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Research article

On Hankel transforms of generalized Bessel matrix polynomials

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Abstract: The present article deals with the evaluation of the Hankel transforms involving Bessel matrix functions in the kernel. Moreover, these transforms are associated with products of certain elementary functions and generalized Bessel matrix polynomials. As applications, many useful special cases are discussed. Further, the current results are more general to the previous one. In addition to, these results are yielded to more results in the modern integral transforms with special matrix functions.

Keywords: matrix Hankel transforms; generalized Bessel matrix polynomials; Bessel matrix functions
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1. Introduction

Recently, many researchers have introduced and discussed the several integral transforms (See, e.g., Fourier transform, Laplace transform, Mellin transform, Hankel transform, etc.) with various special functions (also with the new generalized special matrix functions) as kernels. These transforms play important roles not only in mathematics but also in physics, dynamical systems and engineering disciplines (See, e.g., [1–13] and the references therein).

Hankel transforms (also designated as Fourier-Bessel transforms) are type of integral transforms that involving Bessel functions as the kernel arises naturally in radial problems formulated in cylindrical polar coordinates (See [14–16]). The classical Hankel transformation defined by

$$G\{f(\xi);x\} = \int_0^\infty \sqrt{\xi x}\, J_n(\xi x)\, f(\xi)\, d\xi, \tag{1.1}$$

where $x > 0$. and $J_n(x)$ is the Bessel function of order n (See [6]).

Later on, Hankel transforms are useful tools for solving various sorts of problems in electromagnetic fields, for one-dimensional layered earth model, in the dipole antenna radiation in conductive medium and in solving boundary value problems formulated in cylindrical coordinates (See, for instance, [14–24]).

In the current study, we define the Hankel transforms and its inverse involving Bessel matrix functions [25–27] in the kernel. Moreover, we evaluate some new integrals of matrix functions involving generalized Bessel matrix polynomials [25, 28]. Interesting special cases of the main results are also deduced. The present work is a very useful in the study of boundary value problems, electromechanical problems, statistic theory, numerical calculations and computer science.

2. Definitions and lemmas

In this section, we have enclosed some basic definitions and lemmas which are useful in our main results.

Let $\mathbb{C}^d$ denote the d-dimensional complex vector space and $\mathbb{C}^{d\times d}$ denote the set of all square matrices with d rows and d columns with entries are complex numbers. I and $\mathbf{0}$ stand for the identity matrix and the null matrix in $\mathbb{C}^{d\times d}$, respectively.

Definition 2.1. (See [29]) For a matrix N in $\mathbb{C}^{d\times d}$, $\sigma(N)$ is the spectrum of N, the set of all eigenvalues of N and

$$\vartheta(N) = \max\{Re(\xi) : \xi \in \sigma(N)\}, \quad \widetilde{\vartheta}(N) = \min\{Re(\xi) : \xi \in \sigma(N)\}, \tag{2.1}$$

where $\vartheta(N)$ is referred to as the spectral abscissa of N and $\widetilde{\vartheta}(N) = -\beta(-N)$. A matrix N is said to be a positive stable if and only if $\widetilde{\vartheta}(N) > 0$.

Definition 2.2. [25, 27, 29] The logarithmic norm of a matrix N in $\mathbb{C}^{d\times d}$ is defined as

$$\beta(N) = \lim_{\varsigma\to 0} \frac{\|I + \varsigma N\| - 1}{\varsigma} = \max\{\xi : \xi \in \sigma(\frac{N + N^*}{2})\}, \tag{2.2}$$

where

$$\|N\| = \sup_{y\neq 0}\{\frac{\|Ny\|}{\|y\|}\} = \sup\{\|Ny\|, \ \|y\| = 1\}.$$

Suppose that the number $\widetilde{\beta}(N)$ is such that

$$\widetilde{\beta}(N) = -\beta(-N) = \min\{\xi : \xi \in \sigma(\frac{N + N^*}{2})\}. \tag{2.3}$$

where N^* is the transposed conjugate of N.

The reciprocal gamma function denoted by $\Gamma^{-1}(\xi) = \frac{1}{\Gamma(\xi)}$ is an entire function of the complex variable ξ. Then the image of $\Gamma^{-1}(\xi)$ acting on N denoted by $\Gamma^{-1}(N)$ is a well-defined matrix and invertible as well as

$$N + nI \quad \text{is invertible for all integers} \ \ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}. \tag{2.4}$$

By applying the matrix functional calculus, for a matrix N is positive stable in $\mathbb{C}^{d\times d}$, then from [18,26], the Pochhammer symbol of a matrix argument defined by

$$(N)_n = \begin{cases} N(N+I)\dots(N+(n-1)I) = \Gamma^{-1}(N)\Gamma(N+nI), & n \geq 1, \\ I, & n = 0. \end{cases} \tag{2.5}$$

where $\Gamma(N)$ is the gamma matrix function [26,27]

$$\Gamma(N) = \int_0^\infty e^{-u}u^{N-I}du; \quad u^{N-I} = \exp((N-I)\ln u). \tag{2.6}$$

Remark 2.1. Note that $(-N)_n = (-1)^n\,\Gamma(N+I)\Gamma^{-1}(N+(1-n)I)$, and if $N = -mI$, where m is a positive integer, then $(N)_n = \mathbf{0}$ whenever $n > m$.

Definition 2.3. [25,26] Let h and k be finite positive integers, the generalized hypergeometric matrix function is defined by the matrix power series

$${}_h\mathbf{F}_k\left[\mathbf{M};\mathbf{N};\xi\right] = \sum_{m=0}^{\infty}\prod_{i=1}^{h}(M_i)_m\prod_{j=1}^{k}[(N_j)_m]^{-1}\frac{\xi^m}{m!}, \tag{2.7}$$

where $\mathbf{M} = M_i,\quad 1 \leq i \leq h$ and $\mathbf{N} = N_j,\quad 1 \leq j \leq k$ are commuting matrices in $\mathbb{C}^{d\times d}$ with $N_j + mI$ are invertible for all integers $m \in \mathbb{N}_0$ and $1 \leq i \leq h$. More details, Abdalla discussed regions of convergence and properties of (2.7) in [25,26].

Note that for $h = 2,\ k = 1$, we get the Gauss hypergeometric matrix function ${}_2\mathbf{F}_1$ (See [25,26]).

Lemma 2.1. *[30] The following formula holds:*

$${}_2\mathbf{F}_1[-nI, M; N; 1] = (N-M)_n\,[(N)_n]^{-1},$$

where M, N and $N-M$ are positive stable and commuting matrices in $\mathbb{C}^{d\times d}$ and N satisfies the condition (2.4).

Definition 2.4. [25–27] The Bessel matrix function $J_S(u)$ of the first kind associate to S is defined in the form

$$\begin{aligned} J_S(u) &= \sum_{m=0}^{\infty}\frac{(-1)^m}{(m)!}\Gamma^{-1}(S+(m+1)I)(\frac{u}{2})^{S+2mI} \\ &= (\frac{u}{2})^S\ \Gamma^{-1}(S+I)\ {}_0\mathbf{F}_1(-;S+I,\frac{-u^2}{4}), \end{aligned} \tag{2.8}$$

where S is a matrix in $\mathbb{C}^{d\times d}$ satisfying the condition

$$\upsilon \text{ is not a negative integer for every } \upsilon \in \sigma(S). \tag{2.9}$$

Definition 2.5. [25, 28, 31] Let M and N be commuting matrices in $\mathbb{C}^{d\times d}$ such that N is an invertible matrix. For any natural number $n \in \mathbb{N}_0$, the n-th generalized Bessel matrix polynomial $\mathcal{B}_n(u; M, N)$ is defined by

$$\begin{aligned}\mathcal{B}_n(u; M, N) &= \sum_{m=0}^{n} \frac{(-1)^m}{m!}(-nI)_m(M + (n-1)I)_m(u\, N^{-1})^m \\ &= {}_2\mathbf{F}_0\left[\begin{matrix} -nI, M + (n-1)I \\ - \end{matrix} ; -u\, N^{-1}\right].\end{aligned} \tag{2.10}$$

The following lemmas are needed to find certain integral representations of the Bessel matrix function and generalized Bessel matrix polynomials

Lemma 2.2. *For $u, v, \lambda, \delta \in \mathbb{C}$, $v > 0$, $Re(\delta) > 0$, $Re(\lambda) > 0$ and S is a positive stable matrix in $\mathbb{C}^{d\times d}$ such that $S + I$ is an invertible matrix in $\mathbb{C}^{d\times d}$ and $\widetilde{\beta}(S + \lambda I) > -1$, the following formula holds:*

$$\begin{aligned}&\int_0^\infty u^{\lambda-\frac{1}{2}}\, e^{-\delta u^2}\, J_S(uv)\, \sqrt{uv}\, du \\ &= 2^{-(S+I)}\, v^{S+\frac{1}{2}I}\, \delta^{-\frac{1}{2}(S+(\lambda+1)I)}\, \Gamma^{-1}(S + I)\, \Gamma(\frac{1}{2}(S + (\lambda+1)I)) \\ &\times\ {}_1\mathbf{F}_1\left[\begin{matrix} \frac{1}{2}(S + (\lambda+1)I), \\ S + I \end{matrix} ; -\frac{v^2}{4\delta}\right],\end{aligned} \tag{2.11}$$

where $J_S(u)$ is Bessel matrix function given in (2.8).

Proof. To prove (2.11), let the left hand side equal to:

$$\begin{aligned}LHS &= \int_0^\infty u^{\lambda-\frac{1}{2}}\, e^{-\delta u^2}\, J_S(uv)\, \sqrt{uv}\, du \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\, \Gamma^{-1}(S + (m+1)I)\, (\frac{v}{2})^{S+2mI}\, v^{\frac{1}{2}} \\ &\times \int_0^\infty e^{-\delta u^2}\, u^{S+(\lambda+2m)I} du.\end{aligned}$$

Setting $w = \delta u^2$ we have

$$\begin{aligned}LHS &= \sum_{m=0}^{\infty} \frac{(-1)^m}{2m!}\, \Gamma^{-1}(S + (m+1)I)\, (\frac{v}{2})^{S+2mI}\, v^{\frac{1}{2}}\, \delta^{-mI-\frac{1}{2}(S+(\lambda+1)I)} \\ &\times \int_0^\infty e^{-w}\, w^{mI+\frac{1}{2}(S+(\lambda-1)I)}\, dw \\ &= v^{S+\frac{1}{2}I}\, 2^{-(S+I)}\, \delta^{-\frac{1}{2}(S+(\lambda+1)I)} \\ &\times \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\, \Gamma(\frac{1}{2}(S + (\lambda+1)I + mI)\, \Gamma^{-1}(S + (m+1)I)\, (v^2/4\delta)^m.\end{aligned}$$

This completes the proof of Eq (2.11) asserted by Lemma 2.2. □

Lemma 2.3. *Let S, M and P be positive stable and commuting matrices in $\mathbb{C}^{d\times d}$ such that $(1+n)I-S$ and $(2-n)I-(S+M)$, be invertible matrices. Then, we have the integral representation of generalized Bessel matrix polynomials as follows:*

$$\begin{aligned}
&\int_0^\infty u^{S-I}\, \mathcal{B}_n(1;M,\mu u)\, \mathcal{B}_m(1;P,\nu u)\, e^{-\mu u}\, du \\
=&(-1)^n \mu^{-S}\, \Gamma(S)\, (S+M-I)_n\, [(I-S)_n]^{-1} \\
&\times {}_3\mathbf{F}_2\left[\begin{array}{c} -mI, P+(m-1)I, 2I-S-M \\ \\ (1+n)I-S, (2-n)I-(S+M) \end{array}; \mu/\nu\right],
\end{aligned} \tag{2.12}$$

where $\mu,\nu \in \mathbb{C}$, $Re(\mu)>0, Re(\nu)>0, \widetilde{\beta}(S+M-(1-n)I)>0$ and $\widetilde{\beta}((1+n)I-S)>0$.

Proof. Expanding the two Bessel matrix polynomials in (2.12) by the series (2.10) and interchanging the order of integration and summation, we observe that

$$\begin{aligned}
LHS =& \sum_{k=0}^{n} (-nI)_k\, (M+(n-1)I)_k\, \frac{(-1/\mu)^k}{k!} \\
&\times \sum_{h=0}^{m} (-mI)_h\, (P+(m-1)I)_h\, \frac{(-1/\nu)^h}{h!} \\
&\times \int_0^\infty u^{S-(h+k+1)I}\, e^{-\mu u}\, du.
\end{aligned}$$

Putting $\tau = \mu u$, we have

$$\begin{aligned}
LHS =& \sum_{k=0}^{n}\sum_{h=0}^{m} (-nI)_k\, (M+(n-1)I)_k\, (-mI)_h\, (P+(m-1)I)_h\, \frac{(-1/\nu)^h\, (-1/\mu)^k}{h!\, k!} \\
&\times \mu^{-(S-(k+h)I)} \int_0^\infty \tau^{S-(k+h+1)I}\, e^{-\tau}\, d\tau \\
=& \sum_{k=0}^{n}\sum_{h=0}^{m} (-nI)_k\, (M+(n-1)I)_k\, (-mI)_h\, (P+(m-1)I)_h \\
&\times \frac{(-1)^h\, (-1/\mu)^k}{h!\, k!}\, \mu^{-(S-kI)}\, \Gamma(S-(h+k)I) \\
=& \mu^{-S} \sum_{h=0}^{m} (-mI)_h\, (P+(m-1)I)_h\, \frac{(-\mu/\nu)^h}{h!} \\
&\times \sum_{k=0}^{n} (-nI)_k\, (M+(n-1)I)_k\, \frac{\Gamma(S-hI)\, [((1+h)I-S)_k]^{-1}}{k!} \\
=& \mu^{-S}\, \Gamma(S)\, \Gamma(I-S)\Gamma(2I-(S+M))\, \Gamma^{-1}((1+n)I-S)\, \Gamma^{-1}((2-n)I-(S+M)) \\
&\times \sum_{h=0}^{m} (-mI)_h\, (P+(m-1)I)_h\, (2I-(S+M))_h \\
&\times [((1+n)I-S)_h]^{-1}\, [((2-n)I-(S+M))_h]^{-1} \frac{(\mu/\nu)^h}{h!}.
\end{aligned}$$

We thus arrive at the desired result (2.12). □

3. Matrix version of Hankel transforms and main theorems

3.1. Matrix Hankel transforms

We begin this section with defining a matrix analogue of Hankel transform and its inverse as follows.

Let us consider the generalization of the Hankel integral transform and its inverse by help of the Bessel matrix function $J_S(w)$ of the first kind associate to the matrix $S \in \mathbb{C}^{d\times d}$ in the following definition:

Definition 3.1. (Matrix Hankel Transforms) Let S be a matrix in $\mathbb{C}^{d\times d}$ satisfying (2.9) and let $\Phi(u)$ be a function defined for $u \geq 0$. The Hankel transform involving Bessel matrix function as kernel of $\Phi(u)$ is defined as

$$\Xi_S(v) \equiv \mathcal{H}_S\{\Phi(u); v\} \equiv \int_0^\infty \Phi(u)\ \sqrt{uv}\ J_S(uv)\, du, \tag{3.1}$$

where $v > 0$ and $J_S(uv)$ is Bessel matrix function of the first kind defined in (2.8).

If $\widetilde{\beta}(S) > 1/2$, Hankel's repeated integral immediately gives the inversion formula

$$\Phi(u) = \mathcal{H}_S^{-1}\{\Xi_S(v); u\} \equiv \int_0^\infty \Xi_S(v)\ \sqrt{uv}\ J_S(uv)\, dv. \tag{3.2}$$

Remark 3.1. If the matrix $S \in \mathbb{C}^{1\times 1} = \mathbb{C}$, then the Hankel transform and the inverse Hankel transform in Definition 3.1 reduce to Hankel transforms in scalar setting (See [6, 16, 23]).

Remark 3.2. The most important special cases of the Hankel transform correspond to $S = \mathbf{0}$ and $S = I$ are often useful for the solution of problems involving Laplace's equation in an axisymmetric cylindrical geometry (See [6, 20]).

Remark 3.3. The Hankel transform (3.1) and the inverse Hankel transforms (3.2) are useful in diverse engineering and physical problems and their relevant connections with other integral transforms and also with special matrix functions (For instance, see [6, 24, 25, 32–34]).

3.2. Main theorems

Now we give our main theorem, which encompass the matrix analogue of Hankel transforms of functions involving the generalized Bessel matrix polynomials.

Theorem 3.1. *Let S and N be commuting matrices in $\mathbb{C}^{d\times d}$. If*

$$\Phi(u) = u^{S+(2n+\frac{1}{2})I}\ e^{-\sigma u^2}\ \mathcal{B}_n(N; (1-2n)I - S, u^2), \tag{3.3}$$

then, we have

$$\Xi_S(v) = (2\sigma)^{-(S+(2n+1)I)}\ v^{S+(2n+\frac{1}{2})I}\ e^{\frac{-v^2}{4\sigma}}\ \mathcal{B}_n(4\sigma(I-\sigma N); (1-2n)I - S, -v^2), \tag{3.4}$$

where S is a positive stable matrix in $\mathbb{C}^{d\times d}$, $\widetilde{\beta}(S+nI) > -1$, $v > 0$ and $\sigma \in \mathbb{C}$ such that $Re(\sigma) > 0$.

Proof. To prove (3.4) substitute for $\mathcal{B}_n(N;(1-2n)I-S,u^2)$ by its series expansion in (2.10) into (3.1), we consider

$$\begin{aligned}\Xi_S(v) = &\sum_{m=0}^{n} (\frac{(-N)^m}{m!})\ (-nI)_m\ (-(S+nI))_m\\ &\times \int_0^\infty u^{S+(2n-2m+\frac{1}{2})I}\ e^{-\sigma u^2}\ J_S(uv)\ \sqrt{uv}\ du.\end{aligned}$$

Applying Lemma 2.2, we get

$$\begin{aligned}\Xi_S(v) = &\ 2^{-(S+I)}\ v^{S+\frac{1}{2}I}\ \Gamma^{-1}(S+I)\\ &\times \sum_{m=0}^{n} (\sigma^{(m-n-1)I-S}\ \frac{(-N)^m}{m!})\ (-nI)_m\ (-(S+nI))_m\ \Gamma(S+(n+1-m)I)\\ &\times {}_1\mathbf{F}_1\left[\begin{matrix} S+(n+1-m)I, \\ S+I \end{matrix}; -\frac{v^2}{4\sigma}\right]\\ =&2^{-(S+I)}\ v^{S+\frac{1}{2}I}\ \sigma^{-(S+(n+1)I)}\ e^{\frac{-v^2}{4\sigma}}\\ &\times\sum_{m=0}^{n}\frac{(-N\sigma)^m}{m!}\ (-nI)_m\ (-(S+nI))_m\ (S+I)_{n-m}\\ &\times {}_1\mathbf{F}_1\left[\begin{matrix} (-n+m)I, \\ S+I \end{matrix}; -\frac{v^2}{4\sigma}\right].\end{aligned}$$

Applying the matrix analogue of Kummer's transformations in [32], we obtain

$$\begin{aligned}\Xi_S(v) =&2^{-(S+I)}\ v^{S+\frac{1}{2}I}\ (-N\sigma)^n\ \sigma^{-(S+(n+1)I)}\ e^{\frac{-v^2}{4\sigma}}\\ &\times\sum_{m=0}^{\infty}\ \frac{(N\sigma)^{-m}\ (S+I)_m}{(n-m)!}\ (-nI)_{n-m}\ (-(S+nI))_{n-m}\\ &\times {}_1\mathbf{F}_1\left[\begin{matrix} -mI, \\ S+I \end{matrix}; \frac{v^2}{4\sigma}\right]\\ =&2^{-(S+I)}\ v^{S+\frac{1}{2}I}\ (-N\sigma)^n\ \sigma^{-(S+(n+1)I)}\ e^{\frac{-v^2}{4\sigma}}\\ &\times\sum_{m=0}^{\infty}\ \frac{(N\sigma)^{-m}\ (S+I)_n}{m!}\ (-nI)_m\\ &\times {}_1\mathbf{F}_1\left[\begin{matrix} -mI, \\ S+I \end{matrix}; \frac{v^2}{4\sigma}\right]\\ =&2^{-(S+I)}\ v^{S+\frac{1}{2}I}\ (-N\sigma)^n\ \sigma^{-(S+(n+1)I)}\ (S+I)_n\ e^{\frac{-v^2}{4\sigma}}\\ &\times\sum_{m=0}^{n}\ \frac{(-nI)_m\ (N\sigma)^{-m}}{m!}\ \sum_{t=0}^{m}\frac{(-mI)_t\ [(S+I)_t]^{-1}}{t!}\ (\frac{v^2}{4\sigma})^t\end{aligned}$$

Setting $m = t + s$ in the above expression to get

$$\begin{aligned}
\Xi_S(v) =&2^{-(S+I)}\ v^{S+\frac{1}{2}I}\ (-N\sigma)^n\ \sigma^{-(S+(n+1)I)}\ (S+I)_n\ e^{\frac{-v^2}{4\sigma}}\\
&\times\sum_{s=0}^{n-t} \frac{(-nI)_{s+t}\ (N\sigma)^{-(s+t)}}{(s+t)!}\ \sum_{t=0}^{n} \frac{(-(s+t)I)_t\ [(S+I)_t]^{-1}}{t!}\ (\frac{v^2}{4\sigma})^t\\
=&2^{-(S+I)}\ v^{S+\frac{1}{2}I}\ (-N\sigma)^n\ \sigma^{-(S+(n+1)I)}\ (S+I)_n\ e^{\frac{-v^2}{4\sigma}}\\
&\times\sum_{s=0}^{n-t} \frac{(-nI)_t\ ((-n+t)I)_s\ (N\sigma)^{-(s+t)}}{s!}\ \sum_{t=0}^{n} \frac{[(S+I)_t]^{-1}}{t!}\ (-\frac{v^2}{4\sigma})^t\\
=&2^{-(S+I)}\ v^{S+\frac{1}{2}I}\ (-N\sigma)^n\ \sigma^{-(S+(n+1)I)}\ (S+I)_n\ e^{\frac{-v^2}{4\sigma}}\\
&\times\sum_{t=0}^{n} \frac{(-nI)_t\ [(S+I)_t]^{-1}}{t!}\ (-N\sigma)^{-t}\ (\frac{v^2}{4\sigma})^t\\
&\times\sum_{s=0}^{n-t} \frac{((-n+t)I)_s\ (N\sigma)^{-s}}{s!}.
\end{aligned}$$

After changing the order of summation and simplifying yield

$$\begin{aligned}
\Xi_S(v) =&2^{-(S+I)}\ v^{S+\frac{1}{2}I}\ (-N\sigma)^n\ \sigma^{-(S+(n+1)I)}\ (S+I)_n\ e^{\frac{-v^2}{4\sigma}}\\
&\times\sum_{t=0}^{n} \frac{(-nI)_{n-t}\ [(S+I)_{n-t}]^{-1}}{(n-t)!}\ (\frac{-v^2N^{-1}}{4\sigma^2})^{n-t}\ \{I-(N\sigma)^{-1}\}^t\\
=&2^{-(S+2n+1)}\ v^{(S+(2n+\frac{1}{2})I)}\ \sigma^{-(S+(2n+1)I)}\ e^{\frac{-u^2}{4\sigma}}\\
&\times\sum_{t=0}^{n} \frac{(-nI)_t\ (-(S+nI))_t}{t!}\Big\{\frac{-v^2}{4\sigma}(I-(N\sigma)^{-1})\Big\}^t.
\end{aligned}$$

This finalizes the proof of the Theorem 3.1. □

Theorem 3.2. *Let S, P and M be positive stable and commuting matrices in $\mathbb{C}^{d\times d}$, satisfying the condition (2.4). When*

$$\Phi(u) = u^{P+\frac{1}{2}I}\ e^{-\sigma u^2}\ \mathcal{B}_n(1; M, \sigma u^2), \tag{3.5}$$

then, we have

$$\begin{aligned}
\Xi_S(v) =& (-1)^n\ \sigma^{-(\frac{1}{2}(P+S)+I)}\ (2)^{-(S+I)}\ v^{S+\frac{1}{2}I}\\
&\times\Gamma(\frac{1}{2}(P+S))\ (M+\frac{1}{2}(P+S))_n\ \Gamma^{-1}(S+I)\ [(-(\frac{1}{2}(P+S)))_n]^{-1}\\
&\times {}_2\mathbf{F}_2\left[\begin{array}{c} (1-n)I+\frac{1}{2}(P+S), nI+M+\frac{1}{2}(P+S), \\ \\ S+I, M+\frac{1}{2}(P+S)\end{array} ; -\frac{v^2}{4\sigma}\right],
\end{aligned}\tag{3.6}$$

where $\widetilde{\beta}(P) > -\frac{3}{2}$, $v > 0$ and $\sigma \in \mathbb{C}$ such that $Re(\sigma) > 0$.

Proof. By Definition 2.6 and applying (3.1) into (3.5), we obtain

$$\begin{aligned}\Xi_S(v) &= \int_0^\infty u^{P+(2n+\frac{1}{2})I}\ e^{-\sigma u^2}\ \mathcal{B}_n(1;M,\sigma u^2)\ J_S(uv)\ \sqrt{uv}du\\ &=\sum_{m=0}^{n}\frac{(-1/\sigma)^m\ (-nI)_m\ (M+(n-1)I)_m}{m!}\\ &\times\int_0^\infty u^{P+(-2m+\frac{1}{2})I}\ e^{-\sigma u^2}\ J_S(uv)\ \sqrt{uv}du.\end{aligned}$$

Then, by virtue of Lemma 2.2 applied to the above equation we attain

$$\begin{aligned}\Xi_S(v) &= \sum_{m=0}^{n}\frac{(-1/\sigma)^m\ (-nI)_m\ (M+(n-1)I)_m}{m!}\\ &\times 2^{-(S+\frac{1}{2}I)}\ (\sigma)^{\frac{-1}{2}(P+S-(2m-2)I)}\ v^{S+\frac{1}{2}I}\\ &\times\Gamma^{-1}(S+I)\ \Gamma(\frac{1}{2}(P+S-(2m-2)I))\\ &\times {}_1\mathbf{F}_1\left[\begin{array}{c}\frac{1}{2}(P+S)+(1-m)I,\\ \\ S+I\end{array};-\frac{v^2}{4\sigma}\right]\\ &= 2^{-(S+I)}\ (\sigma)^{\frac{-1}{2}(P+S+I)}\ v^{S+\frac{1}{2}I}\ \Gamma^{-1}(S+I)\ \Gamma(\frac{1}{2}(P+S+I))\\ &\times\sum_{m=0}^{n}\frac{(-nI)_m\ (M+(n-1)I)_m\ [(-\frac{1}{2}(P+S))_m]^{-1}}{m!}\\ &\times\sum_{s=0}^{\infty}\Gamma(\frac{1}{2}(P+S)+(1-m+s)I)\Gamma(S+I)\\ &\times\Gamma^{-1}(\frac{1}{2}(P+S)+(1-m)I)\Gamma^{-1}(S+(s+1)I)\ \frac{(\frac{-v^2}{4\sigma})^s}{s!}.\end{aligned}$$

Applying Lemma 2.1 and after simplification, we thus obtain the desired result as follows

$$\begin{aligned}\Xi_S(v) &= 2^{-(S+I)}\ (\sigma)^{-(\frac{1}{2}(P+S)+I)}\ v^{S+\frac{1}{2}I}\ \Gamma^{-1}(S+I)\ \Gamma(\frac{1}{2}(P+S)+I)\\ &\times\sum_{s=0}^{\infty}(\frac{1}{2}(P+S)+I)_s\ [(S+I)_s]^{-1}\ \frac{(\frac{-v^2}{4\sigma})^s}{s!}\\ &\times\sum_{m=0}^{n}\frac{(-nI)_m\ (M+(n-1)I)_m\ [(-\frac{1}{2}(P+S)-sI)_m]^{-1}}{m!}\\ &=2^{-(S+I)}\ (\sigma)^{-(\frac{1}{2}(P+S)+I)}\ v^{S+\frac{1}{2}I}\ \Gamma^{-1}(S+I)\ \Gamma(\frac{1}{2}(P+S)+I)\ (-1)^n\\ &\times(\frac{1}{2}(P+S+)+M)_n[(-\frac{1}{2}(P+S))_n]^{-1}\\ &\times\sum_{s=0}^{\infty}(\frac{1}{2}(P+S)+(1-n)I)_s\ (\frac{1}{2}(P+S)+M+nI)_s\\ &\times[(S+I)_s]^{-1}\ [(\frac{1}{2}(P+S)+M)_s]^{-1}\ \frac{(\frac{-v^2}{4\sigma})^s}{s!}.\end{aligned}$$

□

Next, we consider some interesting special cases of the Theorem 3.2 in the following corollary:

Corollary 3.1. • *If $S = P$, (3.6) gives:*

$$\Xi_S(v) = (-1)^n\, 2^{-\frac{1}{2}}\, (\sigma)^{-(S+I)}\, (\frac{1}{2}v)^{S+\frac{1}{2}I}\, (S+M)_n\, [(-S)_n]^{-1} \times {}_2\boldsymbol{F}_2\left[\begin{matrix} S+I(1-n), S+M+nI \\ S+M, S+I \end{matrix} ; \frac{-v^2}{4\sigma}\right],$$

where $\widetilde{\beta}(S) > -\frac{3}{2}$, and $Re(\sigma) > 0$.

• *If $S = P = M$, (3.6) gives:*

$$\Xi_S(v) = (-1)^n\, (2\sigma)^{-\frac{1}{2}}\, (\frac{1}{2\sigma}v)^{S+\frac{1}{2}I}\, (2S)_n\, [(-S)_n]^{-1} \times {}_2\boldsymbol{F}_2\left[\begin{matrix} S+I(1-n), 2S+nI \\ 2S, S+I \end{matrix} ; \frac{-v^2}{4\sigma}\right],$$

where $\widetilde{\beta}(S) > -\frac{3}{2}$ and $Re(\sigma) > 0$.

• *If $S = P = M$ and $\sigma = \frac{1}{2}$, (3.6) gives:*

$$\Xi_S(v) = (-1)^n\, v^{S+\frac{1}{2}I}\, (2S)_n\, [(-S)_n]^{-1} \times {}_2\boldsymbol{F}_2\left[\begin{matrix} S+I(1-n), 2S+nI \\ 2S, S+I \end{matrix} ; \frac{-v^2}{2}\right],$$

where $\widetilde{\beta}(S) > -\frac{3}{2}$.

• *If $P = S + 2nI$, (3.6) gives:*

$$\Xi_S(v) = (\frac{v}{2\sigma})^{S+(2n+1)I}\, \frac{1}{\sqrt{v}}\, (S+I)_n\, [-(S+nI)_n]^{-1}\, [(S+M+nI)_n]^{-1} \times e^{\frac{-v^2}{4\sigma}}\; {}_2\boldsymbol{F}_0\left[\begin{matrix} -nI, I(1-2n)-S-M \\ - \end{matrix} ; \frac{-4\sigma}{v^2}\right],$$

where $\widetilde{\beta}(S) > -1$ and $v > 0$.

• *Taking $S = \frac{1}{2}$ in (3.6) to get:*

$$\begin{aligned}\Xi_S(v) &= \int_0^\infty u^{P+\frac{1}{2}I}\, e^{-\sigma u^2}\mathcal{B}_n(1; M, \sigma u^2)\, \sin(uv)\, du \\ &= \frac{1}{\pi}(\sigma)^{\frac{-P}{2}-\frac{I}{4}}\, v\, \sqrt{2}\, [(\frac{-P}{2}-\frac{I}{4})_n]^{-1}\, (M+\frac{P}{2}+\frac{I}{4})_n\, \Gamma(\frac{P}{2}+\frac{5}{4}I) \\ &\times {}_2\boldsymbol{F}_2\left[\begin{matrix} \frac{P}{2}+(\frac{5}{4}-n)I, M+\frac{P}{2}+(\frac{1}{4}+n)I \\ \frac{3}{2}I, M+\frac{1}{2}P+\frac{1}{4}I \end{matrix} ; -\frac{v^2}{4\sigma}\right],\end{aligned}$$

where $\widetilde{\beta}(P) > -3/2$ *and* $Re(\sigma) > 0$.

- *Taking* $S = \frac{-1}{2}$ *in (3.6) to get:*

$$\begin{aligned}\Xi_S(v) &= \int_0^\infty u^{P+\frac{1}{2}I}\, e^{-\sigma u^2} \mathcal{B}_n(1; M, \sigma u^2)\, \cos(uv)\, du\\ &= 2^{\frac{-1}{2}}\, (\sigma)^{\frac{-P}{2}-\frac{3}{4}I}\, [(\frac{-P}{2}+\frac{I}{4})_n]^{-1}\, (M+\frac{P}{2}-\frac{I}{4})_n\, \Gamma(\frac{P}{2}+\frac{3}{4}I)\\ &\times {}_2\boldsymbol{F}_2\left[\begin{matrix}\frac{P}{2}+(\frac{3}{4}-n)I, M+\frac{P}{2}-(\frac{1}{4}-n)I\\ \frac{1}{2}I, M+\frac{1}{2}P-\frac{1}{4}I\end{matrix}; -\frac{v^2}{4\sigma}\right],\end{aligned}$$

where $\widetilde{\beta}(P) > -3/2$ *and* $Re(\sigma) > 0$.

Remark 3.4. The last two special cases in Corollary 3.1, present the formulae of the Fourier sine and cosine transforms when $S = \pm\frac{1}{2}$ in (3.1).

Theorem 3.3. *Let* $\mathcal{B}_n(u; M, N)$ *be given in (2.10). If*

$$\Phi(u) = \mathcal{B}_n(\sigma u^2; M, N), \tag{3.7}$$

then, we have

$$\begin{aligned}\Xi_S(v) =& 2^{\frac{1}{2}}\, v^{-1}\, \Gamma(\frac{1}{2}S+\frac{3}{4}I)\, \Gamma^{-1}(\frac{1}{2}S+\frac{1}{4}I)\\ &\times {}_4\mathbf{F}_0\left[\begin{matrix}-nI, M+(n-1)I, \frac{1}{2}S+\frac{3}{4}I, \frac{-1}{2}S+\frac{3}{4}I,\\ -\end{matrix}; 4\sigma(v^2N)^{-1}\right],\end{aligned} \tag{3.8}$$

where $\widetilde{\beta}(S) > -1/2$ *and* $v > 0$.

Proof. To establish Theorem 3.3, from (3.7) into (3.1), we consider the following integral

$$\begin{aligned}\Xi_S(v) &= \int_0^\infty \mathcal{B}_n(\sigma u^2; M, N)\, J_S(uv)\, \sqrt{uv}\, du\\ &= \sum_{m=0}^{n}(-nI)_m\, (M+(n-1)I)_m\, \frac{(-\sigma N^{-1})^m}{m!} \int_0^\infty u^{2m}\, J_S(uv)\, \sqrt{uv}\, du\\ &= \sum_{m=0}^{n}(-nI)_m\, (M+(n-1)I)_m\, \frac{(-\sigma N^{-1})^m}{m!}\\ &\times 2^{2m+\frac{1}{2}}\, v^{-2m-1}\, \Gamma(\frac{1}{2}(S+(2m+3/2)I))\, \Gamma^{-1}(\frac{1}{2}(S-(2m-1/2)I))\\ &= \sqrt{2}\, v^{-1}\, \Gamma(\frac{1}{2}S+3/4I)\, \Gamma^{-1}(\frac{1}{2}S+1/4I) \sum_{m=0}^{n}(-nI)_m\, (M+(n-1)I)_m\\ &\times(\frac{1}{2}S+3/4I)_m\, [(\frac{1}{2}S+1/4I)_{-m}]^{-1}\, \frac{(-4\sigma N^{-1}\, v^{-2})^m}{m!}\\ &= \sqrt{2}\, v^{-1}\, \Gamma(\frac{1}{2}S+3/4I)\, \Gamma^{-1}(\frac{1}{2}S+1/4I) \sum_{m=0}^{n}(-nI)_m\, (M+(n-1)I)_m\\ &\times(\frac{1}{2}S+3/4I)_m\, (\frac{-1}{2}S+3/4I)_m\, \frac{(-4\sigma N^{-1}\, v^{-2})^m}{m!},\end{aligned}$$

which, in terms of (2.7), yields our desired result (3.8). □

Theorem 3.4. *Let $\mathcal{B}_n(u; M, N)$ be given in (2.10) and S, P, M and N be positive stable and commuting matrices in $\mathbb{C}^{d\times d}$. When*

$$\Phi(u) = u^{P+\frac{1}{2}I}\ e^{-\sigma u^2}\ \mathcal{B}_n(\lambda u^2; M, N), \tag{3.9}$$

then, we have

$$\begin{aligned} \Xi_S(v) =& 2^{-(S+I)}\ v^{P+\frac{1}{2}I}\ \sigma^{-(\frac{1}{2}(P+S)+I)} \\ &\times \sum_{k=0}^{\infty} \Gamma(\frac{1}{2}(P+S)+(1+k)I)\ \Gamma^{-1}(S+(1+k)I)\ \frac{(\frac{-v^2}{4\sigma})^k}{k!} \\ &\times {}_3\boldsymbol{F}_0\left[\begin{matrix} -nI, M+(n-1)I, \frac{1}{2}(P+S)+(1+k)I \\ - \end{matrix}; -\lambda\,(\sigma N)^{-1}\right], \end{aligned} \tag{3.10}$$

where $\widetilde{\beta}(P+S) > -2,\ v > 0$ and $\lambda > 0$.

Proof. Inserting (2.10) in (3.9) and from the relation (3.1), we find that

$$\begin{aligned} \Xi_S(v) &= \int_0^{\infty} u^{P+\frac{1}{2}I}\ e^{-\sigma u^2}\ \mathcal{B}_n(\lambda u^2; M, N)\ J_S(uv)\ \sqrt{uv}\ du \\ &= \sum_{k=0}^{\infty} 2^{-S}\ v^{S+\frac{1}{2}I}\ \Gamma^{-1}(S+(k+1)I)\ \frac{(-v^2/4)^k}{k!} \\ &\times \int_0^{\infty} u^{P+S+(2k+1)I}\ e^{-\sigma u^2}\ \mathcal{B}_n(\lambda u^2; M, N)\ du. \end{aligned}$$

Taking $z = u^2$ and making necessary calculations, we see that

$$\begin{aligned} \Xi_S(v) &= \sum_{k=0}^{\infty} 2^{-(S+I)}\ v^{S+\frac{1}{2}I}\ \Gamma^{-1}(S+(k+1)I)\ \frac{(-v^2/4)^k}{k!} \\ &\times \int_0^{\infty} z^{\frac{1}{2}(P+S)+kI}\ e^{-\sigma z}\ \mathcal{B}_n(\lambda z; M, N)\ dz \\ &= 2^{-(S+I)}\ v^{S+\frac{1}{2}I}\ \sum_{k=0}^{\infty} (\sigma)^{-(\frac{1}{2}(P+S)+(k+1)I)} \\ &\times \Gamma(\frac{1}{2}(P+S)+(k+1)I)\ \Gamma^{-1}(S+(k+1)I)\ \frac{(-v^2/4)^k}{k!} \\ &\times {}_3\mathbf{F}_0\left[\begin{matrix} -nI, M+(n-1)I, \frac{1}{2}(P+S)+(1+k)I \\ - \end{matrix}; -\lambda\,(\sigma N)^{-1}\right] \\ &= 2^{-(S+I)}\ v^{S+\frac{1}{2}I}\ (\sigma)^{-(\frac{1}{2}(P+S)+I)}\ \Gamma(\frac{1}{2}(P+S)+I)\ \Gamma^{-1}(S+I) \\ &\times \sum_{k=0}^{\infty} (\frac{1}{2}(P+S)+I)_k\ [(S+I)_k]^{-1}\ \frac{(-v^2/4\sigma)^k}{k!} \\ &\times {}_3\mathbf{F}_0\left[\begin{matrix} -nI, M+(n-1)I, \frac{1}{2}(P+S)+(1+k)I \\ - \end{matrix}; -\lambda\,(\sigma N)^{-1}\right]. \end{aligned}$$

This completes the proof of (3.10). □

Theorem 3.5. *Let S, P, M and M_1 be positive stable and commuting matrices in $\mathbb{C}^{d\times d}$, such that S, P and M satisfy the condition (2.4). If*

$$\Phi(u) = u^{P+\frac{1}{2}I}\ e^{-\sigma u^2}\ \mathcal{B}_n(1; M, \sigma u^2)\ \mathcal{B}_m(1; M_1, u^2), \tag{3.11}$$

then, we have

$$\begin{aligned}\Xi_S(v) &= (-1)^n\ 2^{-(S+I)}\ v^{P+\frac{1}{2}I}\ \sigma^{-(\frac{1}{2}(P+S)+I)}\\ &\times \Gamma(\frac{1}{2}(P+S)+I)\ \Gamma^{-1}(S+I)\ (M+(\frac{1}{2}(P+S)))_n\ [(I-(\frac{1}{2}(P+S)))_n]^{-1}\\ &\times \sum_{k=0}^{\infty} (M+(1-n)I)_k((\frac{1}{2}(P+S)+M+nI))_k\\ &\times [((\frac{1}{2}(P+S)+M))_k]^{-1}\ [(S+I)_k]^{-1}\ \frac{(-v^2/4)^k}{k!}\\ &\times {}_3\boldsymbol{F}_2\left[\begin{matrix} -mI, M_1+(m-1)I, (1-k)I-\frac{1}{2}(P+S)-M \\ (n-k)I-\frac{1}{2}(P+S), (1-k-n)I-\frac{1}{2}(P+S)-M\end{matrix} ;\sigma\right],\end{aligned} \tag{3.12}$$

where $v > 0$, $Re(\sigma) > 0$ and $\widetilde{\beta}(P) > -1/2$.

Proof. From (3.11) into (3.1) and using (2.10), we have

$$\begin{aligned}\Xi_S(v) &= \int_0^\infty u^{P+\frac{1}{2}I}\ e^{-\sigma u^2}\ \mathcal{B}_n(1; M, \sigma u^2)\ \mathcal{B}_m(1; M_1, u^2)\ J_S(uv)\ \sqrt{uv}\ du\\ &= \sum_{k=0}^{\infty} (-1)^k\ \sqrt{v}\ \Gamma^{-1}(S+(k+1)I)\ \frac{(\frac{1}{2}v)^{S+2kI}}{k!}\\ &\times \int_0^\infty u^{P+S+(2k+1)I}\ e^{-\sigma u^2}\ \mathcal{B}_n(1; M, \sigma u^2)\ \mathcal{B}_m(1; M_1, u^2)\ du.\end{aligned}$$

Putting $u^2 = z$, we get

$$\begin{aligned}\Xi_S(v) &= 2^{-(S+I)}\ v^{(S+\frac{1}{2}I)}\ \sum_{k=0}^{\infty}\Gamma^{-1}(S+(k+1)I)\ \frac{(\frac{-1}{4}v^2)^k}{k!}\\ &\times \int_0^\infty z^{\frac{1}{2}(S+P)+kI}\ e^{-\sigma z}\ \mathcal{B}_n(1; M, \sigma z)\ \mathcal{B}_m(1; M_1, z)\ dz.\end{aligned}$$

Applying Lemma 2.3 and after simplification, we see that

$$\begin{aligned}\Xi_S(v) &= (-1)^n\, 2^{-(S+I)}\, v^{(S+\frac{1}{2}I)}\, (\sigma)^{-(\frac{1}{2}(S+P)+I)}\\ &\times\Gamma(\frac{1}{2}(S+P)+I)\Gamma^{-1}(S+I)\,(M+\frac{1}{2}(S+P))_n\,[(I-\frac{1}{2}(S+P))_n]^{-1}\\ &\times\sum_{k=0}^{\infty}(M+(1-n)I)_k\,(M+(\frac{1}{2}(S+P))+nI)_k\\ &\times[(S+I)_k]^{-1}\,[(M+(\frac{1}{2}(S+P)))_k]^{-1}\frac{(-v^2/4)^k}{k!}\\ &\times{}_3\mathbf{F}_2\left[\begin{array}{c}-mI, M_1+(m-1)I, (1-k)I-\frac{1}{2}(P+S)-M\\ \\ (n-k)I-\frac{1}{2}(P+S), (1-k-n)I-\frac{1}{2}(P+S)-M\end{array};\sigma\right],\end{aligned}$$

which is the claimed result in (3.12). □

Theorem 3.6. *Let $\mathcal{B}_n(u;M,N)$ be given in (2.10). When*

$$\Phi(u) = \log u\,\mathcal{B}_n(\lambda u^2;M,N), \tag{3.13}$$

then, we have

$$\begin{aligned}\Xi_S(v) &= \frac{1}{v\sqrt{2}}\,\Gamma(\frac{1}{2}S+3/4I)\,\Gamma^{-1}(\frac{1}{2}S+1/4I)\\ &\times\sum_{m=0}^{n}(-nI)_m\,(M+(n-1)I)_m\\ &\times(\frac{1}{2}S+3/4I)_m\,(\frac{-1}{2}S+3/4I)_m\,\frac{(4\lambda(Nv^2)^{-1})^m}{m!}\\ &\times\left\{\psi(\frac{1}{2}S+(3/4+m)I)+\psi(\frac{1}{2}S+(1/4-m)I)-\log(v^2/4)\right\}\end{aligned} \tag{3.14}$$

where M,N and S are commuting matrices in $\mathbb{C}^{d\times d}$ such that $\widetilde{\beta}(S)>-3/2$, $\psi(S)$ is the digamma matrix function defined in [25] by

$$\psi(S)=\Gamma^{-1}(S)\Gamma'(S),$$

where $\Gamma^{-1}(S)$ and $\Gamma'(S)$ are reciprocal and derivative of the gamma matrix function, respectively and $v>0$, $\lambda>0$.

Proof. Inserting (3.13) into (3.1) and using (2.10), we observe that

$$\begin{aligned}
\Xi_S(v) =& \int_0^\infty \log u\ \mathcal{B}_n(\lambda u^2; M, N)\ J_S(uv)\ \sqrt{uv}\ du \\
=& \sum_{m=0}^{n} (-nI)_m\ (M + (n-1)I)_m\ \frac{(-\lambda N^{-1})^m}{m!} \\
\times& \int_0^\infty u^{2m}\ \log u\ J_S(uv)\ \sqrt{uv}\ du \\
=& \frac{1}{v\sqrt{2}} \sum_{m=0}^{n} (-nI)_m\ (M + (n-1)I)_m\ \frac{(-4MN^{-1}v^{-2})^m}{m!} \\
\times& \Gamma(\frac{1}{2}S + (m+3/4)I)\ \Gamma^{-1}(\frac{1}{2}S - (m-1/4)I) \\
\times& \left\{\psi(\frac{1}{2}S + (m+3/4)I) + \psi(\frac{1}{2}S - (m-1/4)I) - \log(v^2/4)\right\}.
\end{aligned}$$

Thus, using the above expression, we immediately reach in (3.13). □

Corollary 3.2. • *If* $S = \frac{1}{2}$, *(3.14) gives:*

$$\begin{aligned}
\Xi_S(v) =& \int_0^\infty \log u\ \mathcal{B}_n(\lambda u^2; M, N)\ \sin(uv)\ du \\
=& \left\{\frac{\sqrt{\frac{1}{2}\pi}}{v\sqrt{2\pi}}\right\} \sum_{m=0}^{n} (-nI)_m\ (M + (n-1)I)_m\ (\frac{1}{2})_m\ \frac{(4\lambda(Nv^2)^{-1})^m}{m!} \\
\times& \left\{\psi(1+m) + \psi(1-m) - \log(\frac{v^2}{2})\right\} \\
=& (\frac{1}{2v}) \sum_{m=0}^{n} (-nI)_m\ (M + (n-1)I)_m\ (\frac{1}{2})_m\ \frac{(4\lambda(Nv^2)^{-1})^m}{m!} \\
\times& \left\{\sum_{k=0}^{m}(k+1)^{-1} + \psi(1) + \left\{-(\sum_{k=1}^{m}(1-k)^{-1} - \psi(1))\right\} - \log(\frac{v^2}{4})\right\} \\
=& (\frac{1}{2v}) \sum_{m=0}^{n} (-nI)_m\ (M + (n-1)I)_m\ (\frac{1}{2})_m\ \frac{(4\lambda(Nv^2)^{-1})^m}{m!} \\
\times& \left\{\sum_{k=1}^{m}(\frac{-2k}{(1-k^2)}) - 2\gamma - \log(\frac{v^2}{4})\right\},
\end{aligned}$$

where $-\gamma = 0.5777215665$.

• *If* $S = -\frac{1}{2}$, *(3.14) gives:*

$$\Xi_S(v) = \int_0^\infty \log u\ \mathcal{B}_n(\lambda u^2; M, N)\ \cos(uv)\ du = \boldsymbol{0}.$$

4. Conclusions

The theory of integral transforms are played a very crucial role in the area of mathematical analysis, mathematical physics and engineering sciences.

One of these transforms is the Hankel transform that is more suitable for the problems that are defined in terms of polar coordinate variables. It should be noted that, the kernel of the Hankel transform is the Bessel function, perhaps, for this reason, in some literature, this transform is called Bessel transformation or Fourier−Bessel transform. In addition to, the Hankel transforms are natural generalizations of Fourier transforms.

In this work, Hankel transforms containing Bessel matrix functions as kernels are proposed. Then we provided some matrix Hankel integrals of generalized Bessel matrix polynomials together with ceratin elementary matrix functions, exponential function, and logarithmic function. Further, we gave the matrix versions of those results for Hankel transforms (or formulas) involving a variety of functions and polynomials (See, e.g., [35, Chapter VIII], also see [6, Chapter 7]).

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Conflict of interest

This work does not have any conflicts of interest.

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