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Research article

A new weak convergence non-monotonic self-adaptive iterative scheme for solving equilibrium problems

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Abstract: A number of methods have been proposed to solve the equilibrium problems, one of which is an extragradient method that is particularly interesting and effective. In this paper, we introduce a modified subgradient extragradient method to solve the equilibrium problems in a real Hilbert space. The proposed method uses a non-monotonic step size rule based on local bi-function information instead of its Lipschitz-type constant or other line search method and is capable of solving pseudo-monotone equilibrium problems. Our method only needs to solve a strongly convex programming problem per iteration. Applications of the designed algorithm are presented in order to solve fixed-point problems and variational inequalities. Finally, several computational experiments are studied to confirm the effectiveness of the proposed method. The results of our study include many similar literature studies and detailed numerical studies also show their potential usefulness.

Keywords: equilibrium problem; non-monotonic step size rule; Lipschitz-type conditions; subgradient extragradient method; fixed point problems; variational inequalities
Mathematics Subject Classification: 47J25, 47H09, 47H06, 47J05

1. Introduction

Assume that C is a non-empty, convex and closed subset of real Hilbert space $\mathcal{H}$ and $f : \mathcal{H}\times\mathcal{H} \to \mathbb{R}$ be a bi-function with $f(v, v) = 0$, for every $v \in C$. A *equilibrium problem* (EP) for f on C is stated in the following manner:

$$\text{Find } \wp^* \in C \text{ such that } f(\wp^*, v) \geq 0, \ \ \forall\, v \in C. \tag{EP}$$

In this paper, we study a novel numerical method to solve equilibrium problem based on the following hypothesis. A bi-function $f : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is called to be (see [4, 6]):

(c1) *pseudo-monotone* on C if

$$f(v_1, v_2) \geq 0 \Longrightarrow f(v_2, v_1) \leq 0, \ \ \forall\, v_1, v_2 \in C;$$

(c2) *Lipschitz-type continuous* [23] on C if there exist two constants $c_1, c_2 > 0$ such that

$$f(v_1, v_3) \leq f(v_1, v_2) + f(v_2, v_3) + c_1\|v_1 - v_2\|^2 + c_2\|v_2 - v_3\|^2, \ \ \forall\, v_1, v_2, v_3 \in C;$$

(c3) $\limsup_{n\to\infty} f(v_n, v) \leq f(v^*, v)$ for all $v \in C$ and $\{v_n\} \subset C$ satisfies $v_n \rightharpoonup v^*$;

(c4) $f(v_1, \cdot)$ is sub-differentiable and convex on $\mathcal{H}$ for each fixed $v_1 \in \mathcal{H}$.

Furthermore, $Ep(f, C)$ denotes the solution set of the problem (EP). To the best of our understanding, the term "equilibrium problem" is introduced in 1992 by Muu and Oettli [27] and has been further studied by Blum and Oettli [6]. The problem (EP) is also known as the Ky Fan inequality due to his contribution [10]. In particular, the equilibrium problem is a general mathematical framework in the sense that it brings together different mathematical problems, i.e., the fixed point problems, the vector and scalar minimization problems, the variational inequality problems, the complementarity problems, the saddle point problems, the Nash equilibrium problems in non-cooperative games and the inverse optimization problems [5, 6, 20, 27]. A comprehensive study on equilibria and the detailed description of numerical methods for equilibrium problems can be found in [5, 6, 9, 17]. Iterative schemes are useful mechanisms for finding an estimated solution to the equilibrium problems. Many methods have been used to solve the problem (EP) in real Hilbert spaces, i.e., proximal point-like methods [12, 18, 19, 26], the extragradient-like methods [2, 15, 16, 22, 30, 31, 33, 34, 37–39, 41, 42, 44] and others in [1, 3, 11, 14, 24, 25, 28, 32, 43].

Meanwhile, by using the Korpelevich extragradient method [21] Flaam et al. [12] and Quoc et al. [31] was set up the following method for the solution of equilibrium problems involving pseudo-monotone and Lipschitz-type bi-function:

$$\begin{cases} u_n \in C, \\ v_n = \arg\min_{v\in C}\{\chi f(u_n, v) + \frac{1}{2}\|u_n - v\|^2\}, \\ u_{n+1} = \arg\min_{v\in C}\{\chi f(v_n, v) + \frac{1}{2}\|u_n - v\|^2\}, \end{cases} \tag{1.1}$$

where $0 < \chi < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$ and c_1, c_1 are Lipschitz-type constants. The iterative schemes in [12, 31] are usually recognised as the extragradient method initially due to the result of Korpelevich in [21] to solve the saddle point problems.

It is important to note that most previously established methods used a constant step size rule that is dependent on the Lipschitz-type constants of the bi-functions [14, 22, 31]. This can lead to some limitations in applications because the Lipschitz-type constants are usually unknown or complicated to estimate. Very recently, Hieu et al. [16] introduced the modifications of the gradient method for solving pseudo-monotone equilibrium problems with the new step size rule. But the step size rule is non-increasing, monotone, and the methods in [16] may depend on the choice of initial step size.

A natural question arises, i.e., "Is it possible to introduce a new weakly convergent extragradient algorithm with non-monotone step size rule for approximate the solution of problem (EP) involving pseudo-monotone bi-function that does not depend on the Lipschitz-type constants of the bi-function"?

In this work, we provide a positive answer to the above question, i.e., the gradient methods still hold in case of non-monotonic step size rule for solving equilibrium problems associated with pseudo-monotone bi-functions. Inspired by the works of Censor et al. [8] and Hieu et al. [16], we introduce a new extragradient-type method for evaluating a numerical solution of the problem (EP) in the context of infinite-dimensional real Hilbert spaces. Our main contributions in this study are as follows:

- We introduce a subgradient extragradient method with a non-monotone step size rule to solve the (EP) equilibrium problem in a real Hilbert space. In our suggested method, we do assume that the bi-function in the (EP) equilibrium problem is pseudo-monotone, that involves monotone bi-function.
- A weak convergence result is provided, i.e., that the sequence of iterates generated by our proposed method is weakly convergent to a solution of the problem (EP) in the setting of mild condition on the bi-functions and the iterative control parameters.
- We give numerical descriptions of our method for confirming the theoretical findings and comparing the results in [Algorithm 1 in [16]] and [Algorithm 2 in [16]]. Our numerical findings show that our method is efficient and effective compared to the current ones.

The remaining part of this paper is organized as follows. Section 2 recalls some basic definitions and lemmas to be used later. Section 3 introduces a new subgradient extragradient algorithm and provides its weak convergence result, which combines a non-monotonic step size rule. Section 4, provide the applications of our results to the particular classes of the problem (EP). Finally, many numerical results are reported to explain the behaviour of the new algorithm and also to compare it with existing algorithms.

2. Preliminaries

A *metric projection* $P_C(v_1)$ of $v_1 \in \mathcal{H}$ onto a closed and convex subset C of $\mathcal{H}$ is defined by

$$P_C(v_1) = \arg\min\{\|v_2 - v_1\| : v_2 \in C\}.$$

The following are the key properties of projection mapping.

Lemma 2.1. [13] *Let* $P_C : \mathcal{H} \to C$ *be a metric projection such that*

(i)
$$\|v_1 - P_C(v_2)\|^2 + \|P_C(v_2) - v_2\|^2 \leq \|v_1 - v_2\|^2, \;\; v_1 \in C, v_2 \in \mathcal{H}.$$

(ii) $v_3 = P_C(v_1)$ *if and only if*

$$\langle v_1 - v_3, v_2 - v_3 \rangle \leq 0, \ \forall \, v_2 \in C.$$

(iii)

$$\|v_1 - P_C(v_1)\| \leq \|v_1 - v_2\|, \ v_2 \in C, v_1 \in \mathcal{H}.$$

A *normal cone of* C at $v_1 \in C$ is define by

$$N_C(v_1) = \{v_3 \in \mathcal{H} : \langle v_3, v_2 - v_1 \rangle \leq 0, \ \forall \, v_2 \in C\}.$$

Let $\daleth : C \to \mathbb{R}$ be a convex function and *sub-differential of* $\daleth$ at $v_1 \in C$ is defined by

$$\partial \daleth(v_1) = \{v_3 \in \mathcal{H} : \daleth(v_2) - \daleth(v_1) \geq \langle v_3, v_2 - v_1 \rangle, \ \forall \, v_2 \in C\}.$$

Lemma 2.2. [36] *Let* $\daleth : C \to \mathbb{R}$ *be a sub-differentiable, convex and lower semi-continuous function on C. An element* $v_1 \in C$ *is a minimizer of a function* $\daleth$ *iff*

$$0 \in \partial \daleth(v_1) + N_C(v_1),$$

where $\partial \daleth(v_1)$ *denotes the sub-differential of* $\daleth$ *at* $v_1 \in C$ *and* $N_C(v_1)$ *is the normal cone of C at* v_1.

Lemma 2.3. [29] *Let C be a non-empty subset of* $\mathcal{H}$ *and* $\{v_n\}$ *is a sequence in* $\mathcal{H}$ *such that the following two conditions are held:*

(i) *for every* $v_1 \in C$, $\lim_{n\to\infty} \|v_n - v_1\|$ *exists;*
(ii) *every sequentially weak cluster point of* $\{v_n\}$ *is in C.*

Then, $\{v_n\}$ *converges weakly to a point in C.*

Lemma 2.4. [35] *Let* $\{l_n\}$ *and* $\{m_n\}$ *are sequences of non-negative real numbers meet the following inequality*

$$l_{n+1} \leq l_n + m_n, \ \forall \, n \in \mathbb{N}.$$

If $\sum_n m_n < \infty$, *then* $\lim_{n\to\infty} l_n$ *exists.*

3. Main results

Now, we introduce Popov's subgradient extragradient algorithm with non-monotonic step size rule. The following is a detailed proposed method.

Lemma 3.1. *If* $u_{n+1} = u_n = v_n$ *in Algorithm 1. Then,* v_n *is the solution of the problem* (EP).

Proof. By using expression (3.9), we have

$$\chi_n f(v_n, v) - \chi_n f(v_n, u_{n+1}) \geq \langle u_n - u_{n+1}, v - u_{n+1} \rangle, \ \forall \, v \in \mathcal{H}_n. \tag{3.2}$$

By taking $u_{n+1} = u_n = v_n$ and $\chi_n > 0$, we have

$$f(v_n, v) \geq 0, \ \forall \, v \in \mathcal{H}_n. \tag{3.3}$$

Since $C \subset \mathcal{H}_n$, thus implies that v_n is the solution of the problem (EP). □

Algorithm 1 (Non-monotonic Self-adaptive Popov's subgradient extragradient method)

Step 0: Let $u_0, v_0 \in C, \chi_0 > 0$, $\mu \in (0, \frac{1}{3})$ and choose a non-negative real sequence $\{\varphi_n\}$ such that $\sum_n^{\infty} \varphi_n < +\infty$. Set

$$\begin{cases} u_1 = \arg\min_{v \in C}\{\chi_0 f(v_0, v) + \frac{1}{2}\|u_0 - v\|^2\}, \\ v_1 = \arg\min_{v \in C}\{\chi_0 f(v_0, v) + \frac{1}{2}\|u_1 - v\|^2\}. \end{cases}$$

Step 1: Given u_n, v_{n-1} and v_n are known for $n \geq 1$. Firstly choose $\omega_{n-1} \in \partial_2 f(v_{n-1}, v_n)$ satisfying

$$u_n - \chi_n \omega_{n-1} - v_n \in N_C(v_n)$$

and construct a half-space

$$\mathcal{H}_n = \{z \in \mathcal{H} : \langle u_n - \chi_n \omega_{n-1} - v_n, z - v_n \rangle \leq 0\}.$$

Step 2: Compute

$$u_{n+1} = \arg\min_{v \in \mathcal{H}_n}\{\chi_n f(v_n, v) + \frac{1}{2}\|u_n - v\|^2\}.$$

Step 3: Keep updating the step size rule in the following way:

$$\chi_{n+1} = \begin{cases} \min\left\{\chi_n + \varphi_n, \ \frac{\mu\|v_{n-1} - v_n\|^2 + \mu\|u_{n+1} - v_n\|^2}{2[f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n) - f(v_n, u_{n+1})]}\right\} \\ \quad \text{if} \quad f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n) - f(v_n, u_{n+1}) > 0, \\ \chi_n + \varphi_n, \quad\quad else. \end{cases} \tag{3.1}$$

Step 4: Compute

$$v_{n+1} = \arg\min_{v \in C}\{\chi_{n+1} f(v_n, v) + \frac{1}{2}\|u_{n+1} - v\|^2\}.$$

If $u_{n+1} = u_n = v_n$, then STOP. Otherwise, set $n := n + 1$ and go back to **Step 1**.

Lemma 3.2. *A sequence* $\{\chi_n\}$ *generated by* (3.1) *is convergent to* χ *and* $\Phi = \sum_{n=1}^{+\infty} \varphi_n$ *such that*

$$\min\left\{\frac{\mu}{\max\{2c_1, 2c_2\}}, \chi_0\right\} \le \chi \le \chi_0 + \Phi.$$

Proof. Due to the Lipschitz-type condition on a bi-function f with two positive consonants c_1 and c_2. Let $f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n) - f(v_n, u_{n+1}) > 0$ such that

$$\begin{aligned}\frac{\mu(\|v_{n-1} - v_n\|^2 + \|u_{n+1} - v_n\|^2)}{2[f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n) - f(v_n, u_{n+1})]} &\ge \frac{\mu(\|v_{n-1} - v_n\|^2 + \|u_{n+1} - v_n\|^2)}{2[c_1\|v_{n-1} - v_n\|^2 + c_2\|u_{n+1} - v_n\|^2]} \\ &\ge \frac{\mu}{2\max\{c_1, c_2\}}.\end{aligned} \tag{3.4}$$

By using mathematical induction on the definition of χ_{n+1}, we have

$$\min\left\{\frac{\mu}{\max\{2c_1, 2c_2\}}, \chi_0\right\} \le \chi_n \le \chi_0 + \Phi.$$

Set

$$[\chi_{n+1} - \chi_n]^+ := \max\{0, \chi_{n+1} - \chi_n\} \quad \text{and} \quad [\chi_{n+1} - \chi_n]^- := \max\{0, -(\chi_{n+1} - \chi_n)\}.$$

By the definition of $\{\chi_n\}$ we get

$$\sum_{n=1}^{+\infty}(\chi_{n+1} - \chi_n)^+ = \sum_{n=1}^{+\infty}\max\{0, \chi_{n+1} - \chi_n\} \le \Phi < +\infty. \tag{3.5}$$

That is, the series $\sum_{n=1}^{+\infty}(\chi_{n+1} - \chi_n)^+$ is convergent. Next, we need to prove the convergence of the series $\sum_{n=1}^{+\infty}(\chi_{n+1} - \chi_n)^-$. Let consider that $\sum_{n=1}^{+\infty}(\chi_{n+1} - \chi_n)^- = +\infty$. Due to this fact that $\chi_{n+1} - \chi_n = (\chi_{n+1} - \chi_n)^+ - (\chi_{n+1} - \chi_n)^-$. Assume that

$$\chi_{k+1} - \chi_0 = \sum_{n=0}^{k}(\chi_{n+1} - \chi_n) = \sum_{n=0}^{k}(\chi_{n+1} - \chi_n)^+ - \sum_{n=0}^{k}(\chi_{n+1} - \chi_n)^-. \tag{3.6}$$

By letting $k \to +\infty$ in (3.6), we have $\chi_k \to -\infty$ as $k \to \infty$. That is a contradiction. Due to the convergence of the series $\sum_{n=0}^{k}(\chi_{n+1} - \chi_n)^+$ and $\sum_{n=0}^{k}(\chi_{n+1} - \chi_n)^-$ taking $k \to +\infty$ in (3.6), we obtain $\lim_{n\to\infty}\chi_n = \chi$. This completes the proof. □

Theorem 3.3. *Assume that* $\{u_n\}$ *be a sequence generated by Algorithm 1 and the conditions* (c1)–(c4) *are satisfied. Then,* $\{u_n\}$ *weakly converges to* $\wp^*$. *Moreover,* $\lim_{n\to\infty} P_{EP(f,\,C)}(u_n) = \wp^*$.

Proof. From Lemma 2.2, we can write

$$0 \in \partial_2\Big\{\chi_n f(v_n, v) + \frac{1}{2}\|u_n - v\|^2\Big\}(u_{n+1}) + N_{\mathcal{H}_n}(u_{n+1}).$$

Therefore, for $\upsilon \in \partial f(v_n, u_{n+1})$ there is a $\overline{\upsilon} \in N_{\mathcal{H}_n(}u_{n+1})$ such that

$$\chi_n \upsilon + u_{n+1} - u_n + \overline{\upsilon} = 0.$$

The above implies that

$$\langle u_n - u_{n+1}, v - u_{n+1} \rangle = \chi_n \langle \upsilon, v - u_{n+1} \rangle + \langle \overline{\upsilon}, v - u_{n+1} \rangle, \ \forall\, v \in \mathcal{H}_n.$$

By given that $\overline{\upsilon} \in N_{\mathcal{H}_n}(u_{n+1})$ we have $\langle \overline{\upsilon}, v - u_{n+1} \rangle \leq 0$, for all $v \in \mathcal{H}_n$. Thus, we have

$$\langle u_n - u_{n+1}, v - u_{n+1} \rangle \leq \chi_n \langle \upsilon, v - u_{n+1} \rangle, \ \forall\, v \in \mathcal{H}_n. \tag{3.7}$$

Due to $\upsilon \in \partial f(v_n, u_{n+1})$ we have

$$f(v_n, v) - f(v_n, u_{n+1}) \geq \langle \upsilon, v - u_{n+1} \rangle, \ \forall\, v \in \mathcal{H}. \tag{3.8}$$

By combining (3.7) and (3.8) we get

$$\chi_n f(v_n, v) - \chi_n f(v_n, u_{n+1}) \geq \langle u_n - u_{n+1}, v - u_{n+1} \rangle, \ \forall\, v \in \mathcal{H}_n. \tag{3.9}$$

Due to the definition of half-space H_n implies that

$$\langle u_n - \chi_n \upsilon_n - v_n, u_{n+1} - v_n \rangle \leq 0.$$

Thus, implies that

$$\chi_n \langle \upsilon_n, u_{n+1} - v_n \rangle \geq \langle u_n - v_n, u_{n+1} - v_n \rangle. \tag{3.10}$$

Due to $\upsilon_n \in \partial f(v_{n-1}, v_n)$ and substitute $v = u_{n+1}$ such that

$$f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n) \geq \langle \upsilon_n, u_{n+1} - v_n \rangle, \ \forall\, v \in \mathcal{H}. \tag{3.11}$$

By combining (3.10) with (3.11), we have

$$\chi_n \{ f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n) \} \geq \langle u_n - v_n, u_{n+1} - v_n \rangle. \tag{3.12}$$

By substituting $v = \wp^*$ in (3.9) we obtain

$$\chi_n f(v_n, \wp^*) - \chi_n f(v_n, u_{n+1}) \geq \langle u_n - u_{n+1}, \wp^* - u_{n+1} \rangle. \tag{3.13}$$

Since $f(\wp^*, v_n) \geq 0$ and due to condition (c1) implies that $f(v_n, \wp^*) \leq 0$. Thus, we have

$$\langle u_n - u_{n+1}, u_{n+1} - \wp^* \rangle \geq \chi_n f(v_n, u_{n+1}). \tag{3.14}$$

Due to the definition of χ_{n+1} we get

$$f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n) - f(v_n, u_{n+1}) \leq \frac{\mu(\|v_{n-1} - v_n\|^2 + \|u_{n+1} - v_n\|^2)}{2\chi_{n+1}}$$

with $\chi_n > 0$, provides that

$$\chi_n f(v_n, u_{n+1}) \geq \chi_n f(v_{n-1}, u_{n+1}) - \chi_n f(v_{n-1}, v_n) - \frac{\chi_n \mu(\|v_{n-1} - v_n\|^2 + \|u_{n+1} - v_n\|^2)}{2\chi_{n+1}}. \tag{3.15}$$

By combining (3.14) and (3.15) we accomplish the following

$$\langle u_n - u_{n+1}, u_{n+1} - \wp^* \rangle \geq \chi_n \{f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n)\} - \frac{\mu \chi_n}{2\chi_{n+1}} \|v_{n-1} - v_n\|^2 - \frac{\mu \chi_n}{2\chi_{n+1}} \|u_{n+1} - v_n\|^2. \tag{3.16}$$

Due to (3.12) and (3.16), we obtain

$$\langle u_n - u_{n+1}, u_{n+1} - \wp^* \rangle \geq \langle u_n - v_n, u_{n+1} - v_n \rangle - \frac{\mu \chi_n}{2\chi_{n+1}} \|v_{n-1} - v_n\|^2 - \frac{\mu \chi_n}{2\chi_{n+1}} \|u_{n+1} - v_n\|^2. \tag{3.17}$$

In addition, we have the following formulas:

$$-2\langle u_n - u_{n+1}, u_{n+1} - \wp^* \rangle = -\|u_n - \wp^*\|^2 + \|u_{n+1} - u_n\|^2 + \|u_{n+1} - \wp^*\|^2,$$

$$2\langle v_n - u_n, v_n - u_{n+1} \rangle = \|u_n - v_n\|^2 + \|u_{n+1} - v_n\|^2 - \|u_n - u_{n+1}\|^2,$$

and

$$\|v_{n-1} - v_n\|^2 \leq \left(\|v_{n-1} - u_n\| + \|u_n - v_n\|\right)^2 \leq 2\|v_{n-1} - u_n\|^2 + 2\|u_n - v_n\|^2.$$

Using the facts and the expression (3.17), we have

$$\|u_{n+1} - \wp^*\|^2 \leq \|u_n - \wp^*\|^2 - \Big(1 - \frac{2\mu\chi_n}{\chi_{n+1}}\Big)\|u_n - v_n\|^2 + \frac{2\mu\chi_n}{\chi_{n+1}}\|u_n - v_{n-1}\|^2 - \Big(1 - \frac{\mu\chi_n}{\chi_{n+1}}\Big)\|u_{n+1} - v_n\|^2. \tag{3.18}$$

To prove the boundedness, let fixed a number $m \geq n_0$ and using (3.18), for every numbers $n_0, n_0 + 1, \cdots, m$, such that

$$\begin{aligned}
\|u_{m+1} - \wp^*\|^2 &\leq \|u_{n_0} - \wp^*\|^2 - \sum_{k=n_0}^{m} \Big(1 - \frac{2\mu\chi_k}{\chi_{k+1}}\Big)\|u_k - v_k\|^2 - \sum_{k=n_0}^{m} \|u_{k+1} - v_k\|^2 \\
&\quad + \sum_{k=n_0}^{m} \frac{\mu\chi_k}{\chi_{k+1}}\|u_{k+1} - v_k\|^2 + \sum_{k=n_0}^{m} \frac{2\mu\chi_k}{\chi_{k+1}}\|u_k - v_{k-1}\|^2 \\
&\leq \|u_{n_0} - \wp^*\|^2 - \sum_{k=n_0}^{m} \Big(1 - \frac{2\mu\chi_k}{\chi_{k+1}}\Big)\|u_k - v_k\|^2 + \frac{2\mu\chi_{n_0}}{\chi_{n_0+1}}\|u_{n_0} - v_{n_0-1}\|^2 \\
&\quad - \sum_{k=n_0}^{m} \Big(1 - \frac{\mu\chi_k}{\chi_{k+1}} - \frac{2\mu\chi_k}{\chi_{k+1}}\Big)\|u_{k+1} - v_k\|^2 \\
&\leq \|u_{n_0} - \wp^*\|^2 + \frac{2\mu\chi_{n_0}}{\chi_{n_0+1}}\|u_{n_0} - v_{n_0-1}\|^2.
\end{aligned} \tag{3.19}$$

Thus, expression (3.19) is obtained due to the following facts. Due to definition χ_{n+1} we have $\frac{\chi_n}{\chi_{n+1}} \longrightarrow 1$ as $n \longrightarrow \infty$, and due to $\mu \in (0, \frac{1}{3})$ there exist a fixed number $\epsilon \in (0, 1 - 3\mu)$ such that

$$\lim_{n\to\infty} \Big(1 - \frac{2\mu\chi_n}{\chi_{n+1}}\Big) = 1 - 2\mu > 1 - 3\mu > \epsilon > 0,$$

$$\lim_{n\to\infty} \Big(1 - \frac{\mu\chi_n}{\chi_{n+1}} - \frac{2\mu\chi_{n+1}}{\chi_{n+2}}\Big) = 1 - 3\mu > \epsilon > 0.$$

Due to the above facts there exists a fixed natural number $n_0 \in \mathbb{N}$ such that

$$\Big(1 - \frac{2\mu\chi_n}{\chi_{n+1}}\Big) > \epsilon > 0 \quad \text{and} \quad \Big(1 - \frac{\mu\chi_n}{\chi_{n+1}} - \frac{2\mu\chi_{n+1}}{\chi_{n+2}}\Big) > \epsilon > 0, \ \forall\, n \geq n_0.$$

The relation (3.19) imply that the sequence $\{u_n\}$ is bounded. The following results are also deduced:

$$\sum_n \|u_n - v_n\|^2 < +\infty \Longrightarrow \lim_{n\to\infty} \|u_n - v_n\| = 0, \tag{3.20}$$

$$\sum_n \|u_{n+1} - v_n\|^2 < +\infty \Longrightarrow \lim_{n\to\infty} \|u_{n+1} - v_n\| = 0. \tag{3.21}$$

By Lemma 2.4 and due to expressions (3.18) and (3.21), we obtain

$$\lim_{n\to\infty} \|u_n - \wp^*\| = l, \ \text{ for some finite } l > 0. \tag{3.22}$$

Furthermore, due to the expressions (3.20) and (3.21) infer that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|v_{n+1} - v_n\| = 0. \tag{3.23}$$

The remaining part of the proof is to show that each cluster point of sequence $\{u_n\}$ belongs to the solution set $EP(f, C)$. Let us take a point $\hat{u}$ to be a weak cluster point of $\{u_n\}$. It implies that there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\} \rightharpoonup \hat{u}$. Due to $\|u_n - v_n\| \longrightarrow 0$, we have $\{v_{n_k}\} \rightharpoonup \hat{u}$. Due to expression (3.9), we have

$$\chi_{n_k} f(v_{n_k}, v) \geq \chi_{n_k} f(v_{n_k}, u_{n_k+1}) + \langle u_{n_k} - u_{n_k+1}, v - u_{n_k+1}\rangle. \tag{3.24}$$

By expression (3.15), we obtain

$$\begin{aligned} \chi_{n_k} f(v_{n_k}, u_{n_k+1}) \geq{}& \chi_{n_k} f(v_{n_k-1}, u_{n_k+1}) - \chi_{n_k} f(v_{n_k-1}, v_{n_k}) \\ & - \frac{\chi_{n_k}\mu(\|v_{n_k-1} - v_{n_k}\|^2 + \|u_{n_k+1} - v_{n_k}\|^2)}{2\chi_{n_k+1}}. \end{aligned} \tag{3.25}$$

Combining the relations (3.24), (3.25) and (3.12) we write

$$\begin{aligned} \chi_{n_k} f(v_{n_k}, v) \geq{}& \langle u_{n_k} - v_{n_k}, u_{n_k+1} - v_{n_k}\rangle - \frac{\mu\chi_{n_k}}{2\chi_{n_k+1}}\|v_{n_k-1} - v_{n_k}\|^2 \\ & - \frac{\mu\chi_{n_k}}{2\chi_{n_k+1}}\|v_{n_k} - u_{n_k+1}\|^2 + \langle u_{n_k} - u_{n_k+1}, v - u_{n_k+1}\rangle, \end{aligned} \tag{3.26}$$

where v is be an arbitrary element in $\mathcal{H}_n$. By using the boundedness of the sequence and expressions (3.20), (3.21) and (3.23) that right-hand of the last inequality goes to zero. By the use of $\chi_{n_k} \geq \chi > 0$, we obtain

$$0 \leq \limsup_{k\to\infty} f(v_{n_k}, v) \leq f(\hat{u}, v), \;\; \forall\, v \in \mathcal{H}_n.$$

Given that $C \subset \mathcal{H}_n$ that is $f(\hat{u}, v) \geq 0$, for all $v \in C$. It gives that $\hat{u} \in EP(f, C)$. Then, Lemma 2.3, guarantees that $\{u_n\}$ and $\{v_n\}$ weakly converge to $\wp^*$ as $n \to \infty$.

The second part of the proof is to show that $\lim_{n\to\infty} P_{EP(f,C)}(u_n) = \wp^*$. Let consider $p_n := P_{EP(f,C)}(u_n)$ for every $n \in \mathbb{N}$. Given that $\wp^* \in EP(f, C)$, we have

$$\|p_n\| \leq \|p_n - u_n\| + \|u_n\| \leq \|\wp^* - u_n\| + \|u_n\|. \tag{3.27}$$

Due to the above expression we obtain the boundedness of the sequence $\{p_n\}$. Due to the expression (3.18) for every $n \geq n_0$, we can infer that

$$\|u_{n+1} - p_{n+1}\|^2 \leq \|u_{n+1} - p_n\|^2 \leq \|u_n - p_n\|^2 + \frac{2\mu\chi_n}{\chi_{n+1}}\|u_n - v_{n-1}\|^2, \;\; \forall\, n \geq n_0. \tag{3.28}$$

By using expression (3.28) and Lemma 2.4 gives the existence of $\lim_{n\to\infty} \|u_n - p_n\|$. Consider that

$$\begin{aligned} \|p_n - u_m\|^2 &\leq \|p_n - u_{m-1}\|^2 + \frac{2\mu\chi_{m-1}}{\chi_m}\|u_{m-1} - v_{m-2}\|^2 \\ &\leq \cdots \leq \|p_n - u_n\|^2 + \sum_{k=n}^{m-1} \frac{2\mu\chi_k}{\chi_{k+1}}\|u_k - v_{k-1}\|^2. \end{aligned} \tag{3.29}$$

Next, to show that $\{p_n\}$ is a Cauchy sequence. For this, let $p_m, p_n \in EP(f, C)$, for $m > n \geq n_0$, and Lemma 2.1(i) and (3.29) such that

$$\begin{aligned} \|p_n - p_m\|^2 &\leq \|p_n - u_m\|^2 - \|p_m - u_m\|^2 \\ &\leq \|p_n - u_n\|^2 + \sum_{k=n}^{m-1} \frac{2\mu\chi_k}{\chi_{k+1}}\|u_k - v_{k-1}\|^2 - \|p_m - u_m\|^2. \end{aligned} \tag{3.30}$$

By the use of $\lim_{n\to\infty} \|p_n - u_n\|$ and the summability of $\sum_n \|u_n - v_{n-1}\|$ implies that $\lim_{n\to\infty} \|p_n - p_m\| = 0$, for every $m \geq n$. As a results $\{p_n\}$ is a Cauchy sequence. Since $EP(f, C)$ is closed set and thus implies that $\{p_n\}$ converges strongly to $p^* \in EP(f, C)$. Now, we need to prove that $p^* = \wp^*$. By Lemma 2.1(ii) and $\wp^*, p^* \in EP(f, C)$, we have

$$\langle u_n - p_n, \wp^* - p_n\rangle \leq 0. \tag{3.31}$$

Since $p_n \to p^*$ and $u_n \rightharpoonup \wp^*$, we have

$$\langle \wp^* - p^*, \wp^* - p^*\rangle \leq 0,$$

that implies that $\wp^* = p^* = \lim_{n\to\infty} P_{EP(f,C)}(u_n)$. Furthermore, $\|u_n - v_n\| \to 0$, as $n \to \infty$ implies that $\lim_{n\to\infty} P_{EP(f,C)}(v_n) = \wp^*$.

□

It is worth noting that under the presumptions of Lipschitz-type continuity and pseudo-monotonicity, there is still a need to solve two minimization problems on C. If the set C is simple enough so that minimization problem onto it can be easily solved, then this method is particularly effective; but in case of C is a more general closed and convex set, then a minimal distance problem has to be figure out twice in order to obtain the next iterate. This could have a serious impact on the efficiency of the extra-gradient method. On other hand, the subgradient extragradient method involves the replacement of the second minimization problem onto C by a specific subgradient projection.

By the use of Algorithm 1 and Theorem 3.3, we obtain the modification of the Algorithm 1 in [31] with non-monotonic step size rule.

Corollary 3.4. *Assume that $f : C \times C \to \mathbb{R}$ is a bi-function satisfies the conditions* (c1)–(c4). *Choose $u_0, v_{-1}, v_0 \in C$, $\chi_0 > 0$, $\mu \in (0, \frac{1}{3})$ and select a non-negative real sequence $\{\varphi_n\}$ such that $\sum_n^{\infty} \varphi_n < +\infty$. Let $\{u_n\}$ be the sequence generated in the following way:*

$$\begin{cases} u_{n+1} = \arg\min_{v \in C}\{\chi_n f(v_n, v) + \frac{1}{2}\|u_n - v\|^2\}, \\ v_{n+1} = \arg\min_{v \in C}\{\chi_{n+1} f(v_n, v) + \frac{1}{2}\|u_{n+1} - v\|^2\}, \end{cases} \tag{3.32}$$

where

$$\chi_{n+1} = \begin{cases} \min\left\{\chi_n + \varphi_n, \ \frac{\mu\|v_{n-1} - v_n\|^2 + \mu\|u_{n+1} - v_n\|^2}{2[f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n) - f(v_n, u_{n+1})]}\right\} \\ \quad \textit{if} \quad f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n) - f(v_n, u_{n+1}) > 0, \\ \chi_n + \varphi_n \qquad\qquad\qquad\qquad\qquad\qquad \textit{else}. \end{cases}$$

Then, the sequence $\{u_n\}$ converges weakly to some $\wp^* \in Ep(f, C)$.

4. Applications

In this section, we derive some results to solve variational inequalities and the fixed point problems. In the last few years, variational inequalities have attracted a great deal of attention from both researchers and readers. It is well known that variational inequalities cover a number of subjects such as partial differential equations, optimal control, optimization techniques, applied mathematics, engineering, finance and operational science. On the other hand, the solution of a several problems in pure and applied mathematics is the fixed point of some mapping $\mathcal{S}$. As a result, many iterative schemes in the field of numerical analysis and approximation theory use the accomplishment of approximating to the fixed point of mapping. The significance of fixed point theory primarily lies in the fact that many of the equations that emerge in the various physical phenomena can be converted into fixed point formulae or inclusions.

The problem of classical *variational inequalities* for an operator $\mathcal{F} : C \to \mathcal{H}$ is defined in the following manner: Find $\wp^* \in C$ such that

$$\langle \mathcal{F}(\wp^*), v - \wp^* \rangle \geq 0, \ \forall \, v \in C. \tag{VIP}$$

Suppose that the following conditions are met in order to obtain the convergence results of variational inequalities. A mapping $\mathcal{F} : C \to \mathcal{H}$ is said to be

($\mathcal{F}1$) *pseudo-monotone* on C such that

$$\langle \mathcal{F}(v_1), v_2 - v_1 \rangle \geq 0 \Longrightarrow \langle \mathcal{F}(v_2), v_1 - v_2 \rangle \leq 0, \ \forall\, v_1, v_2 \in C;$$

($\mathcal{F}2$) *Lipschitz continuous* on C with constant $L > 0$ such that

$$\|\mathcal{F}(v_1) - \mathcal{F}(v_2)\| \leq L\|v_1 - v_2\|, \ \forall\, v_1, v_2 \in C;$$

($\mathcal{F}3$) $\limsup_{n\to\infty} \langle \mathcal{F}(u_n), v - u_n \rangle \leq \langle \mathcal{F}(p), v - p \rangle$, for all $v \in C$ and $\{u_n\} \subset C$ satisfying $u_n \rightharpoonup p$.

Remark 4.1. Let a bi-function $f : C \times C \to \mathbb{R}$ is defined by $f(u, v) := \langle \mathcal{F}(u), v - u \rangle$ for all $u, v \in C$. Then, problem (EP) turns into the problem of variational inequalities where $L = 2c_1 = 2c_2$.

Corollary 4.1. *Let a mapping $\mathcal{F} : C \to \mathcal{H}$ meet the conditions ($\mathcal{F}1$)–($\mathcal{F}3$) and the solution set $VI(\mathcal{F}, C)$ is non-empty. Choose $u_0, v_0 \in C, \chi_0 > 0$, $\mu \in (0, \frac{1}{3})$ and choose a non-negative real sequence $\{\varphi_n\}$ such that $\sum_n^{\infty} \varphi_n < +\infty$. Then, the sequence $\{u_n\}$ is generated in the following way:*
(i) *Set*

$$\begin{cases} u_1 = P_C(u_0 - \chi_0 \mathcal{F}(v_0)), \\ v_1 = P_C(u_1 - \chi_0 \mathcal{F}(v_0)). \end{cases}$$

(ii) *Compute*

$$\begin{cases} u_{n+1} = P_{\mathcal{H}_n}(u_n - \chi_n \mathcal{F}(v_n)), \\ v_{n+1} = P_C(u_{n+1} - \chi_{n+1} \mathcal{F}(v_n)), \end{cases}$$

where $\mathcal{H}_n = \{z \in \mathcal{H} : \langle u_n - \chi_n \mathcal{F}(v_{n-1}) - v_n, z - v_n \rangle \leq 0\}$ *and*

$$\chi_{n+1} = \begin{cases} \min\left\{\chi_n + \varphi_n, \frac{\mu\|v_{n-1} - v_n\|^2 + \mu\|u_{n+1} - v_n\|^2}{2\langle \mathcal{F}(v_{n-1}) - \mathcal{F}(v_n), u_{n+1} - v_n \rangle}\right\} \\ \qquad\qquad \langle \mathcal{F}(v_{n-1}) - \mathcal{F}(v_n), u_{n+1} - v_n \rangle > 0, \\ \chi_n + \varphi_n \qquad\qquad else. \end{cases}$$

Then, $\{u_n\}$ weakly converges to $\wp^* \in VI(\mathcal{F}, C)$.

Corollary 4.2. *Let a mapping $\mathcal{F} : C \to \mathcal{H}$ meet the conditions ($\mathcal{F}1$)–($\mathcal{F}3$) and the solution set $VI(\mathcal{F}, C)$ is non-empty. Choose $u_0, v_{-1}, v_0 \in C$, $\chi_0 > 0$, $\mu \in (0, \frac{1}{3})$ and choose a non-negative real sequence $\{\varphi_n\}$ such that $\sum_n^{\infty} \varphi_n < +\infty$. Then, the sequence $\{u_n\}$ is generated in the following way:*

$$\begin{cases} u_{n+1} = P_C(u_n - \chi_n \mathcal{F}(v_n)), \\ v_{n+1} = P_C(u_{n+1} - \chi_{n+1} \mathcal{F}(v_n)), \end{cases}$$

where

$$\chi_{n+1} = \begin{cases} \min\left\{\chi_n + \varphi_n, \frac{\mu\|v_{n-1} - v_n\|^2 + \mu\|u_{n+1} - v_n\|^2}{2\langle \mathcal{F}(v_{n-1}) - \mathcal{F}(v_n), u_{n+1} - v_n \rangle}\right\} \\ \qquad\qquad \langle \mathcal{F}(v_{n-1}) - \mathcal{F}(v_n), u_{n+1} - v_n \rangle > 0, \\ \chi_n + \varphi_n \qquad\qquad else. \end{cases}$$

Then, the sequence $\{u_n\}$ converges weakly to $\wp^* \in VI(\mathcal{F}, C)$.

The problem of *fixed point* for a mapping $\mathcal{S} : C \to \mathcal{H}$ is defined in the following manner: Find $\wp^* \in C$ such that

$$\mathcal{S}(\wp^*) = \wp^*. \tag{FPP}$$

Consider the following conditions are satisfied in order to achieve the convergence analysis of fixed point algorithms: A mapping $\mathcal{S} : C \to \mathcal{H}$ is called to be

($\mathcal{S}$1) *κ-strict pseudo-contraction* [7] on C if

$$\|\mathcal{S}v_1 - \mathcal{S}v_2\|^2 \leq \|v_1 - v_2\|^2 + \kappa\|(v_1 - \mathcal{S}v_1) - (v_2 - \mathcal{S}v_2)\|^2, \ \forall\, v_1, v_2 \in C;$$

($\mathcal{S}$2) *sequentially weakly continuous* on C if

$$\mathcal{S}(v_n) \rightharpoonup \mathcal{S}(v) \text{ for any sequence in } C \text{ satisfying } v_n \rightharpoonup v.$$

Remark 4.2. Let $f : C \times C \to \mathbb{R}$ is defined by $f(u, v) := \langle u - \mathcal{S}u, v - u\rangle$ for all $u, v \in C$. Then, the problem (EP) turns into the problem of fixed point where $2c_1 = 2c_2 = \frac{3-2\kappa}{1-\kappa}$.

Corollary 4.3. *Let a mapping $\mathcal{S} : C \to \mathcal{H}$ meet the conditions ($\mathcal{S}$1) and ($\mathcal{S}$2) and the solution set $Fix(\mathcal{S}, C)$ is non-empty. Choose $u_0, v_0 \in C, \chi_0 > 0$, $\mu \in (0, \frac{1}{3})$ and choose a non-negative real sequence $\{\varphi_n\}$ such that $\sum_n^\infty \varphi_n < +\infty$. Then, the sequence $\{u_n\}$ is generated in the following way:*
(i) *Set*

$$\begin{cases} u_1 = P_C[u_0 - \chi_0(v_0 - \mathcal{S}(v_0))], \\ v_1 = P_C[u_1 - \chi_0(v_0 - \mathcal{S}(v_0))]. \end{cases}$$

(ii) *Compute*

$$\begin{cases} u_{n+1} = P_{\mathcal{H}_n}[u_n - \chi_n(v_n - \mathcal{S}(v_n))], \\ v_{n+1} = P_C[u_{n+1} - \chi_{n+1}(v_n - \mathcal{S}(v_n))], \end{cases}$$

where $\mathcal{H}_n = \{z \in \mathcal{H} : \langle(1 - \chi_n)u_n + \chi_n\mathcal{S}(v_{n-1}) - v_n, z - v_n\rangle \leq 0\}$ and

$$\chi_{n+1} = \begin{cases} \min\left\{\chi_n + \varphi_n, \ \frac{\mu\|v_{n-1}-v_n\|^2+\mu\|u_{n+1}-v_n\|^2}{2\left\langle (v_{n-1}-v_n)-[\mathcal{S}(v_{n-1})-\mathcal{S}(v_n)],\, u_{n+1}-v_n\right\rangle}\right\} \\ \quad \text{if} \quad \left\langle (v_{n-1} - v_n) - [\mathcal{S}(v_{n-1}) - \mathcal{S}(v_n)],\ u_{n+1} - v_n\right\rangle > 0, \\ \chi_n + \varphi_n \qquad\qquad \text{else}. \end{cases}$$

Then, the sequence $\{u_n\}$ converges weakly to $\wp^* \in Fix(\mathcal{S}, C)$.

Corollary 4.4. *Let a mapping $\mathcal{S} : C \to \mathcal{H}$ meet the conditions ($\mathcal{S}$1) and ($\mathcal{S}$2) and the solution set $Fix(\mathcal{S}, C)$ is non-empty. Choose $u_0, v_{-1}, v_0 \in C, \chi_0 > 0$, $\mu \in (0, \frac{1}{3})$ and choose a non-negative real sequence $\{\varphi_n\}$ such that $\sum_n^\infty \varphi_n < +\infty$. Then, the sequence $\{u_n\}$ is generated in the following way:*

$$\begin{cases} u_{n+1} = P_C[u_n - \chi_n(v_n - \mathcal{S}(v_n))], \\ v_{n+1} = P_C[u_{n+1} - \chi_{n+1}(v_n - \mathcal{S}(v_n))], \end{cases}$$

where

$$\chi_{n+1} = \begin{cases} \min\left\{\chi_n + \varphi_n, \ \frac{\mu\|v_{n-1}-v_n\|^2+\mu\|u_{n+1}-v_n\|^2}{2\left\langle (v_{n-1}-v_n)-[\mathcal{S}(v_{n-1})-\mathcal{S}(v_n)],\, u_{n+1}-v_n\right\rangle}\right\} \\ \quad \text{if} \quad \left\langle (v_{n-1} - v_n) - [\mathcal{S}(v_{n-1}) - \mathcal{S}(v_n)],\ u_{n+1} - v_n\right\rangle > 0, \\ \chi_n + \varphi_n \qquad\qquad \text{else}. \end{cases}$$

Then, the sequence $\{u_n\}$ converges weakly to $\wp^* \in Fix(\mathcal{S}, C)$.

5. Numerical illustrations

In this section, we provide a numerical example to show the implementations of the proposed method. All computations are done in MATLAB R2018b and run on HP Core(TM)i5-6200 (7.78 GB usable) RAM 8.00 GB laptop.

Example 5.1. Consider a test problem where a bi-function f is defined as follows

$$f(u, v) := (Pu + Qv + r)^T (v - u)$$

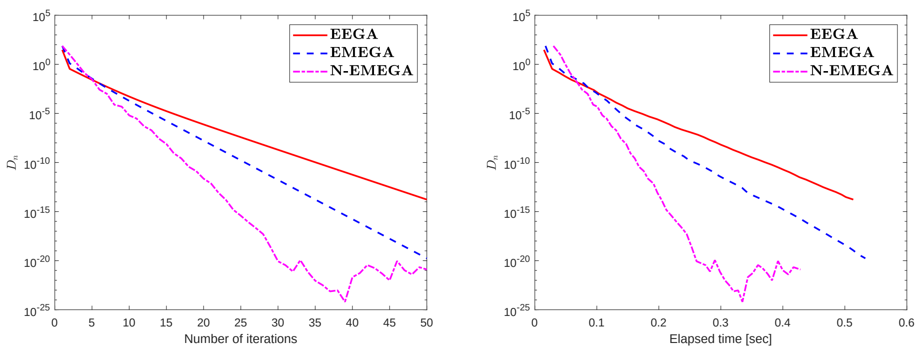
where $P = (p_{ij})_{N\times N}$ and $Q = (q_{ij})_{N\times N}$ are $N \times N$ symmetric positive semi-definite matrices such that $P - Q$ is also positive semi-definite and $r \in \mathbb{R}^N$. The bi-function f has the form of the one arising from a Nash-Cournot oligopolistic electricity market equilibrium model [31] and that f is Lipschitz-type continuous with constants $c_1 = c_2 = \frac{1}{2}\|P - Q\|$ and the positive semi-definition of $P - Q$ gives that f is pseudo-monotone. P and Q are matrices of the form: Choose two diagonal matrices D_1 and D_2 having entries from $[0, N]$ and $[-N, 0]$, respectively. Set $Q = B_1 + B_1^T$ while $B_1 = O_1 D_1 O_1^T$ and $O_1 = RandOrthMat(N)$. Set $P = Q - S$ while $S = B_2 + B_2^T$ and $B_2 = O_2 D_2 O_2^T$ and $O_2 = RandOrthMat(N)$. Moreover, vector r generated randomly in $[-N, N]$.

Experiment 1: In this experiment, the numerical performance of Algorithm 1 with Algorithm 1 in [16] and Algorithm 2 in [16] is provided by letting the starting points u_0, v_{-1}, v_0 are randomly generated in $[-N, N]$. We assume the feasible set in the following manner:

$$C := \{u \in \mathbb{R}^N : -10 \le u_i \le 10\}.$$

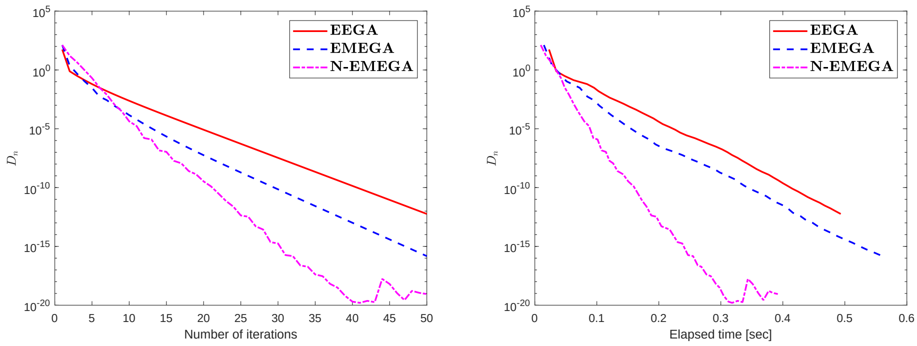
Figures 1–5 have shown a number of results obtained by taking different number of firms. The values of the control parameters are taken as follows:

(i) Algorithm 1 in [16] (**EEGA**): $\chi_0 = 0.25, \mu = 0.33, D_n = \|u_n - v_n\|^2$.
(ii) Algorithm 2 in [16] (**EMEGA**): $\chi_0 = 0.25, \mu = 0.33, D_n = \max\{\|u_{n+1} - v_n\|^2, \|u_n - v_n\|^2\}$.
(iii) Algorithm 1 (**N-EMEGA**): $\chi_0 = 0.25, \mu = 0.33, D_n = \max\{\|u_{n+1} - v_n\|^2, \|u_n - v_n\|^2\}, \varphi_n = \frac{100}{(n+1)^2}$.



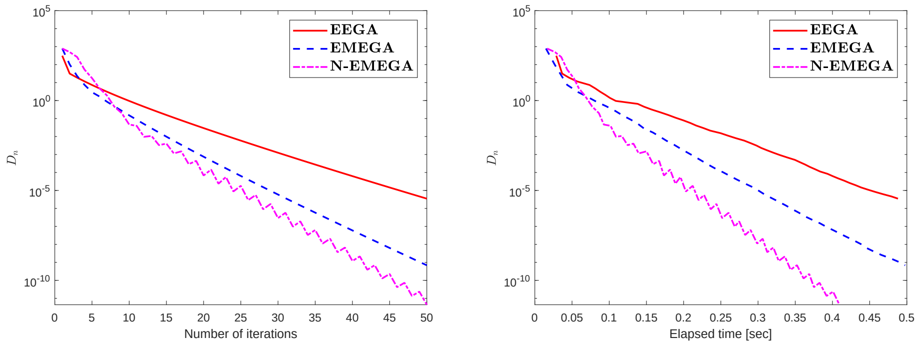
(a) (b)

Figure 1. Example 5.1: Algorithm 1 numerical comparison with Algorithm 1 in [16] and Algorithm 2 in [16] in $\mathbb{R}^5$ for first 50 iterations.



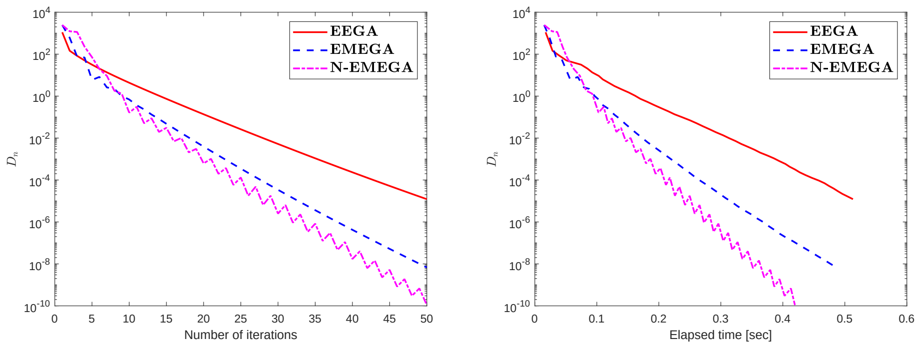
(a) **(b)**

Figure 2. Example 5.1: Algorithm 1 numerical comparison with Algorithm 1 in [16] and Algorithm 2 in [16] in $\mathbb{R}^{10}$ for first 50 iterations.



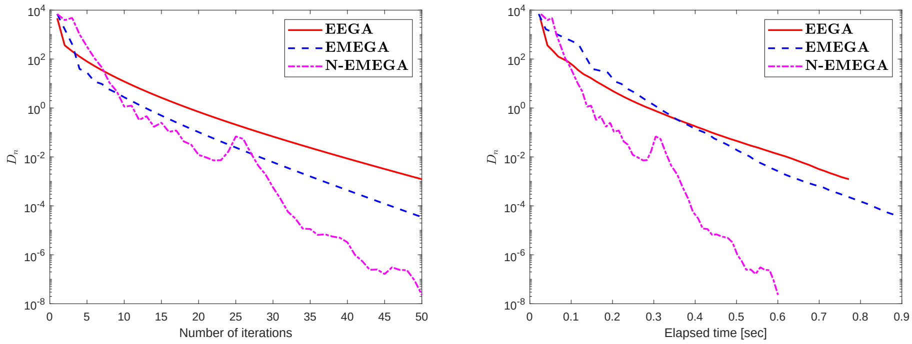
(a) **(b)**

Figure 3. Example 5.1: Algorithm 1 numerical comparison with Algorithm 1 in [16] and Algorithm 2 in [16] in $\mathbb{R}^{20}$ for first 50 iterations.



(a) **(b)**

Figure 4. Example 5.1: Algorithm 1 numerical comparison with Algorithm 1 in [16] and Algorithm 2 in [16] in $\mathbb{R}^{50}$ for first 50 iterations.



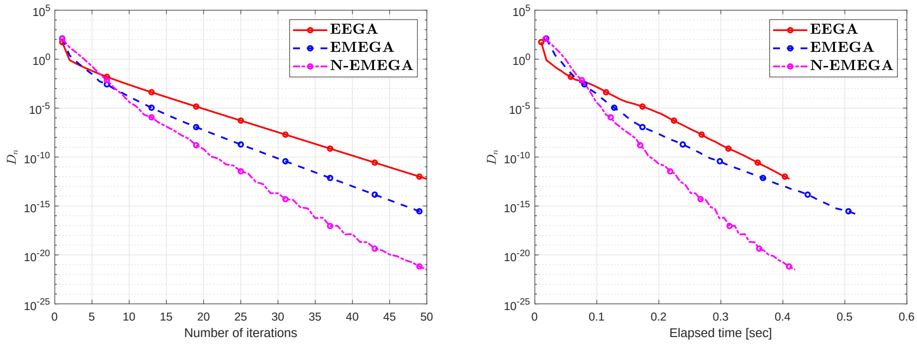
(a) **(b)**

Figure 5. Example 5.1: Algorithm 1 numerical comparison with Algorithm 1 in [16] and Algorithm 2 in [16] in $\mathbb{R}^{100}$ for first 50 iterations.

Experiment 2: In this experiment, the numerical performance of Algorithm 1 with Algorithm 1 in [16] and Algorithm 2 in [16] is considered by taking the starting points u_0, v_{-1}, v_0 are randomly generated in $[-N, N]$. We consider the feasible set as follows:

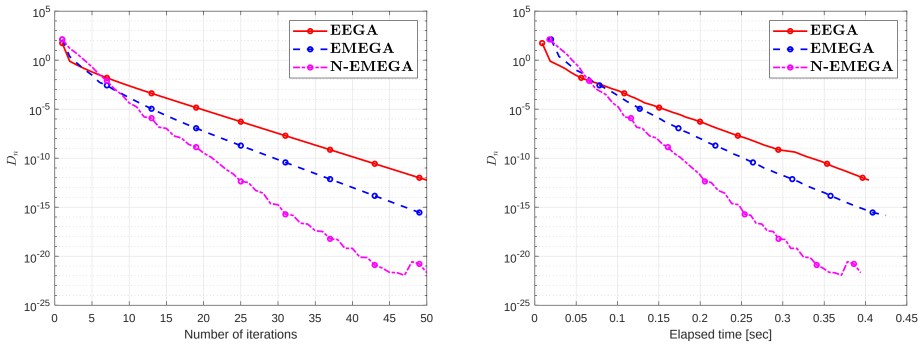
$$C := \{u \in \mathbb{R}^{10} : -M \leq u_i \leq M\}.$$

Figures 6–10 have shown a number of results by letting different different feasible sets based on the length of values given to M. Values of the control parameters are same as in Experiment 1.



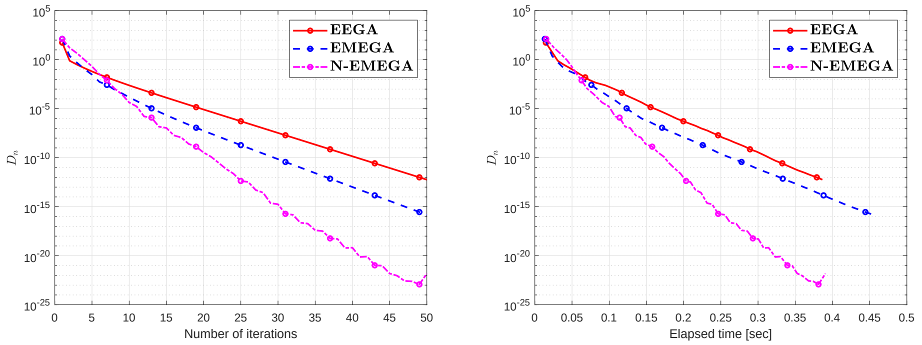
(a) **(b)**

Figure 6. Example 5.1: Algorithm 1 numerical comparison with Algorithm 1 in [16] and Algorithm 2 in [16] for first 50 iterations and $M = 20$.



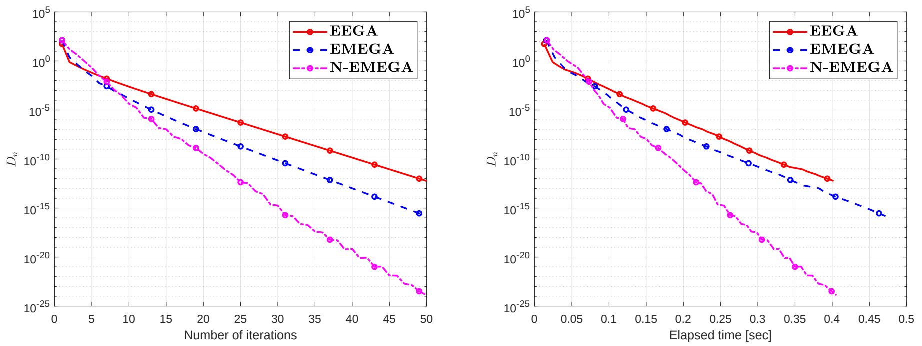
(a) (b)

Figure 7. Example 5.1: Algorithm 1 numerical comparison with Algorithm 1 in [16] and Algorithm 2 in [16] for first 50 iterations and $M = 30$.



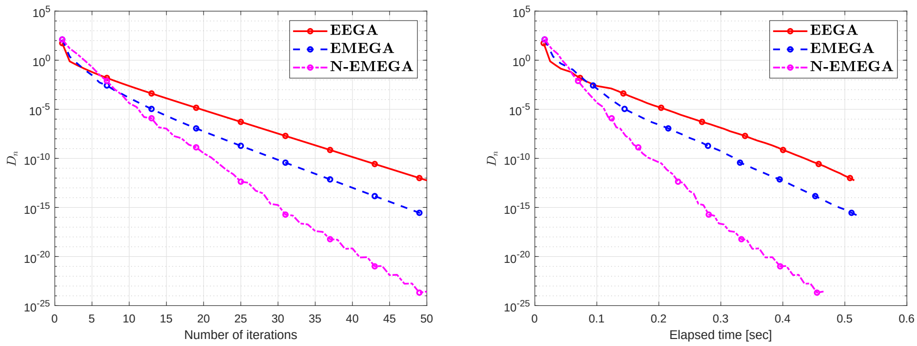
(a) (b)

Figure 8. Example 5.1: Algorithm 1 numerical comparison with Algorithm 1 in [16] and Algorithm 2 in [16] for first 50 iterations and $M = 50$.



(a) (b)

Figure 9. Example 5.1: Algorithm 1 numerical comparison with Algorithm 1 in [16] and Algorithm 2 in [16] for first 50 iterations and $M = 100$.



(a) **(b)**

Figure 10. Example 5.1: Algorithm 1 numerical comparison with Algorithm 1 in [16] and Algorithm 2 in [16] for first 50 iterations and $M = 500$.

Remark 5.1. The following observations are derived from the above discussed experiments.

(i) As we increase the value of N, it is easy to note that more iterations and elapsed time is required in case of all three algorithms.
(ii) Each algorithm has shown the same numerical behaviour corresponding to a different values of M.
(iii) In both experiments, we have fixed the values of control parameters χ_0 and μ. But still, there is an influence of these parameters on the efficiency of the three algorithms.
(iv) Starting points entries are generated randomly from the interval $[-N, N]$, so we can deduce that there is not too much difference in the efficiency of the algorithms.

Example 5.2. Let $f : C \times C \to \mathbb{R}$ be a bi-function defined by

$$f(u, v) = \sum_{i=2}^{5} (v_i - u_i)\|u\|, \;\; \forall\, u, v \in \mathbb{R}^5,$$

where $C \subset \mathbb{R}^5$ is taken as follows:

$$C = \{(u_1, \cdots, u_5) : u_1 \geq -1, u_i \geq 1, i = 2, \cdots, 5\}.$$

Thus, f is Lipschitz-type continuous with $c_1 = c_2 = 2$ and satisfies the conditions (c1)–(c4). Figures 11–16 and Tables 1–9 have shown a number of results by letting different starting points $u_0 = v_0$ and $v_{-1} = (1, 1, 1, 1, 1)^T$. The selection of control parameters are taken as follows:

(i) Algorithm 1 in [16] (**EEGA**): $\chi_0 = 0.15, \mu = 0.20, D_n = \|u_n - v_n\|^2$.
(ii) Algorithm 2 in [16] (**EMEGA**): $\chi_0 = 0.15, \mu = 0.20, D_n = \max\{\|u_{n+1} - v_n\|^2, \|u_n - v_n\|^2\}$.
(iii) Algorithm 1 (**N-EMEGA**): $\chi_0 = 0.15, \mu = 0.20, D_n = \max\{\|u_{n+1} - v_n\|^2, \|u_n - v_n\|^2\}, \varphi_n = \frac{100}{(n+1)^{1.2}}$.

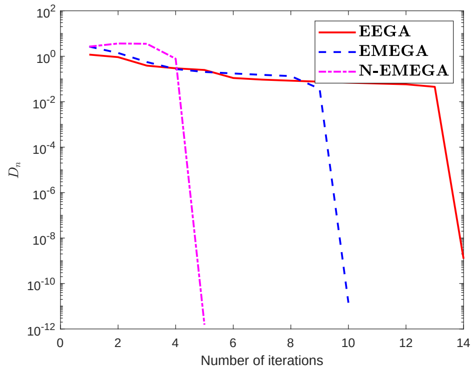


Figure 11. Example 5.2: Algorithm 1 numerical comparison with Algorithm 1 in [16] and Algorithm 2 in [16] and $u_0 = (2, 3, 2, 5, 2)^T$.

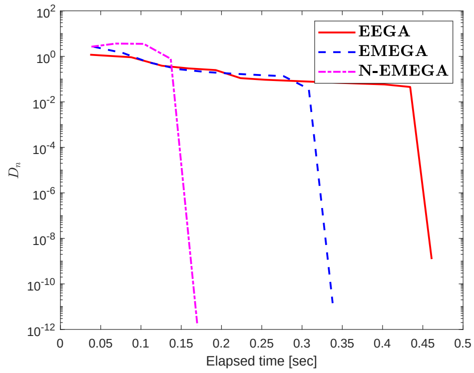


Figure 12. Example 5.2: Algorithm 1 numerical comparison with Algorithm 1 in [16] and Algorithm 2 in [16] and $u_0 = (2, 3, 2, 5, 2)^T$.

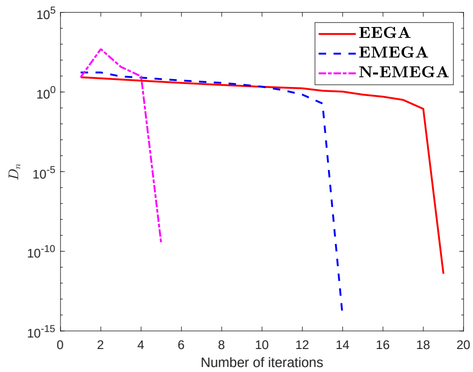


Figure 13. Example 5.2: Algorithm 1 numerical comparison with Algorithm 1 in [16] and Algorithm 2 in [16] and $u_0 = (11, 12, 13, 14, 15)^T$.

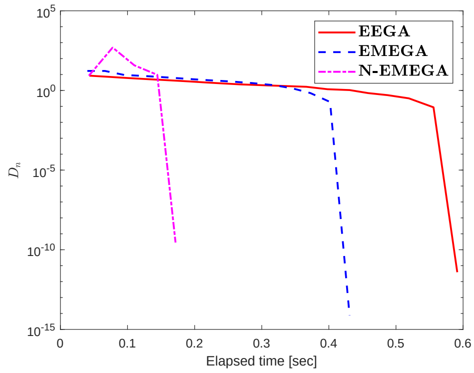


Figure 14. Example 5.2: Algorithm 1 numerical comparison with Algorithm 1 in [16] and Algorithm 2 in [16] and $u_0 = (11, 12, 13, 14, 15)^T$.

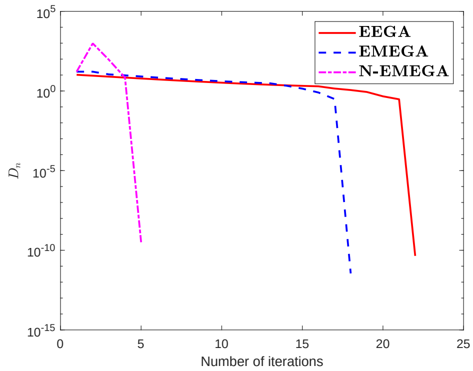


Figure 15. Example 5.2: Algorithm 1 numerical comparison with Algorithm 1 in [16] and Algorithm 2 in [16] and $u_0 = (16, 17, 18, 19, 20)^T$.

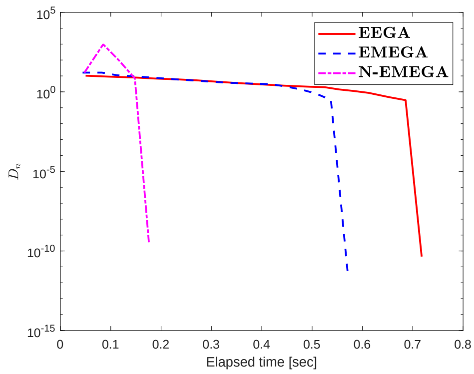


Figure 16. Example 5.2: Algorithm 1 numerical comparison with Algorithm 1 in [16] and Algorithm 2 in [16] and $u_0 = (16, 17, 18, 19, 20)^T$.

Table 1. Example 5.2: Numerical study of Algorithm 1 in [16] and $u_0 = v_0 = (2, 3, 2, 5, 2)^T$.

Iter (n)	u_1	u_2	u_3	u_4	u_5
1	1.99999999292158	2.53245814873752	1.53245818645621	4.53245813010642	1.53245818676069
2	2.00000011151705	2.11762606678515	1.11763031801202	4.11762580226311	1.11763033019828
3	2.00000022898706	1.73795364146178	1.00000161505322	3.73795296286347	1.00000161506305
4	2.00000022120447	1.38845439177892	1.00000005740891	3.38845365463313	1.00000005740869
5	2.00000034004423	1.06557396293431	1.00000128588123	3.06556029300624	1.00000128587826
6	2.00000045824156	1.00000178954353	1.00000136873721	2.76083393107771	1.00000136874456
7	2.00000057622215	1.00000145135853	1.00000145135665	2.47209464809527	1.00000145135926
8	2.00000069487109	1.00000153490405	1.00000153490405	2.19771753584690	1.00000153490991
9	2.00000081292748	1.00000161782998	1.00000161782998	1.93607697655271	1.00000161782998
10	2.00000093081741	1.00000169870621	1.00000169870621	1.68561726473276	1.00000169870702
11	2.00000104965820	1.00000177590098	1.00000177590098	1.44484291183716	1.00000177590088
12	2.00000116829872	1.00000184759923	1.00000184759923	1.21230955893295	1.00000184759923
13	2.00000128670663	1.00000176776014	1.00000176776014	1.00003698145475	1.00000176776014
14	2.00000140519764	1.00000190626625	1.00000190626625	1.00000190660812	1.00000190626625
CPU time is seconds	0.461059				

Table 2. Example 5.2: Numerical study of Algorithm 2 in [16] and $u_0 = v_0 = (2, 3, 2, 5, 2)^T$.

Iter (n)	u_1	u_2	u_3	u_4	u_5
1	2.00000013326561	2.18612074183894	1.18612282556376	4.18612049721514	1.18612279418525
2	2.00000012891999	1.65955376427167	1.00000005896249	3.65955348694164	1.00000005896250
3	2.00000025034999	1.16334131567859	1.00000081898995	3.16333837745867	1.00000081899339
4	2.00000036920083	1.00000145833391	1.00000090449102	2.71252257931897	1.00000090448955
5	2.00000036125328	1.00000004795846	1.00000004795854	2.29483778798085	1.00000004795835
6	2.00000035244232	1.00000005148000	1.00000005148000	1.90555466748076	1.00000005147995
7	2.00000034466122	1.00000005479632	1.00000005479632	1.53962195519060	1.00000005479633
8	2.00000046276893	1.00000118941881	1.00000118941881	1.19229605250082	1.00000118941881
8	2.00000058139100	1.00000121900942	1.00000121900942	1.00000325388395	1.00000121900942
10	2.00000057315663	1.00000005914572	1.00000005914572	1.00000005914607	1.00000005914572
CPU time is seconds	0.338142				

Table 3. Example 5.2: Numerical study of Algorithm 1 and $u_0 = v_0 = (2, 3, 2, 5, 2)^T$.

Iter (n)	u_1	u_2	u_3	u_4	u_5
1	2.00000013326561	2.18612074183894	1.18612282556376	4.18612049721514	1.18612279418525
2	2.00000013455268	1.36460283566350	1.00000003149421	3.36460250671401	1.00000003149352
3	2.00000011482073	1.00000001788559	1.00000001348744	1.88160631941336	1.00000001348743
4	2.00000010595314	1.00000000966688	1.00000000966688	1.00000001684686	1.00000000966688
5	2.00000134334259	1.00000003929131	1.00000003929131	1.00000003929116	1.00000003929131
CPU time is seconds	0.171018				

Table 4. Example 5.2: Numerical study of Algorithm 1 in [16] and $u_0 = v_0 = (11, 12, 13, 14, 15)^T$.

Iter (n)	u_1	u_2	u_3	u_4	u_5
1	11.0083274696798	10.6811303864260	11.6801640613791	12.6793629750437	13.6786881306881
2	11.0083275330855	9.46039733827078	10.4594308558907	11.4586297053150	12.4579548430796
3	11.0083276530072	8.33713834651744	9.33617182858615	10.3353706506051	11.3346957695711
4	11.0083277343258	7.30133200844278	8.30036541799450	9.29956421756527	10.2988893099399
5	11.0083278190382	6.34372806289202	7.34276140765877	8.34196016581329	9.34128521495189
6	11.0083279019944	5.45576321667805	6.45479647791941	7.45399516735627	8.45332016046481
7	11.0083279882304	4.62948228561651	5.62851541496775	6.62771401890804	7.62703895444391
8	11.0083280755389	3.85746424959976	4.85649716787690	5.85569565201102	6.85502051425570
9	11.0083281656242	3.13275234368878	4.13178486452440	5.13098317346056	6.13030794265829
10	11.0083280840588	2.44878602508279	3.44781853106201	4.44701684141454	5.44634159183119
11	11.0083280034533	1.79934062805675	2.79837313222168	3.79757141990420	4.79689616393140
12	11.0083279547318	1.17846486794016	2.17749518340738	3.17669331742426	4.17601799659064
13	11.0083278742395	1.00000004757510	1.57800049103264	2.57719859769294	3.57652326328809
14	11.0083278256693	1.00000068611937	1.00009306645122	1.99421071894621	2.99353517092297
15	11.0083277456809	1.00000003497425	1.00000003497990	1.42197225240730	2.42129667613383
16	11.0083276970838	1.00000071694298	1.00000071694298	1.00000339055939	1.85649270080839
17	11.0083276483865	1.00000072203277	1.00000072203277	1.00000072203601	1.29557614755808
18	11.0083275995927	1.00000072400720	1.00000072400720	1.00000072400720	1.00000160277522
19	11.0083275192397	1.00000003577656	1.00000003577656	1.00000003577656	1.00000003577662
CPU time is seconds	0.5913360				

Table 5. Example 5.2: Numerical study of Algorithm 2 in [16] and $u_0 = v_0 = (11, 12, 13, 14, 15)^T$.

Iter (n)	u_1	u_2	u_3	u_4	u_5
1	11.0000002191261	9.95317310793062	10.9531732304009	11.9531733205376	12.9531734800135
2	11.0000002799103	8.41968215176090	9.41968212182442	10.4196821511721	11.4196823113368
3	11.0000003757963	7.00694297746949	8.00694290556055	9.00694285743516	10.0069430122541
4	11.0000004676307	5.74052178140028	6.74052167076998	7.74052157164007	8.74052167549556
5	11.0000005555938	4.59295024012139	5.59295000972051	6.59294981333090	7.59294986188480
6	11.0000006461726	3.54569141262373	4.54569092548914	5.54569057094979	6.54569055797946
7	11.0000005636409	2.58093544238862	3.58093492944650	4.58093455274703	5.58093452821783
8	11.0000004861456	1.68219898426140	2.68219846980729	3.68219808356875	4.68219803770429
9	11.0000004403918	1.00000274608732	1.83279288297032	2.83279222154370	3.83279208890202
10	11.0000003906827	1.00000049029577	1.01698299449785	2.01695624709248	3.01695589889476
11	11.0000003422224	1.00000050567032	1.00000051683185	1.22102513910523	2.22102316620642
12	11.0000002623398	1.00000002545898	1.00000002545898	1.00000003543451	1.43517885006596
13	11.0000001819360	1.00000002556380	1.00000002556380	1.00000002556381	1.00000005775335
14	11.0000001016463	1.00000002556377	1.00000002556377	1.00000002556377	1.00000002556381
CPU time is seconds	0.43083970				

Table 6. Example 5.2: Numerical study of Algorithm 1 and $u_0 = v_0 = (11, 12, 13, 14, 15)^T$.

Iter (n)	u_1	u_2	u_3	u_4	u_5
1	11.0083274629695	10.5484537388725	11.5474617255091	12.5466412663252	13.5459514498880
2	11.0083276185507	3.96764097934650	4.96664888405123	5.96582810850523	6.96513834195619
3	11.0083275532871	1.00000004860214	1.58685349650335	2.58603261386029	3.58534278265138
4	11.0083276806728	1.00000012097834	1.00000012454813	1.00000013114579	1.00000013848224
5	11.0083444517378	1.00000075648718	1.00000075648718	1.00000075648718	1.00000075648718
CPU time is seconds	0.17152080				

Table 7. Example 5.2: Numerical study of Algorithm 1 in [16] and $u_0 = v_0 = (16, 17, 18, 19, 20)^T$.

Iter (n)	u_1	u_2	u_3	u_4	u_5
1	16.0000000013767	15.5020222401273	16.5020222118653	17.5020221970684	18.5020221786608
2	16.0058802001436	14.1116372710683	15.1110969612532	16.1106281594581	17.1102174892337
3	16.0058801904244	12.8042615537855	13.8037211116071	14.8032522736792	15.8028415734253
4	16.0058801915895	11.5804530958265	12.5799126361565	13.5794437796837	14.5790330658984
5	16.0058801930046	10.4330387588241	11.4324982706068	12.4320293910037	13.4316186547883
6	16.0058801911966	9.35528788508582	10.3547473661296	11.3542784569216	12.3538676960017
7	16.0058801923848	8.34087215529711	9.34033160363169	10.3398626526737	11.3394518630316
8	16.0058801898677	7.38382749785093	8.38328688725377	9.38281789891371	10.3824070750717
9	16.0058801910898	6.47851791708288	7.47797723028635	8.47750820727257	9.47709733257917
10	16.0058801895559	5.61960103412642	6.61906024743004	7.61859117222833	8.61818025692468
10	16.0058801876984	4.80199514404989	5.80145424177028	6.80098508720808	7.80057411372804
12	16.0058801852835	4.02084749996001	5.02030642485596	6.01983715223405	7.01942610530475
13	16.0058801849295	3.27150365568317	4.27096233474183	5.27049290928867	6.27008176709151
14	16.0058800668210	2.54947670625746	3.54893536577234	4.54846593086610	5.54805477579454
15	16.0058799494501	1.85042125995414	2.84987989935855	3.84941044992968	4.84899928842712
16	16.0058798549794	1.17010506203917	2.16956132113662	3.16909171155296	4.16868048550728
17	16.0058797378985	1.00000004031720	1.50291432705136	2.50244467796094	3.50203343743755
18	16.0058796421017	1.00000061458189	1.00000306162503	1.84567072952209	2.84525921740702
19	16.0058795463691	1.00000062071493	1.00000062071877	1.19526960512418	2.19485615373208
20	16.0058794279464	1.00000003093119	1.00000003093119	1.00000004433604	1.54792700963939
21	16.0058793322001	1.00000062580317	1.00000062580317	1.00000062580351	1.00000660727976
22	16.0058792141231	1.00000003101359	1.00000003101359	1.00000003101359	1.00000003101391
CPU time is seconds	0.717764				

Table 8. Example 5.2: Numerical study of Algorithm 2 in [16] and $u_0 = v_0 = (16, 17, 18, 19, 20)^T$.

Iter (n)	u_1	u_2	u_3	u_4	u_5
1	16.0058801988836	14.9884857687756	15.9880089794114	16.9875918008410	17.9872237315845
2	16.0058804192340	13.3304303383913	14.3299539081241	15.3295366987460	16.3291687457091
3	16.0058804597842	11.7851467866923	12.7846703641280	13.7842531659238	14.7838851913108
4	16.0058804574841	10.3599980017078	11.3595215452945	12.3591043267546	13.3587363256131
5	16.0058804636136	9.04026961635667	10.0397931290522	11.0393758826717	12.0390078590822
6	16.0058804648713	7.81429167353663	8.81381514980023	9.81339786506092	10.8130298073808
7	16.0058804625316	6.67102397861150	7.67054738301011	8.67013004597202	9.66976194765656
8	16.0058804596168	5.60015507357128	6.59967838358266	7.59926098359060	8.59889284081691
9	16.0058804577286	4.59198717756650	5.59151036450655	6.59109287577367	7.59072467296010
10	16.0058804552157	3.63734384495638	4.63686684477603	5.63644921688181	6.63608093771071
11	16.0058803370804	2.72747863417141	3.72700161491462	4.72658398832986	5.72621569812753
12	16.0058802212637	1.85399031175675	2.85351328314349	3.85309564688082	4.85272733841624
13	16.0058801262526	1.00878949243540	2.00820134706940	3.00778352312963	4.00741512553108
14	16.0058800314488	1.00000049206834	1.18174196108997	2.18132198761579	3.18095341626987
15	16.0058799125931	1.00000002455318	1.00000003160922	1.36650997277422	2.36614136890780
16	16.0058797943905	1.00000002473897	1.00000002473896	1.00000004528977	1.55744674820487
17	16.0058796984322	1.00000049895431	1.00000049895431	1.00000049895425	1.00000170933540
18	16.0058795803551	1.00000002480596	1.00000002480596	1.00000002480596	1.00000002480596
CPU time is seconds	0.571045				

Table 9. Example 5.2: Numerical study of Algorithm 1 and $u_0 = v_0 = (16, 17, 18, 19, 20)^T$.

Iter (n)	u_1	u_2	u_3	u_4	u_5
1	16.0058801988836	14.9884857687756	15.9880089794114	16.9875918008410	17.9872237315845
2	16.0058803853176	5.32463230780648	6.32415516325652	7.32373778506375	8.32336954685412
3	16.0058802569477	1.00000055670446	1.27586886589585	2.27545093869651	3.27508290241116
4	16.0059092138812	1.00000715534348	1.00000718401277	1.00000728953418	1.00000739829730
5	16.0059208418506	1.00000047513698	1.00000047513698	1.00000047513698	1.00000047513698
CPU time is seconds	0.177573				

Example 5.3. Suppose that $f : \mathcal{H} \times \mathcal{H} \to \mathcal{R}$ is defined by

$$f(u,v) = (5 - \|u\|)\langle u, v-u\rangle, \ \forall\, u,v \in \mathcal{H},$$

where $\mathcal{H} = l_2$ is a real Hilbert space consisting the elements of the form of square-summable sequences and $C = \{u \in \mathcal{H} : \|u\| \leq 3\}$. The bi-function f is Lipschitz-type continuous and value of Lipschitz-constants are $c_1 = c_2 = \frac{11}{2}$. It can easily to shown that the bi-function f is not monotone but pseudo-monotone (for more details see [40]). Table 10 have shown a number of results by letting different starting points. The control parameters conditions are taken as follows:

(i) Algorithm 1 in [16] (**EEGA**): $\chi_0 = 0.18, \mu = 0.55, D_n = \|u_n - v_n\|^2$.
(ii) Algorithm 2 in [16] (**EMEGA**): $\chi_0 = 0.18, \mu = 0.35, D_n = \max\{\|u_{n+1} - v_n\|^2, \|u_n - v_n\|^2\}$.
(iii) Algorithm 1 (**N-EMEGA**): $\chi_0 = 0.18, \mu = 0.55, D_n = \max\{\|u_{n+1} - v_n\|^2, \|u_n - v_n\|^2\}, \varphi_n = \frac{100}{(n+1)^2}$.

Table 10. Numerical results values for Example 5.3.

	Number of Iterations			Execution Time in Seconds		
$u_0 = v_0 = v_{-1}$	**EEGA**	**EMEGA**	**N-EMEGA**	**EEGA**	**EMEGA**	**N-EMEGA**
$(1, 1, \cdots, 1_{5000}, 0, 0, \cdots)$	29	23	11	1.4536274	1.1248362	0.9483582
$(1, 2, \cdots, 5000, 0, 0, \cdots)$	35	28	19	2.1658472	2.1009362	1.2749593
$(5, 5, \cdots, 5_{10000}, 0, 0, \cdots)$	33	24	17	2.0494028	1.8493720	1.0027373

6. Conclusions

The paper has introduced a new modified subgradient extragradient method to approximate the solution of the equilibrium problem in Hilbert spaces. A non-monotonic step size rule has been added that is not dependent on the Lipschitz-type constant information. A weak convergence theorem is well established under mild bi-functional conditions. Applications of the proposed algorithm are presented to solve variational inequalities and fixed-point problems. Many experiments have been reported to demonstrate the numerical behaviour of the proposed algorithm and to compare it to other algorithms.

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Conflict of interest

No potential conflict of interest was reported by the author(s).

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