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Research article

Monotonicity and symmetry of positive solution for 1-Laplace equation

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Abstract: In this paper we deal with a Dirichlet problem for an elliptic equation involving the 1-Laplace operator. Under suitable assumptions on the nonlinearity we show that there exists a symmetric, monotonic and positive solution via the moving plane method. We shall show a priori estimates for some positive solutions.

Keywords: 1-Laplace operator; BV space; symmetry of solutions; moving plane method; mountain pass lemma
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1. Introduction and main results

We are interested in the symmetry and monotonicity of solutions to the problem

$$\begin{cases} -\Delta_1 u = f(u), & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_1 u = \operatorname{div}(\frac{Du}{|Du|})$, Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, and strictly convex. The purpose of the paper is to investigate a priori estimates and symmetric properties of the solutions when the domain is assumed to have symmetric properties and f is supposed to satisfy the following conditions (H_1), (H_2) and (H_4). We also assume that f satisfies the following conditions (H_3) and (H_5) to use mountain pass lemma to get a nontrivial solution.
(H_1): $f : [0, +\infty)$ is a locally Lipschitz continuous function and $f(s) \geq 0$ for $\forall\ s \in [0, +\infty)$.
(H_2): $f(s) \leq C_1(1 + s^{1^*-1})$, for $\forall\ s \in [0, +\infty)$, with $1^* = \frac{N}{N-1}$ and a constant $C_1 > 0$.
(H_3): There exists $\theta > 1$, and $k_0 > 0$ such that

$$0 < \theta F(s) \leq s f(s),\ \ s \geq k_0.$$

(H_4): There exists a constant $C_2 > 0$ such that

$$\liminf_{s\to+\infty} \frac{1^*F(s) - sf(s)}{sf(s)} \geq C_2,$$

where $F(s) = \int_0^s f(t)dt$.
(H_5): There exists a constant $\alpha \in (0, \frac{1}{N-1})$ such that

$$\lim_{s\to 0} \frac{|f(s)|}{s^\alpha} < \infty.$$

We point out that the similar p-Laplace problems ($p > 1$) have many applications and have been studied for a long time, more precisely, Dirichlet problems for the p-Laplace operator,

$$\begin{cases} -\Delta_p u = f(u), & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

In the case $p = 2$, the problem (1.2) $-\Delta_p u = f(u)$ has been widely studied. Gidas and Spruck [27] prove a priori bounds for nonlinearities f for $N \geq 3$ behave as a subcritical power at infinity, introducing the blow up method together with Liouville type theorems for solutions in $\mathbb{R}^N$. Figneiredo, Lions and Nussbaum [19] consider the existence and a priori estimates of positive solutions of the problem (1.2) when f satisfies the superlinear grow at infinity. They prove a priori bound for positive solutions of the problem (1.2) under the hypothesis $\lim_{s\to\infty} \frac{f(s)}{s^{\frac{N+2}{N-2}}} = 0$, together with the monotonic results by Gidas, Ni and Nirenberg [28] obtained by the Alexandrov-Serrin moving plane method [37]. The moving plane method has been improved and simplified by Beresticky and Nirenberg [7] with the aid of the maximum principle in small domain. With the help of the blow up procedure, Azizieh and Clément [5] prove a priori estimates for the problem (1.2) in the case of Ω being a strictly convex domain and f satisfying some suitable assumption. Damascelli and Pacella [14, 15] apply the moving plane method to prove some monotonic and symmetric results for the p-Laplace equation in the singular case $1 < p < 2$, also see [6, 13]. The results are later extended to the case $p > 2$ in the papers [12, 17, 18]. Damascelli and Pardo [16] used the technique introduced in [19] that allowed to give the a priori estimates for solutions in case $1 < p < N$, case $p = N$, and case $p > N$. Esposito, Montoro and Sciunzi [24] study symmetric and monotonic properties of singular positive solutions to the problem (1.2) via moving plane method under suitable assumptions on f. However, all the above mentioned papers can not deal with the case $p = 1$. In this paper, we can extend the case $p > 1$ to the case $p = 1$.

Obviously, the problem of Δ_1 is different from Δ_p ($p > 1$). When $p = 1$, it is necessary to replace $W^{1,1}$ by BV, the space of functions of bounded variation. A function $u \in L^1(\Omega)$ is called a function of bounded variation, whose partial derivatives in the sense of distribution are Radon measures. We point out that the space $W^{1,p}(\Omega)$ is reflexive, however, the space $BV(\Omega)$ is not reflexive, so that we can not follow the arguments on Δ_p. The 1-Laplace operator Δ_1 introduces some extra difficulties and special features. The first difficulty occurs by defining the quotient $\frac{Du}{|Du|}$, Du being just a Radon measure. To deal with the 1-Laplacian operator, we need the theory of pairing of L^∞ divergence measure vector fields (see the pioneering works [3, 4, 8]).

Demengel [21] is concerned with existence of solution in $BV(\Omega)$ to the problem $-\mathrm{div} z + z\mathrm{sign} u = f|u|^{1^*-2}u$ with $z \cdot \nabla u = \nabla u$ in Ω and $-z \cdot \gamma = u$ on $\partial\Omega$. Demengel [22] is devoted to the elliptic equations with 1-Laplacian operator

$$\begin{cases} -\Delta_1 u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

and introduces the concept of locally almost 1-harmonic functions in Ω. The comparison principle, the first eigenvalue and related eigenfunctions for the 1-Laplacian operator are established in [22]. Kawohl and Schuricht [30] consider a number of problems that are associated with the 1-Laplace operator Δ_1, the formal limit of the p-Laplace operator as $p \to 1$, by investigating the underlying variational problem. Since the corresponding solution typically belongs to BV and not to $W^{1,1}$, they have to study the minimizers of the functionals containing the total variation. In particular, they look for constrained minimizers subject to a prescribed L^1 norm which can be considered as an eigenvalue problem for the 1-Laplace operator. Degiovanni and Magrone [20] are concerned with the problem (1.3) with $f(x,u) = \lambda\frac{u}{|u|} + |u|^{1^*-2}u$. It is proved that for every $\lambda \geq \lambda_1$, the problem (1.3) admits a nontrivial solution by the non-standard linking methods. Salas and Segura de León [35] study the problem (1.3) with $f(x,u)$ satisfying subcritical growth; i.e., $|f(x,u)| \leq C(1+|u|^q)$ with $0 < q < 1^* - 1$. They prove that for the problem (1.3) there exists at least two nontrivial solutions, one nonnegative and one nonpositive, by using known existence results for the p-Laplacian ($p > 1$) and considering the limit as $p \to 1^+$. De Cicco, Giachetti, Oliva and Petitta [9] study the existence and regularity of special distributional nonnegative solutions to the boundary value singular problem (1.3) with $f(x,u) = h(u)g(x)$. They show existence of nonnegative solutions to (1.3) with $u^{\max\{1,\gamma\}} \in BV(\Omega)$. These solutions are obtained as a limit as $p \to 1^+$ of nonnegative solutions of the p-Laplacian problems $-\Delta_p u_p = h(u_p)g$ with $u_p = 0$ on $\partial\Omega$. We also refer to [33–36, 38] for the a priori estimates and gradient estimates of solutions. In this paper we can study the monotonicity and symmetry of positive solution to the 1-Laplace problem and show the a priori estimates for the solution.

By the theory of pairing of L^∞ divergence measure vector fields, we introduce the following definition of solutions to the problem (1.1).

Definition 1.1. We say that $u \in BV_{loc}(\Omega)$, $u > 0$, is a solution to problem (1.1) if there exists a vector field $z \in \mathcal{DM}^\infty(\Omega)$ with $\|z\|_{L^\infty} \leq 1$ such that

$$-\operatorname{div} z = f(u), \quad \text{in } \mathcal{D}'(\Omega), \tag{1.4}$$

$$(z, Du) = |Du| \text{ as measures in } \Omega, \tag{1.5}$$

$$[z,\gamma] \in \operatorname{sign}(-u) \text{ on } \partial\Omega, \tag{1.6}$$

where γ is the unit exterior normal on $\partial\Omega$, and the spaces $BV_{loc}(\Omega)$ and $\mathcal{DM}^\infty(\Omega)$ are given in Section 2.

To state more precisely some known result about the monotonicity and symmetry of solutions of the problem (1.1), we need some notations. Let ν be a direction in $\mathbb{R}^N$. For a real number μ we define

$$T_\mu^\nu = \{x \in \mathbb{R}^N \mid x \cdot \nu = \mu\}$$

$$\Omega_\mu^\nu = \{x \in \Omega \mid x \cdot \nu < \mu\}$$

$$x_\mu^\nu = R_\mu^\nu(x) = x + 2(\mu - x \cdot \nu)\nu, \ x \in \mathbb{R}^N$$

and

$$a(\nu) = \inf_{x\in\Omega} x \cdot \nu. \tag{1.7}$$

If $\mu > a(\nu)$ then Ω^ν_μ is nonempty, thus we set

$$(\Omega^\nu_\mu)' = R^\nu_\mu(\Omega^\nu_\mu).$$

Following [6] and [12–18], we observe that $\mu - a(\nu)$ small then $(\Omega^\nu_\mu)'$ is contained in Ω and will remain in it, at least until one of the following occurs:
(A) $(\Omega^\nu_\mu)'$ becomes internally tangent to $\partial\Omega$.
(B) T^ν_μ is orthogonal to $\partial\Omega$.
Let $\Pi_1(\nu)$ be the set of those $\mu > a(\nu)$ such that for each $\eta < \mu$ none of the conditions (A) and (B) holds and define

$$\mu_1(\nu) = \sup \Pi_1(\nu). \tag{1.8}$$

Moreover, let

$$\Pi_2(\nu) = \{\mu > a(\nu) \mid (\Omega^\nu_\eta)' \subset \Omega, \forall \eta \in (a(\nu), \mu]\}$$

and

$$\mu_2(\nu) = \sup \Pi_2(\nu). \tag{1.9}$$

Since Ω is supposed to be smooth, note that neither $\Pi_1(\nu)$ nor $\Pi_2(\nu)$ are empty and $\Pi_1(\nu) \subset \Pi_2(\nu)$, so that $\mu_1(\nu) \leq \mu_2(\nu)$.

We deal with solutions to the problem (1.1) in the sense of Definition 1.1. Our main result is stated as follows.

Theorem 1.2. Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, which is strictly convex. Assume the nonlinearity f satisfies the conditions $(H_1) - (H_5)$. Then there exists a nontrivial positive solution u to the problem (1.1) in the sense Definition 1.1, bounded in $L^\infty(\Omega)$ (i.e., $u \in L^\infty(\Omega)$), and for any direction ν and for μ in the interval $(a(\nu), \mu_1(\nu)]$,

$$u(x) \leq u(x^\nu_\mu), \text{ a.e. } x \in \Omega^\nu_\mu, \tag{1.10}$$

where $a(\nu)$ and $\mu_1(\nu)$ are given by (1.7) and (1.8) respectively.

If f is locally Lipschitz continuous in the closed interval $[0, +\infty)$, the condition (1.10) holds for any μ in the interval $(a(\nu), \mu_2(\nu)]$.

Corollary 1.3. Let the smooth bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be strictly convex with respect to a direction ν and symmetric with respect to the hyperplane $T^\nu_0 = \{x \in \mathbb{R}^N \mid x \cdot \nu = 0\}$. Assume that the nonlinearity f satisfies the conditions $(H_1) - (H_5)$, which is locally Lipschitz continuous in the closed interval $[0, +\infty)$ and strictly positive in $(0, +\infty)$. Then there exists a nontrivial positive solution u to the problem (1.1) in the sense Definition 1.1, bounded in $L^\infty(\Omega)$, almost everywhere symmetric, i.e., $u(x) = u(x^\nu_0)$ and nondecreasing in the ν-direction a.e. in Ω^ν_0.

Remark 1.4. Since the moving plane procedure can be performed in the same way but in the opposite direction, then it is obvious that Corollary 1.3 is obtained by Theorem 1.2 (see Corollary 2.4 of [16]).

2. Preliminaries on BV space

Throughout this paper, Ω denotes an bounded subset of $\mathbb{R}^N$ with Lipschitz boundary. The symbol $|\Omega|$ stands for its N dimensional Lebesgue measure and $H^{N-1}(E)$ for the $N-1$ dimensional Hausdorff measure of a set $E \subset \mathbb{R}^N$. An outward normal with vector $\gamma = \gamma(x)$ is defined for H^{N-1} a.e. $x \in \partial\Omega$. We will denote by $W_0^{1,p}(\Omega)$ the usual Sobolev space, of measureable functions having weak gradient in $L^p(\Omega;\mathbb{R}^N)$ and zero trace on $\partial\Omega$. If $1 < p < N$, denote by $p^* = \frac{Np}{N-p}$ its critical Sobolev exponent. $BV(\Omega)$ will denote the space of functions of bounded variation

$$BV(\Omega) = \{v \in L^1(\Omega) \mid Dv \text{ is a bounded Radon measure}\}$$

where $Dv : \Omega \to \mathbb{R}^N$ is the distributional gradient of u. It is endowed with the norm by

$$\|v\|_{BV} = \int_\Omega |Dv| + \int_\Omega |v| dx,$$

where

$$\int_\Omega |Dv| = \sup\{\int_\Omega v \operatorname{div}\varphi dx \mid \varphi \in C_0^1(\Omega;\mathbb{R}^N), |\varphi(x)| \le 1,\ x \in \Omega\}.$$

$BV(\Omega)$ is a Banach space which is non-reflexive and non-separable. The notion of a trace on the boundary can be extended to functions $v \in BV(\Omega)$ and this fact allows us to write $v|_{\partial\Omega}$. Moreover, the trace defines a linear bounded operator $i : BV(\Omega) \hookrightarrow L^1(\partial\Omega)$ which is onto. By the trace, we have an equivalent norm on $BV(\Omega)$

$$\|v\|_{BV} = \int_\Omega |Dv| + \int_{\partial\Omega} |v| dH^{N-1},$$

where H^{N-1} denotes the $N-1$ dimensional Hausdorff measure. We will often use this norm in what follows. In addition, the following continuous embeddings hold

$$BV(\Omega) \hookrightarrow L^m(\Omega),\ 1 \le m \le \frac{N}{N-1},$$

which are compact for $1 \le m < \frac{N}{N-1}$ (see for instance [25,41]). We denote by $\mathcal{M}(\Omega)$ the space of Radon measures with finite total variation over Ω, by

$$\mathcal{DM}^\infty(\Omega) = \{z \in L^\infty(\Omega;\mathbb{R}^N) \mid \operatorname{div} z \in \mathcal{M}(\Omega)\}$$

and by

$$\mathcal{DM}^\infty_{loc}(\Omega) = \{z \in L^\infty(\Omega;\mathbb{R}^N) \mid \operatorname{div} z \in \mathcal{M}(\Omega'),\ \Omega' \subset\subset \Omega\}.$$

The theory of L^∞ divergence measure vector fields is due to Anzellotti [4] and Chen and Frid [8]. We define the following distribution (z, Dv)

$$\langle (z, Dv), \varphi \rangle = -\int_\Omega v\varphi \operatorname{div} z dx - \int_\Omega v z \cdot \nabla\varphi dx \tag{2.1}$$

for $\forall\ \varphi \in C_c^1(\Omega)$. In Anzellotti's theory we need some compatibility conditions, such as $\operatorname{div} z \in L^1(\Omega)$ and $v \in BV(\Omega) \cap L^\infty(\Omega)$ or $\operatorname{div} z$ a Radon measure with finite total variation and $v \in BV(\Omega) \cap L^\infty(\Omega) \cap$

$C(\Omega)$.
Lemma 2.1 ([34, 35]). Let $v \in BV_{loc}(\Omega) \cap L^1(\Omega, \mu)$ and $z \in \mathcal{DM}^{\infty}_{loc}(\Omega)$. Then the distribution (z, Dv) defined in (2.1) previously satisfies

$$|\langle (z, Dv), \varphi \rangle| \leq \|\varphi\|_{L^\infty} \|z\|_{L^\infty(U)} \int_U |Dv|,$$

for all open set $U \subset\subset \Omega$ and all $\varphi \in C^1_c(U)$.
Lemma 2.2 ([34, 35]). The distribution (z, Dv) is a Radon measure. It and its total variation $|(z, Dv)|$ are absolutely continuous with respect to the measure $|Dv|$ and

$$|\int_B (z, Dv)| \leq \int_B |(z, Dv)| \leq \|z\|_{L^\infty(U)} \int_B |Dv|,$$

holds for all Borel sets B and for all open sets U such that $B \subset U \subset \Omega$.
Lemma 2.3 ([10, 11, 34]). Let $z \in \mathcal{DM}^{\infty}_{loc}(\Omega)$ and let $v \in BV(\Omega) \cap L^\infty(\Omega)$. Then $zv \in \mathcal{DM}^{\infty}_{loc}(\Omega)$. Moreover, the following formula holds in the sense of measures

$$\operatorname{div}(z, v) = (\operatorname{div} z)v + (z, Dv).$$

It follows from Anzellotti's theory that every $z \in \mathcal{DM}^{\infty}(\Omega)$ has a weak trace on $\partial\Omega$ of the normal component of z which is denoted by $[z, \gamma]$ with γ the unit exterior normal on $\partial\Omega$, which satisfies

$$\|[z, \gamma]\|_{L^\infty(\partial\Omega)} \leq \|z\|_{L^\infty},$$

and

$$v[z, \gamma] = [vz, \gamma]$$

for all $z \in \mathcal{DM}^{\infty}(\Omega)$ and $v \in BV(\Omega) \cap L^\infty(\Omega)$.
Lemma 2.4 (Green formula [10, 11, 34]). Let $z \in \mathcal{DM}^{\infty}_{loc}(\Omega)$, $\varpi = \operatorname{div} z$ and $v \in BV(\Omega)$ and assume $v \in L^1(\Omega, \mu)$. Then $vz \in \mathcal{DM}^{\infty}(\Omega)$ and the following holds

$$\int_\Omega v d\varpi + \int_\Omega (z, Dv) = \int_{\partial\Omega} [vz, \gamma] dH^{N-1}.$$

Lemma 2.5 ([34, 35]). Let $z \in \mathcal{DM}^{\infty}_{loc}(\Omega)$ and $v \in BV(\Omega) \cap L^\infty(\Omega)$. If $vz \in \mathcal{DM}^{\infty}(\Omega)$, then

$$|[vz, \gamma]| \leq |v|_{\partial\Omega} \|z\|_{L^\infty(\Omega)}, \; H^{N-1} \text{ a.e. on } \partial\Omega.$$

3. Weak solution to p-Laplacian problem

Let $p_0 := \min\{\theta, \frac{N}{N-1}\}$, with $\theta > 1$ given by (H_3). For each $1 < p < p_0$, let us consider the following problem

$$\begin{cases} -\Delta_p w = f(w), & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$, $1 < p < p_0$ and $f : [0, +\infty) \to \mathbb{R}$ satisfies the conditions $(H_1) - (H_5)$. We need the following propositions and a priori estimates of p-Laplace equation to prove Theorem 1.2.

Definition 3.1. We say $u_p \in W_0^{1,p}(\Omega)$, $u_p \geq 0$, is a weak solution to the problem (3.1) in the sense that

$$\int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\varphi dx = \int_\Omega f(u_p)\varphi dx, \tag{3.2}$$

for $\forall\ \varphi \in W_0^{1,p}(\Omega)$.

If $u_p \in W^{1,p}(\Omega)$ is a weak solution of the problem (3.1) with f satisfying the critical growth, then $u_p \in C^{1,\alpha}(\Omega)$ with $\alpha \in (0, 1)$ (see [23, 31, 40]), so that we suppose from the beginning a C^1 regularity for the solution. Next, we recall some results on the monotonicity and estimates of solutions for the p-Laplace equation. One can refer to [1, 16, 19, 29, 32] for the proof of the following Proposition 3.2–3.7.

Proposition 3.2 ([16]). Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, $1 < p < \infty$, $f : [0, \infty) \to \mathbb{R}$ a continuous function which is locally Lipschitz continuous in $(0, \infty)$ and strictly positive in $(0, \infty)$ if $p > 2$. Let $w \in C^1(\overline{\Omega})$ be a weak solution of (3.1). Then for any direction ν and for μ in the interval $(a(\nu), \mu_1(\nu)]$, we have

$$w(x) \leq w(x_\mu^\nu),\ \text{a.e. } x \in \Omega_\mu^\nu. \tag{3.3}$$

If f is locally Lipschitz continuous in the closed interval $[0, +\infty)$, then (3.3) holds for any μ in the interval $(a(\nu), \mu_2(\nu)]$, where $a(\nu)$, $\mu_1(\nu)$ and $\mu_2(\nu)$ are given by (1.7), (1.8) and (1.9).

Proposition 3.3 ([16, 19]). Let Ω be a strictly convex bounded smooth domain, and define $\Omega_\delta = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \delta\}$, for $\delta > 0$. Then the following result holds for a weak solution $w \in C^1(\Omega)$ of the problem (3.1) with f satisfying the condition (H_1)

$$\begin{cases} \exists\ \sigma, \varepsilon > 0 \text{ depending only on } \Omega,\ \text{such that } \forall\ x \in \Omega \setminus \Omega_\varepsilon \text{ there} \\ \text{is a part of a cone } I_x \text{ with} \\ (i)\ w(\xi) \geq w(x),\ \forall\ \xi \in I_x, \\ (ii)\ I_x \subset \Omega_{\frac{\varepsilon}{2}}, \\ (iii)\ |I_x| \geq \sigma. \end{cases}$$

I_x is a part of a cone K_x with vertex in x, where all the K_x are congruent to a fixed cone K, and if $x \in \Omega \setminus \Omega_{\frac{\varepsilon}{2}}$, then $I_x = K_x \cap \Omega_{\frac{\varepsilon}{2}}$.

Proposition 3.4 ([32]). Let us define

$$\lambda_1 = \inf_{w \in W_0^{1,p_0}(\Omega)} \{\int_\Omega |\nabla w|^{p_0} dx \mid \int_\Omega |w|^{p_0} dx = 1\},\ \text{with } p_0 = \min\{\theta, \frac{N}{N-1}\} > 1,$$

where θ is given by (H_3). Then, λ_1 is the first eigenvalue of the operator $-\Delta_{p_0}$ ($\lambda_1 \leq \lambda$ for any eigenvalue λ), it is simple, i.e., there is only an eigenfunction up to multiplication by a constant, and it is isolated. Moreover a first eigenfunction does not change sign in Ω and by the strong maximum principle it is in fact either strictly positive or strictly negative in Ω. So we can select a unique eigenfunction ϕ_1 such that

$$\int_\Omega \phi_1^{p_0} dx = 1,\ \text{and } \phi_1 > 0 \text{ in } \Omega.$$

The following extension of the Picone's identity for the p-Laplacian has been proved in [1].
Proposition 3.5 (Picone's identity [1])**.** Let $v_1, v_2 \geq 0$ be differentiable functions in an open set Ω, with $v_2 > 0$ and $p > 1$. Set

$$L(v_1, v_2) = |\nabla v_1|^p + (p-1)\frac{v_1^p}{v_2^p}|\nabla v_2|^p - p\frac{v_1^{p-1}}{v_2^{p-1}}|\nabla v_2|^{p-2}\nabla v_1 \cdot \nabla v_2$$

and

$$R(v_1, v_2) = |\nabla v_1|^p - |\nabla v_2|^{p-2}\nabla(\frac{v_1^p}{v_2^{p-1}}) \cdot \nabla v_2.$$

Then $R(v_1, v_2) = L(v_1, v_2) \geq 0$.
As a consequence we have

$$|\nabla v_2|^{p-2}\nabla(\frac{v_1^p}{v_2^{p-1}}) \cdot \nabla v_2 \leq |\nabla v_1|^p.$$

The following extension of the Pohozaev's identity for the p-Laplacian has been given by [29].
Proposition 3.6 (Pohozaev's identity for p-Laplace [29])**.** Let $w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $p > 1$, be a weak solution of the problem

$$\begin{cases} -\Delta_p w = f(w), & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$ and $f : [0, +\infty) \to \mathbb{R}$ is a continuous function. Denote $F(w) = \int_0^w f(s)ds$. Then

$$N\int_\Omega F(w)dx - \frac{N-p}{p}\int_\Omega f(w)wdx = \frac{p-1}{p}\int_{\partial\Omega}|\frac{\partial w}{\partial\gamma}|^p(x\cdot\gamma)dH^{N-1},$$

where γ is the unit exterior normal on $\partial\Omega$.

We need also local $W^{1,\infty}(\Omega)$ result at the boundary. This result follows from the global estimates by Lieberman [31] extending the local interior estimates by Dibenedetto [23].
Proposition 3.7 ([16])**.** Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, and $w \in C^1(\overline{\Omega})$ be a solution of the problem

$$\begin{cases} -\Delta_p w = h, & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega, \end{cases}$$

with $h \in L^{(p^*)'}(\Omega)$. For $\delta > 0$, let $\Omega_\delta = \{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) > \delta\}$ and suppose that $w, h \in L^\infty(\Omega \setminus \Omega_\delta)$ with

$$\|h\|_{L^\infty(\Omega\setminus\Omega_\delta)} \leq M \text{ and } \|w\|_{L^\infty(\Omega\setminus\Omega_\delta)} \leq M.$$

Then there exists a constant $C > 0$ only depending on M and δ such that

$$\|\nabla w\|_{L^\infty(\partial\Omega)} \leq C.$$

Next, we will give the estimate of the solution for the problem (3.1).

Theorem 3.8. If u_p is a weak solution to the problem (3.1) and f satisfies the conditions $(H_2)-(H_4)$, then u_p satisfies

$$\|u_p\|_{W_0^{1,p}(\Omega)} \le C', \tag{3.4}$$

where the constant $C' > 0$ is not dependent on p.

Proof. By $1 < p < p_0$, Proposition 3.4, Proposition 3.5 with $v_2 = u_p$, $v_1 = \phi_1$ and Young's inequality, we have

$$\begin{aligned}
\int_\Omega \frac{f(u_p)}{u_p^{p-1}}\phi_1^p dx &= \int_\Omega -\mathrm{div}(|\nabla u_p|^{p-2}\nabla u_p)\frac{\phi_1^p}{u_p^{p-1}}dx \\
&= \int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla(\frac{\phi_1^p}{u_p^{p-1}})dx \\
&\le \int_\Omega |\nabla \phi_1|^p dx \\
&\le \frac{p}{p_0}\int_\Omega |\nabla \phi_1|^{p_0} dx + \frac{p_0-p}{p_0}|\Omega| \\
&\le \int_\Omega |\nabla \phi_1|^{p_0} dx + |\Omega| \\
&\le \lambda_1 + |\Omega|.
\end{aligned} \tag{3.5}$$

By the condition (H_3), there exists a constant $C_3 > 0$ such that

$$s^{\theta-1} \le C_3 f(s), \text{ for } s \ge k_1,$$

that is

$$s^{\theta-p} \le C_3\frac{f(s)}{s^{p-1}}, \text{ for } s \ge k_1, \tag{3.6}$$

where $k_1 = \max\{k_0, 1\}$ and k_0 is given by (H_3).
Indeed, from (H_3), it holds

$$\frac{\theta}{t} \le \frac{f(t)}{F(t)}, \text{ for } t \ge k_0. \tag{3.7}$$

Setting $k_1 = \max\{k_0, 1\}$ and integrating the above inequality (3.7) with respect to t on the interval $[k_1, s]$, one has

$$\theta \ln\frac{s}{k_1} \le \ln\frac{F(s)}{F(k_1)}, \text{ for } s \ge k_1.$$

That is

$$F(s) \ge F(k_1)(\frac{s}{k_1})^\theta, \text{ for } s \ge k_1. \tag{3.8}$$

Setting $C_3 := \frac{k_1^\theta}{\theta F(k_1)}$ in (3.8), we get

$$F(s) \ge \frac{s^\theta}{\theta C_3}, \text{ for } s \ge k_1. \tag{3.9}$$

Considering (3.9) and $sf(s) \geq \theta F(s)$, for $s \geq k_1$, one gets the inequality (3.6).
Now, taking into account (3.5), (3.6) with $s = u_p$ and Young's inequality, we get

$$\begin{aligned}
\int_\Omega u_p^{\theta-p}\phi_1^p dx &= \int_{\{0\leq u_p\leq k_1\}} u_p^{\theta-p}\phi_1^p dx + \int_{\{u_p>k_1\}} u_p^{\theta-p}\phi_1^p dx \\
&\leq k_1^{\theta-p}\int_{\{0\leq u_p\leq k_1\}} \phi_1^p dx + C_3\int_{\{u_p>k_1\}} \frac{f(u_p)}{u_p^{p-1}}\phi_1^p dx \\
&\leq k_1^{\theta-p}\int_\Omega \phi_1^p dx + C_3\int_{\{u_p>k_1\}} \frac{f(u_p)}{u_p^{p-1}}\phi_1^p dx \\
&= k_1^{\theta-p}\int_\Omega \phi_1^p dx + C_3\int_\Omega \frac{f(u_p)}{u_p^{p-1}}\phi_1^p dx - C_3\int_{\{0<u_p\leq k_1\}} \frac{f(u_p)}{u_p^{p-1}}\phi_1^p dx \\
&\leq k_1^{\theta-p}\int_\Omega \phi_1^p dx + (\lambda_1+|\Omega|)C_3\int_\Omega \phi_1^p dx \\
&\leq (k_1^{\theta-p} + (\lambda_1+|\Omega|)C_3)\int_\Omega \phi_1^p dx \\
&\leq (k_1^{\theta-p} + (\lambda_1+|\Omega|)C_3)(\frac{p}{p_0}\int_\Omega \phi_1^{p_0} dx + \frac{p_0-p}{p_0}|\Omega|) \\
&\leq (k_1^\tau + (\lambda_1+|\Omega|)C_3)(|\Omega|+1) := C_4,
\end{aligned} \tag{3.10}$$

where $\lambda_1+|\Omega|$ is given by (3.5) and $-C_3\int_{\{0<u_p\leq k_0\}}\frac{f(u_p)}{u_p^{p-1}}\phi_1^p dx \leq 0$ is given by the condition (H_1) ($f(s) \geq 0$, for all $s \geq 0$) respectively, and the last inequality is given by Proposition 3.4 with $\int_\Omega \phi_1^{p_0}dx = 1$ and $k_1 = \max\{k_0, 1\} \geq 1$. By Proposition 3.3 and (3.10), for any $x \in \Omega \setminus \Omega_\delta$, we have that

$$\begin{aligned}
\sigma(\inf_{x\in\Omega_{\frac{\delta}{2}}}\phi_1^p)[u_p(x)]^{\theta-p} &\leq \int_{I_x}[u_p(y)]^{\theta-p}\phi_1^p(y)dy \\
&\leq \int_\Omega [u_p(y)]^{\theta-p}\phi_1^p(y)dy \\
&\leq C_4,
\end{aligned}$$

i.e.,

$$\begin{aligned}
u_p(x) &\leq (\frac{C_4}{\sigma\inf_{x\in\Omega_{\frac{\delta}{2}}}\phi_1^p})^{\frac{1}{\theta-p}} \\
&= (\frac{C_4}{\sigma(\inf_{x\in\Omega_{\frac{\delta}{2}}}\phi_1)^p})^{\frac{1}{\theta-p}} \\
&= (\frac{C_4}{\sigma})^{\frac{1}{\theta-p}}(\inf_{x\in\Omega_{\frac{\delta}{2}}}\phi_1)^{-\frac{p}{\theta-p}} \\
&\leq (\frac{C_4}{\sigma}+1)^{\frac{1}{\theta-p_0}}[(\inf_{x\in\Omega_{\frac{\delta}{2}}}\phi_1)^{-\frac{1}{\theta-1}} + (\inf_{x\in\Omega_{\frac{\delta}{2}}}\phi_1)^{-\frac{p_0}{\theta-p_0}}] := C_5,
\end{aligned} \tag{3.11}$$

where the constant C_5 may be depend on C_4, σ, θ, p_0 and ϕ_1 by (3.11), but are independent of p. Estimate (3.11) gives the uniform L^∞ bounds near the boundary: $\exists\, \delta > 0$ and $C_5 > 0$ such that

$$||u_p||_{L^\infty(\Omega\setminus\Omega_\delta)} \leq C_5, \tag{3.12}$$

for $\forall\, u_p \in W_0^{1,p}(\Omega)$ satisfying the problem (3.1). On the other hand, from the condition (H_2) and (3.12), we have

$$\|f(u_p)\|_{L^\infty(\Omega\setminus\Omega_\delta)} \le C_1(1 + \|u_p\|^{\frac{1}{N-1}}_{L^\infty(\Omega\setminus\Omega_\delta)}) \le C_6. \tag{3.13}$$

It is clear that $f(u_p(\cdot)) \in L^{(p^*)'}$ is given by the condition (H_2) and Sobolev embedding. By Proposition 3.7, (3.12) and (3.13), we get

$$\|\frac{\partial u_p}{\partial\gamma}\|_{L^\infty(\partial\Omega)} \le C_7, \tag{3.14}$$

where the constant $C_7 > 0$ is only depending on C_5, C_6 and δ. By Proposition 3.6 (Pohozaev's identity)

$$p^*\int_\Omega F(s)dx - \int_\Omega f(u_p)u_p dx = \frac{p-1}{N-p}\int_{\partial\Omega}|\frac{\partial u_p}{\partial\gamma}|^p(x\cdot\gamma)dH^{N-1},$$

$p^* = \frac{Np}{N-p} > \frac{N}{N-1} = 1^*$ and (H_4), there exists a large enough constant $k_2 > 0$ such that

$$\begin{aligned} f(s)s &\le C_2(1^*F(s) - f(s)s) \\ &\le C_2(p^*F(s) - f(s)s) \end{aligned} \tag{3.15}$$

as $s \ge k_2$, so that by the condition (H_2) and taking $s = u_p$ in (3.15)

$$\begin{aligned} \int_\Omega |\nabla u_p|^p dx &= \int_\Omega f(u_p)u_p dx \\ &= \int_{\{0<u_p\le k_2\}} f(u_p)u_p dx + \int_{\{u_p>k_2\}} f(u_p)u_p dx \\ &\le C_1k_2(1 + k_2^{\frac{1}{N-1}})|\Omega| + C_2\frac{p-1}{N-p}C_7|\partial\Omega| \\ &\le C_1k_2(1 + k_2^{\frac{1}{N-1}})|\Omega| + C_2\frac{p_0-1}{N-p_0}C_7|\partial\Omega| := C_8. \end{aligned} \tag{3.16}$$

That is

$$\|u_p\|_{W_0^{1,p}(\Omega)} \le C_8^{\frac{1}{p}} \le C_8 + 1 := C'.$$

From the definitions of C_5, C_6, C_7 and C_8, i.e., (3.11)–(3.14) and (3.16), we obtain that the constant C' is not dependent on p. The proof of Theorem 3.8 is completed.

The following existence result holds.

Theorem 3.9. Let f satisfy the conditions (H_1), (H_2), (H_3) and (H_5). Then there exists a nontrivial positive solution u_p to the problem (3.1).

Proof. By the conditions (H_1), (H_2), (H_3) and (H_5), it is well known that there exists a nontrivial solution $u_p \ge 0$ to the problem (3.1). The positive solution u_p is obtained using the mountain pass lemma by Ambrosetti and Rabinowitz [2] for the following truncated functional $J_p^+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$J_p^+(w) = \frac{1}{p}\int_\Omega |\nabla w|^p dx - \int_\Omega F_+(w)dx, \tag{3.17}$$

where $F_+(s) = \int_0^s f_+(t)dt$ and

$$f_+(s) = \begin{cases} f(s), & s \ge 0, \\ 0, & s < 0. \end{cases} \tag{3.18}$$

We claim that J_p^+ satisfies the structure of mountain pass lemma and the $(P-S)$ condition.
Indeed, by the condition (H_5), 0 is a local minimum of J_p^+. From the condition (H_3), there exist two constants $\widetilde{C}, \widehat{C} > 0$, such that

$$F_+(s) \geq \widetilde{C}s^\theta - \widehat{C},$$

for all $s \in [0, +\infty)$ with $\theta > 1$. This implies that

$$J_p^+(w) \leq \frac{1}{p}\|w\|^p_{W_0^{1,p}} - \widetilde{C}\|w\|^\theta_{L^\theta} + \widehat{C}|\Omega|, \tag{3.19}$$

for $\forall\ w \in W_0^{1,p}(\Omega)$. We can choose a $w_0 \in W_0^{1,p}(\Omega)$ and $\|w_0\|_{W_0^{1,p}} = 1$ such that

$$J_p^+(tw_0) \leq \frac{t^p}{p} - \widetilde{C}t^\theta\|w_0\|^\theta_{L^\theta} + \widehat{C}|\Omega| \to -\infty,$$

as $t \to +\infty$, with $1 < p < p_0 := \min\{\theta, \frac{N}{N-1}\}$. Whence there exists a large number $t_0 > 0$ such that

$$J_p^+(t_0w_0) < 0. \tag{3.20}$$

We set $e := t_0w_0 \in W_0^{1,p}(\Omega)$. Since (H_2) and the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{1^*}(\Omega)$, $1^* = \frac{N}{N-1} < \frac{Np}{N-p} = p^*$, is compact, we obtain that f_+ satisfies the subcritical grow, i.e.,

$$|f_+(s)| \leq C_1(1 + s^{1^*-1}),\ \text{with}\ 1^* < p^*. \tag{3.21}$$

Considering (3.21) and (H_3), J_p^+ satisfies the $(P-S)$ condition.

4. The proof of Theorem 1.2

In this section we prove our main results concerning the case $p = 1$, namely Theorem 1.2. Under the same assumption of Theorem 1.2, we divide the proof into few steps.
Step 1. Existence of a solution u and a field z.
Step 2. $(z, Du) = |Du|$ as measures in Ω.
Step 3. $[z, \gamma] \in \text{sign}(-u)$ on $\partial\Omega$.
Step 4. The monotonicity of solution u.
Step 5. $u \in L^\infty(\Omega)$.
Step 6. u is nontrivial.

Step 1. Existence of a solution u for the problem (1.1) and existence of a field $z \in \mathcal{DM}^\infty(\Omega)$ satisfying (1.4) and $\|z\|_{L^\infty} \leq 1$.
Proof of Step 1: From Theorem 3.8, we obtain that u_p is bounded in $W_0^{1,p}(\Omega) \hookrightarrow L^m(\Omega)$, with $1 \leq m \leq \frac{N}{N-1} < p^* = \frac{Np}{N-p}$, $1 < p < p_0 < 2 \leq N$.

$$u_p \to u \text{ strongly in } L^m(\Omega), \tag{4.1}$$

$$u_p(x) \to u(x) \text{ a.e. } x \in \Omega, \tag{4.2}$$

$$\exists\ g \in L^m(\Omega),\ \text{such that}\ |u_p(x)| \leq g(x), \tag{4.3}$$

as $p \to 1^+$.
Next, we will show that there exists a vector field z satisfying (1.4). Recalling Theorem 3.8, we obtain that $\{u_p\}$ is bounded in $W_0^{1,p}(\Omega) \subset BV(\Omega)$. So that for $1 \le r < p' = \frac{p}{p-1}$, we have

$$\begin{aligned}\int_\Omega ||\nabla u_p|^{p-2}\nabla u_p|^r dx &= \int_\Omega |\nabla u_p|^{r(p-1)} dx \\ &\le (\int_\Omega |\nabla u_p|^p dx)^{\frac{r}{p'}} |\Omega|^{1-\frac{r}{p'}},\end{aligned}$$

and thus

$$\||\nabla u_p|^{p-2}\nabla u_p\|_{L^r(\Omega)} \le C_8^{\frac{1}{p'}} |\Omega|^{\frac{1}{r}-\frac{1}{p'}}, \tag{4.4}$$

where the constant C_8 is given by (3.16). This implies that $|\nabla u_p|^{p-2}\nabla u_p$ is bounded in $L^r(\Omega; \mathbb{R}^N)$ with respect to p. Then there exists $z_r \in L^r(\Omega; \mathbb{R}^N)$ such that

$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup z_r, \text{ weakly in } L^r(\Omega; \mathbb{R}^N), \tag{4.5}$$

as $p \to 1^+$. A standard diagonal argument shows that there exists a unique vector field z which is defined on Ω independently of r, such that

$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup z, \text{ weakly in } L^r(\Omega; \mathbb{R}^N), \tag{4.6}$$

as $p \to 1^+$. By applying the semicontinuity of the L^r norm the previous inequality (4.4) implies

$$\|z\|_{L^r(\Omega)} \le \liminf_{p\to 1^+} \||\nabla u_p|^{p-2}\nabla u_p\|_{L^r} \le |\Omega|^{\frac{1}{r}}, \ \forall\, r < \infty,$$

so that, letting $r \to \infty$ we have $z \in L^\infty(\Omega; \mathbb{R}^N)$ and

$$\|z\|_{L^\infty(\Omega;\mathbb{R}^N)} \le 1.$$

Using $\varphi \in C_c^1(\Omega)$ with $\varphi \ge 0$ as a test function in (3.1), we have

$$\int_\Omega |\nabla u_p|^{p-1}\nabla u_p \nabla\varphi dx = \int_\Omega f(u_p)\varphi dx. \tag{4.7}$$

Taking $p \to 1^+$ in the left hand side of (4.7) and by (4.6), we get

$$\lim_{p\to 1^+} \int_\Omega |\nabla u_p|^{p-1}\nabla u_p \nabla\varphi dx = \int_\Omega z \cdot \nabla\varphi dx, \tag{4.8}$$

for $\forall\ \varphi \in C_c^1(\Omega)$. On the other hand, thanks to (4.2) and $f(s)$ a locally Lipschitz continuous function, we have

$$f(u_p(x)) \to f(u(x)), \text{ a.e. } x \in \Omega.$$

Moreover, we deduce from (H_2) and (4.3) that

$$|f(u_p(\cdot))| \le C_1(1 + |u_p(\cdot)|^{\frac{1}{N-1}}) \le C_1(1 + |g(\cdot)|^{\frac{1}{N-1}}) \in L^N(\Omega).$$

Consequently, by the Dominated Convergence Theorem, we get

$$\lim_{p\to 1^+}\int_\Omega f(u_p(x))\varphi(x)dx=\int_\Omega f(u(x))\varphi(x)dx, \tag{4.9}$$

for $\forall\ \varphi\in C_c^1(\Omega)$. Therefore, (4.7), (4.8) and (4.9) imply that

$$-\mathrm{div}z=f(u)\ \text{in}\ \mathcal{D}'(\Omega). \tag{4.10}$$

Step 2. $(z,Du)=|Du|$ as measures in Ω.
Before proving $(z,Du)=|Du|$, we need the following lemma for which one can refer to [9].
Lemma 4.1 ([9])**.** Under the same assumptions of Theorem 1.2, the following identity holds

$$-\int_\Omega u\varphi\mathrm{div}z dx=\int_\Omega f(u)u\varphi dx, \tag{4.11}$$

for $\forall\ \varphi\in C_c^1(\Omega)$.
Proof of Step 2: We take $u_p\varphi\in W_0^{1,p}(\Omega)$ as a test function in (3.1) with $0\le\varphi\in C_c^1(\Omega)$, $\max_{x\in\Omega}|\varphi(x)|=M_0$ and get

$$\int_\Omega|\nabla u_p|^p\varphi dx+\int_\Omega u_p|\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi dx=\int_\Omega f(u_p)u_p\varphi dx. \tag{4.12}$$

By Young's inequality and Fatou's Lemma, we estimate the first integral term in (4.12)

$$\begin{aligned}\int_\Omega|Du|\varphi dx &\le \liminf_{p\to 1^+}\int_\Omega|\nabla u_p|\varphi dx\\ &\le \liminf_{p\to 1^+}[\frac{1}{p}\int_\Omega|\nabla u_p|^p\varphi dx+\frac{p-1}{p}\int_\Omega\varphi dx]\\ &= \liminf_{p\to 1^+}\int_\Omega|\nabla u_p|^p\varphi dx\end{aligned} \tag{4.13}$$

On the other hand, by (4.6) we have

$$\lim_{p\to 1^+}\int_\Omega u_p|\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi dx=\int_\Omega uz\cdot\nabla\varphi dx. \tag{4.14}$$

From

$$|f(u_p)u_p\varphi|\le M_0C_1|u_p|(1+|u_p|^{\frac{1}{N-1}})\le M_0C_1|g(\cdot)|(1+|g(\cdot)|^{\frac{1}{N-1}})\in L^1(\Omega),$$

and the Dominated Convergence Theorem, we obtain the right hand side of (4.12) is as follows

$$\lim_{p\to 1^+}\int_\Omega f(u_p)u_p\varphi dx=\int_\Omega f(u)u\varphi dx. \tag{4.15}$$

From (4.12)–(4.15), we have

$$\int_\Omega|Du|\varphi dx+\int_\Omega uz\cdot\nabla\varphi dx\le\int_\Omega f(u)u\varphi dx. \tag{4.16}$$

By (4.16) and Lemma 4.1, we also have

$$\int_\Omega |Du|\varphi dx + \int_\Omega uz \cdot \nabla\varphi dx \le -\int_\Omega u\varphi \mathrm{div} z dx.$$

Therefore, by (2.1), we get

$$\int_\Omega |Du|\varphi dx \le -\int_\Omega uz \cdot \nabla\varphi dx - \int_\Omega u\varphi \mathrm{div} z dx = \int_\Omega (z, Du)\varphi dx.$$

The arbitrariness of φ implies that

$$|Du| \le (z, Du)$$

as measures in Ω. On the other hand, since $\|z\|_{L^\infty} \le 1$, and

$$(z, Du) \le \|z\|_{L^\infty}|Du| \le |Du|$$

as measures in Ω, we have

$$|Du| = (z, Du).$$

Step 3. The boundary condition $[z, \gamma] \in \mathrm{sign}(-u)$ on $\partial\Omega$.
Proof of Step 3: It is easy to check that this fact is equivalent to show

$$\int_{\partial\Omega} (|u| + u[z, \gamma]) dH^{N-1} = 0. \tag{4.17}$$

Choosing u_p as a test function in (3.1), we have

$$\int_\Omega |\nabla u_p|^p dx = \int_\Omega f(u_p)u_p dx.$$

Since $u_p \in W_0^{1,p}(\Omega)$ is bounded, by the fact that $u_p = 0$ on $\partial\Omega$ and Young's inequality, we get

$$\begin{aligned} \int_\Omega |\nabla u_p| dx + \int_{\partial\Omega} |u_p| dH^{N-1} &\le \frac{1}{p}\int_\Omega |\nabla u_p|^p dx + \frac{p-1}{p}|\Omega| \\ &= \frac{1}{p}\int_\Omega f(u_p)u_p dx + \frac{p-1}{p}|\Omega|. \end{aligned} \tag{4.18}$$

We use the lower semicontinuity (4.18) to pass to the limit as $p \to 1^+$ and obtain

$$\begin{aligned} \int_\Omega |Du| dx + \int_{\partial\Omega} |u| dH^{N-1} &\le \liminf_{p\to 1^+}\left(\int_\Omega |\nabla u_p| dx + \int_{\partial\Omega} |u_p| dH^{N-1}\right) \\ &\le \liminf_{p\to 1^+}\left[\frac{1}{p}\int_\Omega f(u_p)u_p dx + \frac{p-1}{p}|\Omega|\right] \\ &= \int_\Omega f(u)u dx, \end{aligned} \tag{4.19}$$

where the last equality is given by the Dominated Convergence Theorem and

$$|f(u_p)u_p| \le C_1|u_p|(1 + |u_p|^{\frac{1}{N-1}}) \le C_1|g(\cdot)|(1 + |g(\cdot)|^{\frac{1}{N-1}}) \in L^1(\Omega).$$

Furthermore, by Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned}\int_\Omega f(u)udx &= -\int_\Omega u\text{div}zdx \\ &= \int_\Omega (z, Du)dx - \int_{\partial\Omega} u[z,\gamma]dH^{N-1}.\end{aligned} \tag{4.20}$$

From $(z, Du) = |Du|$, (4.19) and (4.20), we have

$$\int_{\partial\Omega} (|u| + u[z,\gamma])dH^{N-1} \le 0. \tag{4.21}$$

The inequality (4.21) and $|u| \ge |u|\|z\|_{L^\infty} \ge |u[z,\gamma]| \ge -u[z,\gamma]$ give the desired equality (4.17) and we conclude that

$$[z,\gamma] \in \text{ sign}(-u) \text{ on } \partial\Omega.$$

Step 4. The monotonicity of the solution u of problem (1.1).
Proof of Step 4: By Proposition 3.2, we obtain u_p satisfies the following result. For any direction ν and μ in the interval $(a(\nu), \mu_1(\nu)]$, then

$$u_p(x) \le u_p(x^\nu_\mu), \ \forall\ x \in \Omega^\nu_\mu, \tag{4.22}$$

where $a(\nu)$ and $\mu_1(\nu)$ are given by (1.7) and (1.8). Considering this fact and $u_p(x) \to u(x)$ a.e. in Ω, taking $p \to 1^+$ in (4.22), we have

$$u(x) \le u(x^\nu_\mu), \text{ a.e. } x \in \Omega^\nu_\mu. \tag{4.23}$$

We get the result of monotonicity for the solution u. Inequality (4.23) also holds for any $\mu \in (a(\nu), \mu_2(\nu)]$ by Proposition 3.2, if f is locally Lipschitz continuous, and $a(\nu)$ and $\mu_2(\nu)$ are given by (1.7) and (1.9).

Step 5. The boundedness of the solution u, i.e., $u \in L^\infty(\Omega)$.
Before proving $u \in L^\infty(\Omega)$, we need to prove the following lemma.
Lemma 4.2. For every $\varepsilon > 0$ there exists $k_3 > 0$ which does not depend on p, such that

$$\int_{A_k} (1 + u_p^{\frac{1}{N-1}})^N dx < \varepsilon \tag{4.24}$$

for every $k \ge k_3$ and $\forall\ p \in (1, p_0)$, with $A_k = \{x \in \Omega \mid u_p(x) > k\}$.
Proof of Lemma 4.2: Using Sobolev embedding $W_0^{1,p}(\Omega) \subset BV(\Omega) \hookrightarrow L^{\frac{N}{N-1}}(\Omega)$, Theorem 3.8 and Holder's inequality, we obtain that

$$\begin{aligned}|A_k|^{\frac{N-1}{N}} &\le \frac{1}{k}\left(\int_{A_k} u_p^{\frac{N}{N-1}} dx\right)^{\frac{N-1}{N}} \\ &\le \frac{1}{k} S_1 \int_{A_k} |\nabla u_p| dx\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{S_1}{k}|A_k|^{\frac{p-1}{p}}(\int_{A_k}|\nabla u_p|^p dx)^{\frac{1}{p}} \\
&\leq \frac{S_1}{k}|\Omega|^{\frac{p-1}{p}}C_8^{\frac{1}{p}} \\
&\leq \frac{S_1}{k}(1+|\Omega|)(C_8+1), \qquad (4.25)
\end{aligned}
$$

where S_1 is given by the best Sobolev constant

$$S_1 = \frac{\{\Gamma(1+\frac{N}{2})\}^{\frac{1}{N}}}{\sqrt{\pi}N},$$

see [26, 39], and $|A_k|$ stands for its N dimensional Lebesgue measure. Inequality (4.25) implies that $\lim_{k\to\infty}|A_k| = 0$. It holds that for $\forall\ \varepsilon > 0$, there exists a large number $k_4 > 0$ such that

$$|A_k| < \frac{\varepsilon}{2^N},\ \text{for all } k \geq k_4. \qquad (4.26)$$

On the other hand, by Sobolev embedding $u_p \in W_0^{1,p}(\Omega) \subset BV(\Omega) \hookrightarrow L^{\frac{N}{N-1}}(\Omega)$, Theorem 3.8 and (4.3), we get

$$u_p \in L^{\frac{N}{N-1}}(\Omega)$$

and

$$0 \leq \int_{A_k}|u_p(x)|^{\frac{N}{N-1}}dx \leq \int_{A_k}|g(x)|^{\frac{N}{N-1}}dx, \qquad (4.27)$$

which implies that $u_p(x) < \infty$ a.e. in Ω. Considering (4.27), $\lim_{k\to\infty}|A_k| = 0$ and by absolute continuity of integrable function, we have

$$\lim_{k\to\infty}\int_{A_k}|u_p(x)|^{\frac{N}{N-1}}dx \leq \lim_{k\to\infty}\int_{A_k}|g(x)|^{\frac{N}{N-1}}dx = 0. \qquad (4.28)$$

From (4.28), for $\forall\ \varepsilon > 0$, $\exists\ k_5 > 0$ large enough (not depend on p) and $\delta > 0$ small enough such that as $k \geq k_5$, we have $|A_k| < \delta$ and

$$\int_{A_k}|u_p(x)|^{\frac{N}{N-1}}dx \leq \int_{A_k}|g(x)|^{\frac{N}{N-1}}dx < \frac{\varepsilon}{2^N}, \qquad (4.29)$$

From (4.26) and (4.29), we obtain

$$
\begin{aligned}
\int_{A_k}(1+u_p^{\frac{1}{N-1}})^N dx &\leq 2^{N-1}(|A_k| + \int_{A_k}|u_p(x)|^{\frac{N}{N-1}}dx) \\
&\leq 2^{N-1}(\frac{\varepsilon}{2^N}+\frac{\varepsilon}{2^N}) = \varepsilon,
\end{aligned}
$$

for all $k \geq k_3 := \max\{k_4, k_5\}$. The proof of Lemma 4.2 is completed.

Proof of Step 5: Next, we would like to use Stampacchia truncation [38] to prove the boundedness of the positive solution u. For every $k > 0$, we define the auxiliary function $G_k : [0,\infty) \to \mathbb{R}$ as

$$G_k(s) = \begin{cases} s-k, & s > k, \\ 0, & 0 < s \leq k. \end{cases} \qquad (4.30)$$

Then, choosing $G_k(u_p)$ as a test function in (3.1), we get

$$\int_\Omega |\nabla G_k(u_p)|^p dx = \int_\Omega f(u_p)G_k(u_p)dx. \tag{4.31}$$

By (4.31), (H_2), Sobolev embedding, Young's inequality and Holder's inequality, we have

$$\begin{aligned}
(\int_\Omega G_k(u_p)^{\frac{N}{N-1}}dx)^{\frac{N-1}{N}} &\le S_1\int_\Omega |\nabla G_k(u_p)|dx \\
&\le \frac{S_1}{p}\int_\Omega |\nabla G_k(u_p)|^p dx + \frac{S_1(p-1)}{p}|\Omega| \\
&= \frac{S_1}{p}\int_\Omega f(u_p)G_k(u_p)dx + \frac{S_1(p-1)}{p}|\Omega| \\
&\le \frac{S_1}{p}C_1\int_\Omega (1+u_p^{\frac{1}{N-1}})G_k(u_p)dx + \frac{S_1(p-1)}{p}|\Omega| \\
&\le S_1C_1[\int_{A_k}(1+u_p^{\frac{1}{N-1}})^N dx]^{\frac{1}{N}}(\int_{A_k}G_k(u_p)^{\frac{N}{N-1}}dx)^{\frac{N-1}{N}} \\
&\quad + \frac{S_1(p-1)}{p}|\Omega|.
\end{aligned} \tag{4.32}$$

By Lemma 4.2 and taking $\varepsilon = \frac{1}{(2C_1S_1)^N}$, there exists $k_3 > 0$ which does not depend on p, such that

$$\int_{A_k}(1+u_p^{\frac{1}{N-1}})^N dx < \frac{1}{(2C_1S_1)^N}, \tag{4.33}$$

for all $k \ge k_3$ and $p \in (1, p_0)$. Consequently, from (4.32) and (4.33) we obtain

$$0 \le \int_\Omega G_k(u_p)^{\frac{N}{N-1}}dx \le [\frac{2S_1(p-1)|\Omega|}{p}]^{\frac{N}{N-1}}. \tag{4.34}$$

Since $u_p(x) \to u(x)$ a.e. $x \in \Omega$ and Fatou's Lemma, we can pass to the limit on $p \to 1^+$ in (4.34), to conclude that

$$\int_\Omega (u(x)-k)^{\frac{N}{N-1}}dx = 0,$$

for $\forall\ k \ge k_3 > 0$. Thus $u \in L^\infty(\Omega)$.

Step 6. u is nontrivial.

Proof of Step 6: For $\forall\ v \in BV(\Omega)$, we define the functional $J^+ : BV(\Omega) \to \mathbb{R}$ as

$$J^+(v) = \int_\Omega |Dv| + \int_{\partial\Omega}|v|dH^{N-1} - \int_\Omega F_+(v)dx,$$

where $F_+(s) = \int_0^s f_+(t)dt$ and f_+ is given by (3.18).

We will say that $v_0 \in BV(\Omega)$ is a critical point of J^+ if there exists $z \in \mathcal{DM}^\infty(\Omega)$ with $\|z\|_{L^\infty} \le 1$ such that

$$-\int_\Omega \varphi \operatorname{div} z dx = \int_\Omega f(v_0)\varphi dx, \text{ for all } \varphi \in C_c^1(\Omega),$$

$$(z, Dv_0) = |Dv_0| \text{ as measures in } \Omega,$$

$$[z, \gamma] \in \text{sign}(-v_0) \text{ on } \partial\Omega,$$

where γ is the unit exterior normal on $\partial\Omega$. The critical points of J^+ coincide with solutions to the problem (1.1) in the sense Definition 1.1, for which one can refer to [9] or [35].

We shall show that 0 is a local minimum of J^+.

Indeed, by the condition (H_5), there exists small enough $\delta > 0$ such that

$$|f(s)| \le C_9 |s|^\alpha,$$

for $\forall\ |s| \in (0, \delta)$ and for some constant $C_9 > 0$ with $\alpha \in (0, \frac{1}{N-1})$. Moreover, by the definition of $F_+(s)$, we have

$$F_+(s) = \int_0^s f_+(t)dt \le \int_0^s |f(t)|dt \le \frac{C_9}{1+\alpha}|s|^{1+\alpha} \tag{4.35}$$

for $\forall\ |s| \in (0, \delta)$. By (4.35) and the norm $\|v\|_{BV} = \int_\Omega |Dv| + \int_{\partial\Omega} |v| dH^{N-1}$, $v \in BV(\Omega)$, it holds

$$\begin{aligned} J^+(v) &= \|v\|_{BV} - \int_\Omega F_+(v)dx \\ &\ge \|v\|_{BV} - \frac{C_9}{1+\alpha}\int_\Omega |v|^{1+\alpha}dx \\ &\ge \|v\|_{BV} - C_{10}\|v\|_{BV}^{1+\alpha}, \end{aligned}$$

where the last inequality is given by the embedding $BV(\Omega) \hookrightarrow L^{1+\alpha}(\Omega)$, $\alpha \in (0, \frac{1}{N-1})$. Choosing a positive constant $\rho < \min\{\delta, (\frac{1}{2C_{10}})^{\frac{1}{\alpha}}\}$, we obtain

$$J^+(v) \ge \frac{1}{2}\|v\|_{BV} > 0, \tag{4.36}$$

for $\forall\ v \in BV(\Omega)$ and $\|v\|_{BV} \le \rho$. This implies that 0 is a local minimum of J^+.
Now, we introduce the auxiliary functional

$$I_p(w) = J_p^+(w) + \frac{p-1}{p}|\Omega|, \tag{4.37}$$

where J_p^+ is given by (3.17). By Young's inequality and (4.18), we can fix $p \in (1, p_0)$ and obtain

$$\begin{aligned} I_p(w) &= J_p^+(w) + \frac{p-1}{p}|\Omega| \\ &= \frac{1}{p}\int_\Omega |\nabla w|^p dx - \int_\Omega F_+(w)dx + \frac{p-1}{p}|\Omega| \\ &\ge \int_\Omega |\nabla w|dx + \int_{\partial\Omega} |w| dH^{N-1} - \int_\Omega F_+(w)dx = J^+(w), \end{aligned} \tag{4.38}$$

for $\forall\ w \in W_0^{1,p}(\Omega) \subset BV(\Omega)$, with $p_0 = \min\{\theta, \frac{N}{N-1}\}$. From (4.38) and (3.19), one gets

$$J^+(w) \le J_p^+(w) + \frac{p-1}{p}|\Omega| \le \frac{1}{p}\|w\|^p_{W_0^{1,p}} - \widetilde{C}\|w\|^\theta_{L^\theta} + (\frac{p-1}{p} + \widehat{C})|\Omega|, \tag{4.39}$$

for all $w \in W_0^{1,p}(\Omega)$ with $p < p_0 < \theta$. Recalling the structure of mountain pass lemma in Theorem 3.9, we can deduce that there exists $e = t_0 w_0 \in W_0^{1,p}(\Omega) \subset BV(\Omega)$ and $\|e\|_{BV} > \rho$ such that $J(e) < 0$ by (3.20).

Obviously, the critical points of I_p are identical with the critical points of J_p^+. Then u_p given by Theorem 3.9 is a critical point of J_p^+, and also a critical point of I_p, which implies that the critical point u_p satisfies

$$I_p(u_p) = \inf_{\eta\in\Gamma_p} \max_{t\in[0,1]} I_p(\eta(t)), \tag{4.40}$$

where $\Gamma_p = \{\eta \in C([0,1], W_0^{1,p}(\Omega)) \mid \eta(0) = 0, \eta(1) = e\}$. Considering any path $\eta \in \Gamma_p$ and the continuity of the map $t \to I_p(\eta(t))$, there exists $t_0 > 0$ such that $\|\eta(t_0)\|_{BV} = \rho$. From (4.36), (4.38), (4.40) and $\|\eta(t_0)\|_{BV} = \rho$, we obtain that

$$I_p(u_p) = \inf_{\eta\in\Gamma_p} \max_{t\in[0,1]} I_p(\eta(t)) \geq \frac{\rho}{2}. \tag{4.41}$$

On the other hand, choosing u_p as a test function in (3.1), by the Dominated Convergence Theorem, (4.2) and (4.20), we have

$$\begin{aligned} \lim_{p\to1^+} \frac{1}{p}\int_\Omega |\nabla u_p|^p dx &= \lim_{p\to1^+} \frac{1}{p}\int_\Omega f(u_p)u_p dx \\ &= \int_\Omega f(u)u dx \\ &= \int_\Omega (z, Du) - \int_{\partial\Omega} u[z,\gamma] dH^{N-1} \\ &= \int_\Omega |Du| + \int_{\partial\Omega} |u| dH^{N-1}, \end{aligned} \tag{4.42}$$

where the last equality is given by Step 2 and Step 3. From (H_2), (4.2) and (4.3), we can apply the Dominated Convergence Theorem to obtain

$$\lim_{p\to1^+} \int_\Omega F_+(u_p)dx = \int_\Omega F_+(u)dx. \tag{4.43}$$

By (4.37), (4.42) and (4.43), we can get

$$\lim_{p\to1^+} I_p(u_p) = \lim_{p\to1^+} [J_p^+(u_p) + \frac{p-1}{p}|\Omega|] = \lim_{p\to1^+} J_p^+(u_p) = J^+(u). \tag{4.44}$$

Summarizing (4.41) and (4.44) we obtain that

$$J^+(u) \geq \frac{\rho}{2} > 0,$$

with $0 < \rho < \min\{\delta, (\frac{1}{2C_{10}})^{\frac{1}{\alpha}}\}$, and then u is nontrivial, because $J^+(0) = 0$.
The proof of Theorem 1.2 is completed.

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Conflict of interest

For the publication of this article, no conflict of interest among the authors is disclosed.

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