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Research article

The k-subconnectedness of planar graphs

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Abstract: A graph G with at least $2k$ vertices is called k-subconnected if, for any $2k$ vertices $x_1, x_2, \cdots, x_{2k}$ in G, there are k independent paths joining the $2k$ vertices in pairs in G. In this paper, we prove that a k-connected planar graph with at least $2k$ vertices is k-subconnected for $k = 1, 2$; a 4-connected planar graph is k-subconnected for each k such that $1 \leq k \leq v/2$, where v is the number of vertices of G; and a 3-connected planar graph G with at least $2k$ vertices is k-subconnected for $k = 4, 5, 6$. The bounds of k-subconnectedness are sharp.

Keywords: k-connected graph; independent paths; planar graph; k-subconnected graph; component
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1. Introduction and terminology

Connectivity is an important property of graphs. It has been extensively studied (see [1]). A graph $G = (V, E)$ is called *k-connected* ($k \geq 1$) (*k-edge-connected*) if, for any subset $S \subseteq V(G)$ ($S \subseteq E(G)$) with $|S| < k$, $G - S$ is connected. The *connectivity* $\kappa(G)$ (*edge connectivity* $\lambda(G)$) is the order (size) of minimum cutset (edge cutset) $S \subseteq V(G)$ ($S \subseteq E(G)$). When G is a complete graph K_n, we define that $\kappa(G) = n - 1$.

In recent years, conditional connectivities attract researchers' attention. For example, Peroche [2] studied several sorts of connectivities, including cyclic edge (vertex) connectivity, and their relations. A *cyclic edge (vertex) cutset* S of G is an edge (vertex) cutset whose deletion disconnects G such that at least two of the components of $G - S$ contain a cycle respectively. The *cyclic edge (vertex) connectivity*, denoted by $c\lambda(G)$ ($c\kappa(G)$), is the cardinality of a minimum cyclic edge (vertex) cutset of G. Dvořǎk, Kára, Král and Pangrác [3] obtained the first efficient algorithm to determine the cyclic edge connectivity of cubic graphs. Lou and Wang [4] obtained the first efficient algorithm to determine the cyclic edge connectivity for k-regular graphs. Then Lou and Liang [5] improved the algorithm to have time complexity $O(k^9 V^6)$. Lou [6] also obtained a square time algorithm to determine the cyclic edge connectivity of planar graphs. In [7], Liang, Lou and Zhang obtained the first efficient algorithm to determine the cyclic vertex connectivity of cubic graphs. Liang and Lou [8] also showed that there is an efficient algorithm to determine the cyclic vertex connectivity for k-regular graphs with any fixed k.

Another related concept is *linkage*. Let G be a graph with at least $2k$ vertices. If, for any $2k$ vertices $u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_k$, there are k disjoint paths P_i from u_i to v_i $(i = 1, 2, \cdots, k)$ in G, then G is called *k-linked*. Thomassen [9] mentioned that a necessary condition for G to be k-linked is that G is $(2k-1)$-connected. But this condition is not sufficient unless $k = 1$. He also gave a complete characterization of 2-linked graphs. Bollobás and Thomason [10] proved that if $\kappa(G) \geq 22k$, then G is k-linked. Kawarabayashi, Kostochka and Yu [11] proved that every 2k-connected graph with average degree at least $12k$ is k-linked.

In [12], Qin, Lou, Zhu and Liang introduced the new concept of k-subconnected graphs. Let G be a graph with at least $2k$ vertices. If, for any $2k$ vertices $v_1, v_2, \cdots, v_{2k}$ in G, there are k vertex-disjoint paths joining $v_1, v_2, \cdots, v_{2k}$ in pairs, then G is called *k-subconnected* . If G is k-subconnected and $\nu(G) \geq 3k-1$, then G is called a *properly k-subconnected graph*. In [12], Qin et al. showed that a properly k-subconnected graph is also a properly $(k-1)$-subconnected graph. But only when $\nu(G) \geq 3k-1$, that G is k-subconnected implies that G is $(k-1)$-subconnected. Qin et al. [12] also gave a sufficient condition for a graph to be k-subconnected and a necessary and sufficient condition for a graph to be a properly k-subconnected graph (see Lemmas 1 and 2 and Corollary 3 in this paper).

If G has at least $2k$ vertices, that G is k-linked implies that G is k-connected, while that G is k-connected implies that G is k-subconnected (see Lemma 6 in this paper). Also in a k-connected graph G, deleting arbitrarily some edges from G, the resulting graph H is still k-subconnected. So a graph H to be k-subconnected is a spanning substructure of a k-connected graph G. To study k-subconnected graphs may help to know more properties in the structure of k-connected graphs. Notice that a k-connected graph may have a spanning substructure to be m-subconnected for $m > k$.

K-subconnected graphs have some background in matching theory. The proof of the necessary and sufficient condition [12] for properly k-subconnected graphs uses similar technique to matching theory.

Let S be a subset of $V(G)$ of a graph G. We denote by $G[S]$ the induced subgraph of G on S. We also denote by $\omega(G)$ *the number of components of* G. We also use $\nu(G)$ and $\varepsilon(G)$ to denote $|V(G)|$ and $|E(G)|$. If G is a planar graph, we denote by $\phi(G)$ *the number of faces* in the planar embedding of G. Let H be a graph. A *subdivision* of H is a graph H' obtained by replacing some edges by paths respectively in H. For other terminology and notation not defined in this paper, the reader is referred to [13].

2. Preliminary results

In this section, we shall present some known results and some straightforward corollaries of the known results which will be used in the proof of our main theorems.

Lemma 1 (Theorem 1 of [12]). Let G be a connected graph with at least $2k$ vertices. Then G is k-subconnected if, for any cutset $S \subseteq V(G)$ with $|S| \leq k-1$, $\omega(G-S) \leq |S|+1$.

Lemma 2 (Theorem 2 of [12]). Let G be a connected graph with at least $3k-1$ vertices. If G is a properly k-subconnected graph,then, for any cutset $S \subseteq V(G)$ with $|S| \leq k-1$, $\omega(G-S) \leq |S|+1$.

Only when $\nu \geq 3k-1$, that G is k-subconnected implies that G is (k-1)-subconnected. Here is an counterexample. Let $S = K_{k-2}$ be a complete graph of $k-2$ vertices, let H be k copies of K_2, and let G be a graph with $V(G) = V(S) \cup V(H)$ and $E(G) = E(S) \cup E(H) \cup \{uv|u \in V(S), v \in V(H)\}$. Then $\nu(G) = 3k - 2$, and G is not (k-1)-subconnected since we can choose $2(k\text{-}1)$ vertices by taking one vertex from each copy of K_2 in H and taking all vertices of S, then these $2(k\text{-}1)$ vertices cannot be joined by k-1 independent paths in pairs. But G is k-subconnected since when we take any $2k$ vertices from G, some pairs of vertices will be taken from several same K_2 's in H, and then the $2k$ vertices can be joined by k independent paths in pairs.

Corollary 3 (Theorem 3 of [12]). Let G be a connected graph with at least $3k - 1$ vertices. Then G is a properly k-subconnected graph if and only if, for any cutset $S \subseteq V(G)$ with $|S| \leq k - 1$, $\omega(G - S) \leq |S| + 1$.

Lemma 4 ([14]). Every 4-connected planar graph is Hamiltonian.

Lemma 5. If a graph G has a Hamilton path, then G is k-subconnected for each k such that $1 \leq k \leq \nu(G)/2$.

Proof. Let P be a Hamilton path in G. Let v_i, $i = 1, 2, \cdots, 2k$, be any $2k$ vertices in $V(G)$. Without loss of generality, assume that $v_1, v_2, \cdots, v_{2k}$ appear on P in turn. Then there are k paths P_i on P from v_{2i-1} to v_{2i}, $i = 1, 2, \cdots, k$, respectively. So G is k-subconnected.

Lemma 6. A k-connected graph G with at least $2k$ vertices is k-subconnected.

Proof. Let G be a k-connected graph with at least $2k$ vertices. Then G does not have a cutset $S \subseteq V(G)$ with $|S| \leq k - 1$, so the statement that, for any cutset $S \subseteq V(G)$ with $|S| \leq k - 1$, $\omega(G - S) \leq |S| + 1$ is true. By Lemma 1, G is k-subconnected.

3. The k-subconnectedness of planar graphs

In this section, we shall show the k-subconnectedness of planar graphs with different connectivities, and show the bounds of k-subconnectedness are sharp.

Corollary 7. A 1-connected planar graph G with at least 2 vertices is 1-subconnected.

Proof. By Lemma 6, the result follows.

Corollary 8. A 2-connected planar graph G with at least 4 vertices is 2-subconnected.

Proof. By Lemma 6, the result follows.

Theorem 9. A 4-connected planar graph G is k-subconnected for each k such that $1 \leq k \leq \nu(G)/2$.

Proof. By Lemma 4, G has a Hamilton cycle C, and then has a Hamilton path P. By Lemma 5, the result follows.

Theorem 10. A 3-connected planar graph G with at least $2k$ vertices is k-subconnected for $k = 4, 5, 6$.

Proof. Suppose that G is a 3-connected planar graph with at least $2k$ vertices which is not k-subconnected. By Lemma 1, there is a cutset $S \subseteq V(G)$ with $|S| \leq k - 1 \leq$ such that $\omega(G - S) \geq |S| + 2$. Since G is 3-connected, there is no cutset with less than 3 vertices and so $|S| \geq 3$. On the other hand, $k = 4, 5, 6$, so $|S| \leq 5$. Thus let us consider three cases.

In the first case, $|S| = 3$. By our assumption, $\omega(G - S) \geq |S| + 2$, let $C_1, C_2, \cdots, C_5$ be different components of $G - S$, and $S = \{x_1, x_2, x_3\}$. Since G is 3-connected, every C_i is adjacent to each x_j $(1 \leq i \leq 5, 1 \leq j \leq 3)$. Contract every C_i to a vertex $C_i^{'}$ $(i = 1, 2, \cdots, 5)$ to obtain a planar graph G' as G is planar. Then $G'[\{x_1, x_2, x_3\} \cup \{C_1^{'}, C_2^{'}, C_3^{'}\}]$ contains a $K_{3,3}$, which contradicts the fact that G' is a planar graph.

In the second case, $|S| = 4$. By our assumption, $\omega(G - S) \geq |S| + 2$, let $C_1, C_2, \cdots, C_6$ be different components of $G - S$ and $S = \{x_1, x_2, x_3, x_4\}$. Contract every C_i to a vertex $C_i^{'}$ $(i = 1, 2, \cdots, 6)$ to obtain a planar graph G' as G is planar. Since G is 3-connected, each $C_i^{'}$ is adjacent to at least 3 vertices in S $(1 \leq i \leq 6)$. (In the whole proof, we shall consider that $C_i^{'}$ is adjacent to only 3 vertices in S, and we shall neglect other vertices in S which are possibly adjacent to $C_i^{'}$). Since the number of 3-vertex-combinations in S is $C(4,3) = 4$, but $C_1^{'}, C_2^{'}, \cdots, C_6^{'}$ have 6 vertices, by the Pigeonhole Principle, there are two vertices in $\{C_1^{'}, C_2^{'}, \cdots, C_6^{'}\}$ which are adjacent to the same three vertices in S. Without loss of generality, assume that $C_1^{'}$ and $C_2^{'}$ are both adjacent to x_1, x_2, x_3. If there is another $C_i^{'}$ $(3 \leq i \leq 6)$ adjacent to x_1, x_2, x_3, say $C_3^{'}$, then $G'[\{x_1, x_2, x_3\} \cup \{C_1^{'}, C_2^{'}, C_3^{'}\}]$ contains a $K_{3,3}$, which contradicts the fact that G' is planar (which also contradicts the assumption that G is a planar graph because G has a subgraph which can be contracted to a $K_{3,3}$). So $C_i^{'}$ cannot be adjacent to x_1, x_2, x_3 at the same time $(i = 3, 4, 5, 6)$.

Suppose $C_3^{'}$ is adjacent to x_2, x_3, x_4. If one of $C_4^{'}, C_5^{'}, C_6^{'}$ is adjacent to both x_1 and x_4, say $C_4^{'}$, then $C_3^{'}$ is connected to x_1 by path $C_3^{'}x_4C_4^{'}x_1$, so $G'[\{x_1, x_2, x_3, x_4\} \cup \{C_1^{'}, C_2^{'}, C_3^{'}, C_4^{'}\}]$ contains a subdivision of $K_{3,3}$, contradicting the fact that G' is planar. Since $C_4^{'}, C_5^{'}, C_6^{'}$ are all not adjacent to x_1, x_2, x_3 at the same time, they are all adjacent to x_4. But each of them cannot be adjacent to both x_1 and x_4. So they are all not adjacent to x_1. Hence $C_4^{'}, C_5^{'}, C_6^{'}$ are all adjacent to x_2, x_3, x_4 at the same time. Then $G'[\{x_2, x_3, x_4\} \cup \{C_4^{'}, C_5^{'}, C_6^{'}\}]$ contains a $K_{3,3}$, contradicting the fact that G' is a planar graph.

The cases that $C_3^{'}$ is adjacent to x_1, x_3, x_4 or x_1, x_2, x_4 are similar.

In the third case, $|S| = 5$. By our assumption, $\omega(G - S) \geq |S| + 2$, let $C_1, C_2, \cdots, C_7$ be different components of $G - S$ and $S = \{x_1, x_2, \cdots, x_5\}$. Since G is planar, contracting C_i to a vertex $C_i^{'}$ $(i = 1, 2, \cdots, 7)$, we obtain a planar graph G'. Also since G is 3-connected, every $C_i^{'}$ is adjacent to at least 3 vertices in S $(1 \leq i \leq 7)$.

Case 1. There are two of $C_i^{'}$ $(i = 1, 2, \cdots, 7)$ adjacent to the same three vertices in S. Without loss of generality, assume that $C_1^{'}$ and $C_2^{'}$ are both adjacent to x_1, x_2, x_3 at the same time.

If there is another vertex $C_i^{'}$ $(3 \leq i \leq 7)$ adjacent to x_1, x_2, x_3 at the same time, say $C_3^{'}$. Then $G'[\{x_1, x_2, x_3\} \cup \{C_1^{'}, C_2^{'}, C_3^{'}\}]$ contains a $K_{3,3}$, contradicting the fact that G' is planar. So $C_3^{'}$ cannot be adjacent to x_1, x_2, x_3 at the same time. Without loss of generality, we have two subcases.

Suppose $C_3^{'}$ is only adjacent to two vertices in $\{x_1, x_2, x_3\}$, say x_2 and x_3. Then $C_3^{'}$ must be adjacent to one of x_4 and x_5 as G is 3-connected. Without loss of generality, assume that $C_3^{'}$ is also adjacent to x_4. If one of $C_i^{'}$ $(i = 4, 5, 6, 7)$ is adjacent to both x_1 and x_4, say $C_4^{'}$. Then $C_3^{'}$ is connected to x_1 by path $C_3^{'}x_4C_4^{'}x_1$, hence $G'[\{x_1, x_2, x_3, x_4\} \cup \{C_1^{'}, C_2^{'}, C_3^{'}, C_4^{'}\}]$ contains a subdivision of $K_{3,3}$, contradicting the fact that G' is planar. So none of $C_i^{'}$ $(i = 4, 5, 6, 7)$ is adjacent to both x_1 and x_4.

Case (1.1). Suppose that $C_4^{'}$ is adjacent to three vertices in $\{x_1, x_2, x_3, x_4\}$.

If $C_4^{'}$ is adjacent to x_1, x_2, x_3 at the same time, then the case is similar to that $C_3^{'}$ is adjacent to x_1, x_2, x_3 at the same time, and we have a contradiction. So $C_4^{'}$ is not adjacent to x_1, x_2, x_3 at the same time. If $C_4^{'}$ is adjacent to x_1, since $C_4^{'}$ is adjacent to three vertices in $\{x_1, x_2, x_3, x_4\}$ but not x_1, x_2, x_3, so $C_4^{'}$ is adjacent to x_1 and x_4, by the argument above, we have a contradiction. Hence $C_4^{'}$ can be adjacent only to x_2, x_3, x_4.

Now x_1 and x_4 are symmetric, while x_2 and x_3 are symmetric in $G^{'}\left[\{x_1, x_2, x_3, x_4\} \cup \{C_1^{'}, C_2^{'}, C_3^{'}, C_4^{'}\}\right]$. Then $C_i^{'}(i = 5, 6, 7)$ must be adjacent to x_5, we have two cases as follows.

Notice that now there are not i and j, $5 \leq i \neq j \leq 7$, such that $C_i^{'}$ is adjacent to x_1 and $C_j^{'}$ is adjacent to x_4, otherwise $C_3^{'}$ is connected to x_1 by path $C_3^{'}x_4C_j^{'}x_5C_i^{'}x_1$, then $G^{'}\left[\{x_1, x_2, x_3, x_4, x_5\} \cup \{C_1^{'}, C_2^{'}, C_3^{'}, C_i^{'}, C_j^{'}\}\right]$ contains a subdivision of $K_{3,3}$, contrary to the fact that $G^{'}$

is planar.

Suppose C_5' is adjacent to x_3, x_4, x_5. If C_6' is also adjacent to x_3, x_4, x_5, then C_7' cannot be adjacent to x_3, x_4, x_5, otherwise $G'[\{x_3, x_4, x_5\} \cup \{C_5', C_6', C_7'\}]$ contains a $K_{3,3}$, a contradiction. So suppose C_7' is adjacent to x_2, x_4, x_5, then C_7' is connected to x_3 by path $C_7'x_2C_2'x_3$, and then $G'[\{x_2, x_3, x_4, x_5\} \cup \{C_2', C_5', C_6', C_7'\}]$ contains a subdivision of $K_{3,3}$, a contradiction. So suppose C_7' is adjacent to x_2, x_3, x_5. But C_7' is connected to x_4 by path $C_7'x_2C_3'x_4$, then $G'[\{x_2, x_3, x_4, x_5\} \cup \{C_3', C_5', C_6', C_7'\}]$ contains a subdivision of $K_{3,3}$, a contradiction again. If C_6' is adjacent to x_2, x_3, x_5, suppose C_7' is adjacent to x_3, x_4, x_5, then this case is similar to that C_6' is adjacent to x_3, x_4, x_5 and C_7' is adjacent to x_2, x_3, x_5. Then suppose C_7' is adjacent to x_2, x_3, x_5. Now C_3' is connected to x_5 by path $C_3'x_4C_5'x_5$, so $G'[\{x_2, x_3, x_4, x_5\} \cup \{C_3', C_5', C_6', C_7'\}]$ contains a subdivision of $K_{3,3}$, a contradiction. Now suppose C_7' is adjacent to x_2, x_4, x_5. Then C_6' is connected to x_4 by path $C_6'x_5C_7'x_4$, hence $G'[\{x_2, x_3, x_4, x_5\} \cup \{C_3', C_4', C_6', C_7'\}]$ contains a subdivision of $K_{3,3}$, a contradiction. The remaining case is that C_6' is adjacent to x_2, x_4, x_5. Now the cases that C_7' is adjacent to x_2, x_4, x_5 and that C_7' is adjacent to x_3, x_4, x_5 are symmetric, we only discuss the former. Then C_5' is connected to x_2 by path $C_5'x_3C_4'x_2$, so $G'\left[\{x_2, x_3, x_4, x_5\} \cup \{C_4', C_5', C_6', C_7'\}\right]$ contains a subdivision of $K_{3,3}$, a contradiction. The remaining case is that C_7' is adjacent to x_2, x_3, x_5. Now C_7' is connected to x_4 by path $C_7'x_5C_6'x_4$. Hence $G'\left[\{x_2, x_3, x_4, x_5\} \cup \{C_3', C_4', C_6', C_7'\}\right]$ contains a subdivision of $K_{3,3}$, a contradiction.

Suppose C_5' is adjacent to x_2, x_3, x_5. If C_6' and C_7' are both adjacent to x_2, x_3, x_5, then $G'\left[\{x_2, x_3, x_5\} \cup \{C_5', C_6', C_7'\}\right]$ contains a $K_{3,3}$, contradicting the fact that G' is planar. So one of C_5', C_6',C_7' is adjacent to x_2, x_3, x_5, the other two are adjacent to x_3, x_4, x_5; or one of C_5', C_6',C_7' is adjacent to x_3, x_4, x_5, the other two are adjacent to x_2, x_3, x_5; or C_5' is adjacent to x_2, x_3, x_5, C_6' is adjacent to x_3, x_4, x_5 and C_7' is adjacent to x_2, x_4, x_5. These three cases are symmetric to cases discussed above. (Notice that the roles of C_5', C_6' and C_7' are symmetric.)

Case (1.2). Now suppose $\{x_2, x_3, x_4\}$ - $N(C_4') \neq \emptyset$.

Notice that $\{x_2, x_3, x_4\}$ - $N(C_i') \neq \emptyset$ for $5 \leq i \leq 7$, otherwise the C_i' $(5 \leq i \leq 7)$ is similar to C_4' as discussed above, and the other three in $\{C_4', C_5', C_6', C_7'\}$ are similar to C_5', C_6', C_7', by the same argument as above, we obtain a contradiction. Also since C_4', C_5', C_6', C_7' each cannot be adjacent to both x_1 and x_4, all of C_4', C_5', C_6', C_7' must be adjacent to x_5. Now x_1 and x_4 are not symmetric but x_2 and x_3 are symmetric in $G'[\{x_1, x_2, x_3, x_4\} \cup \{C_1', C_2', C_3'\}]$.

First, suppose C_4' is adjacent to x_4 and x_3 besides x_5. Then suppose C_5' is adjacent to x_4 and x_3. If C_6' is also adjacent to x_4 and x_3, then $G'[\{x_3, x_4, x_5\} \cup \{C_4', C_5', C_6'\}]$ contains $K_{3,3}$, a contradiction. Then suppose C_6' is adjacent to x_4, x_2, now C_6' is connected to x_3 by path $C_6'x_2C_2'x_3$, and $G'[\{x_2, x_3, x_4, x_5\} \cup \{C_2', C_4', C_5', C_6'\}]$ contains a subdivision of $K_{3,3}$, a contradiction. Now suppose C_6' is adjacent to x_4, x_1, then C_6' is connected to x_3 by path $C_6'x_1C_2'x_3$, and $G'[\{x_1, x_3, x_4, x_5\} \cup \{C_2', C_4', C_5', C_6'\}]$ contains a subdivision of $K_{3,3}$ a contradiction. Then suppose C_6' is adjacent to x_3, x_2. Now C_6' is connected to x_4 by path $C_6'x_2C_3'x_4$, so $G'[\{x_2, x_3, x_4, x_5\} \cup \{C_3', C_4', C_5', C_6'\}]$ contains a subdivision of $K_{3,3}$, a contradiction. Next we suppose C_6' is adjacent to x_3, x_1, then C_6' is connected to x_2 by path $C_6'x_5C_4'x_4C_3'x_2$, and $G'[\{x_1, x_2, x_3, x_4, x_5\} \cup \{C_1', C_2', C_3', C_4', C_6'\}]$ contains a subdivision of $K_{3,3}$, a contradiction. Then suppose C_6' is adjacent to x_2, x_1, now C_6' is connected to x_3 by path $C_6'x_5C_4'x_3$, and $G'[\{x_1, x_2, x_3, x_5\} \cup \{C_1', C_2', C_4', C_6'\}]$ contains a subdivision of $K_{3,3}$, a contradiction. Now we suppose C_5' is adjacent to x_4, x_2. By the symmetry of the roles of C_4', C_5', C_6', C_7' , we only consider the cases of C_6' as follows. Suppose C_6' is adjacent to x_4, x_2. Then C_4' is connected to x_2 by path $C'_4x_3C'_3x_2$, and $G'[\{x_2, x_3, x_4, x_5\} \cup \{C'_3, C'_4, C'_5, C'_6\}]$ contains a subdivision of $K_{3,3}$, a contradiction. Then suppose C'_6 is adjacent to x_4, x_1. Now C'_3 is connected to x_1 by path $C'_3x_4C'_6x_1$, and $G'[\{x_1, x_2, x_3, x_4\} \cup \{C'_1, C'_2, C'_3, C'_6\}]$ contains a subdivision of $K_{3,3}$, a contradiction. Now suppose C'_6 is adjacent to x_3, x_2. Then we consider C'_7. By the symmetry of the roles of C'_7 and C'_6, we only consider the cases that C'_7 is adjacent to $\{x_3, x_2\}, \{x_3, x_1\}, \{x_2, x_1\}$ respectively. If C'_7 is adjacent to x_3, x_2, then C'_5 is connected to x_3 by path $C'_5x_4C'_4x_3$, and $G'[\{x_2, x_3, x_4, x_5\} \cup \{C'_4, C'_5, C'_6, C'_7\}]$ contains a subdivision of $K_{3,3}$, a contradiction. If C'_7 is adjacent to x_3, x_1, then C'_3 is connected to x_1 by path $C'_3x_4C'_5x_5C'_7x_1$, and $G'[\{x_1, x_2, x_3, x_4, x_5\} \cup \{C'_1, C'_2, C'_3, C'_5, C'_7\}]$ contains a subdivision of $K_{3,3}$, a contradiction. If C'_7 is adjacent to x_2, x_1, then C'_7 is connected to x_3 by path $C'_7x_5C'_4x_3$, and $G'[\{x_1, x_2, x_3, x_5\} \cup \{C'_1, C'_2, C'_4, C'_7\}]$ contains a subdivision of $K_{3,3}$, a contradiction. Now suppose C'_6 is adjacent to x_3, x_1. Then C'_3 is connected to x_1 by path $C'_3x_4C'_4x_5C'_6x_1$, and $G'[\{x_1, x_2, x_3, x_4, x_5\} \cup \{C'_1, C'_2, C'_3, C'_4, C'_6\}]$ contains a subdivision of $K_{3,3}$, a contradiction. Now we suppose C'_6 is adjacent to x_2, x_1. Then C'_3 is connected to x_1 by path $C'_3x_4C'_4x_5C'_6x_1$, and $G'[\{x_1, x_2, x_3, x_4, x_5\} \cup \{C'_1, C'_2, C'_3, C'_4, C'_6\}]$ contains a subdivision of $K_{3,3}$, a contradiction. Now suppose C'_5 is adjacent to x_4, x_1. Then C'_3 is connected to x_1 by path $C'_3x_4C'_4x_5C'_5x_1$, and $G'[\{x_1, x_2, x_3, x_4, x_5\} \cup \{C'_1, C'_2, C'_3, C'_4, C'_5\}]$ contains a subdivision of $K_{3,3}$, a contradiction. Then we suppose C'_5 is adjacent to x_3, x_2. By symmetry of C'_5 and C'_6, we only need to consider the following cases. Suppose C'_6 is adjacent to x_3, x_2. Then C'_3 is connected to x_5 by path $C'_3x_4C'_4x_5$, and $G'[\{x_2, x_3, x_4, x_5\} \cup \{C'_3, C'_4, C'_5, C'_6\}]$ contains a subdivision of $K_{3,3}$, a contradiction. Now suppose C'_6 is adjacent to x_3, x_1, then C'_3 is connected to x_1 by path $C'_3x_4C'_4x_5C'_6x_1$, and $G'[\{x_1, x_2, x_3, x_4, x_5\} \cup \{C'_1, C'_2, C'_3, C'_4, C'_6\}]$ contains a subdivision of $K_{3,3}$, a contradiction. Then suppose C'_6 is adjacent to x_2, x_1, by the same argument as last case, we also obtain a contradiction. Now we suppose C'_5 is adjacent to x_3, x_1. Then C'_5 is connected to x_2 by path $C'_5x_5C'_4x_4C'_3x_2$, and $G'[\{x_1, x_2, x_3, x_4, x_5\} \cup \{C'_1, C'_2, C'_3, C'_4, C'_5\}]$ contains a subdivision of $K_{3,3}$, a contradiction. So suppose C'_5 is adjacent to x_2, x_1. Then C'_5 is connected to x_3 by path $C'_5x_5C'_4x_3$, and $G'[\{x_1, x_2, x_3, x_5\} \cup \{C'_1, C'_2, C'_4, C'_5\}]$ contains a subdivision of $K_{3,3}$, a contradiction. Since the case that C'_4 is adjacent to x_4, x_2 is symmetric to the case that C'_4 is adjacent to x_4, x_3, we do not discuss this case. The case that C'_4 is adjacent to x_4, x_1 is excluded by the discussion before Case (1.1). So we suppose C'_4 is adjacent to x_3, x_2 now. By symmetry, we only need to consider the following cases. Suppose C'_5 is adjacent to x_3, x_2, then suppose C'_6 is adjacent to x_3, x_2. Then $G'[\{x_2, x_3, x_5\} \cup \{C'_4, C'_5, C'_6\}]$ contains a $K_{3,3}$, a contradiction. Then suppose C'_6 is adjacent to x_3, x_1. Now C'_6 is connected to x_2 by path $C'_6x_1C'_1x_2$, and $G'[\{x_1, x_2, x_3, x_5\} \cup \{C'_1, C'_4, C'_5, C'_6\}]$ contains a subdivision of $K_{3,3}$, a contradiction. So suppose that C'_6 is adjacent to x_2, x_1, then C'_6 is connected to x_3 by path $C'_6x_1C'_1x_3$, and $G'[\{x_1, x_2, x_3, x_5\} \cup \{C'_1, C'_4, C'_5, C'_6\}]$ contains a subdivision of $K_{3,3}$, a contradiction. Now suppose C'_5 is adjacent to x_3, x_1, then suppose C'_6 is adjacent to x_3, x_1. Then C'_4 is connected to x_1 by path $C'_4x_2C'_1x_1$, and $G'[\{x_1, x_2, x_3, x_5\} \cup \{C'_1, C'_4, C'_5, C'_6\}]$ contains a subdivision of $K_{3,3}$, a contradiction. So suppose C'_6 is adjacent to x_2, x_1. Now C'_6 is connected to x_3 by path $C'_6x_5C'_4x_3$, and $G'[\{x_1, x_2, x_3, x_5\} \cup \{C'_1, C'_2, C'_4, C'_6\}]$ contains a subdivision of $K_{3,3}$, a contradiction. Then suppose C'_5 is adjacent to x_2, x_1, and C'_6 can be adjacent only to x_2, x_1, by the same argument as last case, we can obtain a contradiction. Now suppose C'_4 is adjacent to x_3, x_1, then suppose C'_5 and C'_6 are both adjacent to x_3, x_1. But $G'[\{x_1, x_3, x_5\} \cup \{C'_4, C'_5, C'_6\}]$ contains $K_{3,3}$, a contradiction. So we suppose C'_6 is adjacent to x_2, x_1 subject to the above assumption of C'_4 and C'_5. Now C'_6 is connected to x_3 by path $C'_6x_5C'_4x_3$, and $G'[\{x_1, x_2, x_3, x_5\} \cup \{C'_1, C'_2, C'_4, C'_6\}]$ contains a subdivision of $K_{3,3}$, a contradiction. Then suppose C_5' is adjacent to x_2, x_1. Substituting C_5' for C_6' in the discussion of last case, we can obtain a contradiction. The remaining case is that C_4', C_5', C_6' are all adjacent to x_2, x_1. Then $G'[\{x_1, x_2, x_5\} \cup \{C_4', C_5', C_6'\}]$ contains $K_{3,3}$, a contradiction.

Now we come back to the discussion of the first paragraph in Case 1. We have the second subcase as follows.

Suppose C_3' is adjacent to only one vertex in $\{x_1, x_2, x_3\}$. By the symmetry of the roles of $C_3', C_4', \cdots, C_7'$, we can assume that each of $C_3', C_4', \cdots, C_7'$ is adjacent to only one vertex in $\{x_1, x_2, x_3\}$. So all of $C_3', C_4', \cdots, C_7'$ are adjacent to both x_4 and x_5, and one of x_1, x_2, x_3. But x_1, x_2, x_3 have only 3 vertices and $C_3', C_4', \cdots, C_7'$ have 5 vertices, by the Pigeonhole Principle, there are C_i' and C_j' $(3 \leq i \neq j \leq 7)$ adjacent to x_4, x_5 and the same x_r in $\{x_1, x_2, x_3\}$, and there is a C_k' $(k \neq i, j$ and $3 \leq k \leq 7)$ such that C_k' is adjacent to x_4, x_5. We use C_i' and C_j' to replace C_1' and C_2', and C_k' to replace C_3', then Case 1 still happens. The proof is the same as before.

Now suppose that Case 1 does not happen. Then, for any two vertices C_i' and C_j' $(1 \leq i < j \leq 7)$, $|N(C_i') \cap N(C_j')| \leq 2$.

Case 2. Suppose there are two vertices C_i' and C_j' $(1 \leq i < j \leq 7)$ such that $|N(C_i') \cap N(C_j')| = 2$.

Without loss of generality, assume that $N(C_1') = \{x_1, x_2, x_3\}$, $N(C_2') = \{x_2, x_3, x_4\}$ such that $|N(C_1') \cap N(C_2')| = 2$.

Case (2.1). C_1', C_2', C_3', C_4' are adjacent to only vertices in $\{x_1, x_2, x_3, x_4\}$.

Since G is 3-connected and Case 1 does not happen, then $N(C_3') = \{x_1, x_2, x_4\}$ and $N(C_4') = \{x_1, x_3, x_4\}$. In $G'[\{x_1, x_2, x_3, x_4\} \cup \{C_1', C_2', C_3', C_4'\}]$, any two of x_1, x_2, x_3, x_4 are symmetric, and any two of C_1', C_2', C_3', C_4' are symmetric. Since Case 1 does not happen, $N(C_j')$ is not contained in $\{x_1, x_2, x_3, x_4\}$ $(j = 5, 6, 7)$. So C_j' must be adjacent to x_5 $(j = 5, 6, 7)$. By the symmetry of any two of x_1, x_2, x_3, x_4, we can assume that C_5' is adjacent to x_3, x_4. Also by the assumption of Case 2, C_6' is adjacent to x_1 and x_2, or adjacent to x_2 and x_3. In the first case, $\{x_1, x_2\}$ and $\{x_3, x_4\}$ are symmetric, then C_1' is adjacent to x_1, x_2, x_3, C_2' is adjacent to x_2, x_3, and C_2' is connected to x_1 by path $C_2'x_4C_3'x_1$, C_6' is adjacent to x_1, x_2, and C_6' is connected to x_3 by path $C_6'x_5C_5'x_3$. So $G'[\{x_1, x_2, x_3, x_4, x_5\} \cup \{C_1', C_2', C_3', C_5', C_6'\}]$ contains a subdivision of $K_{3,3}$, a contradiction. In the second case, C_2' is adjacent to x_2, x_3, x_4, C_4' is adjacent to x_3, x_4, and is connected to x_2 by path $C_4'x_1C_1'x_2$, C_5' is adjacent to x_3, x_4, and is connected to x_2 by path $C_5'x_5C_6'x_2$. So $G'[\{x_1, x_2, x_3, x_4, x_5\} \cup \{C_1', C_2', C_4', C_5', C_6'\}]$ contains a subdivision of $K_{3,3}$, a contradiction.

Case (2.2). $N(C_i') \subseteq \{x_1, x_2, x_3, x_4\}$ for $i = 1, 2, 3$ but $N(C_j') \nsubseteq \{x_1, x_2, x_3, x_4\}$ for $4 \leq j \leq 7$.

Then $N(C_3') = \{x_1, x_2, x_4\}$. (The case that $N(C_3') = \{x_1, x_3, x_4\}$ is symmetric, and the proof is similar). As the neighbourhood of C_i' $(i = 4, 5, 6, 7)$ is not contained in $\{x_1, x_2, x_3, x_4\}$, C_i' is adjacent to x_5 for $i = 4, 5, 6, 7$. In $G'[\{x_1, x_2, x_3, x_4\} \cup \{C_1', C_2', C_3'\}]$, any two of x_1, x_3, x_4 are symmetric, and any two of C_1', C_2', C_3' are symmetric. By the assumption of Case 2, we have the following four cases.

Case (2.2.1). Suppose $N(C_4') = \{x_3, x_4, x_5\}$, $N(C_5') = \{x_1, x_4, x_5\}$, and $N(C_6') = \{x_1, x_3, x_5\}$.

Notice that any two of x_1, x_3, x_4 are symmetric, since Case (2.1) does not happen, $N(C_7')$ is not contained in $\{x_1, x_3, x_4, x_5\}$, without loss of generality, assume that $N(C_7') = \{x_2, x_3, x_5\}$. Now C_2' is adjacent to x_2, x_3, x_4, C_1' is adjacent to x_2, x_3, and is connected to x_4 by path $C_1'x_1C_3'x_4$, C_7' is adjacent to x_2, x_3, and is connected to x_4 by path $C_7'x_5C_4'x_4$. So $G'[\{x_1, x_2, x_3, x_4, x_5\} \cup \{C_1', C_2', C_3', C_4', C_7'\}]$ contains a subdivision of $K_{3,3}$, a contradiction.

Case (2.2.2). Suppose $N(C_4') = \{x_3, x_4, x_5\}$, $N(C_5') = \{x_1, x_3, x_5\}$.

Now x_1 and x_4 are symmetric. By the assumption of Case 2, as Case (2.2.1) does not hold, we have two cases: $N(C_6^{'}) = \{x_2, x_4, x_5\}$; or $N(C_6^{'}) = \{x_2, x_3, x_5\}$. Suppose $N(C_6^{'}) = \{x_2, x_4, x_5\}$. Then $C_2^{'}$ is adjacent to x_2, x_3, x_4, $C_1^{'}$ is adjacent to x_2, x_3, and is connected to x_4 by path $C_1^{'}x_1C_3^{'}x_4$, $C_6^{'}$ is adjacent to x_2, x_4, and is connected to x_3 by path $C_6^{'}x_5C_5^{'}x_3$. So $G'[\{x_1, x_2, x_3, x_4, x_5\} \cup \{C_1^{'}, C_2^{'}, C_3^{'}, C_5^{'}, C_6^{'}\}]$ contains a subdivision of $K_{3,3}$, contradicting to the fact that G' is planar. Suppose $N(C_6^{'}) = \{x_2, x_3, x_5\}$. Then $C_2^{'}$ is adjacent to x_2, x_3, x_4, $C_1^{'}$ is adjacent to x_2, x_3, and is connected to x_4 by path $C_1^{'}x_1\ C_3^{'}x_4$, $C_6^{'}$ is adjacent to x_2, x_3, and is connected to x_4 by path $C_6^{'}x_5C_4^{'}x_4$. So $G'[\{x_1, x_2, x_3, x_4, x_5\} \cup \{C_1^{'}, C_2^{'}, C_3^{'}, C_4^{'}, C_6^{'}\}]$ contains a subdivision of $K_{3,3}$, a contradiction.

Case (2.2.3). Suppose $N(C_4^{'}) = \{x_3, x_4, x_5\}$.

Now x_3 and x_4 are symmetric. We have two cases: (1) $N(C_5^{'}) = \{x_2, x_3, x_5\}$, $N(C_6^{'}) = \{x_2, x_4, x_5\}$; (2) $N(C_5^{'}) = \{x_2, x_3, x_5\}$, $N(C_6^{'}) = \{x_1, x_2, x_5\}$.

In the first case, $C_2^{'}$ is adjacent to x_2, x_3, x_4, $C_1^{'}$ is adjacent to x_2, x_3, and is connected to x_4 by path $C_1^{'}x_1C_3^{'}x_4$, $C_6^{'}$ is adjacent to x_2, x_4, and is connected to x_3 by path $C_6^{'}x_5C_5^{'}x_3$. So $G'[\{x_1, x_2, x_3, x_4, x_5\} \cup \{C_1^{'}, C_2^{'}, C_3^{'}, C_5^{'}, C_6^{'}\}]$ contains a subdivision of $K_{3,3}$, a contradiction.

In the second case, $C_1^{'}$ is adjacent to x_1, x_2, x_3, $C_2^{'}$ is adjacent to x_2, x_3, and is connected to x_1 by path $C_2^{'}x_4C_3^{'}x_1$, $C_6^{'}$ is adjacent to x_1, x_2, and is connected to x_3 by path $C_6^{'}x_5C_4^{'}x_3$. So $G'[\{x_1, x_2, x_3, x_4, x_5\} \cup \{C_1^{'}, C_2^{'}, C_3^{'}, C_4^{'}, C_6^{'}\}]$ contains a subdivision of $K_{3,3}$, a contradiction.

There remains the last case in Case (2.2) as follows.

Case (2.2.4). Suppose $N(C_4^{'}) = \{x_2, x_3, x_5\}$, $N(C_5^{'}) = \{x_2, x_4, x_5\}$, $N(C_6^{'}) = \{x_1, x_2, x_5\}$.

Now $C_1^{'}$ is adjacent to x_1, x_2, x_3, $C_2^{'}$ is adjacent to x_2, x_3, and is connected to x_1 by path $C_2^{'}x_4C_3^{'}x_1$, $C_6^{'}$ is adjacent to x_1, x_2, and is connected to x_3 by path $C_6^{'}x_5C_4^{'}x_3$. So $G'[\{x_1, x_2, x_3, x_4, x_5\} \cup \{C_1^{'}, C_2^{'}, C_3^{'}, C_4^{'}, C_6^{'}\}]$ contains a subdivision of $K_{3,3}$, a contradiction.

Case (2.3). Suppose only $C_1^{'}, C_2^{'}$ are adjacent only to vertices in $\{x_1, x_2, x_3, x_4\}$.

Since $N(C_i^{'})$ is not contained in $\{x_1, x_2, x_3, x_4\}$, $C_i^{'}$ is adjacent to x_5 for $i = 3, 4, \cdots, 7$. Now x_1 and x_4 are symmetric, x_2 and x_3 are symmetric in $G'[\{x_1, x_2, x_3, x_4\} \cup \{C_1^{'}, C_2^{'}\}]$. By the assumption of Case 2, each $C_i^{'}$ is adjacent to exactly two vertices in $\{x_1, x_2, x_3, x_4\}$ besides x_5 ($i = 3, 4, \cdots, 7$), and $C_i^{'}$ and $C_j^{'}$ ($3 \le i < j \le 7$) are not adjacent to the same two vertices in $\{x_1, x_2, x_3, x_4\}$. Since the number of combinations of two vertices in $\{x_1, x_2, x_3, x_4\}$ is totally $C(4, 2) = \frac{4\times3}{2!} = 6$, there is exactly one combination of two vertices which are not adjacent to the same $C_i^{'}$ ($3 \le i \le 7$), and for each of the other combinations of two vertices in $\{x_1, x_2, x_3, x_4\}$, the two vertices are both adjacent to one $C_i^{'}$ ($3 \le i \le 7$). Considering the symmetry of the roles of $C_i^{'}$ ($i = 3, 4, \cdots, 7$), there are six cases to take five combinations from the totally six combinations of two vertices in $\{x_1, x_2, x_3, x_4\}$ such that the two vertices of each of the five combinations are both adjacent to a $C_i^{'}$ ($3 \le i \le 7$). Also considering the symmetry of x_1 and x_4, and x_2 and x_3, there remains 3 cases as follows.

Case (2.3.1). No $C_i^{'}$ ($3 \le i \le 7$) is adjacent to both x_1 and x_4.

Since the roles of $C_3^{'}, C_4^{'}, \cdots, C_7^{'}$ are symmetric, without loss of generality, assume that $N(C_3^{'}) = \{x_1, x_2, x_5\}$, $N(C_4^{'}) = \{x_3, x_4, x_5\}$, $N(C_5^{'}) = \{x_2, x_4, x_5\}$, $N(C_6^{'}) = \{x_1, x_3, x_5\}$, $N(C_7^{'}) = \{x_2, x_3, x_5\}$.

Now $C_7^{'}$ is adjacent to x_2, x_3, x_5, $C_4^{'}$ is adjacent to x_3, x_5, and is connected to x_2 by path $C_4^{'}x_4C_2^{'}x_2$, $C_3^{'}$ is adjacent to x_2, x_5, and is connected to x_3 by path $C_3^{'}x_1C_6^{'}x_3$. So $G'[\{x_1, x_2, x_3, x_4, x_5\} \cup \{C_2^{'}, C_3^{'}, C_4^{'}, C_6^{'}, C_7^{'}\}]$ contains a subdivision of $K_{3,3}$, a contradiction.

Case (2.3.2). No $C_i^{'}$ ($3 \le i \le 7$) is adjacent to both x_1 and x_2. (For x_1, x_3; x_2, x_4;x_3, x_4, the discussion is similar.)

Since the roles of $C_3^{'}, C_4^{'}, \cdots, C_7^{'}$ are symmetric, without loss of generality, assume that $N(C_3^{'}) =$

$\{x_1, x_4, x_5\}$, $N(C_4^{'}) = \{x_1, x_3, x_5\}$, $N(C_5^{'}) = \{x_2, x_4, x_5\}$, $N(C_6^{'}) = \{x_3, x_4, x_5\}$, $N(C_7^{'}) = \{x_2, x_3, x_5\}$.

Now $C_2^{'}$ is adjacent to x_2, x_3, x_4, $C_1^{'}$ is adjacent to x_2, x_3, and is connected to x_4 by path $C_1^{'}x_1C_3^{'}x_4$, $C_5^{'}$ is adjacent to x_2, x_4, and is connected to x_3 by path $C_5^{'}x_5C_7^{'}x_3$. So $G'[\{x_1, x_2, x_3, x_4, x_5\} \cup \{C_1^{'}, C_2^{'}, C_3^{'}, C_5^{'}, C_7^{'}\}]$ contains a subdivision of $K_{3,3}$, a contradiction.
Case (2.3.3). No $C_i^{'}$ $(3 \leq i \leq 7)$ is adjacent to both x_2 and x_3.

Since the roles of $C_3^{'}, C_4^{'}, \cdots, C_7^{'}$ are symmetric, without loss of generality, assume that $N(C_3^{'}) = \{x_1, x_4, x_5\}$, $N(C_4^{'}) = \{x_1, x_2, x_5\}$, $N(C_5^{'}) = \{x_1, x_3, x_5\}$, $N(C_6^{'}) = \{x_2, x_4, x_5\}$, $N(C_7^{'}) = \{x_3, x_4, x_5\}$.

Now $C_1^{'}$ is adjacent to x_1, x_2, x_3, $C_2^{'}$ is adjacent to x_2, x_3, and is connected to x_1 by path $C_2^{'}x_4C_3^{'}x_1$, $C_4^{'}$ is adjacent to x_1, x_2, and is connected to x_3 by path $C_4^{'}x_5C_5^{'}x_3$. So $G'[\{x_1, x_2, x_3, x_4, x_5\} \cup \{C_1^{'}, C_2^{'}, C_3^{'}, C_4^{'}, C_5^{'}\}]$ contains a subdivision of $K_{3,3}$, a contradiction.
Case 3. Suppose that, for any two vertices $C_i^{'}$ and $C_j^{'}$ $(1 \leq i < j \leq 7)$, $|N(C_i^{'}) \cap N(C_j^{'})| \leq 1$ and Cases 1 and 2 do not hold.

Since $|N(C_i^{'})| = |N(C_j^{'})| = 3$ and $|S| = 5$, $|N(C_i^{'}) \cap N(C_j^{'})| = 1$ $(1 \leq i < j \leq 7)$. Without loss of generality, assume that $N(C_1^{'}) = \{x_1, x_2, x_3\}$, and $N(C_2^{'}) = \{x_3, x_4, x_5\}$. Then $N(C_1^{'}) \cap N(C_2^{'}) = \{x_3\}$. By the assumption of Case 3, $|N(C_3^{'}) \cap N(C_1^{'})| = 1$, then there are two vertices of $N(C_3^{'})$ which are not in $\{x_1, x_2, x_3\}$, hence $|N(C_3^{'}) \cap N(C_2^{'})| \geq 2$, which contradicts the assumption of Case 3.

In all cases discussed above, we can always obtain contradiction. So $\omega(G - S) \geq |S| + 2$ does not hold. By Lemma 1, when $\nu(G) \geq 2k$, G is k-subconnected for $k = 4, 5, 6$.
Remark 1. Now we give some counterexamples to show the sharpness of Corollaries 7 and 8, and Theorem 10. Let H be a connected planar graph, let G_1, G_2, G_3 be three copies of H, and let v be a vertex not in G_i $(i = 1, 2, 3)$. Let G be the graph such that v is joined to G_i $(i = 1, 2, 3)$ by an edge respectively. Then G is a 1-connected planar graph, but G is not 2-subconnected since we take a vertex v_i in G_i $(i = 1, 2, 3)$ and let $v_4 = v$, then there are not two independent paths joining v_1, v_2, v_3, v_4 in two pairs in G. So Corollary 7 is sharp.

Let H be a planar embedding of a 2-connected planar graph, let G_1, G_2, G_3, G_4 be four copies of H, let v_5 and v_6 be two vertices not in G_i $(i = 1, 2, 3, 4)$. Let G be the graph such that v_5 and v_6 are joined to two different vertices on the outer face of G_i $(i = 1, 2, 3, 4)$ by edges respectively. Then G is a 2-connected planar graph, but is not 3-subconnected since we take a vertex v_i in G_i $(i = 1, 2, 3, 4)$, then there are not three independent paths joining $v_1, v_2, v_3, v_4, v_5, v_6$ in three pairs. So Corollary 8 is sharp.

Let G_0 be a triangle with vertex set $\{v_4, v_5, v_6\}$, then insert a vertex v_i into a triangle inner face of G_{i-1} and join v_i to every vertex on the face by an edge respectively to obtain G_i for $i = 1, 2, 3$. Let $G = G_3$. Notice that $\nu(G) = 6$, $\varepsilon(G_0) = 3$, and each time when we insert a vertex, the number of edges increases by 3. So $\varepsilon(G) = 3 + 3 + 3 + 3 = 12$. By the Euler's Formula, $\phi = \varepsilon - \nu + 2 = 12 - 6 + 2 = 8$. Let H be a planar embedding of a 3-connected planar graph. Then put a copy of H into every face of G and join each vertex on the triangle face of G to a distinct vertex on the outer face of H to obtain a planar graph G'. Then G' is a 3-connected planar graph with $\nu(G') \geq 2k$ for $k = 7$, but G' is not 7-subconnected since we have a cutset $S = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, but $G^{'} - S$ has 8 copies of H (8 components) and $\omega(G^{'} - S) = 8 \geq |S| + 2$. By Lemma 1, the conclusion holds. So Theorem 10 is sharp.
Remark 2. For a 3-connected planar graph G with at least $2k$ vertices, by Lemma 6, G is obviously k-subconnected for $k = 1, 2, 3$.

4. Conclusions

In the last section, we prove the k-subconnectivity of k'-connected planar graphs for $k' = 1, 2, \cdots, 5$.

Since a k-subconnected graph is a spanning substructure of a k-connected graph, in the future, we can work on the number of edges deleted from a k-connected graph such that the resulting graph is still k-subconnected.

We may also extend the k-subconnectivity of planar graphs to find the subconnectivity of general graphs with a higher genus.

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