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Research article

Infinitely many sign-changing solutions for a semilinear elliptic equation with variable exponent

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Abstract: This paper is devoted to study a class of semilinear elliptic equations with variable exponent. By means of perturbation technique, variational methods and a priori estimation, the existence of infinitely many sign-changing solutions to this class of problem is obtained.

Keywords: semilinear elliptic equation; variable exponent; variational methods; a priori estimate; sign-changing solution
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1. Introduction and main result

Let $0 \in \Omega \subset \mathbb{R}^N (N \geq 3)$ be a bounded domain with smooth boundary $\partial\Omega$. In this paper, we are interested in establishing the multiplicity of sign-changing solutions to the following semilinear elliptic equations with variable exponent

$$\begin{cases} -\Delta u = |u|^{q(x)-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $q(x)$ satisfies the following assumptions.

(Q_1) $q \in C(\overline{\Omega})$, $q(0) = 2$ and $2 < q(x) \leq \max_{x\in\overline{\Omega}}\{q(x)\} = q^+ < 2^* = \frac{2N}{N-2}$ for $x \neq 0$;

(Q_2) there exist $\alpha \in \left(0, \frac{N+2}{2}\right)$ and $B_{\delta_0} = \{x||x| < \delta_0\} \subset \Omega$ such that $q(x) \geq 2 + |x|^\alpha$ for any $x \in B_{\delta_0}$.

In 1973, Ambrosetti and Rabinowitz in [2] obtained a positive and a negative solution to the following superlinear elliptic problem

$$\begin{cases} -\Delta u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

The existence of the third solution to problem (1.2) was established by Wang in [17]. Castro, Cossio and Neuberger in [6] proved that the third solution to problem (1.2) obtained in [17] changes sign only once. Bartsch and Wang in [3] obtained the existence of sign-changing solution. In addition, Bartsch, Weth and Willem in [4] showed that problem (1.2) possesses a least energy sign-changing solution. In order to study the sign-changing critical points of even functionals, Li and Wang in [11] established a Ljusternik-Schnirelmann theory and showed that problem (1.2) possesses infinitely many sign-changing solutions. Subsequently, the existence of infinitely many sign-changing solutions to problem (1.2) was also obtained by some versions of the symmetric mountain pass lemma(see [15] and [19]).

In fact, these papers required $f(x,t)$ to satisfy the following condition ((AR)-condition, for short)

$$f(x,t)t \geq \theta F(x,t) > 0, \quad \text{for all } x \in \Omega \text{ and } |t| \text{ sufficiently large,}$$

where $\theta > 2$ and $F(x,t) = \int_0^t f(x,s)\,ds$. It is well known that (AR)-condition is important to guarantee the boundedness of Palais-Smale sequence of the Euler-Lagrange functional associated to problem (1.2) which plays a crucial role in applying the critical point theory. For more than 40 years, several researchers studied problem (1.2) trying to drop the above (AR)-condition. For example, a weaker super-quadratic condition ((SQ)-condition, for short) is that

$$\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^2} = \infty \ \text{ uniformly in } \ x \in \Omega.$$

Under (SQ)-condition or some extra assumptions, the existence and multiplicity of nontrivial solution for problem (1.2) were obtained, see [7, 8, 12, 14, 16] and the references therein.

Recently, the special case of problem (1.1) as problem (1.2) is also concerned by some scholars(see [1, 5, 9, 10, 13]). They obtained the existence or multiplicity of the nontrivial solution of problem (1.1) from the discussion the compact embedding from $H_0^1(\Omega)$ to $L^{q(x)}(\Omega)$ with a variable critical or supercritical exponent. In particular, Cao, Li and Liu in [5] obtained that problem (1.1) has infinitely many nodal solutions when $q(x) = 2^* + |x|^\alpha - 2(0 < \alpha < \min\{\frac{N}{2}, N-2\})$ and B_1 is the unit ball in $\mathbb{R}^N$. In addition, Hashizume and Sano in [9] proposed that $ess \inf_{x\in\Omega}\{q(x)\} = 2$ is another critical case. Indeed, if there exists $x_0 \in \Omega$ such that $q(x_0) = \inf_{x\in\Omega}\{q(x)\} = 2$, then the conditions ($AR$) and ($SQ$) do not hold. Therefore, the problem we intend to study is a new phenomenon. To the best of our knowledge, for either p-Laplacian equation(including semilinear elliptic equation) or $p(x)$-Laplacian equation, there are no results in this case. The main difficulty with problem (1.1) is that the corresponding functional may possess unbounded Palais-Smale sequences. To overcome this difficulty, we will use the perturbation technique and the Moser's iteration.

The main result of this paper reads as follows.

Theorem 1.1. *Suppose that* (Q_1) *and* (Q_2) *hold. Then, for every integer* $k \geq 1$*, problem* (1.1) *has* k *sign-changing solutions.*

Remark 1.2. *In [5] and [9], it is crucial to require the space is radially symmetric. However, we do not need the domain to be radial.*

To end this section, we describe the basic ideas in the proof of Theorem 1.1. Noticing that $q(0) = 2$, inspired by [18], we first modify the nonlinear term to guarantee the boundedness of Palais-Smale sequence of the corresponding functional and obtain infinitely many sign-changing solutions of auxiliary problem by a version of the symmetric mountain pass lemma. Subsequently, we use the Moser iteration to obtain the existence of infinitely many sign-changing solutions for problem (1.1).

Throughout this paper, let $B_\delta = \{x||x| < \delta\} \subset \Omega$ and $\Omega_\delta = \Omega\backslash B_\delta$. We use $\|\cdot\|$ to denote the usual norms of $H_0^1(\Omega)$. The letter C stands for positive constant which may take different values at different places.

2. The modified problem

According to $q(0) = 2$, it seems to be difficult to confirm whether the energy function I corresponding to (1.1) satisfies the Palais-Smale condition or not.To apply variational methods, the first step in proving Theorem 1.1 is modifying the nonlinear term to obtain the perturbation equation. Since $q(x)$ is a continuous function and $q^+ < 2^*$, we can choose $r > 0$ such that

$$r < \min\left\{2^* - q^+,\ \frac{1}{4N}\right\}. \tag{2.1}$$

Let $\psi(t) \in C_0^\infty(\mathbb{R}, [0,1])$ be a smooth even function with the following properties: $\psi(t) = 1$ for $|t| \leq 1$, $\psi(t) = 0$ for $|t| \geq 2$ and $\psi(t)$ is monotonically decreasing on the interval $(0, +\infty)$. Define

$$b_\mu(t) = \psi(\mu t), \qquad m_\mu(t) = \int_0^t b_\mu(\tau)d\tau,$$

for $\mu \in (0, 1]$. We will deal with the modified problem

$$\begin{cases} -\Delta u = \left(\frac{u}{m_\mu(u)}\right)^r |u|^{q(x)-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Theorem 2.1. *Suppose that* (Q_1) *and* (Q_2) *hold. Then, for any* $\mu \in (0, 1]$*, problem* (2.2) *has infinitely many sign-changing solutions.*

Let $E := H_0^1(\Omega)$ be the usual Sobolev space endowed with the inner product $\langle u, v\rangle = \displaystyle\int_\Omega \nabla u\nabla v\, dx$ for $u,\ v \in E$ and the norm $\|u\| := \langle u, u\rangle^{\frac{1}{2}}$. Let $\mathcal{P}$ be the positive cone of E, and Y, M be two subspaces of E with $\dim Y < \infty$, $\dim Y - \mathrm{codim} M \geq 1$. For any $\delta > 0$, define $\pm\mathcal{D}(\delta) := \{u \in E : \mathrm{dist}(u, \pm\mathcal{P}) < \delta\}$. Set $\mathcal{D} := \mathcal{D}(\delta) \cup (-\mathcal{D}(\delta))$ and $\mathcal{S} = E\backslash\mathcal{D}$. Let $G \in C^1(E, \mathbb{R})$ and the gradient G' be of the form $G'(u) = u - K_G(u)$, where $K_G : E \to E$ is a continuous operator. Let $\mathcal{K} = \{u \in E : G'(u) = 0\}$

and $\mathcal{K}[a,b] = \{u \in K : G(u) \in [a,b]\}$. We assume that there is another norm $\|\cdot\|_*$ of E such that $\|u\|_* \le C_*\|u\|$ for all $u \in E$, where C_* is a positive constant. Moreover, we assume that $\|u_n - u^*\|_* \to 0$ whenever $u_n \rightharpoonup u^*$ weakly in $(E, \|\cdot\|)$. Write $E = M_1 \oplus M$. Let

$$Q^*(\rho) = \left\{ u \in M : \frac{\|u\|_*^p}{\|u\|^2} + \frac{\|u\|\|u\|_*}{\|u\| + D_*\|u\|_*} = \rho \right\},$$

where $\rho > 0$, $D_* > 0$ and $p > 2$ are fixed constants. Let us assume that $Q^{**} = Q^*(\rho) \cap G^\beta \subset \mathcal{S}$ and $\gamma = \inf_{Q^{**}} G$, where $\beta = \sup_Y G$ and $G^\beta = \{u \in E : G(u) \le \beta\}$. It is easy to see $\beta \ge \gamma$. In addition, we assume that

(A) $K_G(\pm\mathcal{D}(\delta)) \subset \pm\mathcal{D}(\delta)$;

(A_1^*) Assume that for any $a,\ b > 0$, there is a $c_1 = c_1(a,b) > 0$ such that $G(u) \le a$ and $\|u\|_* \le b \Rightarrow \|u\| \le c_1$;

(A_2^*) $\lim\limits_{u\in Y,\ \|u\|\to\infty} G(u) = -\infty$, $\sup_Y G := \beta$.

Now we recall the following Palais-Smale condition and abstract critical point theorem (see Definition 3.3 and Theorem 5.6 in [19]).

Definition 2.2. *The functional G is said to satisfy the* $(w^* - PS)$ *condition if for any sequence* $\{u_n\}$ *such that* $\{G(u_n)\}$ *is bounded and* $G'(u_n) \to 0$, *we have either* $\{u_n\}$ *is bounded and has a convergent subsequence or* $\|G'(u_n)\|\|u_n\| \to \infty$. *In particular, if* $\{G(u_n)\} \to c$, *we say that* $(w^* - PS)_c$ *is satisfied.*

Theorem 2.3. *Assume that* (A), (A_1^*) *and* (A_2^*) *hold. If the even functional G satisfies the* $(w^* - PS)_c$ *condition at level c for each* $c \in [\gamma, \beta]$, *then* $\mathcal{K}[\gamma - \varepsilon, \beta + \varepsilon] \cap (E \setminus (\mathcal{P} \cup (-\mathcal{P}))) \neq \emptyset$ *for all* $\varepsilon > 0$ *small.*

Let $0 < \lambda_1 < \cdots < \lambda_k < \cdots$ denote the distinct Dirichlet eigenvalues of the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Then each λ_k has finite multiplicity. In addition, the principal eigenvalue λ_1 is simple with a positive eigenfunction φ_1, and the eigenfunctions φ_k corresponding to $\lambda_k (k \ge 2)$ are sign-changing. Let N_k denote the eigenspace of λ_k. Then $\dim N_k < \infty$. We fix k and let $E_k := N_1 \oplus N_2 \oplus \cdots \oplus N_k$.

The formal energy functional $I_\mu : H_0^1(\Omega) \to \mathbb{R}$ associated with (2.2) is defined by

$$I_\mu(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \int_\Omega F_\mu(x,u)\, dx,$$

where $f_\mu(x,t) = \left(\frac{t}{m_\mu(t)}\right)^r |t|^{q(x)-2} t$, $F_\mu(x,t) = \displaystyle\int_0^t f_\mu(x,\tau)\, d\tau$. Then $I_\mu \in C^1(E,\mathbb{R})$ and $I'_\mu = id - (-\Delta)^{-1} f_\mu = id - K_{I_\mu}$. Obviously, the critical points of I_μ are just the weak solutions of problem (2.2).

Lemma 2.4. *The function* $F_\mu(x,t)$ *defined above satisfies the following inequalities:*

$$F_\mu(x,t) \le \frac{1}{q(x)} t f_\mu(x,t), \qquad F_\mu(x,t) \le \frac{1}{q(x)+r} t f_\mu(x,t) + C_\mu,$$

for $t > 0$, *where* $C_\mu > 0$ *is a positive constant.*

Proof. Since $b_\mu(t)$ to is monotonically decreasing on the interval $(0, +\infty)$, we have

$$\frac{d}{dt}\left(\frac{t}{m_\mu(t)}\right) = \frac{m_\mu(t) - tb_\mu(t)}{m_\mu^2(t)} = \frac{t(b_\mu(\xi) - b_\mu(t))}{m_\mu^2(t)} \geq 0,$$

for $t > 0$, where $\xi \in (0, t)$. Therefore, $\frac{t}{m_\mu(t)}$ is monotonically increasing on the interval $(0, +\infty)$. Hence, $\frac{f_\mu(x,t)}{t^{q(x)-1}} = \left(\frac{t}{m_\mu(t)}\right)^r$ is also monotonically increasing on the interval $(0, +\infty)$. It follows that

$$F_\mu(x,t) = \int_0^t f_\mu(x,\tau)d\tau \leq \int_0^t \frac{f_\mu(x,t)}{t^{q(x)-1}}\tau^{q(x)-1}d\tau = \frac{1}{q(x)}tf_\mu(x,t), \tag{2.3}$$

for $t > 0$.

By definition of the function m_μ, we have $m_\mu(t) = \frac{A}{\mu}$ for $t \geq \frac{2}{\mu}$, where $A = 1 + \int_1^2 \psi(\tau)d\tau$. For $t > \frac{2}{\mu}$, one has

$$\begin{aligned} F_\mu(x,t) &= \int_0^{\frac{2}{\mu}} f_\mu(x,\tau)\,d\tau + \int_{\frac{2}{\mu}}^t \left(\frac{\mu}{A}\right)^r \tau^{q(x)+r-1}\,d\tau \\ &= \int_0^{\frac{2}{\mu}} \left(f_\mu(x,\tau) - \left(\frac{\mu}{A}\right)^r \tau^{q(x)+r-1}\right) d\tau + \int_0^t \left(\frac{\mu}{A}\right)^r \tau^{q(x)+r-1}\,d\tau \\ &\leq C_\mu + \frac{tf_\mu(x,t)}{q(x)+r}. \end{aligned} \tag{2.4}$$

It implies from (2.3) and (2.4) that

$$F_\mu(x,t) \leq \frac{1}{q(x)+r}tf_\mu(x,t) + C_\mu$$

for $t > 0$. □

Lemma 2.5. *Suppose that (Q_1) and (Q_2) hold. Then, for any $\mu \in (0, 1]$, I_μ satisfies the (PS) condition.*

Proof. Let $\{u_n\}$ be a (PS) sequence of I_μ in E. This means that there exists $C > 0$ such that

$$|I_\mu(u_n)| \leq C, \quad I_\mu'(u_n) \to 0 \text{ as } n \to \infty. \tag{2.5}$$

From (2.1) and Lemma 2.4, we derive that

$$\begin{aligned} & I_\mu(u_n) - \frac{1}{2+r}\langle I_\mu'(u_n), u_n\rangle \\ = \;& \frac{r}{2(2+r)}\|u_n\|^2 + \int_\Omega \left(\frac{1}{2+r} - \frac{1}{q(x)+r}\right) f_\mu(x,u_n)u_n dx - C_\mu \\ \geq \;& \frac{r}{2(2+r)}\|u_n\|^2 - C_\mu, \end{aligned}$$

which implies that $\frac{r}{2(2+r)}\|u_n\|^2 \leq C + C_\mu + o(\|u_n\|)$. We obtain $\{u_n\}$ is bounded in E. Up to a subsequence, we may assume that

$$\begin{cases} u_n \rightharpoonup u, & \text{in } E, \\ u_n \to u, & \text{in } L^s(\Omega), \ 1 \leq s < 2^*. \end{cases}$$

For any integer pair (i, j), one has

$$\|u_i - u_j\|^2 = \langle I_\mu'(u_i) - I_\mu'(u_j), u_i - u_j\rangle + \int_\Omega (f_\mu(x, u_i) - f_\mu(x, u_j))(u_i - u_j)dx.$$

It follows from (2.5) that

$$\langle I_\mu'(u_i) - I_\mu'(u_j), u_i - u_j\rangle \to 0, \quad \text{as} \quad i,\ j \to +\infty. \tag{2.6}$$

It is easy to see that

$$|f_\mu(x, t)| \le |t|^{q(x)-1} + \left(\frac{\mu}{A}\right)^r |t|^{q(x)+r-1}.$$

Note that $2 \le q(x) < q(x) + r \le q^+ + r < 2^*$. It implies that

$$\begin{aligned} &\left|\int_\Omega (f_\mu(x, u_i) - f_\mu(x, u_j))(u_i - u_j)dx\right| \\ \le\ & C \int_\Omega \left(|u_i| + |u_j| + |u_i|^{q^+ + r-1} + |u_j|^{q^+ + r-1}\right)|u_i - u_j| \to 0 \end{aligned} \tag{2.7}$$

as i and j tend to $+\infty$. From (2.6) and (2.7), we have $\|u_i - u_j\| \to 0$ as $i,\ j \to +\infty$, which implies that $\{u_n\}$ contains a strongly convergent subsequence in E. Hence I_μ satisfies the (PS) condition. □

G, Y and M are taken to be I_μ, E_k and $E_{k-1}^\perp$ in Theorem 2.3, respectively. Next we will complete the proof of Theorem 2.1 by verifying the conditions of Theorem 2.3 one by one.

Lemma 2.6. *Suppose that (Q_1) holds. If we replace G, Y and M with I_μ, E_k and $E_{k-1}^\perp$, respectively, then conditions (A_1^*) and (A_2^*) are satisfied.*

Proof. Consider another norm $\|u\|_* = \|u\|_s$ of E, $s \in (2, 2^*)$. Then $\|u\|_s \le C_*\|u\|$ for all $u \in E$, where $C_* > 0$ is a constant and $\|u_n - u^*\|_* \to 0$ whenever $u_n \rightharpoonup u^*$weakly in $(E, \|\cdot\|)$. Define $\beta_k = \sup_{E_k} I_\mu$. Let

$$Q_k^*(\rho) = \left\{ u \in E_{k-1}^\perp : \frac{\|u\|_s^s}{\|u\|^2} + \frac{\|u\|\|u\|_s}{\|u\| + \lambda_k^{\beta_k}\|u\|_s} = \rho \right\},$$

it is easy to obtain that there exists a constant $c_2 > 0$ such that $\|u\|_s \le c_2$ for any $u \in Q_k^*(\rho)$. By assumption (Q_1) and definition of the function m_μ, we have

$$|F_\mu(x, t)| \le \frac{|t|^{q(x)}}{q(x)} + \frac{|t|^{q(x)+r}}{q(x) + r} \le |t|^{q(x)} + |t|^{q(x)+r}.$$

It implies that

$$\left|\int_\Omega F_\mu(x, u)\, dx\right| \le \int_\Omega (|u|^{q(x)} + |u|^{q(x)+r})\, dx. \tag{2.8}$$

By the Sobolev imbedding theorem, it implies from $2 \le q(x) < q(x) + r \le q^+ + r < 2^*$ that

$$\int_\Omega |u|^{q(x)+r}\, dx \le \int_\Omega (|u|^{2+r} + |u|^{q^+ + r})\, dx. \tag{2.9}$$

Set $\Omega_\varepsilon = \{x \in \Omega | 2 \le q(x) < 2+\varepsilon\}$. By the Hölder inequality and the Sobolev imbedding theorem, we have

$$\begin{aligned}\int_\Omega |u|^{q(x)}\,dx &= \int_{\Omega_\varepsilon} |u|^{q(x)}\,dx + \int_{\Omega\setminus\Omega_\varepsilon} |u|^{q(x)}\,dx\\ &\le \int_{\Omega_\varepsilon} (|u|^2 + |u|^{2+\varepsilon})\,dx + \int_{\Omega\setminus\Omega_\varepsilon} (|u|^{2+\varepsilon} + |u|^{q^+})\,dx\\ &\le \int_{\Omega_\varepsilon} |u|^2\,dx + \int_\Omega (|u|^{2+\varepsilon} + |u|^{q^+})\,dx\\ &\le C|\Omega_\varepsilon|^{\frac{2^*-2}{2^*}}\|u\|^2 + \int_\Omega (|u|^{2+\varepsilon} + |u|^{q^+})\,dx. \end{aligned} \tag{2.10}$$

Since $\Omega_0 = \{0\}$, we obtain $|\Omega_\varepsilon| \to 0$ as $\varepsilon \to 0$. Therefore, there exists $\varepsilon_0 > 0$ such that

$$|\Omega_\varepsilon|^{\frac{2^*-2}{2^*}} < \frac{1}{4C} \tag{2.11}$$

for any $\varepsilon \in (0, \varepsilon_0)$. From (2.8)-(2.11), for any $a,\ b > 0$, there is a $c_1 = c_1(a,b) > 0$ such that $I_\mu(u) \le a$ and $\|u\|_{q^++r} \le b \Rightarrow \|u\| \le c_1$. That is, condition (A_1^*) is satisfied.

For $t > \max\left\{1, \frac{2}{\mu}\right\}$, one has

$$\begin{aligned}F_\mu(x,t) &= \int_0^{\frac{2}{\mu}} f_\mu(x,\tau)\,d\tau + \int_{\frac{2}{\mu}}^t \left(\frac{\mu}{A}\right)^r \tau^{q(x)+r-1}\,d\tau\\ &= \int_0^{\frac{2}{\mu}} \left(f_\mu(x,\tau) - \left(\frac{\mu}{A}\right)^r \tau^{q(x)+r-1}\right) d\tau + \int_0^t \left(\frac{\mu}{A}\right)^r \tau^{q(x)+r-1}\,d\tau\\ &\ge \frac{1}{q(x)+r}\left(\frac{\mu}{A}\right)^r t^{q(x)+r}\\ &\ge \frac{1}{q^++r}\left(\frac{\mu}{A}\right)^r t^{2+r}. \end{aligned}$$

Set $Y = E_k$. Noticing that $\dim E_k < \infty$ and all norms of finite dimensional space are equivalent, it implies that

$$\frac{I_\mu(u)}{\|u\|^2} \le \frac{1}{2} - \int_\Omega \frac{F(x,u)}{\|u\|^2}dx \to -\infty$$

as $\|u\| \to \infty$, $u \in E_k$. Therefore, $\lim\limits_{u\in E_k,\ \|u\|\to\infty} I_\mu(u) = -\infty$. So condition (A_2^*) is satisfied. □

Let $Q_k^{**} = Q_k^*(\rho) \cap I_\mu^{\beta_k} \subset \mathcal{S}$ and $\gamma_k = \inf\limits_{Q^{**}} I_\mu$. Set $\mathcal{P} := \{u \in E : u(x) \ge 0$ for a.e $x \in \Omega$. Then, $\mathcal{P}(-\mathcal{P})$ is the positive(negative) cone of E and weakly closed. By Lemma 5.4 in [19], there is a $\eta = \eta(\beta_k) > 0$ such that $dist(Q^{**}, P) = \eta > 0$. We define $\pm\mathcal{D}_0(\delta_0) := \{u \in E : \mathrm{dist}(u, \pm\mathcal{P}) < \delta_0\}$, where δ_0 is determined by the following lemma.

Lemma 2.7. *Under the assumption (Q_1), there is a $\delta_0 \in (0,\eta)$ such that $K_{I_\mu}(\pm\mathcal{D}_0(\delta_0)) \subset \pm\mathcal{D}_0(\delta_0)$. Therefore, condition (A) is satisfied.*

Proof. Write $u^{\pm} = \max\{\pm u, 0\}$. For any $u \in E$ and each $s \in (2.2^*]$, there exists a $C_s > 0$ such that

$$\|u^{\pm}\|_s \le C_s \mathrm{dist}(u, \mp\mathcal{P}). \tag{2.12}$$

Let $v = K_{I_\mu}(u)$. Similar to the derivation of (2.8), (2.9) and (2.10), we have

$$\int_\Omega f_\mu(x, u^+)v^+ dx \le \int_{\Omega_\varepsilon} |u^+||v^+|dx + \int_\Omega \left(|u^+|^{1+\varepsilon} + |u^+|^{q^++r-1}\right)|v^+|dx. \tag{2.13}$$

From (2.12) and (2.13), by the Hölder inequality and the Sobolev imbedding theorem, we obtain

$$\begin{aligned}
&\mathrm{dist}(v, -\mathcal{P})\|v^+\| \\
&\le \|v^+\|^2 \\
&= \langle v^+, v^+\rangle \\
&= \int_\Omega f_\mu(x, u^+)v^+ dx \\
&\le C\left(\left(\int_{\Omega_\varepsilon} |u^+|^2 |dx\right)^{\frac{1}{2}} + \|u^+\|_{2+\varepsilon}^{1+\varepsilon} + \|u^+\|_{q^++r}^{q^++r-1}\right)\|v^+\| \\
&\le C\left(|\Omega_\varepsilon|^{\frac{2^*-2}{2\cdot 2^*}}\|u^+\|_{2^*} + \|u^+\|_{2+\varepsilon}^{1+\varepsilon} + \|u^+\|_{q^++r}^{q^++r-1}\right)\|v^+\| \\
&\le C\left(|\Omega_\varepsilon|^{\frac{2^*-2}{2\cdot 2^*}}\mathrm{dist}(u, -\mathcal{P}) + (\mathrm{dist}(u, -\mathcal{P}))^{1+\varepsilon} + (\mathrm{dist}(u, -\mathcal{P}))^{q^++r-1}\right)\|v^+\|.
\end{aligned}$$

That is,

$$\mathrm{dist}(K_{I_\mu}(u), -\mathcal{P}) \le C\left(|\Omega_\varepsilon|^{\frac{2^*-2}{2\cdot 2^*}}\mathrm{dist}(u, -\mathcal{P}) + (\mathrm{dist}(u, -\mathcal{P}))^{1+\varepsilon} + (\mathrm{dist}(u, -\mathcal{P}))^{q^++r-1}\right).$$

It follows from (2.11) that there exists a $\delta_0 \in (0, \eta)$ such that $\mathrm{dist}(K_{I_\mu}(u), -\mathcal{P}) < \delta_0$ for every $u \in -\mathcal{D}_0(\delta_0)$. Similarly, $\mathrm{dist}(K_{I_\mu}(u), \mathcal{P}) < \delta_0$ for every $u \in \mathcal{D}_0(\delta_0)$. The conclusion follows. □

Now we are in a position to prove the main result of this section.

Proof of Theorem 2.1. By Theorem 2.3, Lemmas 2.5, 2.6 and 2.7, we obtain

$$\mathcal{K}[\gamma_k - \varepsilon, \beta_k + \varepsilon] \cap (E \setminus (\mathcal{P} \cup (-\mathcal{P})) \neq \emptyset$$

for all $\varepsilon > 0$ small. That is, there exists a $u_{k,\mu} \in E \setminus (\mathcal{P} \cup (-\mathcal{P})$ (sign-changing critical point) such that

$$I'_\mu(u_{k,\mu}) = 0, \quad I_\mu(u_{k,\mu}) \in [\gamma_k - 1, \beta_k + 1],$$

where $\gamma_k = \inf\limits_{Q_k^{**}} I_\mu$. Next we show the $\gamma_k \to \infty$ as $k \to \infty$. Recall the Gagliardo-Nirenberg inequality,

$$\|u\|_s \le c_s\|u\|^\alpha\|u\|_2^{1-\alpha}, \quad u \in E. \tag{2.14}$$

where $s \in (2, 2^*)$ and $\alpha \in (0, 1)$ is defined by

$$\frac{1}{s} = \left(\frac{1}{2} - \frac{1}{N}\right)\alpha + \frac{1}{2}(1 - \alpha). \tag{2.15}$$

In addition, for $u \in E_k^{\perp}$, we see that $\|u\|_2 \leq \frac{1}{\lambda_k^{1/2}}\|u\|$. Combine (2.14) with (2.15), we have

$$\|u\|_s^{s-2} \leq c_s^{s-2}\|u\|^{s-2}\lambda_k^{-(1-\alpha)(s-2)/2}, \quad u \in E_k^{\perp}. \tag{2.16}$$

For $u \in Q_k^*(\rho)$, by the Sobolev imbedding theorem, we deduce from (2.16) that

$$\begin{aligned} \rho &= \frac{\|u\|_s^s}{\|u\|^2} + \frac{\|u\|\|u\|_s}{\|u\| + \lambda_k^{\beta_k}\|u\|_s} \\ &\leq \frac{\|u\|\|u\|_s}{2(\|u\|\lambda_k^{\beta_k}\|u\|_s)^{1/2}} + \frac{\|u\|_s^2}{\|u\|^2}\|u\|_s^{s-2} \\ &\leq \frac{(\|u\|\|u\|_s)^{1/2}}{2(\lambda_k^{\beta_k})^{1/2}} + C_*^2\|u\|_s^{s-2} \\ &\leq \frac{C_*^{1/2}\|u\|}{2(\lambda_k^{\beta_k})^{1/2}} + C_*^2\|u\|_s^{s-2} \\ &\leq \frac{C_*^{1/2}\|u\|}{2(\lambda_k^{\beta_k})^{1/2}} + C_*^2 c_s^{s-2}\|u\|^{s-2}\lambda_k^{-(1-\alpha)(s-2)/2} \\ &\leq \max\left\{C_*^{1/2}\lambda_k^{-\beta_k/2}\|u\|, 2C_*^2 c_s^{s-2}\lambda_k^{-(1-\alpha)(s-2)/2}\|u\|^{s-2}\right\}. \end{aligned} \tag{2.17}$$

It implies that

$$\|u\| \geq \Lambda_s^* T_k T, \tag{2.18}$$

where $\Lambda_s^* = \min\left\{C_*^{-1/2}, 2^{-1/(s-2)}C_*^{-2/(s-2)}c_s^{-1}\right\}$, $T_k = \min\left\{\lambda_k^{\beta_k/2}, \lambda_k^{(1-\alpha)/2}\right\}$ and $T = \min\left\{\rho, \rho^{1/(s-2)}\right\}$. From (2.8)-(2.11), we know that

$$\left|\int_\Omega F_\mu(x,u)\,dx\right| \leq \frac{1}{4}\|u\|^2 + C\int_\Omega |u|^{q^++r}\,dx, \quad u \in E.$$

We can choose that $\rho > 0$ such that $\rho < \frac{1}{8C}$. For any $u \in Q_k^*(\rho)$, we see that $\|u\|_s^s/\|u\|^2 \leq \rho$. Therefore, for any $u \in Q_k^*(\rho)$, it implies from (2.18) that

$$\begin{aligned} I_\mu(u) &= \frac{1}{2}\|u\|^2 - \int_\Omega F_\mu(x,u)\,dx \\ &\geq \frac{1}{4}\|u\|^2 - C\int_\Omega |u|^{q^++r}\,dx \\ &\geq \|u\|^2\left(\frac{1}{4} - C\frac{\|u\|_{q^++r}^{q^++r}}{\|u\|^2}\right) \\ &\geq \left(\frac{1}{4} - C\rho\right)\|u\|^2 \\ &\geq \frac{1}{8}\|u\|^2 \\ &\geq \frac{1}{8}(\Lambda_{p^++r}^* T_k T)^2. \end{aligned}$$

Since $\lambda_k \to \infty$ as $k \to \infty$, we obtain $T_k = \min\left\{\lambda_k^{\beta_k/2}, \lambda_k^{(1-\alpha)/2}\right\} \to \infty$ as $k \to \infty$. Therefore, $\gamma_k \to \infty$ as $k \to \infty$. Hence, for any $\mu \in (0, 1]$, problem (2.2) has infinitely many sign-changing solutions. The proof is complete. □

3. A priori estimate and proof of Theorem 1.1

In this section, we will show that solutions of auxiliary problem (2.2) are indeed solutions of original problem (1.1). For this purpose, we need the following uniform L^∞-estimate for critical points of the functional I_μ.

Proposition 3.1. *Suppose that* (Q_1) *and* (Q_2) *hold. If* v *is a critical point of* I_μ *with* $I_\mu(v) \leq L$*, then there exists a positive constant* $M = M(L)$ *independent of* μ *such that* $\|v\|_{L^\infty(\Omega)} \leq M$*.*

In order to prove Proposition 3.1, we need some preliminaries.

Lemma 3.2. *Suppose that* (Q_1) *and* (Q_2) *hold. If* $I_\mu(v) \leq L$ *and* $I'_\mu(v) = 0$*, then, for any* $\delta \in (0, \delta_0)$*, there exists* $C_\delta > 0$ *independent of* μ *such that* $\int_{\Omega_\delta} |\nabla v|^2 dx \leq C_\delta$*.*

Proof. By Lemma 2.4 and (Q_1), we have

$$\begin{aligned}
L &\geq I_\mu(v) - \left\langle I'_\mu(v), \frac{v}{q(x)} \right\rangle \\
&= \int_\Omega \left(\frac{1}{2} - \frac{1}{q(x)}\right)|\nabla v|^2 dx + \int_\Omega \left(\frac{f_\mu(x,v)v}{q(x)} - F_\mu(x,v)\right) dx \\
&\geq \int_\Omega \left(\frac{1}{2} - \frac{1}{q(x)}\right)|\nabla v|^2 dx \\
&\geq \int_{\Omega_\delta} \left(\frac{1}{2} - \frac{1}{q(x)}\right)|\nabla v|^2 dx. \qquad (3.1)
\end{aligned}$$

According to (Q_1), for any $\delta \in (0, \delta_0)$, we know that there exists $m_\delta > 0$ such that $\frac{1}{2} - \frac{1}{q(x)} \geq m_\delta$ for any $x \in \Omega_\delta$. Therefore, we have

$$\int_{\Omega_\delta} |\nabla v|^2 dx \leq m_\delta^{-1} \int_{\Omega_\delta} \left(\frac{1}{2} - \frac{1}{q(x)}\right)|\nabla v|^2 dx \leq m_\delta^{-1} L = C_\delta.$$

The proof is complete. □

Lemma 3.3. *Let* $1 < p < \frac{N}{2}$ *and* $0 < r < R$*. Suppose that the nonnegative functions* $w(x)$ *and* $g(x)$ *satisfy* $g \in L^p(B_R)$ *and*

$$-\Delta w \leq g, \quad \text{in } B_R. \qquad (3.2)$$

Then, we have

$$\|w\|_{L^{\frac{Np}{N-2p}}(B_r)} \leq C\left(\|w\|_{L^{\frac{(N-2)p}{N-2p}}(B_R \setminus B_r)} + \|g\|_{L^p(B_R)}\right), \qquad (3.3)$$

where $C = C(N, p, R, r) > 0$.

Proof. Set $\xi = \frac{N(p-1)}{N-2p}$. Then, we have the following identity

$$\frac{N(1+\xi)}{N-2} = \frac{Np}{N-2p} = \frac{p\xi}{p-1}. \tag{3.4}$$

Let $\varphi \in C_0^\infty(\mathbb{R}^N, [0,1])$ satisfies $\varphi(x) = 1$ for $|x| \le r$ and $\varphi(x) = 0$ for $|x| \ge R$. For any $\theta > 0$, multiply inequality (3.2) by the test function $((w+\theta)^\xi - \theta^\xi)\varphi^2$ and integrate to obtain

$$\int_{B_R} \nabla w \nabla (((w+\theta)^\xi - \theta^\xi)\varphi^2)dx \le \int_{B_R} g((w+\theta)^\xi - \theta^\xi)\varphi^2 dx. \tag{3.5}$$

By the Young inequality, we hace

$$\begin{aligned}
&\int_{B_R} \nabla w \nabla (((w+\theta)^\xi - \theta^\xi)\varphi^2)dx \\
= \;& \xi \int_{B_R} |\nabla w|^2 (w+\theta)^{\xi-1}\varphi^2 dx + 2\int_{B_R} ((w+\theta)^\xi - \theta^\xi)\varphi \nabla w \nabla \varphi dx \\
\ge \;& \frac{4\xi}{(\xi+1)^2} \int_{B_R} \left|\nabla (w+\theta)^{\frac{\xi+1}{2}}\right|^2 \varphi^2 dx - C\int_{B_R} \nabla (w+\theta)^{\frac{\xi+1}{2}} (w+\theta)^{\frac{\xi+1}{2}} \varphi \nabla \varphi dx \\
\ge \;& C\int_{B_R} \left|\nabla \left(\left((w+\theta)^{\frac{\xi+1}{2}} - \theta^{\frac{\xi+1}{2}}\right)\varphi\right)\right|^2 dx - C\int_{B_R} (w+\theta)^{\xi+1}|\nabla\varphi|^2 dx \\
\ge \;& C\left(\int_{B_R} \left(\left((w+\theta)^{\frac{\xi+1}{2}} - \theta^{\frac{\xi+1}{2}}\right)\varphi\right)^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}} \\
& -C\int_{B_R} (w+\theta)^{\xi+1}|\nabla\varphi|^2 dx.
\end{aligned} \tag{3.6}$$

According to $1 < p < \frac{N}{2}$, we have $\frac{2p}{p-1} > \frac{2N}{N-2}$. It implies that

$$\begin{aligned}
\int_{B_R} g((w+\theta)^\xi - \theta^\xi)\varphi^2 dx \;&\le\; \|g\|_{L^p(B_R)} \left(\int_{B_R} ((w+\theta)^\xi - \theta^\xi)^{\frac{p}{p-1}} \varphi^{\frac{2p}{p-1}} dx\right)^{\frac{p-1}{p}} \\
&\le\; \|g\|_{L^p(B_R)} \left(\int_{B_R} ((w+\theta)^\xi)^{\frac{p}{p-1}} \varphi^{\frac{2N}{N-2}} dx\right)^{\frac{p-1}{p}}.
\end{aligned} \tag{3.7}$$

Letting $\theta \to 0$, we conclude from (3.5), (3.6) and (3.7) that

$$\begin{aligned}
&\left(\int_{B_R} w^{\frac{N(\xi+1)}{N-2}} \varphi^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}} \\
\le\; & C\int_{B_R} w^{\xi+1}|\nabla\varphi|^2 dx + C\|g\|_{L^p(B_R)} \left(\int_{B_R} w^{\frac{p\xi}{p-1}} \varphi^{\frac{2N}{N-2}} dx\right)^{\frac{p-1}{p}},
\end{aligned}$$

which implies that

$$\left(\int_{B_R} w^{\frac{N(\xi+1)}{N-2}} \varphi^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}} \le C\int_{B_R} w^{\xi+1}|\nabla\varphi|^2 dx, \tag{3.8}$$

or

$$\left(\int_{B_R} w^{\frac{N(\xi+1)}{N-2}}\varphi^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{N}} \le C\|g\|_{L^p(B_R)}\left(\int_{B_R} w^{\frac{p\xi}{p-1}}\varphi^{\frac{2N}{N-2}}dx\right)^{\frac{p-1}{p}}. \tag{3.9}$$

From (3.4), (3.8) and (3.9), we have

$$\begin{aligned}\left(\int_{B_R} w^{\frac{Np}{N-2p}}\varphi^{\frac{2N}{N-2}}dx\right)^{\frac{N-2p}{Np}} &\le C\left(\int_{B_R} w^{\frac{(N-2)p}{N-2p}}|\nabla\varphi|^2dx\right)^{\frac{N-2p}{(N-2)p}}\\ &= C\left(\int_{B_R\setminus B_r} w^{\frac{(N-2)p}{N-2p}}dx\right)^{\frac{N-2p}{(N-2)p}},\end{aligned}$$

or

$$\left(\int_{B_R} w^{\frac{Np}{N-2p}}\varphi^{\frac{2N}{N-2}}dx\right)^{\frac{N-2p}{Np}} \le C\|g\|_{L^p(B_R)}.$$

Therefore, we obtain

$$\|w\|_{L^{\frac{Np}{N-2p}}(B_r)} \le \left(\int_{B_R} w^{\frac{Np}{N-2p}}\varphi^{\frac{2N}{N-2}}dx\right)^{\frac{N-2p}{Np}} \le C\left(\|w\|_{L^{\frac{(N-2)p}{N-2p}}(B_R\setminus B_r)} + \|g\|_{L^p(B_R)}\right).$$

The proof is complete. □

Lemma 3.4. *Suppose that* (Q_1) *and* (Q_2) *hold. If* $I_\mu(v) \le L$ *and* $I'_\mu(v) = 0$*, then there exist* $\delta_1 \in (0,\ \delta_0)$ *and* $C > 0$ *independent of* μ *such that* $\int_{B_\delta} |\nabla v|^2dx \le C$ *for any* $\delta \in (0,\ \delta_1)$.

Proof. It follows from $I'_\mu(v) = 0$ that v is a solution of problem (2.2). For any $\delta \in (0,\ \delta_0)$ and $B_{2\delta} \subset \Omega$, let $\phi \in C_0^\infty(\Omega, [0,1])$ satisfies $\phi(x) = 1$ for $|x| \le \delta$, $\phi(x) = 0$ for $|x| \ge 2\delta$ and $|\nabla\phi| \le C$ for $x \in \Omega$. Multiply equation (2.2) by $v\phi^2$ and integrate to obtain

$$\int_\Omega \nabla v\nabla(v\phi^2)dx = \int_\Omega k_\mu(x,v)v\phi^2dx \le \int_{B_{2\delta}} k_\mu(x,v)vdx. \tag{3.10}$$

By the Young inequality, we have

$$\begin{aligned}&\int_\Omega \nabla v\nabla(v\varphi^2)dx\\ =\ &\int_\Omega |\nabla v|^2\varphi^2dx + 2\int_\Omega v\varphi\nabla v\nabla\varphi dx\\ \ge\ &\frac{1}{2}\int_\Omega |\nabla v|^2\varphi^2dx - C\int_\Omega v^2|\nabla\phi|^2dx\\ \ge\ &\frac{1}{2}\int_{B_\delta} |\nabla v|^2dx - C\int_{B_{2\delta}} v^2dx.\end{aligned} \tag{3.11}$$

By definition of the function m_μ, we know that $m_\mu(t) = t$ for $t \le \frac{1}{\mu}$ and $m_\mu(t) \ge \frac{1}{\mu}$ for $t > \frac{1}{\mu}$. Therefore, we have

$$|k_\mu(x, v)| \le C\mu^r |v|^{q(x)+r-1} \le C|v|^{q(x)+r-1} \le C|v|^{q^+ + r-1}, \tag{3.12}$$

for any $\mu \in (0,\ 1]$ and $x \in \Omega$. From (3.10), (3.11) and (3.12), we obtain

$$\int_{B_\delta} |\nabla v|^2 dx \le C \int_{B_{2\delta}} k_\mu(x, v) v dx + C \int_{B_{2\delta}} v^2 dx \le C \int_{B_{2\delta}} |v|^{q^+ + r} dx. \tag{3.13}$$

In order to complete our proof, we just need to prove that there exists $\delta_2 > 0$ such that $\int_{B_{\delta_2}} |v|^{q^+ + r} dx \le C$.

By definition of the function k_μ, we obtain $k_\mu(x, t) \ge t^{q(x)-1}$. According to Lemma 2.4, (Q_1) and (Q_2), for any $\delta \in (0,\ \delta_0)$, we have

$$\begin{aligned} L &\ge I_\mu(v) - \frac{1}{2}\langle I_\mu'(v), v\rangle \\ &= \int_\Omega \left(\frac{k_\mu(x, v)v}{2} - K_\mu(x, v) \right) dx \\ &\ge \int_\Omega \left(\frac{1}{2} - \frac{1}{q(x)} \right) k_\mu(x, v) v dx \\ &\ge \frac{1}{2q^+} \int_{B_\delta} |x|^\alpha |v|^{q(x)} dx. \end{aligned} \tag{3.14}$$

Noticing that $N \ge 3$, from (Q_2) and (2.1), we have

$$0 < \frac{\alpha(1 + r)}{2 - (1 + r)} = \frac{\alpha(1 + r)}{1 - r} < \frac{\alpha(4N + 1)}{4N - 1} < \frac{(N + 2)(4N + 1)}{2(4N - 1)} < N.$$

Therefore, we can choose $p \in \left(1, \frac{2N}{N+1}\right)$ satisfying

$$\frac{p(1 - r)}{p - 1} > 2 \quad \text{and} \quad 0 < \frac{p\alpha(1 + r)}{2 + r - p(1 + r)} < \frac{7N + 2}{8} < N. \tag{3.15}$$

Let $q_\delta^+ = \sup\{q(x) | x \in B_\delta\}$. It follows from (Q_1) and (3.15) that there exists $\delta_3 < \min\{1,\ \delta_0\}$ such that

$$q_\delta^+ \le \frac{p(1 - r)}{p - 1} \quad \text{and} \quad 0 < \frac{p\alpha(q_\delta^+ - 1 + r)}{q_\delta^+ - p(q_\delta^+ - 1 + r)} < N \tag{3.16}$$

for any $\delta \in (0,\ \delta_3)$. Using the Young inequality, we deduce from (3.14) and (3.16) that

$$\begin{aligned} \int_{B_\delta} |v|^{p(q(x)+r-1)} dx &= \int_{B_\delta} \left(|x|^\alpha |v|^{q(x)} \right)^{\frac{p(q(x)+r-1)}{q(x)}} \cdot |x|^{-\frac{p\alpha(q(x)+r-1)}{q(x)}} dx \\ &\le C \int_{B_\delta} |x|^\alpha |v|^{q(x)} dx + C \int_{B_\delta} |x|^{-\frac{p\alpha(q(x)+r-1)}{q(x) - p(q(x)+r-1)}} dx \end{aligned}$$

$$
\begin{aligned}
&\leq C\int_{B_\delta}|x|^{\alpha}|v|^{q(x)}dx + C\int_{B_\delta}|x|^{-\frac{p\alpha(q_\delta^+ + r - 1)}{q_\delta^+ - p(q_\delta^+ + r - 1)}}dx \\
&\leq 2q^+CL + C\delta^{N-\frac{p\alpha(q_\delta^+ + r - 1)}{q_\delta^+ - p(q_\delta^+ + r - 1)}}, \qquad (3.17)
\end{aligned}
$$

for any $\delta \in (0,\ \delta_3)$. According to (3.12) and (3.17), for any $\delta \in (0,\ \delta_3)$, we obtain

$$
\|k_\mu(x,v)\|_{L^p(B_\delta)} \leq C_\delta. \qquad (3.18)
$$

Since $-\triangle v = k_\mu(x,v)$ in B_δ. By Lemma 3.2 and Lemma 3.3, for any $\delta' \in (0,\ \delta)$, it implies from (3.18) that

$$
\begin{aligned}
\|v\|_{L^{\frac{Np}{N-2p}}(B_{\delta'})} &\leq C\left(\|v\|_{L^{\frac{(N-2)p}{N-2p}}(B_\delta\backslash B_{\delta'})} + \|k_\mu(x,v)\|_{L^p(B_\delta)}\right) \\
&\leq C\left(\|v\|_{L^{\frac{(N-2)p}{N-2p}}(\Omega_{\delta'})} + \|k_\mu(x,v)\|_{L^p(B_\delta)}\right) \\
&\leq C\left(\int_{\Omega_{\delta'}}|\nabla v|^2dx\right)^{\frac{1}{2^*}} + C\|k_\mu(x,v)\|_{L^p(B_\delta)} \\
&\leq C, \qquad (3.19)
\end{aligned}
$$

where $\delta \in (0,\ \delta_3)$ and $C = C(\delta', \delta, N, p) > 0$ is independent of μ.

If $\zeta_1 = \frac{Np}{N-2p} \geq q^+ + r$, using the Hölder inequality, we are done. Otherwise, using the fact $r < \frac{1}{4N} < \frac{2}{N-2}$ provided by (2.1), we can choose $\sigma_1 \in (0,\ \delta) \subset (0,\ \delta_3)$ such that $\tau_1 = q_{\sigma_1}^+ + r - 1 < \frac{N}{N-2}$. It follows from (3.12) that

$$
|k_\mu(x,v)| \leq C|v|^{\tau_1},
$$

for any $\mu \in (0,\ 1]$ and $x \in B_{\sigma_1}$. Noticing that $\zeta_1 > \frac{N}{N-2}$, we have $p_1 = \frac{\zeta_1}{\tau_1} > 1$. According to (3.19), we obtain $k_\mu(x,v) \in L^{p_1}(B_{\sigma_1})$. Similar to (3.19), we can choose $\sigma_2 \in (0, \sigma_1)$ to obtain

$$
\|v\|_{L^{\zeta_2}(B_{\sigma_2})} \leq C,
$$

where $C = C(\sigma_1, \sigma_2, N, p_1) > 0$ is independent of μ and

$$
\zeta_2 = \frac{Np_1}{N-2p_1} = \frac{N\zeta_1}{N\tau_1 - 2\zeta_1} \geq \frac{N}{(N-2)\tau_1}\zeta_1 = d_1\zeta_1,
$$

here $d_1 = \frac{N}{(N-2)\tau_1} > 1$. If $\zeta_2 \geq q^+ + r$, using the Hölder inequality, we are done. Otherwise, repeating the above process and using a finite number of iterations, we obtain that there exist $\zeta_k > 0$ and $\sigma_k \in (0, \sigma_{k-1})$ such that $\zeta_k \geq 2^* > q^+ + r$ and $\|v\|_{L^{\zeta_k}(B_{\sigma_k})} \leq C$, where $C > 0$ is independent of μ. Using the Hölder inequality, we have

$$
\|v\|_{L^{q^+ + r}(B_{\sigma_k})} \leq C. \qquad (3.20)
$$

Let $\delta_1 = \frac{\zeta_k}{2}$. It implies from (3.13) and (3.20) that

$$
\int_{B_\delta}|\nabla v|^2dx \leq C\int_{B_{2\delta}}|v|^{q^+ + r}dx \leq C\int_{B_{\sigma_k}}|v|^{q^+ + r}dx \leq C,
$$

for any $\delta \in (0, \delta_1)$. □

Proof of Proposition 3.1. By Lemma 3.2 and Lemma 3.4, we obtain that there exists $C > 0$ independent of μ such that

$$\int_\Omega |\nabla v|^2 dx \le C. \tag{3.21}$$

Using the Sobolev embedding theorem, we have

$$\int_\Omega |v|^{2^*} dx \le C\left(\int_\Omega |\nabla v|^2 dx\right)^{\frac{2^*}{2}} \le C. \tag{3.22}$$

Let $s > 0$ and $t = q^+ + r$. According to (3.12), multiply equation (2.2) by v^{2s+1} and integrate to obtain

$$\int_\Omega \nabla v \nabla v^{2s+1} dx = \int_\Omega k_\mu(x, v) v^{2s+1} dx \le C \int_\Omega |v|^{2s+t} dx.$$

It implies that

$$\int_\Omega |\nabla v|^2 v^{2s} dx = \frac{1}{2s+1} \int_\Omega \nabla v \nabla v^{2s+1} dx \le C \int_\Omega |v|^{2s+t} dx. \tag{3.23}$$

On the one hand, by the Sobolev embedding theorem, we have

$$\int_\Omega |\nabla v|^2 v^{2s} dx = \frac{1}{(1+s)^2} \int_\Omega |\nabla v^{1+s}|^2 dx \ge \frac{C}{(1+s)^2} \left(\int_\Omega |v|^{(1+s)2^*} dx\right)^{\frac{2}{2^*}}. \tag{3.24}$$

On the other hand, by the Hölder inequality and (3.22), we have

$$\begin{aligned} \int_\Omega |v|^{2s+t} dx &\le \left(\int_\Omega |v|^{2^*} dx\right)^{\frac{t-2}{2^*}} \left(\int_\Omega |v|^{2(1+s)\frac{2^*}{2^*-t+2}} dx\right)^{\frac{2^*-t+2}{2^*}} \\ &\le C\left(\int_\Omega |v|^{(1+s)\frac{2^*}{d}} dx\right)^{\frac{2d}{2^*}}, \end{aligned} \tag{3.25}$$

where $d = \frac{2^*-t+2}{2} > 1$. According to (3.23), (3.24) and (3.25), we obtain

$$\left(\int_\Omega |v|^{(1+s)2^*} dx\right)^{\frac{2}{2^*}} \le (C(1+s))^2 \left(\int_\Omega |v|^{(1+s)\frac{2^*}{d}} dx\right)^{\frac{2d}{2^*}},$$

which implies that

$$\left(\int_\Omega |v|^{(1+s)2^*} dx\right)^{\frac{1}{(1+s)2^*}} \le (C(1+s))^{\frac{1}{1+s}} \left(\int_\Omega |v|^{(1+s)\frac{2^*}{d}} dx\right)^{\frac{d}{(1+s)2^*}}. \tag{3.26}$$

Now we carry out an iteration process. Set $s_k = d^k - 1$ for $k = 1, 2, \cdots$. By (3.26), we have

$$\left(\int_\Omega |v|^{d^k 2^*} dx\right)^{\frac{1}{d^k 2^*}} \le (Cd^k)^{\frac{1}{d^k}} \left(\int_\Omega |v|^{d^{k-1} 2^*} dx\right)^{\frac{1}{d^{k-1} 2^*}}$$

$$\leq \Pi_{j=1}^{k}(Cd^j)^{\frac{1}{d^j}} \left(\int_\Omega |v|^{2^*} dx\right)^{\frac{1}{2^*}}$$

$$\leq C^{\Sigma_{j=1}^{k} d^{-j}} \cdot d^{\Sigma_{j=1}^{k} jd^{-j}} \left(\int_\Omega |v|^{2^*} dx\right)^{\frac{1}{2^*}}. \qquad (3.27)$$

Since $d > 1$, the series $\sum\limits_{j=1}^{\infty} d^{-j}$ and $\sum\limits_{j=1}^{\infty} jd^{-j}$ are convergent. Letting $k \to \infty$, we conclude from (3.22) and (3.27) that $\|v\|_{L^\infty(\Omega)} \leq M$. The proof is complete. □

Proof of Theorem 1.1. By the proof of Theorem 2.1, for every integer $k \geq 1$, we know that problem (2.2) has k sign-changing solutions $u_{k,\mu}$ satisfying $\gamma_k - 1 < I_\mu(u_{k,\mu}) < \beta_k + 1$. Consider the functional

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \int_\Omega \frac{|u|^{p(x)}}{p(x)} \, dx.$$

By definition of the function f_μ, we obtain $|f_\mu(x,t)| \geq |t|^{q(x)-1}$. It is easy to see that $I_\mu(u) \leq J(u)$. Therefore, there exists a sequence of positive numbers $\{\Upsilon_k\}$ independent of μ such that $\beta_k + 1 \leq \Upsilon_k$. Let $L_k = \max\{\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_k\}$. By Proposition 3.1, there exists a positive constant $M_k = M_k(L_k)$ independent of μ such that $\|u_{k,\mu}\|_{L^\infty(\Omega)} \leq M_k$. By definition of the function m_μ, we have $m_\mu(t) = t$ for $t \leq \frac{1}{\mu}$. Hence, problem (2.2) reduces to problem (1.1) for $|t| \leq \frac{1}{\mu}$. Let $\mu < \frac{1}{2M_k}$. It is easy to see that $u_{k,\mu}$ is indeed a sign-changing solution of problem (1.1). □

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Conflict of interest

The authors declare no conflict of interest.

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