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Research article

New escape conditions with general complex polynomial for fractals via new fixed point iteration

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Abstract: The aim of this paper is to generalize the results regarding fractals and prove escape conditions for general complex polynomial. In this paper we state the orbit of a newly defined iterative scheme and establish the escape criteria in fractal generation for general complex polynomial. We use established escape criteria in algorithms to generate Mandelbrot and Multi-corns sets. In addition, we present some graphs of quadratic, cubic and higher Mandelbrot and Multi-corns sets and discuss how the alteration in parameters make changes in graphs.

Keywords: Mandelbrot set; Multi-corns set; general polynomial; fractal; fixed point
Mathematics Subject Classification: 47H10, 54H25

1. Introduction

Although fractal geometry is closely connected with computer techniques, some people had worked on fractals long before the invention of computers. Those people were British cartographers, who encountered the problem in measuring the length of Britain coast. The coastline measured on a large scale map was approximately half the length of coastline measured on a detailed map. The closer they looked, the more detailed and longer the coastline became. They did not realize that they had discovered one of the main properties of fractals. In 1919, Gaston Julia worked on it and explained how to iterate this function. The resulting sequence of iterates are now called as Julia set and the boundary of this set had infinite length and it was impracticable to sketch by hand in those days. The generation of Mandelbrot set in 1985 with the help of computer made it exceptionally attractive amongst the computer experts of that era. The emerging attraction established under the humongous complexity, intricate aesthetic geometry and fascinating patterns [1]. Moreover, the most phenomenal work was the extension of the quadratic function to higher exponents: $z \longrightarrow z^n + c$ [2]. In 1989 Crowe et al. [3] initially demonstrated the connected locus and graphics of conjugate complex polynomial $\bar{z}^2 + c$. Its visualized graphs were called "tricorn" [4]. Later on the use of fixed point theory in the generation of fractals generalized the Mandelbrot and Julia sets (see in [5, 6]). Now the fixed point theory is a key to fascinating the image encryption or compression and cryptography [7]. Due to diversity and self-similarity in graphics, fractals have been used in art, design and engineering. The use of fractal theory in electrical and electronics engineering enhance the way to develop security control system, radar system, capacitors, radio and antennae for wireless system [8, 9]. Furthermore, architectonic sketches and ornament patterns are also a part of fractals [10].

Some generalizations for Julia and Mandelbrot sets by rational and transcendental complex functions were discussed in [11]. Fractals with quaternions, vectors (i.e. 4D and 3D fractals), bi-complex and tri-complex functions were studied in ([12–14]. Some generalized fractals via one, two and three steps feedback iterations elaborated in [15–20]. The implicit iterative methods were used to develop convergence criterion for fractals in [21–26].

Over the last forty years complex fractals have been studied in the form of Mandelbrot and Julia sets for a complex polynomial $p(z) = z^n + c$ by using different fixed point algorithms. Some researchers extended the established results for the complex polynomial $p(z) = z^n + a_1 z + a_0$ where a_1 and $a_0 \in \mathbb{C}$. They discussed that images change with variation in input parameters (i.e. Variations in α, β and γ). Here, we apply a new iteration (i.e. MM-iteration) proposed in [27] to establish the escape criteria in fractal generation for general complex polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ where all $a_i \in \mathbb{C}$ for $i = 0, 1, 2, \ldots, n$. We use proposed escape criteria in algorithms to generate Mandelbrot and Multi-corns sets. In addition, we present some graphs of quadratic, cubic and higher Mandelbrot and Multi-corns sets and compare their images. We discuss how the alteration in parameters a_i make changes in graphs.

The composition of this paper is as follows: The section 2 presents preliminaries. We establish escape criteria for complex polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ via MM-iteration and generalize algorithms in section 3. We discuss the behavior of Mandelbrot set for different a_i via proposed method in section 4. At the end we add some concluding remarks in section 5.

2. Preliminaries

In this section we present some basic definitions and results.

Definition 1 (Julia set [28])**.** Let $p(z_k) = a_n z_k^n + a_{n-1} z_k^{n-1} + \cdots + a_1 z_k + a_0$ be a complex polynomial with $n \geq 2$. Then the set of points $J_{p_{a_0}}$ in $\mathbb{C}$ is named as filled Julia set. The orbits of points in $J_{p_{a_0}}$ does not moves to ∞ as $k \to \infty$, i.e.,

$$J_{p_{a_0}} = \{z \in \mathbb{C} : \{\left|p_{a_0}^k\right|\}_{k=0}^{\infty} \text{ is bounded}\}, \tag{2.1}$$

where $p_{a_0}^k$ is the $k - th$ iterate of z. The set of boundary points of $J_{p_{a_0}}$ is called simple Julia set.

Definition 2 (Mandelbrot set [29])**.** Let $p(z_k) = a_n z_k^n + a_{n-1} z_k^{n-1} + \cdots + a_1 z_k + a_0$ be a complex polynomial with $n \geq 2$. Then the multitude of all connected Julia sets is called as Mandelbrot set M, i.e.,

$$M = \{a_0 \in \mathbb{C} : J_{p_{a_0}} \text{ is connected}\}, \tag{2.2}$$

equivalently Mandelbrot set is defined as [30]:

$$M = \{a_0 \in \mathbb{C} : \{p^k_{a_0}(\theta)\} \nrightarrow \infty \text{ as } k \to \infty\}, \tag{2.3}$$

where θ is any critical point of p_{a_0}, so mostly authors used $z_0 = \theta$ as an initial guess.

Definition 3 (Multi-corns set [3])**.** Let $p(z_k) = a_n\bar{z}_k{}^n + a_{n-1}\bar{z}_k{}^{n-1} + \cdots + a_1\bar{z}_k + a_0$ be a conjugate complex polynomial with $n \geq 2$. Then the Multi-corns set M' is defined as multitude of all connected Julia sets is called as Multi-corns or Mandelbar set M, i.e.,

$$M' = \{a_0 \in \mathbb{C} : \{p^k_{a_0}(\theta)\} \nrightarrow \infty \text{ as } k \to \infty\}, \tag{2.4}$$

where $p^k_{a_0}(\theta)$ is the $k-th$ iterate of $p^k_{a_0}(\bar{z})$.

As mentioned in introduction section that many fixed point iterative methods have been used for generation of fractal images in past years. Her we aim to use a new iteration named as MM-iteration to visualize Mandelbrot and Multi-corns sets. The MM-iteration is defined as follows:

Definition 4 (MM-iteration [27])**.** Let $p : \mathbb{C} \to \mathbb{C}$ be a complex polynomial of with $p \geq 2$. For any $z_0 \in \mathbb{C}$, the MM-iteration is defined as:

$$\begin{cases} z_{k+1} = p(v_k), \\ v_k = (1 - \alpha_2 - \alpha_3)z_k + \alpha_2 p(z_k) + \alpha_3 p(u_k), \\ u_k = (1 - \alpha_1)z_k + \alpha_1 p(z_k), \end{cases} \tag{2.5}$$

where $\alpha_1, \alpha_2, \alpha_3 \in (0, 1]$ and $k = 0, 1, 2, \ldots$.

Remark 1. The MM-iteration reduces to Picard-Ishikawa iteration when $\alpha_2 = 0$.

Definition 5 (MM-orbit)**.** Let $p(z_k) = a_n z_k^n + a_{n-1}z_k^{n-1} + \cdots + a_1 z_k + a_0$ be a complex polynomial with $n \geq 2$. Then the sequence of iterates $\{z_k\}_{k\in\mathbb{W}}$ from 2.5 is called MM-orbit if its points does not moves to ∞ as $k \to \infty$.

3. Escape conditions

Here we prove some escape criterion for complex polynomial $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ where $n \geq 2$ and all $a_i \in \mathbb{C}$ for $i = 0, 1, 2, \ldots, n$ via MM-iteration. Without escape criterion, we cannot generate fractal because escape criterion is the basic key to run the algorithms. Throughout this section we use $p(z)$ as $p_{a_0}(z)$, $z_0 = z$, $u_0 = u$ and $v_0 = v$.

Theorem 3.1. *Assume that $p_{a_0}(z) = \sum_{i=0}^{n} a_i z^i$ where $n \geq 2$, $a_i \in \mathbb{C}$ for $i = 0, 1, 2, \ldots, n$ be a complex polynomial with $|z| \geq |a_0| > \left(\frac{2(1+|a_1|)}{\alpha_1(\alpha-\beta)}\right)^{\frac{1}{n-1}}$, $|z| \geq |a_0| > \left(\frac{2(1+|a_1|)}{\alpha_1(\alpha-\beta)}\right)^{\frac{1}{n-1}}$ and $|z| \geq |a_0| > \left(\frac{2(1+|a_1|)}{\alpha_1(\alpha-\beta)}\right)^{\frac{1}{n-1}}$ where $n \geq 2, \alpha_1, \alpha_2, \alpha_3 \in (0, 1]$ and $a_0 \in \mathbb{C}$. The sequence of iterates $\{z_k\}_{k\in\mathbb{W}}$ for MM-iteration is define as follows:*

$$\begin{cases} z_{k+1} = p(v_k), \\ v_k = (1 - \alpha_2 - \alpha_3)z_k + \alpha_2 p(z_k) + \alpha_3 p(u_k), \\ u_k = (1 - \alpha_1)z_k + \alpha_1 p(z_k), \end{cases} \tag{3.1}$$

where $\alpha_1, \alpha_2, \alpha_3 \in (0, 1]$ and $k = 0, 1, 2, \ldots$. Then $|z_k| \longrightarrow \infty$ as $k \longrightarrow \infty$.

Proof. Since $p(z) = \sum_{i=0}^{n} a_i z^i$, where $a_i \in \mathbb{C}$ for $i = 0, 1, 2, \ldots, n$, $z_0 = x$, $v_0 = v$ and $u_0 = u$, then

$$|u_k| = |(1-\alpha_1)z_k + \alpha_1 p(z_k)|\,.$$

For $k = 0$, we have

$$\begin{aligned}
|u_0| &= |(1-\alpha_1)z_0 + \alpha_1 p(z_0)|\,. \\
&= \left|(1-\alpha_1)z + \alpha_1\left(\sum_{i=1}^{n} a_i z^i + a_0\right)\right| \\
&\geq \alpha_1\left|\sum_{i=1}^{n} a_i z^i\right| - \alpha_1|a_1 z| - \alpha_1|a_0| - |z| + \alpha_1|z| \\
&\geq \alpha_1\left|\sum_{i=1}^{n} a_i z^i\right| - (1 + |a_1|)\,|z|.
\end{aligned}$$

Because $|z| \geq |a_0|$ and $\alpha_1 < 1$.

$$|u_0| = (1 + |a_1|)\,|z|\left(\alpha_1\left|\sum_{i=2}^{n} a_i z^{i-1}\right| - 1\right).$$

Since $(1 + |a_1|) > 1$, then

$$\begin{aligned}
|u_0| &\geq |z|\left(\frac{\alpha_1\left|\sum_{i=2}^{n} a_i z^{i-1}\right|}{1 + |a_1|} - 1\right) \\
&\geq |z|\left(\frac{\alpha_1\left|z^{p-1}\right|\left(|a_n| - \sum_{i=2}^{n-1}|a_i|\right)}{1 + |a_1|} - 1\right) \\
&= |z|\left(\frac{\alpha_1\left|z^{n-1}\right|(\beta - \gamma)}{1 + |a_1|} - 1\right) \\
|u_0| &\geq \alpha_1\,|z|\,.
\end{aligned}$$

Because $|z| \geq |a_0| > \left(\frac{2(1+|a_1|)}{\alpha_1(\beta-\gamma)}\right)^{\frac{1}{n-1}}$ where $\beta = |a_n|\,, \gamma = \sum\limits_{2}^{n-1}|a_i|$, this produced the situation $|z|\left(\frac{|z|^{n-1}(\alpha_1(\beta-\gamma))}{1+|a_1|} - 1\right) > |z| \geq \alpha_1|z|$.
In second step of MM-iteration we have

$$|v_k| = |(1-\alpha_2-\alpha_3)z_k + \alpha_2 p(z_k) + \alpha_3 p(u_k)|\,, \tag{3.2}$$

For $k = 0$ we get

$$\begin{aligned}
|v_0| &= |(1-\alpha_2-\alpha_3)z_0 + \alpha_2 p(z_0) + \alpha_3 p(u_0)| \\
&\geq \left|\alpha_3\left(\sum_{i=1}^{n} a_i u^i + a_0\right) + \alpha_2\left(\sum_{i=1}^{n} a_i z^i + a_0\right)\right| \\
&- (1-\alpha_2-\alpha_3)\,|z| \\
&\geq \left|\alpha_2 \sum_{i=1}^{n} a_i z^i + \alpha_3 \sum_{i=1}^{n} a_i u^i\right| - \alpha_2\,|a_0| - \alpha_3\,|a_0| \\
&- (1-\alpha_2-\alpha_3)\,|z| \\
&\geq \alpha_3\alpha_1 \left|\sum_{i=1}^{n} a_i z^i\right| - |z|
\end{aligned}$$

Because $|u_0| \geq \alpha_1|z|$ implies $|u^n| \geq \alpha_1|z^n|$ and $|z| \geq a_0 > 2$. Since $\alpha_1\alpha_3 > \alpha_1\alpha_2\alpha_3$, then

$$\begin{aligned}
|v_0| &\geq \alpha_1\alpha_2\alpha_3 \left|\sum_{i=1}^{n} a_i z^i\right| - |z| \\
&\geq \alpha_1\alpha_2\alpha_3 \left|\sum_{i=2}^{n} a_i z^i\right| - (1-|a_1|)\,|z| \\
&\geq |z|\left(\frac{\alpha_1\alpha_2\alpha_3\left|\sum_{i=2}^{n} a_i z^{i-1}\right|}{1+|a_1|} - 1\right) \\
&\geq |z|\left(\frac{\alpha_1\alpha_2\alpha_3\left|z^{n-1}\right|\left(|a_n| - \sum_{i=2}^{n-1}|a_i|\right)}{1+|a_1|} - 1\right) \\
&= |z|\left(\frac{\alpha_1\alpha_2\alpha_3\left|z^{n-1}\right|(\beta-\gamma)}{1+|a_1|} - 1\right) \\
|v_0| &\geq \alpha_1\alpha_2\alpha_3\,|z|\,.
\end{aligned}$$

The last step of MM-iteration is

$$z_{k+1} = p(v_k)$$

For $k = 0$ we have

$$\begin{aligned}
|z_1| &= |p(v_0)| \\
&= |\left(\sum_{i=1}^{n} a_i v^i + a_0\right)| \\
&\geq |\sum_{i=1}^{n} a_i v^i| - |a_0| \\
z_1 &\geq \alpha_1\alpha_2\alpha_3 \left|\sum_{i=1}^{n} a_i z^i\right| - |z|.
\end{aligned}$$

Therefore

$$|z_1| \geq |z| \left(\frac{\alpha_1\alpha_2\alpha_3 \left| z^{n-1} \right| (\alpha - \beta)}{1 + |a_1|} - 1 \right). \tag{3.3}$$

Since $|z| > \left(\frac{2(1+|a_1|)}{\alpha_1(\alpha-\beta)}\right)^{\frac{1}{n-1}}$, $|z| > \left(\frac{2(1+|a_1|)}{\alpha_2(\alpha-\beta)}\right)^{\frac{1}{n-1}}$ and $|z| > \left(\frac{2(1+|a_1|)}{\alpha_3(\alpha-\beta)}\right)^{\frac{1}{n-1}}$, then $|z|^{n-1} > \left(\frac{2(1+|a_1|)}{\alpha_1\alpha_2\alpha_3(\alpha-\beta)}\right)$ and this implies $\frac{\alpha_1\alpha_2\alpha_3(\alpha-\beta)|z|^{n-1}}{1+|a_1|} - 1 > 1$. Therefore there exists $\eta > 0$ such that $\frac{\alpha_1\alpha_2\alpha_3(\alpha-\beta)|z|^{n-1}}{1+|a_1|} - 1 > 1 + \eta$. Consequently

$$|z_1| > (1 + \eta)|z|. \tag{3.4}$$

Iterate upto k^{th} term

$$\begin{aligned} |z_2| &> (1 + \eta)^2|z| \\ |z_3| &> (1 + \eta)^3|z| \\ &\vdots \\ |z_k| &> (1 + \eta)^k|z|. \end{aligned}$$

Hence $|z_k| \to \infty$ as $k \to \infty$.

□

Corollary 3.2. *Assume that*

$$|z_m| > \max\{|a_0|, \zeta_1, \zeta_1, \zeta_1\},$$

for some $m \geq 0$. *Where* $\zeta_1 = \left(\frac{2(1+|a_1|)}{\alpha_1(\alpha-\beta)}\right)^{\frac{1}{n-1}}$, $\zeta_2 = \left(\frac{2(1+|a_1|)}{\alpha_2(\alpha-\beta)}\right)^{\frac{1}{n-1}}$ *and* $\zeta_3 = \left(\frac{2(1+|a_1|)}{\alpha_3(\alpha-\beta)}\right)^{\frac{1}{n-1}}$. *Thus there exists* $\eta > 0$ *such that* $|z_{m+k}| > (1 + \eta)^k|z_k|$ *and* $|z_k| \to \infty$ *as* $k \to \infty$.

4. Mandelbrot and Multi-corns sets

There are some well-known criterion to generate the fractals, for example distance estimator [31], potential function algorithms [32] and escape criteria [33]. In this paper we use escape criterion to sketch some fascinating Mandelbrot and Multi-corns sets. In literature authors fixed maximum number of iteration upto hundred and checked the alteration of images with input parameters (i.e. like $\alpha_1, \alpha_2, \alpha_3$). In this paper we fixed some parameters as follows:

- $K = 100$, (Number of iterations)
- $\alpha_1 = 0.9, \alpha_2 = 0.05$ and $\alpha_3 = 0.005$.

In this section we present two algorithms, one for Mandelbrot set and other for Multi-corns set with proposed iteration. We visualize some Mandelbrot and Multi-corns sets for different values of a_i for $i = 0, 1, 2, .., n$. Since $|z| = |\bar{z}|$, therefore to generate Multi-corns set we just replace $|z|$ with $|\bar{z}|$.

Example 4.1. In this example we present Mandelbrot and Multi-corns sets for a polynomial $p(z) = a_2z^2 + a_1z + a_0$ where $a_2, a_1, a_0 \in \mathbb{C}$ via proposed orbit. The graphs in Figures 1–6 are quadratic Mandelbrot sets and graphs in Figures 7–11 are tri-corns sets (special case of Multi-corns sets). The a_i and area for these images were as follows:

- Figure 1: $a_2 = 1 + \mathbf{i}$, $a_1 = \sin(c)$,$A = [-1.4, 1.4] \times [-1, 1]$,
- Figures 2 and 7: $a_2 = 1 + \mathbf{i}$, $a_1 = 0.5$, $A = [-1, 1]^2$,
- Figures 3 and 8: $a_2 = 1 - \mathbf{i}$, $a_1 = 0.5$, $A = [-1, 1.5]^2$,
- Figures 4 and 9: $a_2 = -1 - \mathbf{i}$, $a_1 = 0.5$, $A = [-1, 1.5] \times [-1.5, 1]$,
- Figures 5 and 10: $a_2 = -1 + \mathbf{i}$, $a_1 = 0.5$, $A = [-1.5, 1] \times [-1.5, 1]$,
- Figures 6 and 11: $a_2 = 1 + \sin(c)$, $a_1 = 1$, $A = [-1.5, 1] \times [-1, 1.5]$.

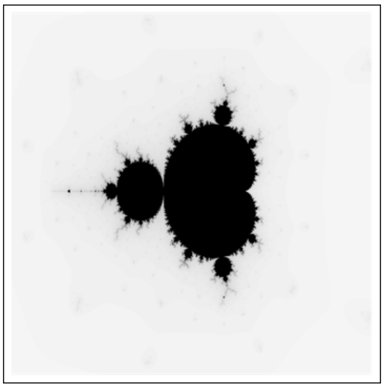

Figure 1. Classical Mandelbrot for $p(z)$ with $n = 2$ in MM-Orbit, where $a_2 = 1+\mathbf{i}$, $a_1 = \sin(c)$ and $A = [-1.4, 1.4] \times [-1, 1]$.

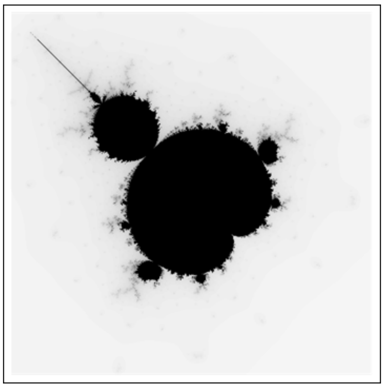

Figure 2. Mandelbrot for $p(z)$ with $n = 2$ in MM-Orbit, where $a_2 = 1 + \mathbf{i}$, $a_1 = 0.5$ and $A = [-1, 1]^2$.

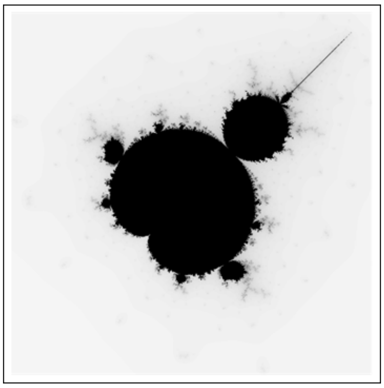

Figure 3. Mandelbrot for $p(z)$ with $n = 2$ in MM-Orbit, where $a_2 = 1 - \mathbf{i}$, $a_1 = 0.5$ and $A = [-1, 1.5]^2$.

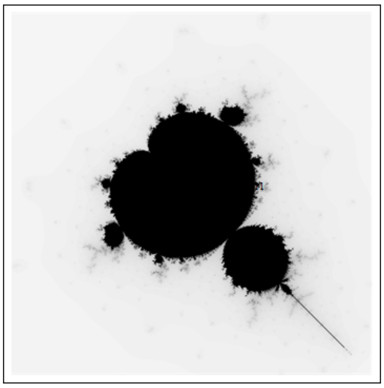

Figure 4. Mandelbrot for $p(z)$ with $n = 2$ in MM-Orbit, where $a_2 = -1 - \mathbf{i}$, $a_1 = 0.5$ and $A = [-1, 1.5] \times [-1.5, 1]$.

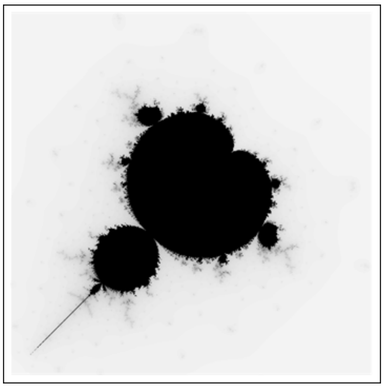

Figure 5. Mandelbrot for $p(z)$ with $n = 2$ in MM-Orbit, where $a_2 = -1 + \mathbf{i}$, $a_1 = 0.5$ and $A = [-1.5, 1] \times [-1.5, 1]$.

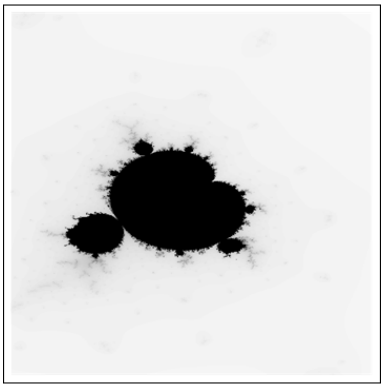

Figure 6. Mandelbrot for $p(z)$ with $n = 2$ in MM-Orbit, where $a_2 = 1 + \sin(c)$, $a_1 = 1$ and $A = [-1.5, 1] \times [-1, 1.5]$.

Figure 7. Tri-corns set for $p(z)$ with $n = 2$ in MM-Orbit, where $a_2 = 1 + \mathbf{i}$, $a_1 = 0.5$ and $A = [-1, 1]^2$.

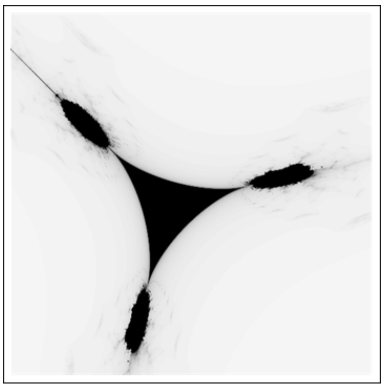

Figure 8. Tri-corns set for $p(z)$ with $n = 2$ in MM-Orbit, where $a_2 = 1 - \mathbf{i}$, $a_1 = 0.5$ and $A = [-1, 1.5]^2$.

Figure 9. Tri-corns set for $p(z)$ with $n = 2$ in MM-Orbit, where $a_2 = -1 - \mathbf{i}$, $a_1 = 0.5$ and $A = [-1, 1.5] \times [-1.5, 1]$.

Figure 10. Tri-corns set for $p(z)$ with $n = 2$ in MM-Orbit, where $a_2 = -1 + \mathbf{i}$, $a_1 = 0.5$ and $A = [-1.5, 1] \times [-1.5, 1]$.

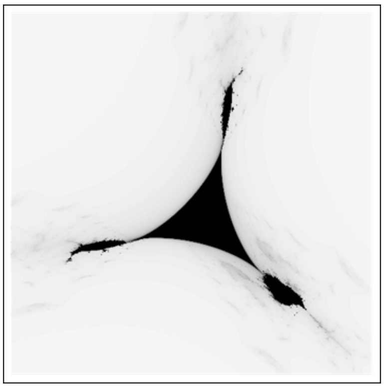

Figure 11. Tri-corns set for $p(z)$ with $n = 2$ in MM-Orbit, where $a_2 = 1 + \sin(c)$, $a_1 = 1$ and $A = [-1.5, 1] \times [-1, 1.5]$.

We observe that the image in 1 is a classical Mandelbrot set with one large bulb symmetry along horizontal axis while the images in 2–5 are also classical Mandelbrot sets but due to alteration in a_2 images rotate clock wise with angle 90^0. Since in classical Mandelbrot set small bulbs symmetry along vertical axis but the image in 6 presents an interesting view like a classical Mandelbrot set is laying on surface. The images in Figures 7–10 are tri-corns sets with three corns, one main large corn and two small corns are twisted. We see the main corn in Figures 7–10 also have symmetrical rotation of 90^0. In Figure 11 the secondary corns rotate clockwise.

Example 4.2. In this example we present the Mandelbrot and Multi-corns sets for a polynomial $p(z) = a_3z^3 + a_2z^2 + a_1z + a_0$ where $a_3, \ldots, a_0 \in \mathbb{C}$ in proposed orbit. We notice that the image in Figure 12 is symmetrical along y-axis and is resemble with classical cubic Mandelbrot set. When we change a_3 the image rotate left and right of vertical axis with angle $\frac{\Pi}{4}$ (see in Figures 13 and 14) and the large bulbs also twisted. In Figures 15 and 16 images also change their positions with changes in a_3's. The image in Figures 17–21 are Multi-corns sets at different a_3. The image in Figure 17 is classical Multi-corns with $n + 1$ corns. We see a little bit in lower corn of image in Figure 18 and in upper corn of image in Figure 19. The images in Figures 20 and 21 have a sharp edge and a disjoint corn each.

- Figures 12 and 17 $a_3 = 2$, $a_2 = 2$, $a_1 = 1$, $a_1 = \frac{1}{3}$, $A = [-1.5, 1.5]^2$,
- Figures 13 and 18 $a_3 = 1 + \mathbf{i}$, $a_2 = 2$, $a_1 = 1$, $a_1 = \frac{1}{3}$, $A = [-1.5, 1.5]^2$,
- Figures 14 and 19 $a_3 = 1 - \mathbf{i}$, $a_2 = 2$, $a_1 = 1$, $a_1 = \frac{1}{3}$, $A = [-1.5, 1.5]^2$,
- Figures 15 and 20 $a_3 = -1 + \mathbf{i}$, $a_2 = 2$, $a_1 = 1$, $a_1 = \frac{1}{3}$, $A = [-1.5, 1.5]^2$,
- Figures 16 and 21 $a_3 = -1 - \mathbf{i}$, $a_2 = 2$, $a_1 = 1$, $a_1 = \frac{1}{3}$, $A = [-1.5, 1.5]^2$.

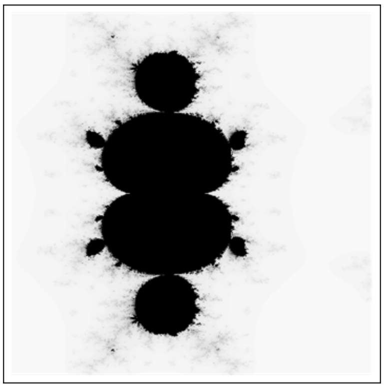

Figure 12. Mandelbrot set for $p(z)$ with $n = 3$ in MM-Orbit, where $a_3 = 2$, $a_2 = 2$, $a_1 = 1$, $a_1 = \frac{1}{3}$ and $A = [-1.5, 1.5]^2$.

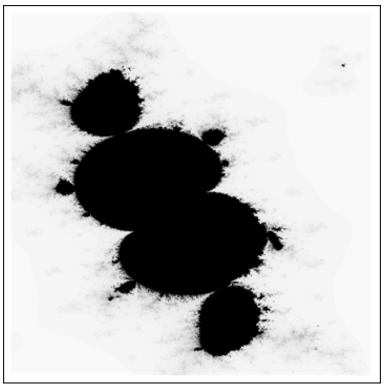

Figure 13. Mandelbrot set for $p(z)$ with $n = 3$ in MM-Orbit, where $a_3 = 1 + \mathbf{i}$, $a_2 = 2$, $a_1 = 1$, $a_1 = \frac{1}{3}$ and $A = [-1.5, 1.5]^2$.

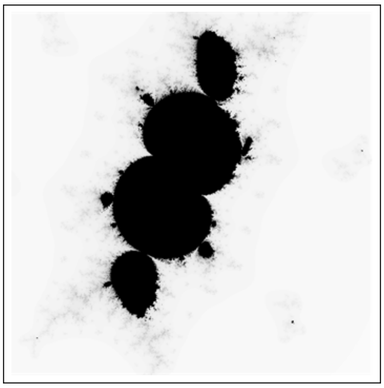

Figure 14. Mandelbrot set for $p(z)$ with $n = 3$ in MM-Orbit, where $a_3 = 1 - \mathbf{i}$, $a_2 = 2$, $a_1 = 1$, $a_1 = \frac{1}{3}$ and $A = [-1.5, 1.5]^2$.

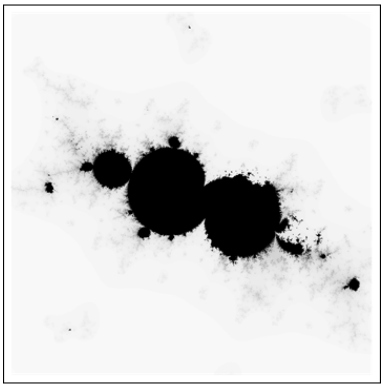

Figure 15. Mandelbrot set for $p(z)$ with $n = 3$ in MM-Orbit, where $a_3 = -1 + \mathbf{i}$, $a_2 = 2$, $a_1 = 1$, $a_1 = \frac{1}{3}$ and $A = [-1.5, 1.5]^2$.

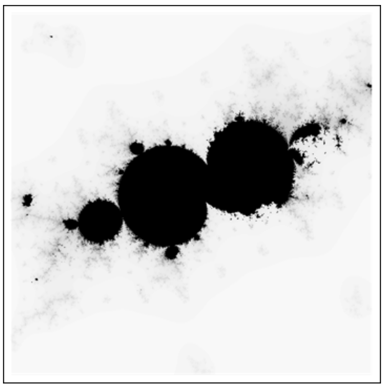

Figure 16. Mandelbrot set for $p(z)$ with $n = 3$ in MM-Orbit, where $a_3 = -1 - \mathbf{i}$, $a_2 = 2$, $a_1 = 1$, $a_1 = \frac{1}{3}$ and $A = [-1.5, 1.5]^2$.

Figure 17. Multi-corns set for $p(z)$ with $n = 3$ in MM-Orbit, where $a_3 = 2$, $a_2 = 2$, $a_1 = 1$, $a_1 = \frac{1}{3}$ and $A = [-1.5, 1.5]^2$.

Figure 18. Multi-corns set for $p(z)$ with $n = 3$ in MM-Orbit, where $a_3 = 1+\mathbf{i}$, $a_2 = 2$, $a_1 = 1$, $a_1 = \frac{1}{3}$ and $A = [-1.5, 1.5]^2$.

Figure 19. Multi-corns set for $p(z)$ with $n = 3$ in MM-Orbit, where $a_3 = 1-\mathbf{i}$, $a_2 = 2$, $a_1 = 1$, $a_1 = \frac{1}{3}$ and $A = [-1.5, 1.5]^2$.

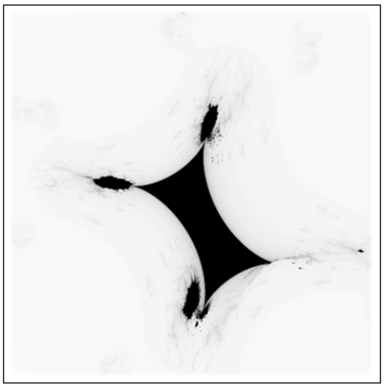

Figure 20. Multi-corns set for $p(z)$ with $n = 3$ in MM-Orbit, where $a_3 = -1 + \mathbf{i}$, $a_2 = 2$, $a_1 = 1$, $a_1 = \frac{1}{3}$ and $A = [-1.5, 1.5]^2$.

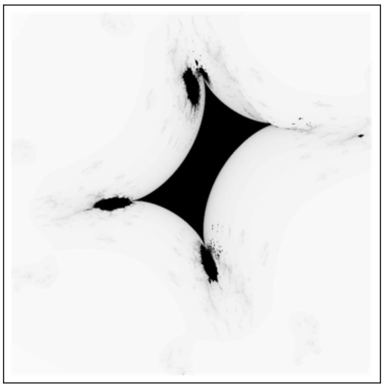

Figure 21. Multi-corns set for $p(z)$ with $n = 3$ in MM-Orbit, where $a_3 = -1 - \mathbf{i}$, $a_2 = 2$, $a_1 = 1$, $a_1 = \frac{1}{3}$ and $A = [-1.5, 1.5]^2$.

Example 4.3. In this example we present the Mandelbrot and Multi-corns sets for a polynomial $p(z) = a_5z^5 + a_4z^4 + a_3z^3 + a_2z^2 + a_1z + a_0$ where $a_5, \ldots, a_0 \in \mathbb{C}$ in proposed orbit. The graphs in Figures 22 and 23 are Mandelbrot sets for $n = 5$ and graphs in Figures 24 and 25 are Multi-corns sets for $n = 5$. Here we also observe that a very small change in a_5 cause drastically changes in images.

- $a_5 = 15, a_4 = 1, a_3 = 0.5, a_2 = \frac{1}{3}$, $a_1 = 0.5$ and $A = [-1, 1]^2$,
- $a_5 = 1 + 14\mathbf{i}, a_4 = 1, a_3 = 0.5, a_2 = \frac{1}{3}$, $a_1 = 0.5$ and $A = [-1, 1]^2$.

Example 4.4. we also present the Mandelbrot and Multi-corns sets for a polynomial $p(z) = a_7z^7 + a_6z^6 + a_5z^5 + a_4z^4 + a_3z^3 + a_2z^2 + a_1z + a_0$ where $a_7, \ldots, a_0 \in \mathbb{C}$ in proposed orbit. The graphs in Figure 26 Mandelbrot sets and graphs in Figure 27 is a multi-corns sets for $a_7 = 70, a_6 = 0, a_5 = \frac{1}{5}, a_4 = \frac{1}{2}, a_3 = \frac{1}{2}, a_2 = \frac{1}{3}$, $a_1 = \frac{1}{25}$ and $A = [-1, 1]^2$.

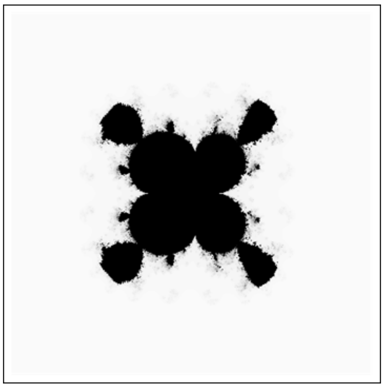

Figure 22. Mandelbrot set for $p(z)$ with $n = 5$ in MM-Orbit, where $a_5 = 15, a_4 = 1, a_3 = 0.5, a_2 = \frac{1}{3}, a_1 = 0.5$ and $A = [-1, 1]^2$.

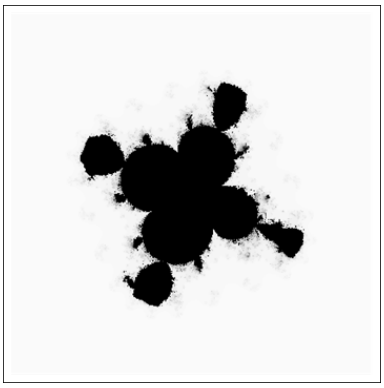

Figure 23. Mandelbrot set for $p(z)$ with $n = 5$ in MM-Orbit, where $a_5 = 1 + 14\mathbf{i}, a_4 = 1, a_3 = 0.5, a_2 = \frac{1}{3}, a_1 = 0.5$ and $A = [-1, 1]^2$.

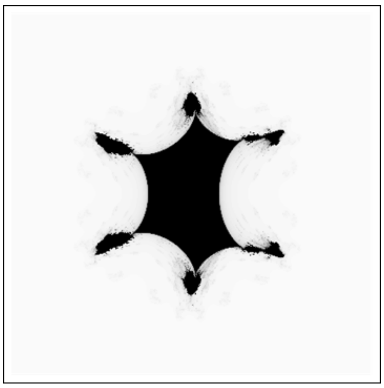

Figure 24. Multi-corns set for $p(z)$ with $n = 5$ in MM-Orbit, where $a_5 = 15, a_4 = 1, a_3 = 0.5, a_2 = \frac{1}{3}, a_1 = 0.5$ and $A = [-1, 1]^2$.

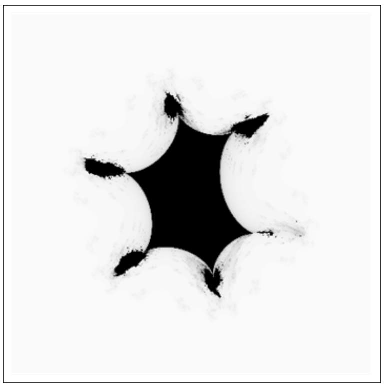

Figure 25. Multi-corns set for $p(z)$ with $n = 5$ in MM-Orbit, where $a_5 = 1 + 14\mathbf{i}, a_4 = 1, a_3 = 0.5, a_2 = \frac{1}{3}, a_1 = 0.5$ and $A = [-1, 1]^2$.

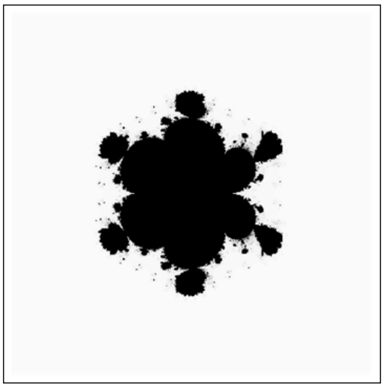

Figure 26. Mandelbrot set for $p(z)$ with $n = 7$ in MM-Orbit, where $a_7 = 70, a_6 = 0, a_5 = \frac{1}{5}, a_4 = \frac{1}{2}, a_3 = \frac{1}{2}, a_2 = \frac{1}{3}, a_1 = \frac{1}{25}$ and $A = [-1, 1]^2$.

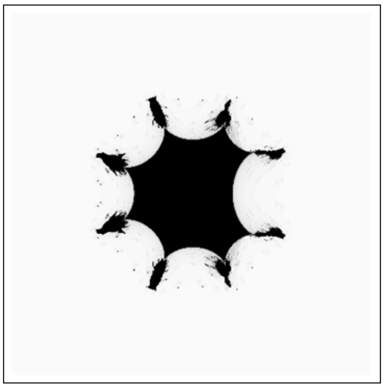

Figure 27. Multi-corns set for $p(z)$ with $n = 7$ in MM-Orbit, where $a_7 = 70, a_6 = 0, a_5 = \frac{1}{5}, a_4 = \frac{1}{2}, a_3 = \frac{1}{2}, a_2 = \frac{1}{3}, a_1 = \frac{1}{25}$ and $A = [-1, 1]^2$.

5. Conclusions

We analyzed a new three step fixed point iteration (i.e. MM-iteration) in the generation of fractals. We established escape conditions for the orbit of MM-iteration. We used the established escape conditions in algorithm to draw some Mandelbrot and Multi-corns sets for $n = 2, 3$ and for higher powers. First time we generated fascinating Mandelbrot and Multi-corns sets for different a_i for $i = 0, 1, 2, .., n$ and compared the images. We observed that the alteration in a_n change the symmetrical angle of images.

Conflict of interest

The authors declare that they do not have any competing interests.

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