, \operatorname{diam}(\mathcal{S}_i) < \varepsilon \right\},$$

where $\operatorname{diam}(S)$ is the diameter of S.

Khojasteh *et al.* [9] introduced the concept of $\mathcal{Z}$-contraction using simulation functions as follows:

Definition 2.2. A function $\mathscr{Z} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is simulation if:

($\mathscr{Z}_1$) $\mathscr{Z}(0, 0) = 0$;
($\mathscr{Z}_2$) $\mathscr{Z}(\kappa_1, \kappa_2) < \kappa_2 - \kappa_1$, for all $\kappa_1, \kappa_2 > 0$;
($\mathscr{Z}_3$) if $\{\kappa_n^*\}$ and $\{\kappa_n\}$ are two sequences in $(0, \infty)$ such that $\lim_{n\to\infty} \kappa_n^* = \lim_{n\to\infty} \kappa_n > 0$, then

$$\lim_{n\to\infty} \mathscr{Z}(\kappa_n^*, \kappa_n) < 0.$$

Roldán-López-de-Hierro *et al.* [12] slightly modified the Definition 2.2 of [9] as follows.

Definition 2.3. A mapping $\mathscr{Z} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is simulation if:

($\mathscr{Z}_1$) $\mathscr{Z}(0,0) = 0$;
($\mathscr{Z}_2$) $\mathscr{Z}(\kappa_1,\kappa_2) < \kappa_2 - \kappa_1$, for all $\kappa_1,\kappa_2 > 0$;
($\mathscr{Z}_3$) if $\{\kappa_n^*\}$, $\{\kappa_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} \kappa_n^* = \lim_{n\to\infty} \kappa_n > 0$ and $\kappa_n < \kappa_n^*$, then

$$\lim_{n\to\infty} \mathscr{Z}(\kappa_n^*,\kappa_n) < 0.$$

Every simulation function in the original Definition 2.2 is also a simulation function in the sense of Definition 2.3, but the converse is not true, see for instance [12]. Note that $\mathcal{Z} = \{\mathscr{Z} : \mathscr{Z}$ is a simulation function in the sense of Definition 2.3$\}$. The following are some examples of simulation functions.

Example 2.4. The mapping $\mathscr{Z} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ defined by:

1. $\mathscr{Z}(\kappa_1,\kappa_2) = \kappa_2 - f(\kappa_1) - \kappa_1$, for all $\kappa_1,\kappa_2 \in \mathbb{R}_+$, where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a lower semi-continuous function such that $f^{-1}(0) = \{0\}$,
2. $\mathscr{Z}(\kappa_1,\kappa_2) = \kappa_2 - \varphi(\kappa_1) - \kappa_1$, for all $\kappa_1,\kappa_2 \in \mathbb{R}_+$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that $\varphi(\kappa_1) = \{0\} \Leftrightarrow \kappa_1 = 0$,
3. $\mathscr{Z}(\kappa_1,\kappa_2) = \kappa_2 \curlyvee (\kappa_2) - \kappa_1$, for all $\kappa_1,\kappa_2 \in \mathbb{R}_+$, where $\curlyvee : \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that $\limsup_{n\to r^+} \curlyvee(\kappa_1) < 1$,
4. $\mathscr{Z}(\kappa_1,\kappa_2) = \phi(\kappa_2) - \kappa_1$, for all $\kappa_1,\kappa_2 \in \mathbb{R}_+$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is an upper semi-continuous function such that $\phi(\kappa_1) < \kappa_1$, for all $\kappa_1 > 0$ and $\phi(0) = 0$,
5. $\mathscr{Z}(\kappa_1,\kappa_2) = \kappa_2 - \int_0^{\kappa_1} \lambda(x)dx$, for all $\kappa_1,\kappa_2 \in \mathbb{R}_+$, where $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that $\int_0^{\epsilon} \lambda(x)dx$ exists and $\int_0^{\epsilon} \lambda(x)dx > \epsilon$, for every $\epsilon > 0$,
6. $\mathscr{Z}(\kappa_1,\kappa_2) = \frac{\kappa_2}{\kappa_2+1} - \kappa_1$, for all $\kappa_1,\kappa_2 \in \mathbb{R}_+$,

are simulation functions.

Cho [3] introduced the notion of $\mathcal{L}$-simulation function as follows:

Definition 2.5. An $\mathcal{L}$-simulation function is a function $\mathscr{L} : [1,\infty) \times [1,\infty) \to \mathbb{R}$ satisfying the following conditions:

($\mathscr{L}_1$) $\mathscr{L}(1,1) = 1$;
($\mathscr{L}_2$) $\mathscr{L}(\kappa_1,\kappa_2) < \frac{\kappa_2}{\kappa_1}$, for all $\kappa_1,\kappa_2 > 1$;
($\mathscr{L}_3$) if $\{\kappa_n\}$ and $\{\kappa_n^*\}$ are two sequences in $(1,\infty)$ such that $\lim_{n\to\infty} \kappa_n = \lim_{n\to\infty} \kappa_n^* > 1$ and $\kappa_n < \kappa_n^*$, then

$$\lim_{n\to\infty} \mathscr{L}(\kappa_n,\kappa_n^*) < 1.$$

Note that $\mathscr{L}(1,1) < 1$, for all $t > 1$.

Example 2.6. The functions $\mathscr{L}_b, \mathscr{L}_w : [1,\infty) \times [1,\infty) \to \mathbb{R}$ defined by

1. $\mathscr{L}_b(\kappa_1, \kappa_2) = \frac{\kappa_2^{\mathfrak{n}}}{\kappa_1}$, for all $\kappa_1, \kappa_2 \geqslant 1$, where $\mathfrak{n} \in (0, 1)$;
2. $\mathscr{L}_w(\kappa_1, \kappa_2) = \frac{\kappa_2}{\kappa_1 \phi(\kappa_2)}$ $\forall \kappa_1, \kappa_2 \geqslant 1$, where $\phi : [1, \infty) \to [1, \infty)$ is lower semicontinuous and nondecreasing with $\phi^{-1}(\{1\}) = 1$,

are $\mathcal{L}$-simulation functions.

Definition 2.7. [8] A continuous non-decreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(t) = 0$ if and only if $t = 0$ is called an altering distance function.

3. Some results on $\mathcal{Z}_{\mathfrak{M}}$-contractions

In this section, we obtain the results on generalized $\mathcal{Z}_{\mathfrak{M}}$-contraction. First we give the following definition.

Definition 3.1. Let $\Lambda \in \mathfrak{K}$. A self-mapping Υ on Λ is called generalized $\mathcal{Z}_{\mathfrak{M}}$-contraction if there exists $\mathscr{X} \in \mathcal{Z}$ such that

$$\mathscr{X}(\mathfrak{M}(\Upsilon(\Lambda_1)), \Delta(\Lambda_1, \Lambda_2)) \geqslant 0, \tag{3.1}$$

where Λ_1 and Λ_2 are subsets of Λ, $\mathfrak{M}(\Lambda_1), \mathfrak{M}(\Upsilon(\Lambda_1)), \mathfrak{M}(\Upsilon(\Lambda_2)) > 0$, $\mathfrak{M}$ is MNC defined in Ξ and

$$\Delta(\Lambda_1, \Lambda_2) = \max\left\{\mathfrak{M}(\Lambda_1), \mathfrak{M}(\Upsilon(\Lambda_1)), \mathfrak{M}(\Upsilon(\Lambda_2)), \frac{1}{2}\mathfrak{M}(\Upsilon(\Lambda_1) \cup \Upsilon(\Lambda_2))\right\}.$$

Using the notion of $\mathcal{Z}_{\mathfrak{M}}$-contraction, we establish the main result of this section.

Theorem 3.2. *Let $\Upsilon : \Lambda \to \Lambda$ be a continuous function, where $\Lambda \in \mathfrak{K}$. Assume that there exists $\mathscr{X} \in \mathcal{Z}$ such that $\mathscr{X}$ non-decreasing function and Υ is a generalized $\mathcal{Z}_{\mathfrak{M}}$-contraction. Then $Fix(\Upsilon) \neq \emptyset$.*

Proof. Define a sequence $\{\Lambda_n\}_{n=0}^{\infty}$ such that

$$\Lambda_0 = \Lambda \text{ and } \Lambda_n = \overline{co}(\Upsilon\Lambda_{n-1}), \text{ for all } n \in \mathbb{N}. \tag{3.2}$$

We need to prove that $\Lambda_{n+1} \subset \Lambda_n$ and $\Upsilon\Lambda_n \subset \Lambda_n$, for all $n \in \mathbb{N}$. For proof of the first inclusion, we use induction. If $n = 1$, then by (3.2), we have $\Lambda_0 = \Lambda$ and $\Lambda_1 = \overline{co}(\Upsilon\Lambda_0) \subset \Lambda_0$. Next, assume that $\Lambda_n \subset \Lambda_{n-1}$, then $\overline{co}(\Upsilon(\Lambda_n)) \subset \overline{co}(\Upsilon(\Lambda_{n-1}))$, using (3.2), we get the first inclusion

$$\Lambda_{n+1} \subset \Lambda_n. \tag{3.3}$$

To obtained the second inclusion, using the inclusion (3.3) we have

$$\Upsilon\Lambda_n \subset \overline{co}(\Upsilon\Lambda_n) = \Lambda_{n+1} \subset \Lambda_n. \tag{3.4}$$

Thus $\Lambda_{n+1} \subset \Lambda_n$ and $\Upsilon\Lambda_n \subset \Lambda_n, \forall n \in \mathbb{N}$.

Now, we discuss two cases, depending on the values of $\mathfrak{M}$. If we consider m as a non-negative integer with $\mathfrak{M}(\Lambda_m) = 0$, then Λ_m is a compact set and hence by Theorem 1.2, Υ has a fixed point in $\Lambda_m \subset \Lambda$. Instead, assume that $\mathfrak{M}(\Lambda_n) > 0$, $\forall n \in \mathbb{N}$. Then on setting $\Lambda_1 = \Lambda_{n+1}$ and $\Lambda_2 = \Lambda_n$ in contraction (3.1), we have

$$\mathscr{X}(\mathfrak{M}(\Upsilon(\Lambda_{n+1})), \Delta(\Lambda_{n+1}, \Lambda_n)) \geqslant 0, \tag{3.5}$$

where

$$\begin{aligned}\Delta(\Lambda_n,\Lambda_{n+1}) &= \max\left\{\mathfrak{M}(\Lambda_n),\mathfrak{M}(\Upsilon(\Lambda_n)),\mathfrak{M}(\Upsilon(\Lambda_{n+1})),\frac{1}{2}\mathfrak{M}(\Upsilon(\Lambda_n)\cup\Upsilon(\Lambda_{n+1}))\right\}\\ &\leqslant \max\left\{\mathfrak{M}(\Lambda_n),\mathfrak{M}(\Lambda_n),\mathfrak{M}(\Lambda_{n+1}),\frac{1}{2}\mathfrak{M}(\Lambda_n\cup\Lambda_{n+1})\right\}\\ &= \max\left\{\mathfrak{M}(\Lambda_n),\mathfrak{M}(\Lambda_n),\mathfrak{M}(\Lambda_{n+1}),\frac{1}{2}\mathfrak{M}(\Lambda_n)\right\}\\ &=\mathfrak{M}(\Lambda_n),\end{aligned}$$

that is,

$$\Delta(\Lambda_n,\Lambda_{n+1}) \leqslant\mathfrak{M}(\Lambda_n). \tag{3.6}$$

Using inequality (3.6) and the axiom $(\mathscr{Z}_2)$ of $\mathscr{Z}$-simulation function, inequality (3.5) becomes,

$$\begin{aligned}0 &\leqslant\mathscr{Z}(\mathfrak{M}(\Upsilon(\Lambda_{n+1})),\Delta(\Lambda_{n+1},\Lambda_n))\\ &\leqslant\mathscr{Z}(\mathfrak{M}(\Lambda_{n+1}),\mathfrak{M}(\Lambda_n))\\ &\leqslant\mathfrak{M}(\Lambda_n)-\mathfrak{M}(\Lambda_{n+1}),\end{aligned} \tag{3.7}$$

that is, $\mathfrak{M}(\Lambda_n) \geqslant \mathfrak{M}(\Lambda_{n+1})$ and hence, $\{\mathfrak{M}(\Lambda_n)\}$ is a decreasing sequence of positive real numbers. Thus, we can find $r \geqslant 0$ such that $\lim_{n\to\infty} \mathfrak{M}(\Lambda_n) = r$. Next, we claim that $r = 0$. To support our claim, suppose that $r \neq 0$, that is, $r > 0$. Let $u_n = \mathfrak{M}(\Lambda_{n+1})$ and $v_n = \mathfrak{M}(\Lambda_n)$, then since $u_n < v_n$, so by the axiom $(\mathscr{Z}_3)$ of $\mathscr{Z}$-simulation function, we have

$$\limsup_{n\to\infty} \mathscr{Z}(\mathfrak{M}(\Lambda_{n+1}),\mathfrak{M}(\Lambda_n)) = \limsup_{n\to\infty} \mathscr{Z}(u_n, v_n) < 0,$$

which is contradiction to (3.7). Thus $r = 0$ and hence $\{\Lambda_n\}$ is a sequence of closed sets in $\mathfrak{B}(\Xi)$ with $\Lambda_{n+1} \subset \Lambda_n$, for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mathfrak{M}(\Lambda_n) = 0$, so the intersection set $\Lambda_\infty = \bigcap_{n=1}^{+\infty} \Lambda_n$ is non-empty, closed and convex subset of Λ. Furthermore, since $\Lambda_\infty \subset \Lambda_n$, for all $n \in \mathbb{N}$, so by Definition 2.1(ii), $\mathfrak{M}(\Lambda_\infty) \leqslant \mathfrak{M}(\Lambda_n)$, for all $n \in \mathbb{N}$. Thus $\mathfrak{M}(\Lambda_\infty) = 0$ and hence $\Lambda_\infty \in \ker\mathfrak{M}$, that is , Λ_∞ is bounded. But Λ_∞ is closed so that Λ_∞ is compact. Therefore by Theorem 1.2, $Fix(\Upsilon) \neq \emptyset$. □

From Theorem 3.2 we obtain the following corollaries. We assume that $\Lambda \in \mathfrak{K}$.

Corollary 3.3. *Let $\Upsilon : \Lambda \to \Lambda$ be a continuous function such that*

$$\psi_1 (\mathfrak{M} (\Upsilon(\Lambda_1))) \leqslant \psi_2 \left(\max\left\{\mathfrak{M}(\Lambda_1),\mathfrak{M}(\Upsilon(\Lambda_1)),\mathfrak{M}(\Upsilon(\Lambda_2)),\frac{1}{2}\mathfrak{M}(\Upsilon(\Lambda_1)\cup\Upsilon(\Lambda_2))\right\}\right),$$

for any non-empty subsets Λ_1 and Λ_2 of Λ, where $\mathfrak{M}$ is MNC. Then $Fix(\Upsilon) \neq \emptyset$.

Corollary 3.4. *Let $\Upsilon : \Lambda \to \Lambda$ be a continuous function such that*

$$\mathfrak{M} (\Upsilon(\Lambda_1)) \leqslant \Delta(\Lambda_1,\Lambda_2) - f (\Delta(\Lambda_1,\Lambda_2)),$$

for any non-empty subsets Λ_1 and Λ_2 of Λ, where $\mathfrak{M}$ is MNC and $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a lower semi-continuous function such that $f^{-1}(0) = \{0\}$. Then $Fix(\Upsilon) \neq \emptyset$.

The conclusion of Corollary 3.4 is true if $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous function such that $f(t) = 0 \Leftrightarrow t = 0$.

Corollary 3.5. *Let $\Upsilon : \Lambda \to \Lambda$ be a continuous function such that*

$$\mathfrak{M}(\Upsilon(\Lambda_1)) \leqslant \curlyvee(\Delta(\Lambda_1, \Lambda_2))\,\Delta(\Lambda_1, \Lambda_2),$$

for any non-empty subsets Λ_1 and Λ_2 of Λ, where $\mathfrak{M}$ is MNC and $\curlyvee : \mathbb{R}_+ \to \mathbb{R}_+$ is a function with $\limsup_{n\to r^+} \curlyvee(t) < 1$. Then $Fix(\Upsilon) \neq \emptyset$.

Corollary 3.6. *Let $\Upsilon : \Lambda \to \Lambda$ be a continuous function such that*

$$\mathfrak{M}(\Upsilon(\Lambda_1)) \leqslant \phi\left(\max\left\{\mathfrak{M}(\Lambda_1), \mathfrak{M}(\Upsilon(\Lambda_1)), \mathfrak{M}(\Upsilon(\Lambda_2)), \frac{1}{2}\mathfrak{M}(\Upsilon(\Lambda_1) \cup \Upsilon(\Lambda_2))\right\}\right),$$

for any non-empty subsets Λ_1 and Λ_2 of Λ, where $\mathfrak{M}$ is MNC and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is an upper semi-continuous mapping with $\phi(t) < t$, $\forall t > 0$ and $\phi(0) = 0$. Then $Fix(\Upsilon) \neq \emptyset$.

Corollary 3.7. *Let $\Upsilon : \Lambda \to \Lambda$ be a continuous function such that*

$$\int_0^{\mathfrak{M}(\Upsilon(\Lambda_1))} \lambda(x)dx \leqslant \max\left\{\mathfrak{M}(\Lambda_1), \mathfrak{M}(\Upsilon(\Lambda_1)), \mathfrak{M}(\Upsilon(\Lambda_2)), \frac{1}{2}\mathfrak{M}(\Upsilon(\Lambda_1) \cup \Upsilon(\Lambda_2))\right\},$$

for any non-empty subsets Λ_1 and Λ_2 of Λ, where $\mathfrak{M}$ is MNC and $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is a mapping such that $\int_0^{\epsilon} \lambda(x)dx$ exists and $\int_0^{\epsilon} \lambda(x)dx > \epsilon$, for every $\epsilon > 0$. Then $Fix(\Upsilon) \neq \emptyset$.

Corollary 3.8. *Let $\Upsilon : \Lambda \to \Lambda$ be a continuous function such that*

$$\mathfrak{M}(\Upsilon(\Lambda_1)) \leqslant \frac{\Delta(\Lambda_1, \Lambda_2)}{1 + \Delta(\Lambda_1, \Lambda_2)},$$

for any non-empty subsets Λ_1 and Λ_2 of Λ, where $\mathfrak{M}$ is MNC. Then $Fix(\Upsilon) \neq \emptyset$.

4. Some results on $\mathcal{L}_{\mathfrak{M}}$-contraction

In this section, we obtain some results on $\mathcal{L}_{\mathfrak{M}}$-contraction. Let us denote by Θ the class of all functions $\theta : (0, \infty) \to (1, \infty)$ that satisfy conditions (θ_1) and (θ_2). First we introduce the notion of $\mathcal{L}_{\mathfrak{M}}$-contraction as:

Definition 4.1. Let $\Lambda^* \in \mathfrak{K}$. A self-mapping Υ on Λ^* is called $\mathcal{L}_{\mathfrak{M}}$-contraction with respect to $\mathscr{L}$ if there exist $\theta \in \Theta$ such that, for all $\Lambda \subset \Lambda^*$ with $\mathfrak{M}(\Lambda) > 0$,

$$\mathscr{L}(\theta(\mathfrak{M}(\Upsilon(\Lambda))), \theta(\mathfrak{M}(\Lambda))) \geqslant 1, \tag{4.1}$$

where $\mathfrak{M}$ is MNC defined in Ξ.

Theorem 4.2. *Let $\Upsilon : \Lambda \to \Lambda$ be a continuous function. Assume that there exist $\theta \in \Theta$ and Υ is $\mathcal{L}_{\mathfrak{M}}$-contraction with respect to $\mathscr{L}$. Then $Fix(\Upsilon) \neq \emptyset$.*

Proof. Define a sequence $\{\Lambda_n\}_{n=0}^{\infty}$ such that

$$\Lambda_0 = \Lambda \text{ and } \Lambda_n = \overline{co}(\Upsilon\Lambda_{n-1}), \text{ for all } n \in \mathbb{N}. \tag{4.2}$$

Then $\Lambda_{n+1} \subset \Lambda_n$ and $\Upsilon\Lambda_n \subset \Lambda_n$, $\forall n \in \mathbb{N}$.

Now, we discuss two cases, depending on the values of $\mathfrak{M}$. If we consider m as a non-negative integer with $\mathfrak{M}(\Lambda_m) = 0$, then Λ_m is a compact set and hence by Theorem 1.2, Υ has a fixed point in $\Lambda_m \subset \Lambda$. Instead, assume that $\mathfrak{M}(\Lambda_n) > 0$, for all $n \in \mathbb{N}$. Then on setting $\Lambda = \Lambda_n$ in contraction (4.1), we have

$$1 \leqslant \mathscr{L}(\theta(\mathfrak{M}(\Upsilon(\Lambda_n))), \theta(\mathfrak{M}(\Lambda_n))) \leqslant \frac{\theta(\mathfrak{M}(\Lambda_n))}{\theta(\mathfrak{M}(\Upsilon(\Lambda_n)))}, \tag{4.3}$$

that is,

$$\theta(\mathfrak{M}(\Upsilon(\Lambda_n))) \leqslant \theta(\mathfrak{M}(\Lambda_n)).$$

Since θ is nondecreasing, so that

$$\mathfrak{M}(\Upsilon(\Lambda_n)) \leqslant \mathfrak{M}(\Lambda_n). \tag{4.4}$$

Now, using inequality (4.4), we have

$$\begin{aligned} \mathfrak{M}(\Lambda_{n+1}) &= \mathfrak{M}(\overline{co}(\Upsilon(\Lambda_n))) \\ &= \mathfrak{M}(\Upsilon(\Lambda_n)) \\ &\leqslant \mathfrak{M}(\Lambda_n), \end{aligned}$$

that is, $\mathfrak{M}(\Lambda_{n+1}) \leqslant \mathfrak{M}(\Lambda_n)$ and hence, $\{\mathfrak{M}(\Lambda_n)\}$ is a decreasing sequence of positive real numbers. Thus, we can find $r \geqslant 0$ with $\lim\limits_{n\to\infty} \mathfrak{M}(\Lambda_n) = r$. Next, we claim that $r = 0$. To support our claim, suppose that $r \neq 0$. Then in view of (θ_2), we get

$$\lim_{n\to\infty} \theta(\mathfrak{M}(\Lambda_n)) \neq 1,$$

which implies that

$$\lim_{n\to\infty} \theta(\mathfrak{M}(\Lambda_n)) > 1. \tag{4.5}$$

Let $u_n = \theta(\mathfrak{M}(\Lambda_{n+1}))$ and $v_n = \theta(\mathfrak{M}(\Lambda_n))$, then since $u_n \leqslant v_n$, so by the axiom $(\mathscr{L}_3)$ of $\mathscr{L}$-simulation function, we have

$$1 \leqslant \limsup_{n\to\infty} \mathscr{L}(\mathfrak{M}(\Lambda_{n+1}), \mathfrak{M}(\Lambda_n)) = \limsup_{n\to\infty} \mathscr{L}(u_n, v_n) < 1,$$

which is contradiction. Thus $r = 0$ and hence $\{\Lambda_n\}$ is a sequence of closed sets from $\mathfrak{B}(\Xi)$ such that $\Lambda_{n+1} \subset \Lambda_n$, for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \mathfrak{M}(\Lambda_n) = 0$, so the intersection set $\Lambda_\infty = \bigcap_{n=1}^{+\infty} \Lambda_n$ is non-empty, closed and convex subset of Λ. Furthermore, since $\Lambda_\infty \subset \Lambda_n$, for all $n \in \mathbb{N}$, so by Definition 2.1(ii), $\mathfrak{M}(\Lambda_\infty) \leqslant \mathfrak{M}(\Lambda_n)$, for all $n \in \mathbb{N}$. Thus $\mathfrak{M}(\Lambda_\infty) = 0$ and hence $\Lambda_\infty \in \ker \mathfrak{M}$, that is, Λ_∞ is bounded. But Λ_∞ is closed so that Λ_∞ is compact. Therefore by Theorem 1.2, $Fix(\Upsilon) \neq \emptyset$. □

By taking $\mathscr{L} = \mathscr{L}_b$ in Theorem 4.2, we obtain the following result.

Corollary 4.3. *Let $\Upsilon : \Lambda^* \to \Lambda^*$ be a continuous functions such that, for all $\Lambda \subset \Lambda^*$ with $\mathfrak{M}(\Lambda) > 0$,*

$$\theta(\mathfrak{M}(\Upsilon(\Lambda))) \leqslant (\theta(\mathfrak{M}(\Lambda)))^k, \tag{4.6}$$

where $\theta \in \Theta$ and $k \in (0, 1)$. Then $Fix(\Upsilon) \neq \varnothing$.

Remark 4.4. Corollary 4.3 is the Darbo type version of Theorem 2.1 in [7].

By taking $\mathscr{L} = \mathscr{L}_w$ in Theorem 4.2, we obtain the next result.

Corollary 4.5. *Let $\Upsilon : \Lambda^* \to \Lambda^*$ be a continuous functions such that, for all $\Lambda \subset \Lambda^*$ with $\mathfrak{M}(\Lambda) > 0$,*

$$\theta(\mathfrak{M}(\Upsilon(\Lambda))) \leqslant \frac{\theta(\mathfrak{M}(\Lambda))}{\phi(\theta(\mathfrak{M}(\Lambda)))}, \tag{4.7}$$

where $\theta \in \Theta$ and $\phi : [1, \infty) \to [1, \infty)$ is lower semi-continuous and nondecreasing with $\phi^{-1}(\{1\}) = 1$. Then $Fix(\Upsilon) \neq \varnothing$.

By taking $\theta(t) = e^t$, for all $t > 0$ in Corollary 4.5, we obtain next result.

Corollary 4.6. *Let $\Upsilon : \Lambda^* \to \Lambda^*$ be a continuous functions such that, for all $\Lambda \subset \Lambda^*$ with $\mathfrak{M}(\Lambda) > 0$,*

$$\mathfrak{M}(\Upsilon(\Lambda)) \leqslant \mathfrak{M}(\Lambda) - \varphi(\mathfrak{M}(\Lambda)), \tag{4.8}$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is lower semi-continuous and nondecreasing with $\varphi^{-1}(\{0\}) = 0$. Then $Fix(\Upsilon) \neq \varnothing$.

Remark 4.7. Corollary 4.6 is the Rhoades's Theorem of Darbo type [13].

5. Some applications

Let $B(a, r)$ be the closed ball with center at a and radius r and B_r be the ball $B(0, r)$. We check the existence of solution to Caputo fractional Volterra–Fredholm integro differential equation

$$^cD^{\wp}\mu(\varkappa) = \mathfrak{g}(\varkappa) + \lambda_1 \int_0^{\varkappa} \mathfrak{J}_1(\varkappa, t)\, \xi_1(t, \mu(t))\, dt + \lambda_2 \int_0^{\mathfrak{T}} \mathfrak{J}_2(\varkappa, t)\, \xi_2(t, \mu(t))\, dt, \tag{5.1}$$

under boundary conditions:

$$a\mu(0) + b\mu(\mathfrak{T}) = \frac{1}{\Gamma(\wp)} \int_0^{\mathfrak{T}} (\mathfrak{T} - t)^{\wp - 1} \mathfrak{J}_3(\varkappa, t) dt, \tag{5.2}$$

where $\wp \in (0, 1]$, ^cD is the Caputo fractional derivative, λ_1, λ_2 are parameters, and $a, b > 0$ are real constants, $\mu, \mathfrak{g} : [0, \mathfrak{T}] \to \mathbb{R}$, $\mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3 : [0, \mathfrak{T}] \times [0, \mathfrak{T}] \to \mathbb{R}$ and $\xi_1, \xi_2 : [0, \mathfrak{T}] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Lemma 5.1. *[10] For $p_l \in \mathbb{R}$, $l = 0, 1, \ldots, r - 1$, we have*

$$I^{\wp}[^cD^{\wp}\mathfrak{h}(\mathrm{t})] = \mathfrak{h}(\mathrm{t}) + p_0 + p_1\mathrm{t} + p_2\mathrm{t}^2 + ... + p_{r-1}\mathrm{t}^{r-1}.$$

Using Lemma 5.1, we can easily establish the following result.

Lemma 5.2. *Problem* (5.1) *is equivalent to the integral equation*

$$\begin{aligned}\mu(\varkappa) =&\frac{1}{\Gamma(\wp)}\int_0^{\varkappa}(\varkappa-\vartheta)^{\wp-1}\mathfrak{g}(\vartheta)d\vartheta-\frac{b}{(a+b)\Gamma(\wp)}\int_0^{\mathfrak{T}}(\mathfrak{T}-\vartheta)^{\wp-1}\mathfrak{g}(\vartheta)d\vartheta\\&+\frac{1}{\Gamma(\wp)}\int_0^{\varkappa}(\varkappa-\vartheta)^{\wp-1}\left(\lambda_1\int_0^{\vartheta}\mathfrak{J}_1(\vartheta,t)\,\xi_1(t,\mu(t))\,dt+\lambda_2\int_0^{\mathfrak{T}}\mathfrak{J}_2(\vartheta,t)\,\xi_2(t,\mu(t))\,dt\right)d\vartheta\\&-\frac{b}{(a+b)\Gamma(\wp)}\int_0^{\mathfrak{T}}(\mathfrak{T}-\vartheta)^{\wp-1}\bigg(\lambda_1\int_0^{\vartheta}\mathfrak{J}_1(\vartheta,t)\,\xi_1(t,\mu(t))\,dt\\&+\lambda_2\int_0^{\mathfrak{T}}\mathfrak{J}_2(\vartheta,t)\,\xi_2(t,\mu(t))\,dt\bigg)d\vartheta+\frac{1}{(a+b)\Gamma(\wp)}\int_0^{\mathfrak{T}}(\mathfrak{T}-t)^{\wp-1}\mathfrak{J}_3(\varkappa,t)\,dt.\end{aligned} \tag{5.3}$$

Proof. Using Lemma 5.1, we obtain

$$\mu(\varkappa)=-c_0+I^{\wp}(\mathfrak{g}(\varkappa))+\lambda_1 I^{\wp}\left(\int_0^{\varkappa}\mathfrak{J}_1(\varkappa,t)\,\xi_1(t,\mu(t))\,dt+\lambda_2\int_0^{\mathfrak{T}}\mathfrak{J}_2(\varkappa,t)\,\xi_2(t,\mu(t))\,dt\right) \tag{5.4}$$

Apply boundary conditions, we deduce that

$$\begin{aligned}c_0 =&\frac{b}{(a+b)\Gamma(\wp)}\int_0^{\mathfrak{T}}(\mathfrak{T}-\vartheta)^{\wp-1}\mathfrak{g}(\vartheta)d\vartheta-\frac{1}{(a+b)\Gamma(\wp)}\int_0^{\mathfrak{T}}(\mathfrak{T}-t)^{\wp-1}\mathfrak{J}_3(\varkappa,t)\,dt\\&+\frac{b}{(a+b)\Gamma(\wp)}\int_0^{\mathfrak{T}}(\mathfrak{T}-\vartheta)^{\wp-1}\left(\lambda_1\int_0^{\vartheta}\mathfrak{J}_1(\vartheta,t)\,\xi_1(t,\mu(t))\,dt+\lambda_2\int_0^{\mathfrak{T}}\mathfrak{J}_2(\vartheta,t)\,\xi_2(t,\mu(t))\,dt\right)d\vartheta.\end{aligned}$$

Thus by substituting the values of c_0 in (5.4), we get integral equation (5.3).

□

Notice that the solution of Eq (5.1) is equivalent to Eq (5.3). Now, we are in a position to present the existence result.

Theorem 5.3. *Let* $\mu,\nu\in B_r,\vartheta,\tau\in[0,\mathfrak{T}]$, *and* $a,b>0$ *be real constants. If* $\mu,\mathfrak{g}:[0,\mathfrak{T}]\to\mathbb{R}$, $\mathfrak{J}_1,\mathfrak{J}_2,\mathfrak{J}_3:[0,\mathfrak{T}]\times[0,\mathfrak{T}]\to\mathbb{R}$ *and* $\xi_1,\xi_2:[0,\mathfrak{T}]\times\mathbb{R}\to\mathbb{R}$ *are continuous functions satisfying the following axioms:*

1. *there exist* $C_i>0, i=1,2$ *such that*

$$|\xi_i(t,\mu(t))-\xi_i(t,\nu(t))|\leqslant C_i\|\mu-\nu\|; \tag{5.5}$$

2. *there exist real numbers* λ_1 *and* λ_2 *with* $|\lambda_1|C_1\mathfrak{R}_1+|\lambda_2|C_2\mathfrak{R}_2<\frac{\Gamma(\wp+1)}{2\mathfrak{T}^p}$ *such that*

$$\frac{2[\|\mathfrak{g}\|+|\lambda_1|\mathfrak{R}_1\mathfrak{F}_1+|\lambda_2|\mathfrak{R}_2\mathfrak{F}_2]+\mathfrak{R}_3}{\mathfrak{T}^{-\wp}\Gamma(\wp+1)-2[|\lambda_1|C_1\mathfrak{R}_1+|\lambda_2|C_2\mathfrak{R}_2]}\leqslant r, \tag{5.6}$$

where $\mathfrak{F}_1=\sup|\xi_1(t,0)|$, $\mathfrak{F}_2=\sup|\xi_2(t,0)|$ *and*

$$\mathfrak{R}_1=\sup\int_0^{\vartheta}|\mathfrak{J}_1(\vartheta,\tau)|\,d\tau<\infty, \tag{5.7}$$

and

$$\Re_i = \sup \int_0^{\mathfrak{T}} |\mathfrak{J}_i(\vartheta,\tau)|\, d\tau < \infty, \;\; i = 2,3. \tag{5.8}$$

Then problem (5.3) *has a solution in* B_r*, equivalently problem* (5.1) *has a solution in* B_r*.*

Proof. Let $B_r = \{\mu \in C([0,\mathfrak{T}],\mathbb{R}) : \|\mu\| \leqslant r\}$. Then, B_r is a non-empty, closed, bounded, and convex subset of $C([0,\mathfrak{T}],\mathbb{R})$. Define the operator $\Upsilon : B_r \to B_r$ by

$$\begin{aligned}
\Upsilon\mu(\varkappa) =& \frac{1}{\Gamma(\wp)} \int_0^{\varkappa} (\varkappa-\vartheta)^{\wp-1} \mathfrak{g}(\vartheta) d\vartheta - \frac{b}{(a+b)\Gamma(\wp)} \int_0^{\mathfrak{T}} (\mathfrak{T}-\vartheta)^{\wp-1} \mathfrak{g}(\vartheta) d\vartheta \\
&+ \frac{1}{\Gamma(\wp)} \int_0^{\varkappa} (\varkappa-\vartheta)^{\wp-1} \left(\lambda_1 \int_0^{\vartheta} \mathfrak{J}_1(\vartheta,t)\,\xi_1(t,\mu(t))\, dt + \lambda_2 \int_0^{\mathfrak{T}} \mathfrak{J}_2(\vartheta,t)\,\xi_2(t,\mu(t))\, dt \right) d\vartheta \\
&- \frac{b}{(a+b)\Gamma(\wp)} \int_0^{\mathfrak{T}} (\mathfrak{T}-\vartheta)^{\wp-1} \Bigg(\lambda_1 \int_0^{\vartheta} \mathfrak{J}_1(\vartheta,t)\,\xi_1(t,\mu(t))\, dt \\
&+ \lambda_2 \int_0^{\mathfrak{T}} \mathfrak{J}_2(\vartheta,t)\,\xi_2(t,\mu(t))\, dt \Bigg) d\vartheta + \frac{1}{(a+b)\Gamma(\wp)} \int_0^{\mathfrak{T}} (\mathfrak{T}-t)^{\wp-1} \mathfrak{J}_3(\varkappa,t)\, dt.
\end{aligned}$$

Our first claim is $\Upsilon : B_r \to B_r$ is well-defined. Let $\mu \in B_r$, for some r. Then for all $\varkappa \in [0,\mathfrak{T}]$, we have

$$\begin{aligned}
|\Upsilon\mu(\varkappa)| \leqslant& \frac{1}{\Gamma(\wp)} \int_0^{\varkappa} (\varkappa-\vartheta)^{\wp-1} |\mathfrak{g}(\vartheta)|\, d\vartheta + \frac{b}{(a+b)\Gamma(\wp)} \int_0^{\mathfrak{T}} (\mathfrak{T}-\vartheta)^{\wp-1} |\mathfrak{g}(\vartheta)|\, d\vartheta \\
&+ \frac{1}{\Gamma(\wp)} \int_0^{\varkappa} (\varkappa-\vartheta)^{\wp-1} \Bigg(|\lambda_1| \int_0^{\vartheta} |\mathfrak{J}_1(\vartheta,t)|\, |\xi_1(t,\mu(t))|\, dt \\
&+ |\lambda_2| \int_0^{\mathfrak{T}} |\mathfrak{J}_2(\vartheta,t)|\, |\xi_2(t,\mu(t))|\, dt \Bigg) d\vartheta + \frac{1}{(a+b)\Gamma(\wp)} \int_0^{\mathfrak{T}} (\mathfrak{T}-t)^{\wp-1} |\mathfrak{J}_3(\varkappa,t)|\, dt \\
&+ \frac{b}{(a+b)\Gamma(\wp)} \int_0^{\mathfrak{T}} (\mathfrak{T}-\vartheta)^{\wp-1} \Bigg(|\lambda_1| \int_0^{\vartheta} |\mathfrak{J}_1(\vartheta,t)|\, |\xi_1(t,\mu(t))|\, dt \\
&+ |\lambda_2| \int_0^{\mathfrak{T}} |\mathfrak{J}_2(\vartheta,t)|\, |\xi_2(t,\mu(t))|\, dt \Bigg) d\vartheta \\
\leqslant& \frac{1}{\Gamma(\wp)} \int_0^{\varkappa} (\varkappa-\vartheta)^{\wp-1} |\mathfrak{g}(\vartheta)|\, d\vartheta + \frac{1}{\Gamma(\wp)} \int_0^{\mathfrak{T}} (\mathfrak{T}-\vartheta)^{\wp-1} |\mathfrak{g}(\vartheta)|\, d\vartheta \\
&+ \frac{1}{\Gamma(\wp)} \int_0^{\varkappa} (\varkappa-\vartheta)^{\wp-1} \Bigg\{ |\lambda_1| \int_0^{\vartheta} |\mathfrak{J}_1(\vartheta,t)| \left(|\xi_1(t,\mu(t)) - \xi_1(t,0)| + |\xi_1(t,0)| \right) dt \\
&+ |\lambda_2| \int_0^{\mathfrak{T}} |\mathfrak{J}_2(\vartheta,t)| \left(|\xi_2(t,\mu(t)) - \xi_2(t,0)| + |\xi_2(t,0)| \right) dt \Bigg\} d\vartheta \\
&+ \frac{1}{\Gamma(\wp)} \int_0^{\mathfrak{T}} (\mathfrak{T}-\vartheta)^{\wp-1} \Bigg\{ |\lambda_1| \int_0^{\vartheta} |\mathfrak{J}_1(\vartheta,t)| \left(|\xi_1(t,\mu(t)) - \xi_1(t,0)| + |\xi_1(t,0)| \right) dt \\
&+ |\lambda_2| \int_0^{\mathfrak{T}} |\mathfrak{J}_2(\vartheta,t)| \left(|\xi_2(t,\mu(t)) - \xi_2(t,0)| + |\xi_2(t,0)| \right) dt \Bigg\} d\vartheta \\
&+ \frac{1}{\Gamma(\wp)} \int_0^{\mathfrak{T}} (\mathfrak{T}-t)^{\wp-1} |\mathfrak{J}_3(\varkappa,t)|\, dt
\end{aligned}$$

$$\leqslant \frac{\|\mathfrak{g}\|}{\Gamma(\wp)}\int_0^{\varkappa}(\varkappa-\vartheta)^{\wp-1}d\vartheta+\frac{\|\mathfrak{g}\|}{\Gamma(\wp)}\int_0^{\mathfrak{T}}(\mathfrak{T}-\vartheta)^{\wp-1}d\vartheta+\frac{\mathfrak{R}_3}{\Gamma(\wp)}\int_0^{\mathfrak{T}}(\mathfrak{T}-t)^{\wp-1}dt$$
$$+\frac{|\lambda_1|\mathfrak{R}_1(C_1\|\mu\|+\mathfrak{F}_1)+|\lambda_2|\mathfrak{R}_2(C_2\|\mu\|+\mathfrak{F}_2)}{\Gamma(\wp)}\int_0^{\varkappa}(\varkappa-\vartheta)^{\wp-1}d\vartheta$$
$$+\frac{\{|\lambda_1|\mathfrak{R}_1(C_1\|\mu\|+\mathfrak{F}_1)+|\lambda_2|\mathfrak{R}_2(C_2\|\mu\|+\mathfrak{F}_2)\}}{\Gamma(\wp)}\int_0^{\mathfrak{T}}(\mathfrak{T}-\vartheta)^{\wp-1}d\vartheta,$$

with the help of (5.6), (5.7) and (5.8), we get

$$|\Upsilon\mu(\varkappa)|\leqslant \frac{\|\mathfrak{g}\|}{\Gamma(\wp)}\frac{\varkappa^{\wp}}{\wp}+\frac{\|\mathfrak{g}\|}{\Gamma(\wp)}\frac{\mathfrak{T}^{\wp}}{\wp}+\frac{|\lambda_1|\mathfrak{R}_1(C_1r+\mathfrak{F}_1)+|\lambda_2|\mathfrak{R}_2(C_2r+\mathfrak{F}_2)}{\Gamma(\wp)}\frac{\varkappa^{\wp}}{\wp}$$
$$+\frac{|\lambda_1|\mathfrak{R}_1(C_1r+\mathfrak{F}_1)+|\lambda_2|\mathfrak{R}_2(C_2r+\mathfrak{F}_2)}{\Gamma(\wp)}\frac{\mathfrak{T}^{\wp}}{\wp}+\frac{r+\mathfrak{R}_3}{\Gamma(\wp)}\frac{\mathfrak{T}^{\wp}}{\wp}$$
$$\leqslant(2\|\mathfrak{g}\|+2|\lambda_1|\mathfrak{R}_1(C_1r+\mathfrak{F}_1)+2|\lambda_2|\mathfrak{R}_2(C_2r+\mathfrak{F}_2)+\mathfrak{R}_3)\frac{\mathfrak{T}^{\wp}}{\Gamma(\wp+1)}$$
$$=\{2(\|\mathfrak{g}\|+|\lambda_1|\mathfrak{R}_1\mathfrak{F}_1+|\lambda_2|\mathfrak{R}_2\mathfrak{F}_2)+2r(|C_1\lambda_1|\mathfrak{R}_1+|\lambda_2|C_2\mathfrak{R}_2)+\mathfrak{R}_3\}\frac{\mathfrak{T}^{\wp}}{\Gamma(\wp+1)}$$
$$\leqslant r.$$

That is, $\|\Upsilon(\mu)\|\leqslant r$, for all $\mu\in B_r$, which implies that $\Upsilon(\mu)\in B_r$ and hence $\Upsilon: B_r\to B_r$ is well-defined. Now, we have to show that $\Upsilon: B_r\to B_r$ is continuous. For this, using (5.7), (5.5) and (5.8), we have

$$\left|\Upsilon\mu(\varkappa)-\Upsilon\nu(\varkappa)\right|$$
$$=\left|\frac{1}{\Gamma(\wp)}\int_0^{\varkappa}(\varkappa-\vartheta)^{\wp-1}\left(\lambda_1\int_0^{\vartheta}\mathfrak{I}_1(\vartheta,t)\xi_1(t,\mu(t))dt+\lambda_2\int_0^{\mathfrak{T}}\mathfrak{I}_2(\vartheta,t)\xi_2(t,\mu(t))dt\right)d\vartheta\right.$$
$$-\frac{b}{(a+b)\Gamma(\wp)}\int_0^{\mathfrak{T}}(\mathfrak{T}-\vartheta)^{\wp-1}\left(\lambda_1\int_0^{\vartheta}\mathfrak{I}_1(\vartheta,t)\xi_1(t,\mu(t))dt+\lambda_2\int_0^{\mathfrak{T}}\mathfrak{I}_2(\vartheta,t)\xi_2(t,\mu(t))dt\right)d\vartheta$$
$$-\frac{1}{\Gamma(\wp)}\int_0^{\varkappa}(\varkappa-\vartheta)^{\wp-1}\left(\lambda_1\int_0^{\vartheta}\mathfrak{I}_1(\vartheta,t)\xi_1(t,\nu(t))dt+\lambda_2\int_0^{\mathfrak{T}}\mathfrak{I}_2(\vartheta,t)\xi_2(t,\nu(t))dt\right)d\vartheta$$
$$\left.+\frac{b}{(a+b)\Gamma(\wp)}\int_0^{\mathfrak{T}}(\mathfrak{T}-\vartheta)^{\wp-1}\left(\lambda_1\int_0^{\vartheta}\mathfrak{I}_1(\vartheta,t)\xi_1(t,\nu(t))dt+\lambda_2\int_0^{\mathfrak{T}}\mathfrak{I}_2(\vartheta,t)\xi_2(t,\nu(t))dt\right)d\vartheta\right|$$
$$\leqslant\frac{1}{\Gamma(\wp)}\int_0^{\varkappa}(\varkappa-\vartheta)^{\wp-1}\left(|\lambda_1|\int_0^{\vartheta}|\mathfrak{I}_1(\vartheta,t)||\xi_1(t,\mu(t))-\xi_1(t,\nu(t))|dt\right)d\vartheta$$
$$+\frac{1}{\Gamma(\wp)}\int_0^{\varkappa}(\varkappa-\vartheta)^{\wp-1}\left(|\lambda_2|\int_0^{\mathfrak{T}}|\mathfrak{I}_2(\vartheta,t)||\xi_2(t,\mu(t))-\xi_2(t,\nu(t))|dt\right)d\vartheta$$
$$+\frac{b}{(a+b)\Gamma(\wp)}\int_0^{\mathfrak{T}}(\mathfrak{T}-\vartheta)^{\wp-1}\left(|\lambda_1|\int_0^{\vartheta}|\mathfrak{I}_1(\vartheta,t)||\xi_1(t,\mu(t))-\xi_1(t,\nu(t))|dt\right)d\vartheta$$
$$+\frac{b}{(a+b)\Gamma(\wp)}\int_0^{\mathfrak{T}}(\mathfrak{T}-\vartheta)^{\wp-1}\left(|\lambda_2|\int_0^{\mathfrak{T}}|\mathfrak{I}_2(\vartheta,t)||\xi_2(t,\mu(t))-\xi_2(t,\nu(t))|dt\right)d\vartheta$$

$$
\begin{aligned}
\leqslant & \frac{C_1\Re_1\left|\lambda_1\right|\|\mu-\nu\|+C_2\Re_2\left|\lambda_2\right|\|\mu-\nu\|}{\Gamma(\wp)} \int_0^{\varkappa}(\varkappa-\vartheta)^{\wp-1} d\vartheta \\
& +\frac{bC_1\Re_1\left|\lambda_1\right|\|\mu-\nu\|+bC_2\Re_2\left|\lambda_2\right|\|\mu-\nu\|}{(a+b)\Gamma(\wp)} \int_0^{\mathfrak{T}}(\mathfrak{T}-\vartheta)^{\wp-1} d\vartheta \\
= & \frac{\left(C_1\Re_1\left|\lambda_1\right|\|\mu-\nu\|+C_2\Re_2\left|\lambda_2\right|\|\mu-\nu\|\right)\varkappa^{\wp}}{\Gamma(\wp+1)} \\
& +\frac{C_1\left(\Re_1\left|\lambda_1\right|\|\mu-\nu\|+C_2\Re_2\left|\lambda_2\right|\|\mu-\nu\|\right)\mathfrak{T}^{\wp}}{\Gamma(\wp+1)} \\
\leqslant & \frac{2\left(C_1\Re_1\left|\lambda_1\right|+C_2\Re_2\left|\lambda_2\right|\right)\mathfrak{T}^{\wp}}{\Gamma(\wp+1)}\|\mu-\nu\| .
\end{aligned}
$$

But $\left|\lambda_1\right| C_1\Re_1+\left|\lambda_2\right| C_2\Re_2<\frac{\Gamma(\wp+1)}{2\mathfrak{T}^p}$, that is $\mathbb{h}=\frac{2(C_1\Re_1|\lambda_1|+C_2\Re_2|\lambda_2|)\mathfrak{T}^{\wp}}{\Gamma(\wp+1)} \in(0,1)$. It follows from above that

$$
\|\Upsilon\mu-\Upsilon\nu\| \leqslant \mathbb{h}\|\mu-\nu\| . \tag{5.9}
$$

That is, $\Upsilon: B_r \rightarrow B_r$ is contraction and hence continuous. Next, we have to show that Υ is $\mathcal{L}_{\mathfrak{m}}$-contraction. Let Λ be any subset of B_r with $\mathfrak{m}(\Lambda)>0$, and $\mu, \nu \in \Lambda$. Then from inequality (5.9), we write

$$
\operatorname{diam}(\Upsilon\Lambda) \leqslant \mathbb{h}\operatorname{diam}(\Lambda) . \tag{5.10}
$$

Now, let define $\theta:(0, \infty) \rightarrow(1, \infty)$ by $\theta(t)=e^t$, then clearly $\theta \in \Theta$. Using inequality (5.10), we have

$$
\begin{aligned}
\theta(\mathfrak{m}(\Upsilon(\Lambda))) & =\theta(\operatorname{diam}(\Upsilon(\Lambda))) \\
& =e^{\operatorname{diam}(\Upsilon(\Lambda))} \\
& \leqslant e^{\mathbb{h}\operatorname{diam}(\Lambda)} \\
& =\left(e^{\operatorname{diam}(\Lambda)}\right)^{\mathbb{h}} \\
& =\left(e^{\mathfrak{m}(\Lambda)}\right)^{\mathbb{h}} \\
& =(\theta(\mathfrak{m}(\Lambda)))^{\mathbb{h}} .
\end{aligned}
$$

Consequently,

$$
\frac{(\theta(\mathfrak{m}(\Lambda)))^{\mathbb{h}}}{\theta(\mathfrak{m}(\Upsilon(\Lambda)))} \geqslant 1 .
$$

Thus for $\mathscr{L}(\kappa_1, \kappa_2)=\frac{\kappa_2^{\mathbb{h}}}{\kappa_1}$, the above inequality becomes

$$
\mathscr{L}(\theta(\mathfrak{m}(\Upsilon(\Lambda))), \theta(\mathfrak{m}(\Lambda))) \geqslant 1,
$$

That is, $\Upsilon: B_r \rightarrow B_r$ is $\mathcal{L}_{\mathfrak{m}}$-contraction and so Theorem 4.2 ensures the existence of a fixed point of Υ in B_r, equivalently, the Eq (5.3) has a solution in B_r. □

To illustrate the Theorem 5.3, we present an example.

Example 5.4. Consider the following Caputo fractional Volterra–Fredholm integro-differential equation

$$ {}^{c}D^{0.9}\mu(\varkappa) = -\frac{\varkappa^3 e^{-\varkappa^4}}{2} - \frac{1}{7}\int_0^{\varkappa} \frac{\varkappa}{3}\sin\left(\frac{\vartheta}{3}\right)\sqrt{2\vartheta + 3[\mu(\vartheta)]^2}\,d\vartheta + \frac{1}{22}\int_0^2 \frac{\varkappa^2\sqrt{5 + 2[\mu(\vartheta)]^2}}{1 + \vartheta\varkappa^2}\,d\vartheta, \quad (5.11) $$

with boundary condition

$$ 5\mu(0) + 3\mu(2) = \frac{1}{\Gamma(0.9)}\int_0^2 (2-\vartheta)^{0.9-1}\frac{s\cos\left(\frac{\vartheta}{2}\right)}{\varkappa^2 + 5}\,d\vartheta, \quad (5.12) $$

Compare Eq (5.11) with Eq (5.1), we get

$$ \begin{aligned} &\lambda_1 = \frac{-1}{7}, \lambda_2 = \frac{1}{22}, a = 5, b = 3, \mathfrak{g}(\varkappa) = -\frac{\varkappa^3 e^{-\varkappa^4}}{2}, \\ &\mathfrak{I}_1(\varkappa, \vartheta) = \frac{\varkappa}{3}\sin\left(\frac{\vartheta}{3}\right), \mathfrak{I}_2(\varkappa, \vartheta) = \frac{\varkappa^2}{1 + \vartheta\varkappa^2}, \mathfrak{I}_3(\varkappa, \vartheta) = \frac{s\cos\left(\frac{\vartheta}{2}\right)}{\varkappa^2 + 5}, \\ &\xi_1(\vartheta, \mu(\vartheta)) = \sqrt{2\vartheta + 3[\mu(\vartheta)]^2}, \xi_2(\vartheta, \mu(\vartheta)) = \sqrt{5 + 2[\mu(\vartheta)]^2}. \end{aligned} $$

Clearly $\mathfrak{g} : [0, 2] \to \mathbb{R}$, $\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_3 : [0, 2] \times [0, 2] \to \mathbb{R}$ and $\xi_1, \xi_2 : [0, 2] \times \mathbb{R} \to \mathbb{R}$ are continuous. Now, we have to verify condition (5.5) of Theorem 5.3. Consider

$$ \begin{aligned} |\xi_1(\vartheta, \mu(\vartheta)) - \xi_1(\vartheta, \nu(\vartheta))| &= \left|\sqrt{2\vartheta + 3[\mu(\vartheta)]^2} - \sqrt{2\vartheta + 3[\nu(\vartheta)]^2}\right| \\ &= \frac{|2\vartheta + 3[\mu(\vartheta)]^2 - 2\vartheta - 3[\nu(\vartheta)]^2|}{\sqrt{2\vartheta + 3[\mu(\vartheta)]^2} + \sqrt{2\vartheta + 3[\nu(\vartheta)]^2}} \\ &\leqslant \frac{3\,|[\mu(\vartheta)]^2 - [\nu(\vartheta)]^2|}{3[|\mu(\vartheta)| + |\nu(\vartheta)|]} \\ &= \frac{|\mu(\vartheta) - \nu(\vartheta)|\,|\mu(\vartheta) + \nu(\vartheta)|}{|\mu(\vartheta)| + |\nu(\vartheta)|} \\ &\leqslant \|\mu - \nu\|. \end{aligned} $$

Similarly,

$$ |\xi_2(\vartheta, \mu(\vartheta)) - \xi_2(\vartheta, \nu(\vartheta))| \leqslant \|\mu - \nu\|. $$

Thus $\xi_1, \xi_2 : \mathbb{R} \to \mathbb{R}$ are Lipschitz with $C_1 = C_2 = 1$.

Next, we have to verify the conditions (5.7) and (5.8) of Theorem 5.3. To do this, we have

$$ \mathfrak{R}_1 = \sup \int_0^{\varkappa} \left|\frac{\varkappa}{3}\sin\left(\frac{\vartheta}{3}\right)\right| d\vartheta = \sup\left(-|\varkappa|\cos\left(\frac{\varkappa}{3}\right)\right) = 0, $$

$$ \mathfrak{R}_2 = \sup \int_0^{\varkappa} \left|\frac{\varkappa^2}{1 + \vartheta\varkappa^2}\right| d\vartheta = \sup\left(\ln(1 + \varkappa^3)\right) \approx 2.197, $$

and

$$\mathfrak{K}_3 = \sup\left(\frac{1}{\varkappa^2+5}\int_0^{\varkappa} \vartheta \cos\left(\frac{\vartheta}{2}\right) d\vartheta\right) = \sup\left(\frac{2\varkappa \sin\left(\frac{\varkappa}{2}\right) + 4\cos\left(\frac{\varkappa}{2}\right) - 4}{\varkappa^2+5}\right) \approx 0.17.$$

Finally, to verify condition (5.6) of Theorem 5.3. Let $B_2 = \{\mu \in C([0,2], \mathbb{R}) : \|\mu\| \leqslant 2\}$, then since $\|\mathfrak{g}\| = 0, \mathfrak{K}_1 = 0, \mathfrak{K}_2 \approx 2.197, \mathfrak{K}_3 = 0.17$, $\mathfrak{F}_1 = 2$, and $\mathfrak{F}_2 = \sqrt{5}$, so we have

$$|\lambda_1| C_1\mathfrak{K}_1 + |\lambda_2| C_2\mathfrak{K}_2 \approx 0.104619 < 0.2576988 \approx \frac{\Gamma(\wp+1)}{2\mathfrak{T}^p},$$

and

$$\frac{2\left[\|\mathfrak{g}\| + |\lambda_1|\mathfrak{K}_1\mathfrak{F}_1 + |\lambda_2|\mathfrak{K}_2\mathfrak{F}_2\right] + \mathfrak{K}_3}{\mathfrak{T}^{-\wp}\Gamma(\wp+1) - 2\left[|\lambda_1| C_1\mathfrak{K}_1 + |\lambda_2| C_2\mathfrak{K}_2\right]} \approx 1.953315 < 2.$$

Thus Theorem 5.3 ensures the existence of a solution of (5.11) in B_2.

6. Conclusions

Darbo type contractions are introduced and fixed point results are established in a Banach space using the concept of measure of non compactness. Various existing results are deduced as corollaries to our main results. Further, our results are applied to prove the existence and uniqueness of solution to the Caputo fractional Volterra–Fredholm integro-differential equation under integral type boundary conditions which is further illustrated by appropriate example. Our study paves the way for further studies on Darbo type contractions and its applications.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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