



Research article

Global well-posedness and large time decay for the d-dimensional tropical climate model

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Abstract: This paper investigates the Cauchy problem on the d-dimensional tropical climate model with fractional hyperviscosity. We establish the small data global well-posedness of solutions to this model with supercritical dissipation. Furthermore, we study the asymptotic stability of these global solutions and obtain the optimal decay rates by using energy method and the method of bootstrapping argument.

Keywords: d-dimensional (dD) tropical climate model; global well-posedness; temporal decay

Mathematics Subject Classification: 35Q35, 35B40, 76D03

1. Introduction

Consider the d-dimensional (dD) tropical climate model with fractional dissipation

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nu \Lambda^{2\alpha} u + \nabla p + \nabla \cdot (v \otimes v) = 0, \\ \partial_t v + u \cdot \nabla v + \mu \Lambda^{2\beta} v + \nabla \theta + v \cdot \nabla u = 0, \\ \partial_t \theta + u \cdot \nabla \theta + \eta \Lambda^{2\gamma} \theta + \nabla \cdot v = 0, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), \theta(x, 0) = \theta_0(x), \end{cases} \quad (1.1)$$

where $(x, t) \in \mathbb{R}^d \times \mathbb{R}^+$ with $d \geq 2$, $u = (u_1(x, t), u_2(x, t), \dots, u_d(x, t))$ is the barotropic mode, $v = (v_1(x, t), v_2(x, t), \dots, v_d(x, t))$ is the first baroclinic mode of vector velocity, $p = p(x, t)$ is the scalar pressure and $\theta = \theta(x, t)$ is the scalar temperature, respectively. $v \otimes v$ denotes the tensor product, namely $v \otimes v = (v_i v_j)$ with $i, j = 1, 2, \dots, d$, the parameters $\nu \geq 0, \mu \geq 0, \eta \geq 0, \alpha > 0, \beta > 0, \gamma > 0$ are real numbers, and $\Lambda = (-\Delta)^{\frac{1}{2}}$ denotes the Zygmund operator. The fractional operator Λ^r is defined via the Fourier transform as

$$\widehat{\Lambda^r f}(\xi) = |\xi|^r \hat{f}(\xi), \quad \xi \in \mathbb{R}^d, \quad r > 0.$$

The inviscid case of system (1.1), namely $\nu = 0$, $\mu = 0$ and $\eta = 0$, was originally derived by Frierson, Majda and Pauluis [7] for large-scale dynamics of precipitation fronts in the tropical atmosphere. The viscous counterpart of system (1.1) with the standard Laplacian can be derived by the same argument from the viscous primitive equations (see, e.g., [12]). The model considered here, namely (1.1), is appended with fractional dissipation terms, which may be relevant in the study of viscous flows in the thinning of atmosphere. Flows in the middle atmosphere traveling upward undergo changes due to the changes of atmospheric properties. The effect of kinematic and thermal diffusion is attenuated by the thinning of atmosphere. This anomalous attenuation can be modeled by using the space fractional Laplacian (see, e.g., [3]).

Considering the 2D tropical climate model (1.1) with fractional dissipation or partial dissipation, the global well-posedness problem has recently attracted considerable attention and significant progress has been made. When there is no thermal diffusion in (1.1), namely $\eta = 0$, Li and Titi in [13] and Dong, Wang, Wu and Zhang in [6] were able to establish the global regularity for the case $\alpha = \beta = 1$ and the case $\alpha + \beta = 2$, respectively. Concerning the case $\nu > 0$, $\mu > 0$ and $\eta > 0$, Ye [21] obtained the global regularity for (1.1) when $\alpha > 0$, $\beta = 1$ and $\gamma = 1$. Recently, the decay estimates were studied by Li and Xiao [11] when $\alpha = \beta = \gamma = 1$. For more results on the 2D tropical climate model, one can refer to [3–5, 14, 15, 22] for more examples.

Concerning to the dD tropical climate model with $d \geq 3$, Ye in [22] proved the global regularity of this model in the case when $\alpha \geq \frac{1}{2} + \frac{d}{4}$, $\alpha + \beta \geq 1 + \frac{d}{2}$ and $\beta \geq 0$. When $\alpha < \frac{1}{2} + \frac{d}{4}$, whether classical solutions to this model, even for the Navier-Stokes equations (namely system (1.1) with $\nu = \theta = 0$), can develop finite time singularities remains outstandingly open.

This paper focuses its attention on the case when $\alpha < \frac{1}{2} + \frac{d}{4}$ with $d \geq 2$. To the best of authors' knowledge, compared with the magnitude of research conducted on the global well-posedness problem of the model (1.1), the large-time behavior of solutions has been studied relatively little. Here we first seek small data global solutions emanating from initial data in almost critical Sobolev space, and then study the temporal decay for these global solutions. More precisely, the first result is the global stability of solutions to (1.1) in $H^s(\mathbb{R}^d)$, which is stated as follows.

Theorem 1.1. *Let $\frac{1}{2} < \alpha, \beta, \gamma < \frac{1}{2} + \frac{d}{4}$ with $d \geq 2$. Assume that $(u_0, v_0, \theta_0) \in H^s(\mathbb{R}^d)$ with $s > 1 + \frac{d}{2} - 2 \min\{\alpha, \beta, \gamma\}$ and $\nabla \cdot u_0 = 0$. Then there exists a positive constant C_0 such that for all $0 < \epsilon < C_0$, if*

$$\|u_0\|_{H^s(\mathbb{R}^d)} + \|v_0\|_{H^s(\mathbb{R}^d)} + \|\theta_0\|_{H^s(\mathbb{R}^d)} < \epsilon, \quad (1.2)$$

then system (1.1) has a unique global solution (u, v, θ) satisfying, for any $T > 0$,

$$(u, v, \theta) \in L^\infty(0, T; H^s(\mathbb{R}^d)), \quad (\Lambda^\alpha u, \Lambda^\beta v, \Lambda^\gamma \theta) \in L^2(0, T; H^s(\mathbb{R}^d)), \quad (1.3)$$

and

$$\|u(t)\|_{H^s(\mathbb{R}^d)} + \|v(t)\|_{H^s(\mathbb{R}^d)} + \|\theta(t)\|_{H^s(\mathbb{R}^d)} < \epsilon. \quad (1.4)$$

Theorem 1.1 shall be proved by using the delicate energy method and fully exploiting the special structure of this model. We remark that, mathematically, system (1.1) is more complex than the magnetohydrodynamic equations ((1.1) with $\theta = 0$ and $\nabla \cdot v = 0$), since it involves the coupling of a divergence-free vector field u and a non-divergence-free vector field v . In particular, the results obtained in this paper also hold for the magnetohydrodynamic equations.

The second result is to explore the long time behavior with explicit decay rates for the global solution itself and its derivative to system (1.1) when the initial data is also in negative Sobolev space $\dot{H}^{-\sigma}(\mathbb{R}^d)$ or negative Besov space $\dot{B}_{2,\infty}^{-\sigma}(\mathbb{R}^d)$, which is stated as in the following theorem.

Theorem 1.2. *Let all the assumptions in Theorem 1.1 hold. Suppose also that $(u_0, v_0, \theta_0) \in \dot{H}^{-\sigma}(\mathbb{R}^d)$ with $0 \leq \sigma < \frac{d}{2}$ or $(u_0, v_0, \theta_0) \in \dot{B}_{2,\infty}^{-\sigma}(\mathbb{R}^d)$ with $0 < \sigma \leq \frac{d}{2}$. Then for $s \geq 1 + \frac{d}{2}$, the global solution (u, v, θ) established in Theorem 1.1 satisfies for all $t > 0$,*

$$\|u(t)\|_{\dot{H}^{-\sigma}(\mathbb{R}^d)} + \|v(t)\|_{\dot{H}^{-\sigma}(\mathbb{R}^d)} + \|\theta(t)\|_{\dot{H}^{-\sigma}(\mathbb{R}^d)} \leq C, \quad (1.5)$$

or

$$\|u(t)\|_{\dot{B}_{2,\infty}^{-\sigma}(\mathbb{R}^d)} + \|v(t)\|_{\dot{B}_{2,\infty}^{-\sigma}(\mathbb{R}^d)} + \|\theta(t)\|_{\dot{B}_{2,\infty}^{-\sigma}(\mathbb{R}^d)} \leq C. \quad (1.6)$$

Moreover, for any real number m with $0 \leq m \leq s$,

$$\|D^m u(t)\|_{L^2(\mathbb{R}^d)} + \|D^m v(t)\|_{L^2(\mathbb{R}^d)} + \|D^m \theta(t)\|_{L^2(\mathbb{R}^d)} \leq C(1+t)^{-\frac{m+\sigma}{2\max\{\alpha,\beta,\gamma\}}}. \quad (1.7)$$

Remark 1.3. *Note that for $\sigma = \frac{d}{p} - \frac{d}{2}$, $L^p(\mathbb{R}^d) \hookrightarrow \dot{H}^{-\sigma}(\mathbb{R}^d)$ when $\sigma \in [0, \frac{d}{2})$ and $p \in (1, 2]$, and $L^p(\mathbb{R}^d) \hookrightarrow \dot{B}_{2,\infty}^{-\sigma}(\mathbb{R}^d)$ when $\sigma \in (0, \frac{d}{2}]$ and $p \in [1, 2)$, thus Theorem 1.2 also holds for $(u_0, v_0, \theta_0) \in L^p(\mathbb{R}^d)$ with $p \in [1, 2]$.*

The proof of Theorem 1.2 is divided into two steps. The first uses energy method to derive the evolution of the negative Sobolev and Besov norms of solutions (u, v, θ) to the system (1.1), and the second establishes the desired results in Theorem 1.2 by the method of bootstrapping argument. We remark that the negative spaces $\dot{H}^{-\sigma}(\mathbb{R}^d)$ and $\dot{B}_{2,\infty}^{-\sigma}(\mathbb{R}^d)$ were introduced to study the decay estimates of the Boltzmann equation by Guo and Wang in [8] and Sohinger and Strain in [18], respectively. The main advantages of these two negative spaces are that the negative Sobolev and Besov norms of solutions are shown to be preserved along time evolution and enhance the decay rates.

The rest of this paper is organized as follows. In Section 2 and Section 3, we give the proofs of Theorem 1.1 and Theorem 1.2, respectively. An appendix containing the Littlewood-Paley decomposition and the definition of Besov spaces is also given for the convenience of the readers. Throughout this manuscript, to simplify the notations, we will write $\int f$ for $\int_{\mathbb{R}^d} f dx$, $\|f\|_{L^p}$ for $\|f\|_{L^p(\mathbb{R}^d)}$, $\|f\|_{\dot{H}^s}$, $\|f\|_{H^s}$ and $\|f\|_{\dot{B}_{2,\infty}^s}$ for $\|f\|_{\dot{H}^s(\mathbb{R}^d)}$, $\|f\|_{H^s(\mathbb{R}^d)}$ and $\|f\|_{\dot{B}_{2,\infty}^s(\mathbb{R}^d)}$ respectively. For simplicity, we set $\nu = 1$, $\mu = 1$ and $\eta = 1$ in the subsequent sections.

2. Proof of the Theorem 1.1

This section is devoted to the proof of Theorem 1.1. For the purpose of proving this theorem, we first present an *a priori* estimate stated in Proposition 2.2 below, which contains a major ingredient in proving this theorem. Then we can prove this theorem by the methods of successive approximations.

As preparations we first give the following calculus inequality involving fractional differential operators (see, e.g., [9, 10]).

Lemma 2.1. *Let $s > 0$. Let $1 < r < \infty$ and $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$ with $q_1, p_2 \in (1, \infty)$ and $p_1, q_2 \in [1, \infty]$. Then*

$$\|\Lambda^s(fg)\|_{L^r} \leq C (\|\Lambda^s f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|\Lambda^s g\|_{L^{q_2}}),$$

where C is a positive constant depending on the indices s, r, p_1, q_1, p_2 and q_2 .

As explained above, we start with an important global *a priori* estimate. More precisely, we have the following proposition.

Proposition 2.2. *Let $\frac{1}{2} < \alpha, \beta, \gamma < \frac{1}{2} + \frac{d}{4}$. Assume that $(u_0, v_0, \theta_0) \in H^s(\mathbb{R}^d)$ with $s > 1 + \frac{d}{2} - 2 \min\{\alpha, \beta, \gamma\}$ and $\nabla \cdot u_0 = 0$. Then any solution (u, v, θ) of the system (1.1) obeys the following differential inequality*

$$\begin{aligned} & \frac{d}{dt} (\|u\|_{H^s}^2 + \|v\|_{H^s}^2 + \|\theta\|_{H^s}^2) + \|\Lambda^\alpha u\|_{H^s}^2 + \|\Lambda^\beta v\|_{H^s}^2 + \|\Lambda^\gamma \theta\|_{H^s}^2 \\ & \leq C (\|u\|_{H^s}^2 + \|v\|_{H^s}^2 + \|\theta\|_{H^s}^2) (\|\Lambda^\alpha u\|_{H^s}^2 + \|\Lambda^\beta v\|_{H^s}^2 + \|\Lambda^\gamma \theta\|_{H^s}^2). \end{aligned} \quad (2.1)$$

Proof. Dotting (1.1)₁, (1.1)₂ and (1.1)₃ by u, v and θ , respectively, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\theta\|_{L^2}^2) + \|\Lambda^\alpha u\|_{L^2}^2 + \|\Lambda^\beta v\|_{L^2}^2 + \|\Lambda^\gamma \theta\|_{L^2}^2 \\ & = - \int \nabla \cdot (v \otimes v) \cdot u - \int \nabla \theta \cdot v - \int v \cdot \nabla u \cdot v - \int \nabla \cdot v \theta \\ & = 0, \end{aligned} \quad (2.2)$$

where we have used the facts that

$$\int \nabla \cdot (v \otimes v) \cdot u + \int v \cdot \nabla u \cdot v = 0,$$

and

$$\int \nabla \theta \cdot v + \int \nabla \cdot v \theta = 0.$$

Applying Λ^s to the first three equations in (1.1), dotting the resulting equations with $\Lambda^s u, \Lambda^s v$ and $\Lambda^s \theta$ respectively, integrating in space domain and adding the results up, one obtains

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s v\|_{L^2}^2 + \|\Lambda^s \theta\|_{L^2}^2) + \|\Lambda^{s+\alpha} u\|_{L^2}^2 + \|\Lambda^{s+\beta} v\|_{L^2}^2 + \|\Lambda^{s+\gamma} \theta\|_{L^2}^2 \\ & = - \int \Lambda^s (u \cdot \nabla u) \cdot \Lambda^s u - \int \Lambda^s \nabla \cdot (v \otimes v) \cdot \Lambda^s u - \int \Lambda^s (u \cdot \nabla v) \cdot \Lambda^s v \\ & \quad - \int \Lambda^s \nabla \theta \cdot \Lambda^s v - \int \Lambda^s (v \cdot \nabla u) \cdot \Lambda^s v - \int \Lambda^s (u \cdot \nabla \theta) \cdot \Lambda^s \theta \\ & \quad - \int \Lambda^s (\nabla \cdot v) \cdot \Lambda^s \theta \\ & = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7. \end{aligned} \quad (2.3)$$

Integration by parts implies

$$I_4 + I_7 = - \int \Lambda^s \nabla \theta \cdot \Lambda^s v - \int \Lambda^s (\nabla \cdot v) \cdot \Lambda^s \theta = 0. \quad (2.4)$$

Applying Hölder's inequality, Lemma 2.1 and Sobolev embedding inequality, we estimate the term I_1 as

$$\begin{aligned} I_1 &= - \int \Lambda^{s-\alpha}(u \cdot \nabla u) \cdot \Lambda^{s+\alpha} u \\ &= - \int \Lambda^{s-\alpha} \nabla \cdot (u \otimes u) \cdot \Lambda^{s+\alpha} u \\ &\leq C \|\Lambda^{s+1-\alpha}(u \otimes u)\|_{L^2} \|\Lambda^{s+\alpha} u\|_{L^2} \\ &\leq C \|u\|_{L^{\frac{d}{2\alpha-1}}} \|\Lambda^{s+1-\alpha} u\|_{L^{\frac{2d}{d+2-4\alpha}}} \|\Lambda^{s+\alpha} u\|_{L^2} \\ &\leq C \|\Lambda^{1+\frac{d}{2}-2\alpha} u\|_{L^2} \|\Lambda^{s+\alpha} u\|_{L^2}^2. \end{aligned}$$

Similarly, we have

$$\begin{aligned} I_2 &\leq C \|\Lambda^{s+1-\alpha}(v \otimes v)\|_{L^2} \|\Lambda^{s+\alpha} u\|_{L^2} \\ &\leq C \|v\|_{L^{\frac{d}{\alpha+\beta-1}}} \|\Lambda^{s+1-\alpha} v\|_{L^{\frac{2d}{d+2-2(\alpha+\beta)}}} \|\Lambda^{s+\alpha} u\|_{L^2} \\ &\leq C \|\Lambda^{1+\frac{d}{2}-(\alpha+\beta)} v\|_{L^2} \|\Lambda^{s+\beta} v\|_{L^2} \|\Lambda^{s+\alpha} u\|_{L^2}. \end{aligned}$$

$$\begin{aligned} I_3 &\leq \|\Lambda^{s+1-\beta}(u \otimes v)\|_{L^2} \|\Lambda^{s+\beta} v\|_{L^2} \\ &\leq C (\|\Lambda^{s+1-\beta} u\|_{L^{\frac{2d}{d+2-2(\alpha+\beta)}}} \|v\|_{L^{\frac{d}{\alpha+\beta-1}}} + \|u\|_{L^{\frac{d}{2\beta-1}}} \|\Lambda^{s+1-\beta} v\|_{L^{\frac{2d}{d+2-4\beta}}}) \|\Lambda^{s+\beta} v\|_{L^2} \\ &\leq C (\|\Lambda^{1+\frac{d}{2}-(\alpha+\beta)} v\|_{L^2} \|\Lambda^{s+\alpha} u\|_{L^2} \|\Lambda^{s+\beta} v\|_{L^2} + \|\Lambda^{1+\frac{d}{2}-2\beta} u\|_{L^2} \|\Lambda^{s+\beta} v\|_{L^2}^2). \end{aligned}$$

$$\begin{aligned} I_6 &\leq \|\Lambda^{s+1-\gamma}(u \otimes \theta)\|_{L^2} \|\Lambda^{s+\gamma} \theta\|_{L^2} \\ &\leq C (\|\Lambda^{s+1-\gamma} u\|_{L^{\frac{2d}{d+2-2(\alpha+\gamma)}}} \|\theta\|_{L^{\frac{d}{\alpha+\gamma-1}}} + \|u\|_{L^{\frac{d}{2\gamma-1}}} \|\Lambda^{s+1-\gamma} \theta\|_{L^{\frac{2d}{d+2-4\gamma}}}) \|\Lambda^{s+\gamma} \theta\|_{L^2} \\ &\leq C (\|\Lambda^{1+\frac{d}{2}-(\alpha+\gamma)} \theta\|_{L^2} \|\Lambda^{s+\alpha} u\|_{L^2} \|\Lambda^{s+\gamma} \theta\|_{L^2} + \|\Lambda^{1+\frac{d}{2}-2\gamma} u\|_{L^2} \|\Lambda^{s+\gamma} \theta\|_{L^2}^2). \end{aligned}$$

We cannot bound I_5 as above, since v is not divergence free. Using Hölder's inequality and Lemma 2.1, we derive that

$$\begin{aligned} I_5 &= - \int \Lambda^{s-\beta}(v \cdot \nabla u) \cdot \Lambda^{s+\beta} v \\ &\leq \|\Lambda^{s-\beta}(v \cdot \nabla u)\|_{L^2} \|\Lambda^{s+\beta} v\|_{L^2} \\ &\leq C (\|\Lambda^{s-\beta} v\|_{L^{\frac{2d}{d-2\beta}}} \|\nabla u\|_{L^{\frac{d}{\beta}}} + \|v\|_{L^{\frac{d}{\alpha+\beta-1}}} \|\Lambda^{s-\beta+1} u\|_{L^{\frac{2d}{d+2-2(\alpha+\beta)}}}) \|\Lambda^{s+\beta} v\|_{L^2} \\ &\leq C (\|\Lambda^s v\|_{L^2} \|\Lambda^{1+\frac{d}{2}-\beta} u\|_{L^2} \|\Lambda^{s+\beta} v\|_{L^2} + \|\Lambda^{1+\frac{d}{2}-(\alpha+\beta)} v\|_{L^2} \|\Lambda^{s+\alpha} u\|_{L^2} \|\Lambda^{s+\beta} v\|_{L^2}). \end{aligned}$$

Combining these bounds and (2.4) with (2.3) together, we get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} (\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s v\|_{L^2}^2 + \|\Lambda^s \theta\|_{L^2}^2) + \|\Lambda^{s+\alpha} u\|_{L^2}^2 + \|\Lambda^{s+\beta} v\|_{L^2}^2 + \|\Lambda^{s+\gamma} \theta\|_{L^2}^2 \\ &\leq C (\|u\|_{H^s} \|\Lambda^\alpha u\|_{H^s}^2 + \|v\|_{H^s} \|\Lambda^\alpha u\|_{H^s} \|\Lambda^\beta v\|_{H^s} + \|u\|_{H^s} \|\Lambda^\beta v\|_{H^s}^2 \\ &\quad + \|\theta\|_{H^s} \|\Lambda^\alpha u\|_{H^s} \|\Lambda^\gamma \theta\|_{H^s} + \|u\|_{H^s} \|\Lambda^\gamma \theta\|_{H^s}^2). \end{aligned} \tag{2.5}$$

Adding (2.2) and (2.5) up, then the Young inequality implies the desired inequality (2.1). Thus the proof of Proposition 2.2 is completed. \square

With Proposition 2.2 at our disposal, we are ready to prove Theorem 1.1.

Proof of the Theorem 1.1. We apply the method of successive approximation. It consists of constructing a successive approximation sequence (u^n, v^n, θ^n) with $n \geq 0$ and showing its convergence to the solution (u, v, θ) of the system (1.1).

Consider successive approximation sequences (u^n, v^n, θ^n) satisfying

$$\begin{cases} u^0 = 0, v^0 = 0, \theta^0 = 0, \\ \partial_t u^{n+1} + u^n \cdot \nabla u^{n+1} + \Lambda^{2\alpha} u^{n+1} + \nabla p^{n+1} + \nabla \cdot v^{n+1} v^n + v^n \cdot \nabla v^{n+1} = 0, \\ \partial_t v^{n+1} + u^n \cdot \nabla v^{n+1} + \Lambda^{2\beta} v^{n+1} + \nabla \theta^{n+1} + v^n \cdot \nabla u^{n+1} = 0, \\ \partial_t \theta^{n+1} + u^n \cdot \nabla \theta^{n+1} + \Lambda^{2\gamma} \theta^{n+1} + \nabla \cdot v^{n+1} = 0, \\ \nabla \cdot u^{n+1} = 0, \\ u^{n+1}(x, 0) = u_0(x), v^{n+1}(x, 0) = v_0(x), \theta^{n+1}(x, 0) = \theta_0(x). \end{cases} \quad (2.6)$$

To show that (u^n, v^n, θ^n) converges, we first prove that there exists a constant $\epsilon > 0$ independent of n , such that for any $T > 0$,

$$\begin{aligned} & \|u^n(t)\|_{H^s}^2 + \|v^n(t)\|_{H^s}^2 + \|\theta^n(t)\|_{H^s}^2 \\ & + \frac{1}{2} \int_0^t (\|\Lambda^\alpha u^n(\tau)\|_{H^s}^2 + \|\Lambda^\beta v^n(\tau)\|_{H^s}^2 + \|\Lambda^\gamma \theta^n(\tau)\|_{H^s}^2) d\tau \\ & \leq \epsilon^2, \end{aligned} \quad (2.7)$$

for all $0 < t \leq T$.

We will prove (2.7) by mathematical induction. Obviously, (2.7) holds for $n = 0$. Assume that (2.7) is true for $n \geq 0$. We start to show it for $n + 1$. We proceed as in the proof of Proposition 2.2. Actually, after going through the steps as in proof of Proposition 2.2, we arrive at

$$\begin{aligned} & \frac{d}{dt} (\|u^{n+1}\|_{H^s}^2 + \|v^{n+1}\|_{H^s}^2 + \|\theta^{n+1}\|_{H^s}^2) + \|\Lambda^\alpha u^{n+1}\|_{H^s}^2 + \|\Lambda^\beta v^{n+1}\|_{H^s}^2 + \|\Lambda^\gamma \theta^{n+1}\|_{H^s}^2 \\ & \leq C (\|u^n\|_{H^s}^2 + \|v^n\|_{H^s}^2 + \|\theta^n\|_{H^s}^2) (\|\Lambda^\alpha u^{n+1}\|_{H^s}^2 + \|\Lambda^\beta v^{n+1}\|_{H^s}^2 + \|\Lambda^\gamma \theta^{n+1}\|_{H^s}^2). \end{aligned} \quad (2.8)$$

Integrating this in $[0, t]$, together with (1.2) and inductive assumption, we derive that

$$\begin{aligned} & \|u^{n+1}(t)\|_{H^s}^2 + \|v^{n+1}(t)\|_{H^s}^2 + \|\theta^{n+1}(t)\|_{H^s}^2 \\ & + \int_0^t (\|\Lambda^\alpha u^{n+1}\|_{H^s}^2 + \|\Lambda^\beta v^{n+1}\|_{H^s}^2 + \|\Lambda^\gamma \theta^{n+1}\|_{H^s}^2)(\tau) d\tau \\ & \leq \|u_0\|_{H^s}^2 + \|v_0\|_{H^s}^2 + \|\theta_0\|_{H^s}^2 \\ & + C \int_0^t (\|u^n\|_{H^s}^2 + \|v^n\|_{H^s}^2 + \|\theta^n\|_{H^s}^2) (\|\Lambda^\alpha u^{n+1}\|_{H^s}^2 + \|\Lambda^\beta v^{n+1}\|_{H^s}^2 + \|\Lambda^\gamma \theta^{n+1}\|_{H^s}^2)(\tau) d\tau \\ & \leq \epsilon^2 + C\epsilon^2 \int_0^t (\|\Lambda^\alpha u^{n+1}\|_{H^s}^2 + \|\Lambda^\beta v^{n+1}\|_{H^s}^2 + \|\Lambda^\gamma \theta^{n+1}\|_{H^s}^2)(\tau) d\tau. \end{aligned}$$

This implies (2.7) holds for $n + 1$ by choosing ϵ sufficiently small such that $\epsilon \leq \frac{1}{\sqrt{2C}}$. Thus (2.7) is true for all $n \geq 0$.

Next we show that (u^n, v^n, θ^n) is a Cauchy sequence in $C([0, T]; H^s)$. Resorting to (2.8) and (2.7), it infers that for all $0 \leq t_1 \leq t_2 \leq T$,

$$\begin{aligned} & |(\|u^n(t_2)\|_{H^s}^2 + \|v^n(t_2)\|_{H^s}^2 + \|\theta^n(t_2)\|_{H^s}^2) - (\|u^n(t_1)\|_{H^s}^2 + \|v^n(t_1)\|_{H^s}^2 + \|\theta^n(t_1)\|_{H^s}^2)| \\ &= \left| \int_{t_1}^{t_2} \frac{d}{d\tau} (\|u^n(\tau)\|_{H^s}^2 + \|v^n(\tau)\|_{H^s}^2 + \|\theta^n(\tau)\|_{H^s}^2) d\tau \right| \\ &\leq C\epsilon^2 \int_{t_1}^{t_2} (\|\Lambda^\alpha u^n(\tau)\|_{H^s}^2 + \|\Lambda^\beta v^n(\tau)\|_{H^s}^2 + \|\Lambda^\gamma \theta^n(\tau)\|_{H^s}^2) d\tau, \end{aligned}$$

which implies that (u^n, v^n, θ^n) is absolutely continuous from $[0, T]$ to H^s or simply $(u^n, v^n, \theta^n) \in C([0, T]; H^s)$.

To prove that (u^n, v^n, θ^n) is a Cauchy sequence, we consider the differences

$$u^{(n+1)} = u^{n+1} - u^n, \quad v^{(n+1)} = v^{n+1} - v^n, \quad \theta^{(n+1)} = \theta^{n+1} - \theta^n, \quad p^{(n+1)} = p^{n+1} - p^n,$$

which satisfy

$$\begin{cases} \partial_t u^{(n+1)} + u^n \cdot \nabla u^{(n+1)} + u^{(n)} \cdot \nabla u^n + \Lambda^{2\alpha} u^{(n+1)} + \nabla p^{(n+1)} \\ \quad + \nabla \cdot v^{(n+1)} v^n + \nabla \cdot v^n v^{(n)} + v^n \cdot \nabla v^{(n+1)} + v^{(n)} \cdot \nabla v^n = 0, \\ \partial_t v^{(n+1)} + u^n \cdot \nabla v^{(n+1)} + u^{(n)} \cdot \nabla v^n + \Lambda^{2\beta} v^{(n+1)} + \nabla \theta^{(n+1)} \\ \quad + v^n \cdot \nabla u^{(n+1)} + v^{(n)} \cdot \nabla u^n = 0, \\ \partial_t \theta^{(n+1)} + u^n \cdot \nabla \theta^{(n+1)} + u^{(n)} \cdot \nabla \theta^n + \Lambda^{2\gamma} \theta^{(n+1)} + \nabla \cdot v^{(n+1)} = 0, \\ \nabla \cdot u^{(n+1)} = 0 \\ u^{(n+1)}(x, 0) = 0, v^{(n+1)}(x, 0) = 0, \theta^{(n+1)}(x, 0) = 0. \end{cases} \quad (2.9)$$

After going through a similar procedure as above, we obtain

$$\begin{aligned} & \frac{d}{dt} (\|u^{(n+1)}\|_{H^s}^2 + \|v^{(n+1)}\|_{H^s}^2 + \|\theta^{(n+1)}\|_{H^s}^2) + \|\Lambda^\alpha u^{(n+1)}\|_{H^s}^2 + \|\Lambda^\beta v^{(n+1)}\|_{H^s}^2 + \|\Lambda^\gamma \theta^{(n+1)}\|_{H^s}^2 \\ & \leq C(\|u^{(n)}\|_{H^s}^2 + \|v^{(n)}\|_{H^s}^2 + \|\theta^{(n)}\|_{H^s}^2)(\|\Lambda^\alpha u^n\|_{H^s}^2 + \|\Lambda^\beta v^n\|_{H^s}^2 + \|\Lambda^\gamma \theta^n\|_{H^s}^2) \\ & \quad + C(\|u^n\|_{H^s}^2 + \|v^n\|_{H^s}^2 + \|\theta^n\|_{H^s}^2)(\|\Lambda^\alpha u^{(n+1)}\|_{H^s}^2 + \|\Lambda^\beta v^{(n+1)}\|_{H^s}^2 + \|\Lambda^\gamma \theta^{(n+1)}\|_{H^s}^2). \end{aligned} \quad (2.10)$$

Integrating this inequality with respect to time, together with (2.7), one infers that for all $0 \leq t \leq T$,

$$\begin{aligned} & \|u^{(n+1)}(t)\|_{H^s}^2 + \|v^{(n+1)}(t)\|_{H^s}^2 + \|\theta^{(n+1)}(t)\|_{H^s}^2 \\ & \quad + \int_0^t (\|\Lambda^\alpha u^{(n+1)}\|_{H^s}^2 + \|\Lambda^\beta v^{(n+1)}\|_{H^s}^2 + \|\Lambda^\gamma \theta^{(n+1)}\|_{H^s}^2)(\tau) d\tau \\ & \leq C\epsilon^2 \sup_{0 \leq \tau \leq t} (\|u^{(n)}(\tau)\|_{H^s}^2 + \|v^{(n)}(\tau)\|_{H^s}^2 + \|\theta^{(n)}(\tau)\|_{H^s}^2) \\ & \quad + C\epsilon^2 \int_0^t (\|\Lambda^\alpha u^{(n+1)}\|_{H^s}^2 + \|\Lambda^\beta v^{(n+1)}\|_{H^s}^2 + \|\Lambda^\gamma \theta^{(n+1)}\|_{H^s}^2)(\tau) d\tau. \end{aligned} \quad (2.11)$$

By choosing $\epsilon > 0$ as above, it follows from (2.11) that

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|u^{(n+1)}(t)\|_{H^s}^2 + \|v^{(n+1)}(t)\|_{H^s}^2 + \|\theta^{(n+1)}(t)\|_{H^s}^2) \\ & \leq \frac{1}{2} \sup_{0 \leq t \leq T} (\|u^{(n)}(t)\|_{H^s}^2 + \|v^{(n)}(t)\|_{H^s}^2 + \|\theta^{(n)}(t)\|_{H^s}^2), \end{aligned} \quad (2.12)$$

which implies that (u^n, v^n, θ^n) is a Cauchy sequence in $C([0, T]; H^s)$. Therefore, the limit function (u, v, θ) satisfying system (1.1) indeed exists in $C([0, T]; H^s)$. Moreover, it obeys

$$\begin{aligned} & \|u(t)\|_{H^s}^2 + \|v(t)\|_{H^s}^2 + \|\theta(t)\|_{H^s}^2 \\ & + \frac{1}{2} \int_0^t (\|\Lambda^\alpha u(\tau)\|_{H^s}^2 + \|\Lambda^\beta v(\tau)\|_{H^s}^2 + \|\Lambda^\gamma \theta(\tau)\|_{H^s}^2) d\tau \\ & \leq \epsilon^2, \end{aligned} \quad (2.13)$$

for all $0 < t < T$.

Finally, we prove the uniqueness. Let (u, v, θ) and $(\tilde{u}, \tilde{v}, \tilde{\theta})$ be two solutions of system (1.1) in the regularity class (2.13). Similar process as the proof of convergence above, we derive that their difference $(\bar{u}, \bar{v}, \bar{\theta})$ with

$$\bar{u} = u - \tilde{u}, \bar{v} = v - \tilde{v}, \bar{\theta} = \theta - \tilde{\theta}$$

satisfies

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|\bar{u}(t)\|_{H^s}^2 + \|\bar{v}(t)\|_{H^s}^2 + \|\bar{\theta}(t)\|_{H^s}^2) \\ & \leq \frac{1}{2} \sup_{0 \leq t \leq T} (\|\bar{u}(t)\|_{H^s}^2 + \|\bar{v}(t)\|_{H^s}^2 + \|\bar{\theta}(t)\|_{H^s}^2). \end{aligned} \quad (2.14)$$

This inequality implies $(\bar{u}, \bar{v}, \bar{\theta}) = 0$ or $(u, v, \theta) = (\tilde{u}, \tilde{v}, \tilde{\theta})$ for all $0 \leq t \leq T$. Thus we complete the proof of Theorem 1.1. \square

3. Proof of the Theorem 1.2

This section proves Theorem 1.2. To this end, we first establish the global *a priori* estimates for the global solution (u, v, θ) of system (1.1) in the negative Sobolev norm $\dot{H}^{-\sigma}$ with $0 \leq \sigma < \frac{d}{2}$ and negative Besov norm $\dot{B}_{2,\infty}^{-\sigma}$ with $0 < \sigma \leq \frac{d}{2}$, respectively. Then we will establish Theorem 1.2 by the method of bootstrapping argument.

As preparations we recall the Hardy-Littlewood-Sobolev inequality for fractional integration and an inequality for homogeneous Besov norm (see [19] and [18] respectively).

Lemma 3.1. *Let $0 \leq \sigma < \frac{d}{2}$ and $1 < p \leq 2$ with $\frac{1}{2} + \frac{\sigma}{d} = \frac{1}{p}$. Then*

$$\|\Lambda^{-\sigma} f\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}. \quad (3.1)$$

Lemma 3.2. *Let $0 < \sigma \leq \frac{d}{2}$ and $1 \leq p < 2$ with $\frac{1}{2} + \frac{\sigma}{d} = \frac{1}{p}$. Then*

$$\|f\|_{\dot{B}_{2,\infty}^{-\sigma}(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}. \quad (3.2)$$

Now we show the global *a priori* estimates for the global solution (u, v, θ) established in Theorem 1.1 in $\dot{H}^{-\sigma}$ with $0 \leq \sigma < \frac{d}{2}$. More precisely, we have the following lemma.

Lemma 3.3. *Let the assumptions stated in Theorem 1.2 hold. Then for $s > \frac{d}{2}$, (u, v, θ) obeys*

$$\begin{aligned} & \frac{d}{dt} (\|u\|_{\dot{H}^{-\sigma}}^2 + \|v\|_{\dot{H}^{-\sigma}}^2 + \|\theta\|_{\dot{H}^{-\sigma}}^2) \\ & \leq C (\|u\|_{L^2}^{\frac{4s+2\sigma-d-2}{2s}} + \|v\|_{L^2}^{\frac{4s+2\sigma-d-2}{2s}} + \|\theta\|_{L^2}^{\frac{4s+2\sigma-d-2}{2s}}) (\|u\|_{\dot{H}^s}^{\frac{d+2-2\sigma}{2s}} + \|v\|_{\dot{H}^s}^{\frac{d+2-2\sigma}{2s}} + \|\theta\|_{\dot{H}^s}^{\frac{d+2-2\sigma}{2s}}) \\ & \quad \times (\|u\|_{\dot{H}^{-\sigma}} + \|v\|_{\dot{H}^{-\sigma}} + \|\theta\|_{\dot{H}^{-\sigma}}). \end{aligned} \quad (3.3)$$

Proof. Applying $\Lambda^{-\sigma}$ to (1.1)₁ – (1.1)₃, and taking the L^2 -inner products with $\Lambda^{-\sigma}u$, $\Lambda^{-\sigma}v$ and $\Lambda^{-\sigma}\theta$ respectively, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\Lambda^{-\sigma}u\|_{L^2}^2 + \|\Lambda^{-\sigma}v\|_{L^2}^2 + \|\Lambda^{-\sigma}\theta\|_{L^2}^2) + (\|\Lambda^{\alpha-\sigma}u\|_{L^2}^2 + \|\Lambda^{\beta-\sigma}v\|_{L^2}^2 + \|\Lambda^{\gamma-\sigma}\theta\|_{L^2}^2) \\ & = - \int \Lambda^{-\sigma}(u \cdot \nabla u) \cdot \Lambda^{-\sigma}u - \int \Lambda^{-\sigma}\nabla \cdot (v \otimes v) \cdot \Lambda^{-\sigma}u - \int \Lambda^{-\sigma}(u \cdot \nabla v) \cdot \Lambda^{-\sigma}v \\ & \quad - \int \Lambda^{-\sigma}(v \cdot \nabla u) \cdot \Lambda^{-\sigma}v - \int \Lambda^{-\sigma}(u \cdot \nabla \theta) \cdot \Lambda^{-\sigma}\theta \\ & := K_1 + K_2 + K_3 + K_4 + K_5, \end{aligned} \quad (3.4)$$

where we have used the fact

$$\int \Lambda^{-\sigma}\nabla\theta \cdot \Lambda^{-\sigma}v + \int \Lambda^{-\sigma}(\nabla \cdot v) \cdot \Lambda^{-\sigma}\theta = 0.$$

Using Hölder's inequality, Lemma 3.1 and the Gagliardo-Nirenberg inequality, we derive that

$$\begin{aligned} K_1 & = - \int \Lambda^{-\sigma}(u \cdot \nabla u) \cdot \Lambda^{-\sigma}u \\ & \leq \|\Lambda^{-\sigma}(u \cdot \nabla u)\|_{L^2} \|\Lambda^{-\sigma}u\|_{L^2} \\ & \leq C \|u \cdot \nabla u\|_{L^{\frac{2d}{d+2\sigma}}} \|\Lambda^{-\sigma}u\|_{L^2} \\ & \leq C \|u\|_{L^{\frac{d}{\sigma}}} \|\nabla u\|_{L^2} \|\Lambda^{-\sigma}u\|_{L^2} \\ & \leq C \|u\|_{L^2}^{\frac{4s+2\sigma-d-2}{2s}} \|\Lambda^s u\|_{L^2}^{\frac{d+2-2\sigma}{2s}} \|\Lambda^{-\sigma}u\|_{L^2}. \end{aligned}$$

Similarly, we have

$$K_2 \leq C \|v\|_{L^2}^{\frac{4s+2\sigma-d-2}{2s}} \|\Lambda^s v\|_{L^2}^{\frac{d+2-2\sigma}{2s}} \|\Lambda^{-\sigma}u\|_{L^2}.$$

Again applying Hölder's inequality, Lemma 3.1 and the Gagliardo-Nirenberg inequality, we derive that

$$\begin{aligned} K_3 & = - \int \Lambda^{-\sigma}(u \cdot \nabla v) \cdot \Lambda^{-\sigma}v \\ & \leq \|\Lambda^{-\sigma}(u \cdot \nabla v)\|_{L^2} \|\Lambda^{-\sigma}v\|_{L^2} \\ & \leq C \|u \cdot \nabla v\|_{L^{\frac{2d}{d+2\sigma}}} \|\Lambda^{-\sigma}v\|_{L^2} \\ & \leq C \|u\|_{L^{\frac{d}{\sigma}}} \|\nabla v\|_{L^2} \|\Lambda^{-\sigma}v\|_{L^2} \\ & \leq C \|u\|_{L^2}^{\frac{2s+2\sigma-d}{2s}} \|\Lambda^s u\|_{L^2}^{\frac{d-2\sigma}{2s}} \|v\|_{L^2}^{\frac{s-1}{s}} \|\Lambda^s v\|_{L^2}^{\frac{1}{s}} \|\Lambda^{-\sigma}v\|_{L^2}. \end{aligned}$$

Similarly, we obtain

$$K_4 \leq C \|v\|_{L^2}^{\frac{2s+2\sigma-d}{2s}} \|\Lambda^s v\|_{L^2}^{\frac{d-2\sigma}{2s}} \|u\|_{L^2}^{\frac{s-1}{s}} \|\Lambda^s u\|_{L^2}^{\frac{1}{s}} \|\Lambda^{-\sigma} v\|_{L^2}.$$

$$K_5 \leq C \|u\|_{L^2}^{\frac{2s+2\sigma-d}{2s}} \|\Lambda^s u\|_{L^2}^{\frac{d-2\sigma}{2s}} \|\theta\|_{L^2}^{\frac{s-1}{s}} \|\Lambda^s \theta\|_{L^2}^{\frac{1}{s}} \|\Lambda^{-\sigma} \theta\|_{L^2}.$$

Inserting the above bounds into (3.4), together with the Young inequality, leads to (3.3). Thus the proof of Lemma 3.3 is completed. \square

Next we establish the global *a priori* estimates for the global solution (u, v, θ) established in Theorem 1.1 in $\dot{B}_{2,\infty}^{-\sigma}$ with $0 < \sigma \leq \frac{d}{2}$, as stated in the following lemma.

Lemma 3.4. *Let the assumptions stated in Theorem 1.2 hold. Then for $s > \frac{d}{2}$, (u, v, θ) obeys*

$$\begin{aligned} & \frac{d}{dt} (\|u\|_{\dot{B}_{2,\infty}^{-\sigma}}^2 + \|v\|_{\dot{B}_{2,\infty}^{-\sigma}}^2 + \|\theta\|_{\dot{B}_{2,\infty}^{-\sigma}}^2) \\ & \leq C (\|u\|_{L^2}^{\frac{4s+2\sigma-d-2}{2s}} + \|v\|_{L^2}^{\frac{4s+2\sigma-d-2}{2s}} + \|\theta\|_{L^2}^{\frac{4s+2\sigma-d-2}{2s}}) (\|u\|_{\dot{H}^s}^{\frac{d+2-2\sigma}{2s}} + \|v\|_{\dot{H}^s}^{\frac{d+2-2\sigma}{2s}} + \|\theta\|_{\dot{H}^s}^{\frac{d+2-2\sigma}{2s}}) \\ & \quad \times (\|u\|_{\dot{B}_{2,\infty}^{-\sigma}} + \|v\|_{\dot{B}_{2,\infty}^{-\sigma}} + \|\theta\|_{\dot{B}_{2,\infty}^{-\sigma}}). \end{aligned} \quad (3.5)$$

Proof. We remark that the argument is similar to the proof of Lemma 3.3, here we give the details for reader's convenience. Applying $\dot{\Delta}_j$, which definition is in the appendix, to (1.1)₁ – (1.1)₃, taking the L^2 -inner products with $\dot{\Delta}_j u$, $\dot{\Delta}_j v$ and $\dot{\Delta}_j \theta$ respectively, multiplying the results by $2^{-2\sigma j}$, and taking the supremum over $j \in \mathbb{Z}$, we conclude that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|u\|_{\dot{B}_{2,\infty}^{-\sigma}}^2 + \|v\|_{\dot{B}_{2,\infty}^{-\sigma}}^2 + \|\theta\|_{\dot{B}_{2,\infty}^{-\sigma}}^2) \\ & \leq \sup_{j \in \mathbb{Z}} 2^{-2\sigma j} \left| \int \dot{\Delta}_j (u \cdot \nabla u) \cdot \dot{\Delta}_j u \right| + \sup_{j \in \mathbb{Z}} 2^{-2\sigma j} \left| \int \dot{\Delta}_j \nabla \cdot (v \otimes v) \cdot \dot{\Delta}_j u \right| \\ & \quad + \sup_{j \in \mathbb{Z}} 2^{-2\sigma j} \left| \int \dot{\Delta}_j (u \cdot \nabla v) \cdot \dot{\Delta}_j v \right| + \sup_{j \in \mathbb{Z}} 2^{-2\sigma j} \left| \int \dot{\Delta}_j (v \cdot \nabla u) \cdot \dot{\Delta}_j v \right| \\ & \quad + \sup_{j \in \mathbb{Z}} 2^{-2\sigma j} \left| \int \dot{\Delta}_j (u \cdot \nabla \theta) \cdot \dot{\Delta}_j \theta \right| \\ & := M_1 + M_2 + M_3 + M_4 + M_5, \end{aligned} \quad (3.6)$$

where we used the fact that

$$\int \dot{\Delta}_j \nabla \theta \cdot \dot{\Delta}_j v + \int \dot{\Delta}_j (\nabla \cdot v) \cdot \dot{\Delta}_j \theta = 0.$$

Applying Hölder's inequality, Lemma 3.2 and the Gagliardo-Nirenberg inequality, one infers that

$$\begin{aligned} M_1 & \leq \|u \cdot \nabla u\|_{\dot{B}_{2,\infty}^{-\sigma}} \|u\|_{\dot{B}_{2,\infty}^{-\sigma}} \\ & \leq \|u \cdot \nabla u\|_{L^{\frac{2d}{d+2\sigma}}} \|u\|_{\dot{B}_{2,\infty}^{-\sigma}} \\ & \leq C \|u\|_{L^{\frac{d}{\sigma}}} \|\nabla u\|_{L^2} \|u\|_{\dot{B}_{2,\infty}^{-\sigma}} \\ & \leq C \|u\|_{L^2}^{\frac{4s+2\sigma-d-2}{2s}} \|\Lambda^s u\|_{L^2}^{\frac{d+2-2\sigma}{2s}} \|u\|_{\dot{B}_{2,\infty}^{-\sigma}}. \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 M_2 &\leq C \|v\|_{L^2}^{\frac{4s+2\sigma-d-2}{2s}} \|\Lambda^s v\|_{L^2}^{\frac{d+2-2\sigma}{2s}} \|u\|_{\dot{B}_{2,\infty}^{-\sigma}}. \\
 M_3 &\leq C \|u\|_{L^2}^{\frac{2s+2\sigma-d}{2s}} \|\Lambda^s u\|_{L^2}^{\frac{d-2\sigma}{2s}} \|v\|_{L^2}^{\frac{s-1}{s}} \|\Lambda^s v\|_{L^2}^{\frac{1}{s}} \|v\|_{\dot{B}_{2,\infty}^{-\sigma}}. \\
 M_4 &\leq C \|v\|_{L^2}^{\frac{2s+2\sigma-d}{2s}} \|\Lambda^s v\|_{L^2}^{\frac{d-2\sigma}{2s}} \|u\|_{L^2}^{\frac{s-1}{s}} \|\Lambda^s u\|_{L^2}^{\frac{1}{s}} \|v\|_{\dot{B}_{2,\infty}^{-\sigma}}. \\
 M_5 &\leq C \|u\|_{L^2}^{\frac{2s+2\sigma-d}{2s}} \|\Lambda^s u\|_{L^2}^{\frac{d-2\sigma}{2s}} \|\theta\|_{L^2}^{\frac{s-1}{s}} \|\Lambda^s \theta\|_{L^2}^{\frac{1}{s}} \|\theta\|_{\dot{B}_{2,\infty}^{-\sigma}}.
 \end{aligned}$$

Then (3.5) eventually follows from the above bounds, (3.6) and the Young inequality. This completes the proof of Lemma 3.4. \square

With Lemma 3.3 and Lemma 3.4 at our disposal, we are ready to prove Theorem 1.2 by the method of bootstrapping argument.

Proof of the Theorem 1.2. We will just focus on the case $(u_0, v_0, \theta_0) \in \dot{H}^{-\sigma}$. The case $(u_0, v_0, \theta_0) \in \dot{B}_{2,\infty}^{-\sigma}$ can be treated similarly. Assume that

$$\|u_0\|_{\dot{H}^{-\sigma}}^2 + \|v_0\|_{\dot{H}^{-\sigma}}^2 + \|\theta_0\|_{\dot{H}^{-\sigma}}^2 = C_0. \quad (3.7)$$

Suppose that for all $t \in [0, T]$,

$$\|u(t)\|_{\dot{H}^{-\sigma}}^2 + \|v(t)\|_{\dot{H}^{-\sigma}}^2 + \|\theta(t)\|_{\dot{H}^{-\sigma}}^2 \leq 2C_0. \quad (3.8)$$

If we can derive that for all $t \in [0, T]$,

$$\|u(t)\|_{\dot{H}^{-\sigma}}^2 + \|v(t)\|_{\dot{H}^{-\sigma}}^2 + \|\theta(t)\|_{\dot{H}^{-\sigma}}^2 \leq \frac{3C_0}{2}, \quad (3.9)$$

then an application of the bootstrapping argument would imply that the solution (u, v, θ) of system (1.1) satisfies (3.9) for all $t \in [0, T]$, which implies (1.5).

With (3.7) and (3.8) at our disposal, we shall show that (3.9) holds. At the same time, the decay estimates (1.5) will be established in this process. Similar as the proof of (2.5), one can show that for $0 \leq m \leq s$,

$$\begin{aligned}
 &\frac{1}{2} \frac{d}{dt} (\|\Lambda^m u\|_{L^2}^2 + \|\Lambda^m v\|_{L^2}^2 + \|\Lambda^m \theta\|_{L^2}^2) + \|\Lambda^{m+\alpha} u\|_{L^2}^2 + \|\Lambda^{m+\beta} v\|_{L^2}^2 + \|\Lambda^{m+\gamma} \theta\|_{L^2}^2 \\
 &\leq C (\|u\|_{H^s} \|\Lambda^\alpha u\|_{H^m}^2 + \|v\|_{H^s} \|\Lambda^\alpha u\|_{H^m} \|\Lambda^\beta v\|_{H^m} + \|u\|_{H^s} \|\Lambda^\beta v\|_{H^m}^2 \\
 &\quad + \|\theta\|_{H^s} \|\Lambda^\alpha u\|_{H^m} \|\Lambda^\gamma \theta\|_{H^m} + \|u\|_{H^s} \|\Lambda^\gamma \theta\|_{H^m}^2).
 \end{aligned} \quad (3.10)$$

Then this inequality together with the Young inequality implies

$$\begin{aligned}
 &\frac{d}{dt} (\|\Lambda^m u\|_{L^2}^2 + \|\Lambda^m v\|_{L^2}^2 + \|\Lambda^m \theta\|_{L^2}^2) + \|\Lambda^{m+\alpha} u\|_{L^2}^2 + \|\Lambda^{m+\beta} v\|_{L^2}^2 + \|\Lambda^{m+\gamma} \theta\|_{L^2}^2 \\
 &\leq C (\|u\|_{H^s}^2 + \|v\|_{H^s}^2 + \|\theta\|_{H^s}^2) (\|\Lambda^{m+\alpha} u\|_{L^2}^2 + \|\Lambda^{m+\beta} v\|_{L^2}^2 + \|\Lambda^{m+\gamma} \theta\|_{L^2}^2).
 \end{aligned} \quad (3.11)$$

Using (1.4) with $\epsilon < \frac{1}{\sqrt{2C}}$, it follows from (3.11) that

$$\frac{d}{dt}(\|\Lambda^m u\|_{L^2}^2 + \|\Lambda^m v\|_{L^2}^2 + \|\Lambda^m \theta\|_{L^2}^2) + \frac{1}{2}(\|\Lambda^{m+\alpha} u\|_{L^2}^2 + \|\Lambda^{m+\beta} v\|_{L^2}^2 + \|\Lambda^{m+\gamma} \theta\|_{L^2}^2) \leq 0. \quad (3.12)$$

Applying the Gagliardo-Nirenberg inequality, together with (3.8), we obtain

$$\begin{aligned} \|\Lambda^m u\|_{L^2} &\leq C \|u\|_{\dot{H}^{-\sigma}}^{\frac{\alpha}{m+\alpha+\sigma}} \|\Lambda^{m+\alpha} u\|_{L^2}^{\frac{m+\sigma}{m+\alpha+\sigma}} \\ &\leq C \|\Lambda^{m+\alpha} u\|_{L^2}^{\frac{m+\sigma}{m+\alpha+\sigma}}. \end{aligned} \quad (3.13)$$

Similarly, we have

$$\|\Lambda^m v\|_{L^2} \leq C \|\Lambda^{m+\beta} v\|_{L^2}^{\frac{m+\sigma}{m+\beta+\sigma}}. \quad (3.14)$$

$$\|\Lambda^m \theta\|_{L^2} \leq C \|\Lambda^{m+\gamma} \theta\|_{L^2}^{\frac{m+\sigma}{m+\gamma+\sigma}}. \quad (3.15)$$

Inserting (3.13)–(3.15) into (3.12), there exists a positive constant $C_1 > 0$ such that

$$\frac{d}{dt}(\|\Lambda^m u\|_{L^2}^2 + \|\Lambda^m v\|_{L^2}^2 + \|\Lambda^m \theta\|_{L^2}^2) + C_1(\|\Lambda^m u\|_{L^2}^2 + \|\Lambda^m v\|_{L^2}^2 + \|\Lambda^m \theta\|_{L^2}^2)^{\frac{m+\alpha+\sigma}{m+\sigma}} \leq 0$$

with $a_0 = \max\{\alpha, \beta, \gamma\}$. Integrating this inequality with respect to time, we derive that

$$\|\Lambda^m u\|_{L^2}^2 + \|\Lambda^m v\|_{L^2}^2 + \|\Lambda^m \theta\|_{L^2}^2 \leq C(1+t)^{-\frac{m+\sigma}{a_0}}, \quad (3.16)$$

which implies (1.7).

Now we start to show (3.9). Integrating (3.3) in $[0, t]$ with $0 < t \leq T$, together with (3.16), (3.7) and (1.4), one infers that

$$\begin{aligned} &\|u(t)\|_{\dot{H}^{-\sigma}}^2 + \|v(t)\|_{\dot{H}^{-\sigma}}^2 + \|\theta(t)\|_{\dot{H}^{-\sigma}}^2 \\ &\leq \|u_0\|_{\dot{H}^{-\sigma}}^2 + \|v_0\|_{\dot{H}^{-\sigma}}^2 + \|\theta_0\|_{\dot{H}^{-\sigma}}^2 + C \sup_{0 \leq \tau \leq t} (\|u(\tau)\|_{\dot{H}^{-\sigma}} + \|v(\tau)\|_{\dot{H}^{-\sigma}} + \|\theta(\tau)\|_{\dot{H}^{-\sigma}}) \\ &\quad \times \int_0^t (\|u(\tau)\|_{L^2}^{\frac{4s+2\sigma-d-2}{2s}} + \|v(\tau)\|_{L^2}^{\frac{4s+2\sigma-d-2}{2s}} + \|\theta(\tau)\|_{L^2}^{\frac{4s+2\sigma-d-2}{2s}}) \\ &\quad \times (\|u(\tau)\|_{\dot{H}^s}^{\frac{d+2-2\sigma}{2s}} + \|v(\tau)\|_{\dot{H}^s}^{\frac{d+2-2\sigma}{2s}} + \|\theta(\tau)\|_{\dot{H}^s}^{\frac{d+2-2\sigma}{2s}}) d\tau \\ &\leq C_0 + C \sup_{0 \leq \tau \leq t} (\|u(\tau)\|_{\dot{H}^{-\sigma}} + \|v(\tau)\|_{\dot{H}^{-\sigma}} + \|\theta(\tau)\|_{\dot{H}^{-\sigma}}) \\ &\quad \times \epsilon^{\epsilon_0} \int_0^t (1+\tau)^{-\left(\frac{\sigma}{2a_0} \left(\frac{4s+2\sigma-d-2}{2s} - \epsilon_0\right) + \frac{(s+\sigma)(d+2-2\sigma)}{4a_0 s}\right)} d\tau \\ &\leq C_0 + C \epsilon^{\epsilon_0} \sup_{0 \leq \tau \leq t} (\|u(\tau)\|_{\dot{H}^{-\sigma}} + \|v(\tau)\|_{\dot{H}^{-\sigma}} + \|\theta(\tau)\|_{\dot{H}^{-\sigma}}), \end{aligned} \quad (3.17)$$

where $\epsilon_0 > 0$ is chosen small enough such that $\frac{\sigma}{2a_0} \left(\frac{4s+2\sigma-d-2}{2s} - \epsilon_0\right) + \frac{(s+\sigma)(d+2-2\sigma)}{4a_0 s} > 1$, which is meaningful since assumptions $\frac{1}{2} < a_0 < \frac{d+2}{4}$ and $0 \leq \sigma < \frac{d}{2}$ implies that $\frac{\sigma(4s+2\sigma-d-2) + (s+\sigma)(d+2-2\sigma)}{4a_0 s} > 1$. By choosing ϵ sufficiently small, then (3.17) together with the Young inequality yields (3.9) for all $t \in [0, T]$, which closes the proof. Thus we complete the proof of Theorem 1.2. \square

4. Appendix

This appendix provides the definition of the Littlewood-Paley decomposition and the definition of Besov spaces. Some related facts used in the previous sections are also included. Materials presented in this appendix can be found in several books and many papers (see, e.g., [1, 2, 16, 17, 20]).

We start with several notation. \mathcal{S} denotes the usual Schwarz class and \mathcal{S}' its dual, the space of tempered distributions. To introduce the Littlewood-Paley decomposition, we write for each $j \in \mathbb{Z}$

$$A_j = \{\xi \in \mathbb{R}^d : 2^{j-1} \leq |\xi| < 2^{j+1}\}.$$

The Littlewood-Paley decomposition asserts the existence of a sequence of functions $\{\Phi_j\}_{j \in \mathbb{Z}} \in \mathcal{S}$ such that

$$\text{supp} \widehat{\Phi}_j \subset A_j, \quad \widehat{\Phi}_j(\xi) = \widehat{\Phi}_0(2^{-j}\xi) \quad \text{or} \quad \Phi_j(x) = 2^{jd} \Phi_0(2^j x),$$

and

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi) = \begin{cases} 1, & \text{if } \xi \in \mathbb{R}^d \setminus \{0\}, \\ 0, & \text{if } \xi = 0. \end{cases}$$

Therefore, for a general function $\psi \in \mathcal{S}$, we have

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi) \widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for } \xi \in \mathbb{R}^d \setminus \{0\}.$$

We now choose $\Psi \in \mathcal{S}$ such that

$$\widehat{\Psi}(\xi) = 1 - \sum_{j=0}^{\infty} \widehat{\Phi}_j(\xi), \quad \xi \in \mathbb{R}^d.$$

Then, for any $\psi \in \mathcal{S}$,

$$\Psi * \psi + \sum_{j=0}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\Psi * f + \sum_{j=0}^{\infty} \Phi_j * f = f \tag{4.1}$$

in \mathcal{S}' for any $f \in \mathcal{S}'$. To define the inhomogeneous Besov space, we set

$$\Delta_j f = \begin{cases} 0, & \text{if } j \leq -2, \\ \Psi * f, & \text{if } j = -1, \\ \Phi_j * f, & \text{if } j = 0, 1, 2, \dots \end{cases} \tag{4.2}$$

To define the homogeneous Besov space, we set

$$\dot{\Delta}_j f = \Phi_j * f, \quad \text{if } j = 0, \pm 1, \pm 2, \dots \tag{4.3}$$

Definition 4.1. The inhomogeneous and homogeneous Besov spaces $B_{p,q}^s$ and $\dot{B}_{p,q}^s$ with $s \in \mathbb{R}$ and $p, q \in [1, \infty]$ consists of $f \in \mathcal{S}'$ satisfying

$$\|f\|_{B_{p,q}^s} \equiv \|2^{js} \|\Delta_j f\|_{L^p} \|l_j^q\| < \infty,$$

and

$$\|f\|_{\dot{B}_{p,q}^s} \equiv \|2^{js} \|\dot{\Delta}_j f\|_{L^p} \|l_j^q\| < \infty,$$

respectively.

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Conflict of interest

The authors declare that they have no conflict of interest.

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