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Research article

Analytical solutions of q-fractional differential equations with proportional derivative

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Abstract: In this paper, we aim to propose a novel q-fractional derivative in the Caputo sense included proportional derivative. To this end, we firstly introduced a new concept of proportional q-derivative and discussed its properties in detail. Then, we add this definition in the concept of Caputo derivative to state a new type of dynamical system with q-calculus. For analytically solving this system, q-Laplace transform has been successfully applied to obtain the solutions. Indeed, the bivariate Mittag-Leffler function has an essential role in this regard. Two illustrative examples are also given in detail.

Keywords: q-calculus; q-proportional; proportional Caputo q-derivative; q-Laplace transform
Mathematics Subject Classification: 26A33, 39A13

1. Introduction

q-Calculus has extensively discussed by many researchers [1–8]. Later, q-difference equations have studied in several applications with various fields [9–18]. The solutions of q-fractional differential equations have investigated in [19–32] and references therein.

To generalize the concept of integration and differentiation, fractional calculus has been widely discussed. One of the main parts of fractional analysis is exploring solutions of fractional differential equations with various types of differentiability. On the other hand, kernels divided this field into the following two cases: 1. Singular kernels (Caputo, Riemann-Liouville, Hadamard, etc), 2. Non-singular kernels (Caputo-Fabrizio, Harmonious, Proportional, etc). Several articles have been published to answer, what kind of differentiability (dynamic system model of fractional order) is suitable for specific problems! One of the main objectives of our article is what will be the method for applying quantum systems of some fractional orders with a combination of singular and non-singular kernels? Finally what will be the attractivity of the solution?

To the best of knowledge of the authors, the q-fractional differential equations with proportional derivative have not been explored. The aim of this paper is firstly defining the concept of proportional q-integral and q-derivative, discussing the well-known properties of such concepts. Besides, we proposed a new definition for a proportional Caputo q-derivative and the related q-integral. To solve a q-dynamical system, we defined a q-Laplace transform to solve the problem analytically and to show the robustness of the model and the approach, we solved two illustrative examples in detail.

2. Preliminaries

For the basic definitions and results, we refer the reader to [8, 30–36].

Let $\mathbb{T}$ be the time scale defined by $\mathbb{T} = \{q^n \big| \, n \in \mathbb{N} \text{ and } 0 < q < 1\} \cup \{0\}$.

The nabla q-derivative of function $g(x) : \mathbb{T} \to \mathbb{R}$ can be defined as

$$\nabla_q g(x) = \frac{g(qx) - g(x)}{(q-1)x}, \qquad \nabla_q g(0) = \lim_{x \to 0} \nabla_q g(x),$$

and for the higher order q-derivatives, we have

$$\nabla_q^0 g = g, \qquad \nabla_q^n g = \nabla_q(\nabla_q^{n-1} g), \quad n \in \mathbb{N}.$$

The formulas for the nabla q-derivative of a sum, a product and a quotient of functions are, respectively;

$$\nabla_q\Big(f(x) + g(x)\Big) = \nabla_q(f(x)) + \nabla_q(g(x)), \tag{2.1}$$

$$\nabla_q\Big(f(x)g(x)\Big) = f(qx)\nabla_q(g(x)) + g(x)\nabla_q(f(x)), \tag{2.2}$$

$$\nabla_q\Big(\frac{f(x)}{g(x)}\Big) = \frac{g(x)\nabla_q(f(x)) - f(x)\nabla_q(g(x))}{g(x)g(qx)}. \tag{2.3}$$

The q-analogue of factorial, for $n = 1, 2, \ldots$, is defined as $[n]_q! = [n]_q \times [n-1]_q \times \cdots \times [1]_q$, where for $t \in \mathbb{R}$, $[t]_q$ is the q-number and is defined as $[t]_q = \frac{1-q^t}{1-q}$.

The q-binomial expansion is defined as

$$(b + x)_q^n = \begin{cases} 1 & n = 0, \\ \prod_{m=0}^{n-1}(b + xq^m) & n = 1, 2, \ldots, \end{cases} \tag{2.4}$$

and for any complex number α, it is also defined as

$$(b + x)_q^\alpha = \frac{(b + x)_q^\infty}{(b + q^\alpha x)_q^\infty},$$

where $(b+x)_q^\infty := \lim_{n\to\infty}\prod_{m=0}^{n}(b+xq^m)$. Furthermore, the q-Taylor series expansion of (2.4) about $x=0$ is given by

$$(b+x)_q^n = \sum_{i=0}^{n}\binom{n}{i}_q b^{n-i}x^i q^{\binom{i}{2}}, \tag{2.5}$$

where $\binom{n}{i}_q = \frac{[n]_q!}{[i]_q![n-i]_q!}$, are called q-binomial coefficients. The interested reader can find some important identities involving q-binomial coefficients in [37].

One can also recall definitions of the q-exponential functions [33, 34, 38] as follows:

$$e_q^x = \frac{1}{(1-(1-q)x)_q^\infty} = \sum_{n=0}^{\infty}\frac{1}{[n]_q!}x^n, \qquad |x|<1, \tag{2.6}$$

$$E_q^x = (1+(1-q)x)_q^\infty = \sum_{n=0}^{\infty}\frac{1}{[n]_q!}x^n q^{\binom{n}{2}}, \qquad x\in\mathbb{C}. \tag{2.7}$$

It can be seen that $e_q^x E_q^{-x} = 1$, $e_{q^{-1}}^x = E_q^x$ and their nabla q-derivative are $\nabla_q e_q^{ax} = a e_q^{ax}$ and $\nabla_q E_q^{ax} = aE_q^{aqx}$. The product of these two q-exponential functions are investigated in a more detailed way in [38–41]. The contribution of the corresponding references can be summarized in the following theorem:

Theorem 1. *For all $x, y \in \mathbb{C}$ the following equation holds*

$$e_q^x\, E_q^y = \sum_{n=0}^{\infty}\frac{1}{[n]_q!}(x+y)_q^n = e_q^{(x+y)_q}, \tag{2.8}$$

where $(x+y)_q^n$ is defined in (2.5).

The nabla q-derivative of $e_q^{(x+y)_q}$ with respect to x and y are respectively, $e_q^{(x+y)_q}$ and $e_q^{(x+qy)_q}$. In the case of $x=y$, the nabla q-derivative of $e_q^{(x+x)_q}$ is $\nabla_q e_q^{(x+x)_q} = (x+x)_q\; e_q^{(x+qx)_q}$.

The nabla q-integral of the function $f(x)$ can be defined as

$$I_{q,0}f(x) = \int_0^x f(v)\;\nabla_q v = x(1-q)\sum_{k=0}^{\infty} q^k f(xq^k), \tag{2.9}$$

and if the lower limit of integral is different from zero, we will have

$$I_{q,a}f(x) = \int_a^x f(v)\;\nabla_q v = \int_0^x f(v)\;\nabla_q v - \int_0^a f(v)\;\nabla_q v. \tag{2.10}$$

The higher order of nabla q-integral is defined by

$$I_{q,a}^0 f = f, \qquad I_{q,a}^n f = I_{q,a}(I_{q,a}^{n-1} f) \quad n = 1,2,3,\ldots.$$

Al-Salam in [42] showed that this higher order q-integral can be represented as a single integral of one variable as

$$I_{q,a}^n f(x) = \frac{1}{[n-1]_q!}\int_a^x (x-qv)_q^{n-1} f(v)\;\nabla_q v \qquad n\in\mathbb{N}. \tag{2.11}$$

Many types of fractional q-derivatives and q-integrals have been proposed by Al-Salam [43] and Agarwal [44], where they all have zero at the lower integration limit. However, it is interesting to have the non-zero at the lower integration limit in some problems, such as solving the fractional order of the q-differential equation or creating the q-Taylor formula.

Definition 1. *For $0 < \alpha < 1$ and $0 < a < x$, the fractional q-integral is*

$$I^{\alpha}_{q,a}f(x) = \frac{1}{\Gamma_q(\alpha)} \int_a^x (x - qv)_q^{\alpha-1} f(v)\, \nabla_q v, \tag{2.12}$$

where $\Gamma_q(x) = \frac{(1-q)_q^{x-1}}{(1-q)^{x-1}}$ is the q-Gamma function.

Based on the fractional q-integral, we can define the fractional q-derivative.

Definition 2. *Let $0 < a < x$ and $\alpha > 0$, then the fractional q-derivative of Riemann–Liouville and Caputo types of order α are respectively:*

$$\nabla^{\alpha}_{q,a}f(x) = \begin{cases} I^{-\alpha}_{q,a}f(x) & \alpha \le 0, \\ \nabla_q^{\lceil\alpha\rceil} I^{\lceil\alpha\rceil-\alpha}_{q,a} f(x) & \alpha > 0, \end{cases} \tag{2.13}$$

and

$$^{c}\nabla^{\alpha}_{q,a}f(x) = \begin{cases} I^{-\alpha}_{q,a}f(x) & \alpha \le 0, \\ I^{\lceil\alpha\rceil-\alpha}_{q,a} \nabla_q^{\lceil\alpha\rceil} f(x) & \alpha > 0, \end{cases} \tag{2.14}$$

where $\lceil\alpha\rceil$ denotes the smallest integer greater or equal to α.

Here are some properties of the fractional q-integrals and q-derivatives.

Lemma 1. *Let $\alpha, \beta \in \mathbb{R}^+$ and $0 < a < x$, then the following equations hold:*

(1) $I^{\beta}_{q,a} I^{\alpha}_{q,a} f(x) = I^{\alpha+\beta}_{q,a} f(x)$,

(2) $\nabla^{\alpha}_{q,a} I^{\alpha}_{q,a} f(x) = f(x)$,

(3) $I^{\alpha}_{q,a} \nabla^{\alpha}_{q,a} f(x) = f(x) - \displaystyle\sum_{i=1}^{\lceil\alpha\rceil} \frac{(x-qa)_q^{\alpha-i}}{\Gamma_q(\alpha-i+1)} \nabla_q^{\alpha-i} f(a)$,

(4) $^{c}\nabla^{\alpha}_{q,a} I^{\alpha}_{q,a} f(x) = f(x)$,

(5) $I^{\alpha}_{q,a}\, {}^{c}\nabla^{\alpha}_{q,a} f(x) = f(x) - \displaystyle\sum_{i=0}^{\lceil\alpha\rceil-1} \frac{(x-qa)_q^{i}}{\Gamma_q(i+1)} \nabla_q^{i} f(a)$.

Hahn ([39] or see [31]) defined the q-analogue of the Laplace transform as

$${}_qL_s\{f(t)\} = \frac{1}{1-q} \int_0^{\frac{1}{s}} E_q^{-\frac{q}{1-q}st} f(t)\, \nabla_q t, \tag{2.15}$$

where $Re(s) > 0$ and E_q^t is defined in (2.7).

In this paper, we give the definition of proportional fractional q-integrals and q-derivatives and to solve them, we need to use the q-Laplace transform, therefore some of the basic properties of the q-Laplace transform are listed in the following lemma.

Lemma 2. *Let $q, \alpha \in (0,1)$, $a, r, t \in \mathbb{R}$, $m, n \in \mathbb{N}$, and let f be an integrable function.*

(1) For $r > -1$, ${}_qL_s\{\frac{t^r}{\Gamma_q(r+1)}\} = \frac{(1-q)^r}{s^{r+1}}$, where $Re(s) > 0$.
(2) ${}_qL_s\{e_q^{\frac{at}{1-q}}\} = \frac{1}{s-a}$.
(3) ${}_qL_s\{\nabla_q^n f(x)\} = (\frac{s}{1-q})^n\ {}_qL_s\{f(x)\} - \sum_{j=1}^{n} \nabla_q^{n-j} f(0)\frac{s^{j-1}}{(1-q)^j}$.
(4) ${}_qL_s\{I_q^n f(t)\} = (\frac{1-q}{s})^n {}_qL_s\{f(t)\}$.
(5) ${}_qL_s\{\sum_{r=0}^{m} f_r(t)\} = \sum_{r=0}^{m} {}_qL_s\{f_r(t)\}$.
(6) ${}_qL_s\{I_q^\alpha f(t)\} = (\frac{1-q}{s})^\alpha {}_qL_s\{f(t)\}$.
(7) ${}_qL_s\{\nabla_q^\alpha f(x)\} = (\frac{s}{1-q})^\alpha\ {}_qL_s\{f(x)\} - \sum_{j=1}^{\lceil\alpha\rceil} \nabla_q^{\alpha-j} f(0)\frac{s^{j-1}}{(1-q)^j}$.
(8) ${}_qL_s\{{}^c\nabla_q^\alpha f(x)\} = (\frac{s}{1-q})^\alpha \left({}_qL_s\{f(x)\} - \sum_{j=0}^{\lceil\alpha\rceil-1} \nabla_q^j f(0)\frac{(1-q)^j}{s^{j+1}}\right)$.

Let us give the definition of bivariate q-Mittag-Leffler function that we will use it in this paper.

Definition 3. *Let $\alpha, \beta, \gamma \in \mathbb{C}$ be complex parameters with $Re(\alpha) > 0$ and $Re(\beta) > 0$.We define the following bivariate q-Mittag-Leffler function for general complex numbers x and y:*

$$ {}_qE_{\alpha,\beta,\gamma}(x,y) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{k+l}{l}\frac{x^k y^l}{\Gamma_q(\alpha k + \beta l + \gamma)}. \tag{2.16}$$

3. Proportional q-derivative and q-integral

Definition of proportional derivative [45] is given by

$$\nabla^\alpha f(t) = k_0(\alpha,t)f'(t) + k_1(\alpha,t)f(t), \tag{3.1}$$

where $\alpha \in (0,1]$ and for any $t \in \mathbb{R}$

$$\lim_{\alpha\to 0^+} k_0(\alpha,t) = 0, \qquad \lim_{\alpha\to 0^+} k_1(\alpha,t) = 1,$$
$$\lim_{\alpha\to 1^-} k_0(\alpha,t) = 1, \qquad \lim_{\alpha\to 1^-} k_1(\alpha,t) = 0,$$

In this section, we study on the q-analouge of the proportional derivative and integral.

3.1. Proportional q-derivative

In the following definition, we propose the q-analouge of the proportional derivative.

Definition 4. *Let $\alpha \in (0,1]$, then the proportional q-derivative is*

$${}^P\nabla_q^\alpha f(t) = \alpha\ \nabla_q f(t) + (1-\alpha)f(t), \tag{3.2}$$

where $\nabla_q f(t)$ is the q-derivative of $f(t)$ with respect to t.

It is easy to see that ${}^{P}\nabla_q^0 f(t) = f(t)$ and ${}^{P}\nabla_q^1 f(t) = \nabla_q f(t)$.

In the next lemma, we are giving some basic properties of the proportional q-derivative.

Lemma 3. *Let $\alpha \in (0,1]$, $a, b, c \in \mathbb{R}$ and the functions f and g are q-differentiable, then*

(1) ${}^{P}\nabla_q^\alpha(af(t) + bg(t)) = a\, {}^{P}\nabla_q^\alpha f(t) + b\, {}^{P}\nabla_q^\alpha g(t),$

(2) ${}^{P}\nabla_q^\alpha c = (1-\alpha)c,$

(3) ${}^{P}\nabla_q^\alpha(f(t)g(t)) = f(t)\, {}^{P}\nabla_q^\alpha g(t) + g(qt)\, {}^{P}\nabla_q^\alpha f(t) - (1-\alpha)f(t)g(qt),$

(4) ${}^{P}\nabla_q^\alpha(f(t)/g(t)) = \dfrac{g(t)\, {}^{P}\nabla_q^\alpha f(t) - f(t)\, {}^{P}\nabla_q^\alpha g(t)}{g(t)g(qt)} + (1-\alpha)\dfrac{f(t)}{g(t)}.$

Proof. Items (a) and (b) follow easily from (3.2). For (c), we use (3.2) and (2.2) to get that

$$\begin{aligned}
{}^{P}\nabla_q^\alpha(f(t)g(t)) &= \alpha\, \nabla_q(f(t)g(t)) + (1-\alpha)(f(t)g(t)) \\
&= \alpha\, (f(t)\nabla_q g(t) + g(qt)\nabla_q f(t)) + (1-\alpha)(f(t)g(t)) \\
&= f(t)\, {}^{P}\nabla_q^\alpha g(t) + g(qt)\, {}^{P}\nabla_q^\alpha f(t) - (1-\alpha)f(t)g(qt).
\end{aligned}$$

The proof of (d) is similar and is omitted. □

3.2. Proportional q-integral

In the below, will give the definition of proportional q-integral which is closely associated with finding the proportional q-antiderivative.

Definition 5. *Let $\alpha \in (0,1]$, then the proportional q-integral is*

$${}^{P}I_q^\alpha f(t) = \frac{1}{\alpha} \int_0^t f(s) e_q^{-\frac{1-\alpha}{\alpha}(t-qs)_q}\, \nabla_q s, \tag{3.3}$$

where $e_q^{(t-s)_q}$ is defined in (2.8).

Theorem 2. *Let ${}^{P}\nabla_q^\alpha$ and ${}^{P}I_q^\alpha$ are the proportional q-derivative and q-integral, respectively, with $\alpha \in (0,1]$. Let the function f be a q-differentiable function. Then*

$${}^{P}\nabla_q^\alpha({}^{P}I_q^\alpha f(t)) = f(t), \tag{3.4}$$

and

$${}^{P}I_q^\alpha({}^{P}\nabla_q^\alpha f(t)) = f(t) - e_q^{-\frac{1-\alpha}{\alpha}t} f(0). \tag{3.5}$$

Proof. To prove the equation in (3.4), we use the linearity property of the proportional q-derivative, to obtain

$${}^{P}\nabla_q^\alpha({}^{P}I_q^\alpha f(t)) = \alpha\, \nabla_q({}^{P}I_q^\alpha f(t)) + (1-\alpha)\, {}^{P}I_q^\alpha f(t). \tag{3.6}$$

To find the q-derivative of ${}^{P}I_q^\alpha f(t)$, we use the definition of proportional q-integral and Eq (2.8), therefore we will have

$$\nabla_q({}^{P}I_q^\alpha f(t)) = \nabla_q\Big(\frac{1}{\alpha} \int_0^t f(s) e_q^{-\frac{1-\alpha}{\alpha}(t-qs)_q}\, \nabla_q s\Big)$$

$$
\begin{aligned}
&= \nabla_q\Big(\frac{1}{\alpha}\, e_q^{-\frac{1-\alpha}{\alpha}t} \int_0^t f(s) E_q^{\frac{1-\alpha}{\alpha}qs}\, \nabla_q s\Big) \\
&= -\frac{1-\alpha}{\alpha^2}\, e_q^{-\frac{1-\alpha}{\alpha}t} \int_0^t f(s) E_q^{\frac{1-\alpha}{\alpha}qs}\, \nabla_q s + \frac{1}{\alpha}\, e_q^{-\frac{1-\alpha}{\alpha}qt} f(t) E_q^{\frac{1-\alpha}{\alpha}qt} \\
&= -\frac{1-\alpha}{\alpha}\, {}^{P}I_q^{\alpha} f(t) + \frac{1}{\alpha} f(t).
\end{aligned} \tag{3.7}
$$

In the third equation, we used the product rule for q-derivative, which is given in Eq (2.2). We will obtain the desired result by substituting Eq (3.7) into (3.6).

To prove the equation in (3.5), we use the linearity property of the proportional q-integral to obtain

$$
{}^{P}I_q^{\alpha}({}^{P}\nabla_q^{\alpha} f(t)) = \alpha\, {}^{P}I_q^{\alpha}(\nabla_q f(t)) + (1-\alpha)\, {}^{P}I_q^{\alpha} f(t). \tag{3.8}
$$

To find the proportional q-intergal of $\nabla_q f(t)$, we use the definition of proportional q-integral and then q-integration by parts, therefore we will have

$$
\begin{aligned}
{}^{P}I_q^{\alpha}(\nabla_q f(t)) &= \frac{1}{\alpha} \int_0^t \nabla_q f(s) e_q^{-\frac{1-\alpha}{\alpha}(t-qs)_q}\, \nabla_q s \\
&= \frac{1}{\alpha}\Big(f(s) e_q^{-\frac{1-\alpha}{\alpha}(t-s)_q}\Big|_{s=0}^{s=t} - \frac{1-\alpha}{\alpha} \int_0^t f(s) e_q^{-\frac{1-\alpha}{\alpha}(t-qs)_q}\, \nabla_q s\Big) \\
&= \frac{1}{\alpha}\Big(f(t) - e_q^{-\frac{1-\alpha}{\alpha}t} f(0) - (1-\alpha)\, {}^{P}I_q^{\alpha} f(t)\Big).
\end{aligned} \tag{3.9}
$$

We will obtain the desired result by substituting Eq (3.9) into (3.8). □

Note that, if $f(0) = 0$, then the proportional q-derivative and q-integral which are given in (3.2) and (3.3), respectively, are inverse of each other.

4. Proportional Caputo q-derivative and its q-integral

If we take the fractional q-integral of the proportional q-derivative, we will see that the result is a linear combination of Caputo fractional q-derivative and fractional q-integral as follow;

$$
I_q^{1-\alpha}({}^{P}\nabla_q^{\alpha} f(t)) = \alpha\, I_q^{1-\alpha}(\nabla_q f(t)) + (1-\alpha)\, I_q^{1-\alpha} f(t) = \alpha\, {}^{c}\nabla_q^{\alpha} f(t) + (1-\alpha)\, I_q^{1-\alpha} f(t). \tag{4.1}
$$

The expression in the right hand side of the above equation, also can be found by replacing $\nabla_q f(s)$ with $\nabla_q^{\alpha} f(s)$ in the integrand of Caputo fractional q-derivative of order α. This interesting result motivate us to give the following definition.

Definition 6. *Let $\alpha \in (0, 1]$, then the proportional Caputo q-derivative is*

$$
{}^{Pc}\nabla_q^{\alpha} f(t) = \frac{1}{\Gamma_q(1-\alpha)} \int_0^t (t-qs)_q^{-\alpha}\, {}^{P}\nabla_q^{\alpha} f(s)\, \nabla_q s, \tag{4.2}
$$

where ${}^{P}\nabla_q^{\alpha} f(s)$ is the proportional q-derivative.

Note that Eq (4.2) also can be written as follow:

$$ {}^{Pc}\nabla_q^{\alpha} f(t) = \alpha \, {}^{c}\nabla_q^{\alpha} f(t) + (1-\alpha)\, I_q^{1-\alpha} f(t), \tag{4.3}$$

where ${}^{c}\nabla_q^{\alpha}$ and I_q^{α} are Caputo fractional q-derivative and fractional q-integral, respectively.

Remark 1. *It is easy to see that in the limit cases of $\alpha \to 0$ and $\alpha \to 1$, we will obtain*

$$\lim_{\alpha\to 0^+} {}^{Pc}\nabla_q^{\alpha} f(t) = I_q f(t), \quad \text{and} \quad \lim_{\alpha\to 1^-} {}^{Pc}\nabla_q^{\alpha} f(t) = \nabla_q f(t).$$

In the below, we give the definition of proportional Caputo q-integral which is closely associated with finding the proportional Caputo q-antiderivative.

Definition 7. *Let $\alpha \in (0,1]$, then the proportional Caputo q-integral is*

$$ {}^{Pc}I_q^{\alpha} f(t) = \frac{1}{\alpha} \int_0^t e_q^{-\frac{1-\alpha}{\alpha}(t-qs)_q} \, {}^{c}\nabla_q^{1-\alpha} f(s)\, \nabla_q s, \tag{4.4}$$

where ${}^{c}\nabla_q^{\alpha}$ is the Caputo fractional q-derivative and $e_q^{(t-s)_q}$ is defined in (2.8).

Theorem 3. *Let ${}^{Pc}\nabla_q^{\alpha}$ and ${}^{Pc}I_q^{\alpha}$ are the proportional Caputo q-derivative and q-integral, respectively, with $\alpha \in (0,1]$. Let the function f be a q-differentiable function. Then*

$$ {}^{Pc}\nabla_q^{\alpha}({}^{Pc}I_q^{\alpha} f(t)) = f(t) - f(0), \tag{4.5}$$

and

$$ {}^{Pc}I_q^{\alpha}({}^{Pc}\nabla_q^{\alpha} f(t)) = f(t) - e_q^{-\frac{1-\alpha}{\alpha}t} f(0). \tag{4.6}$$

Proof. To prove the equation in (4.5), we use Eq (4.3) to obtain

$$ {}^{Pc}\nabla_q^{\alpha}({}^{Pc}I_q^{\alpha} f(t)) = \alpha \, {}^{c}\nabla_q^{\alpha}({}^{Pc}I_q^{\alpha} f(t)) + (1-\alpha)\, I_q^{1-\alpha}({}^{Pc}I_q^{\alpha} f(t)). \tag{4.7}$$

To find the ${}^{c}\nabla_q^{\alpha}({}^{Pc}I_q^{\alpha} f(t))$, we use the definition of proportional Caputo q-integral and equation (2.14), therefore we will have

$$\begin{aligned}
{}^{c}\nabla_q^{\alpha}({}^{Pc}I_q^{\alpha} f(t)) &= I_q^{1-\alpha}\nabla_q\Big(\frac{1}{\alpha}\int_0^t e_q^{-\frac{1-\alpha}{\alpha}(t-qs)_q}\, {}^{c}\nabla_q^{1-\alpha} f(s)\, \nabla_q s\Big) \\
&= I_q^{1-\alpha}\nabla_q\Big(\frac{1}{\alpha} e_q^{-\frac{1-\alpha}{\alpha}t} \int_0^t E_q^{\frac{1-\alpha}{\alpha}qs}\, {}^{c}\nabla_q^{1-\alpha} f(s)\, \nabla_q s\Big) \\
&= I_q^{1-\alpha}\Big(-\frac{1-\alpha}{\alpha^2}\, e_q^{-\frac{1-\alpha}{\alpha}t}\int_0^t E_q^{\frac{1-\alpha}{\alpha}qs}\, {}^{c}\nabla_q^{1-\alpha} f(s)\, \nabla_q s + \frac{1}{\alpha}\, e_q^{-\frac{1-\alpha}{\alpha}qt} E_q^{\frac{1-\alpha}{\alpha}qt}\, {}^{c}\nabla_q^{1-\alpha} f(t)\Big) \\
&= I_q^{1-\alpha}\Big(-\frac{1-\alpha}{\alpha}\, {}^{Pc}I_q^{\alpha} f(t) + \frac{1}{\alpha}\, {}^{c}\nabla_q^{1-\alpha} f(t)\Big) \\
&= -\frac{1-\alpha}{\alpha}\, I_q^{1-\alpha}({}^{Pc}I_q^{\alpha} f(t)) + \frac{1}{\alpha}\,(f(t)-f(0)).
\end{aligned} \tag{4.8}$$

In the third equation, we used the product rule for q-derivative, which is given in Eq (2.2) and in the last equation, we used Lemma 1. So we will obtain the desired result by substituting Eq (4.8) into (4.7).

To prove the equation in (4.6), we use the definition of proportional Caputo q-integral and then Eqs (4.1) and (4.3), therefore we will have

$$\begin{aligned}
{}^{Pc}I_q^{\alpha}({}^{Pc}\nabla_q^{\alpha}f(t)) &= \frac{1}{\alpha}\int_0^t e_q^{-\frac{1-\alpha}{\alpha}(t-qs)_q}\ {}^{c}\nabla_q^{1-\alpha}({}^{Pc}\nabla_q^{\alpha}f(s))\ \nabla_q s \\
&= \frac{1}{\alpha}\int_0^t e_q^{-\frac{1-\alpha}{\alpha}(t-qs)_q}\ {}^{c}\nabla_q^{1-\alpha}(I_q^{1-\alpha}({}^{P}\nabla_q^{\alpha}f(s)))\ \nabla_q s \\
&= \frac{1}{\alpha}\int_0^t e_q^{-\frac{1-\alpha}{\alpha}(t-qs)_q}\ {}^{P}\nabla_q^{\alpha}f(s)\ \nabla_q s \\
&= {}^{P}I_q^{\alpha}({}^{P}\nabla_q^{\alpha}f(t)) = f(t) - e_q^{-\frac{1-\alpha}{\alpha}t}f(0).
\end{aligned}$$

where, in the third row of the above equations, we used Lemma 1 and in the last row, we used Definition 5 and Theorem 2. □

Note that, if $f(0) = 0$, then the proportional Caputo q-derivative and q-integral which are given in (4.2) and (4.4), respectively, are inverse of each other.

Example 1. *Let us try to solve the following simple fractional differential equation:*

$$^{Pc}\nabla_q^{\alpha}f(t) = 0, \qquad f(0) = A.$$

Let us use Eq (4.3) to rewrite the fractional differential equation as

$$\alpha\ {}^{c}\nabla_q^{\alpha}f(t) + (1-\alpha)\ I_q^{1-\alpha}f(t) = 0.$$

Applying the q-Laplace transform to both sides of the equation and then using Lemma 2, we find

$${}_qL_s\{f(t)\}\frac{s^{\alpha}}{(1-q)^{\alpha}}\alpha(1+\frac{1-\alpha}{\alpha}\frac{1-q}{s}) = \alpha\frac{s^{\alpha-1}}{(1-q)^{\alpha}}(f(0)\frac{1}{s}),$$

after substituting the condition $f(0) = A$, we obtain:

$${}_qL_s\{f(t)\} = A\frac{1}{s+\frac{1-\alpha}{\alpha}(1-q)}.$$

Taking the inverse qLaplace transform, we find

$$f(t) = Ae_q^{-\frac{1-\alpha}{\alpha}t}.$$

Example 2. *Let us try to solve, for an arbitrary constant λ,*

$$^{Pc}\nabla_q^{\alpha}f(t) = \lambda f(t), \qquad f(0) = 1.$$

Let us use Eq (4.3) to rewrite the fractional differential equation as

$$\alpha\ {}^{c}\nabla_q^{\alpha}f(t) + (1-\alpha)\ I_q^{1-\alpha}f(t) = \lambda f(t).$$

Applying the q-Laplace transform to both sides of the equation and then using Lemma 2 and the condition $f(0) = 1$, we find

$$\begin{aligned} {}_qL_s\{f(t)\} &= s^{-1}\frac{1}{1-(\frac{\lambda(1-q)^\alpha}{\alpha s^\alpha}-\frac{(1-\alpha)(1-q)}{\alpha s})} \\ &= s^{-1}\sum_{n=0}^{\infty}\Big(\frac{\lambda(1-q)^\alpha}{\alpha s^\alpha}-\frac{(1-\alpha)(1-q)}{\alpha s}\Big)^n \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k(1-\alpha)^k\lambda^{n-k}}{\alpha^n}\frac{(1-q)^{\alpha n-\alpha k+k}}{s^{\alpha n-\alpha k+k+1}}. \end{aligned}$$

Taking the inverse q-Laplace transform term by term, we find that:

$$f(t) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k(1-\alpha)^k\lambda^{n-k}}{\alpha^n}\frac{t^{\alpha n-\alpha k+k}}{\Gamma_q(\alpha n-\alpha k+k+1)}.$$

Relabelling as $l = n-k$ enables the double sum over n and k to be rearranged as an independent double sum over k and l both going from 0 *to* ∞*:*

$$\begin{aligned} f(t) &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{k+l}{k}\frac{(-1)^k(1-\alpha)^k\lambda^l}{\alpha^{k+l}}\frac{t^{\alpha l+k}}{\Gamma_q(\alpha l+k+1)} \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{k+l}{k}(\frac{\lambda t^\alpha}{\alpha})^l(-\frac{1-\alpha}{\alpha}t)^k\frac{1}{\Gamma_q(\alpha l+k+1)}. \end{aligned}$$

This series can be written in terms of the bivariate q-Mittag-Leffler function which is defined in (2.16):

$$f(t) = {}_qE_{\alpha,1,1}(\frac{\lambda t^\alpha}{\alpha},-\frac{1-\alpha}{\alpha}t).$$

5. Conclusions

Due to recently published papers about q-Calculus and fractional differentiability, we introduced a new q-derivative of fractional order, called proportional q-derivative. Some well-known properties of such derivative are discussed in detail using such a new concept, we have proposed fractional q-differential equation with proportional q-derivative. In this regard, using the q-Laplace transform method, we derived the solutions. For further work, we numerically solve such q-dynamic systems, with different singular and non-singular kernels.

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Conflict of interest

The authors declare no conflict of interest.

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