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Research article

On a class of Langevin equations in the frame of Caputo function-dependent-kernel fractional derivatives with antiperiodic boundary conditions

Abdelatif Boutiara[1], Mohammed S. Abdo[2], Manar A. Alqudah[3] and Thabet Abdeljawad[4,5,6,*]

[1] Laboratory of Mathematics and Applied Sciences University of Ghardaia, Algeria
[2] Department of Mathematics, Hodeidah University, Al-Hudaydah, Yemen
[3] Department of Mathematical Sciences, Faculty of Sciences, Princess Nourah bint Abdulrahman University, Riyadh 11671, Saudi Arabia
[4] Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, Saudi Arabia
[5] Department of Medical Research, China Medical University, Taichung 40402, Taiwan
[6] Department of Computer Science and Information Engineering, Asia University, Taichung, Taiwan

* **Correspondence:** Email: tabdeljawad@psu.edu.sa.

Abstract: In this manuscript, we consider a class of nonlinear Langevin equations involving two different fractional orders in the frame of Caputo fractional derivative with respect to another monotonic function ϑ with antiperiodic boundary conditions. The existence and uniqueness results are proved for the suggested problem. Our approach is relying on properties of ϑ-Caputo's derivative, and implementation of Krasnoselskii's and Banach's fixed point theorem. At last, we discuss the Ulam-Hyers stability criteria for a nonlinear fractional Langevin equation. Some examples justifying the results gained are provided. The results are novel and provide extensions to some of the findings known in the literature.

Keywords: ϑ-Caputo-type fractional Langevin equation; existence and U-H stability; fixed point theorem
Mathematics Subject Classification: 26A33, 34A60

1. Introduction

The theme of fractional calculus (FC) has appeared as a broad and interesting research point due to its broad applications in science and engineering. FC is now greatly evolved and embraces a wide scope of interesting findings. To obtain detailed information on applications and recent results about this topic, we refer to [1–5] and the references therein.

Some researchers in the field of FC have realized that innovation for new FDs with many non-singular or singular kernels is very necessary to address the need for more realistic modeling problems in different fields of engineering and science. For instance, we refer to works of Caputo and Fabrizio [6], Losada and Nieto [7] and Atangana-Baleanu [8]. The class of FDs and fractional integrals (FIs) concerning functions is a considerable branch of FC. This class of operators with analytic kernels is a new evolution proposed in [1, 9, 10]. Every one of these operators is appropriate broader to cover various kinds of FC and catch diversified behaviors in fractional models. Joining the previous ideas yields another, significantly wide, which is a class of function-dependent-kernel fractional derivatives. This covers both of the two preceding aforesaid classes see [11, 12].

In another context, the Langevin equations (LEs) were formulated by Paul Langevin in 1908 to describe the development of physical phenomena in fluctuating environments [13]. After that, diverse generalizations of the Langevin equation have been deliberated by many scholars we mention here some works [14–16]. Recently, many researchers have investigated sufficient conditions of the qualitative properties of solutions for the nonlinear fractional LEs involving various types of fractional derivatives (FDs) and by using different types of methods such as standard fixed point theorems (FPTs), Leray-Schauder theory, variational methods, etc., e.g. [17–24]. Some recent results on the qualitative properties of solutions for fractional LEs with the generalized Caputo FDs can be found in [25–30], e.g., Ahmad et al. [25] established the existence results for a nonlinear LE involving a generalized Liouville-Caputo-type

$$\begin{cases} {}^{\rho}_{c}\mathcal{D}^{\alpha_1}_{a^+}\left({}^{\rho}_{c}\mathcal{D}^{\alpha_2}_{a^+}+\lambda\right)z(\varsigma)=\mathcal{F}(\varsigma,z(\varsigma)), \quad \varsigma\in[a,T],\ \lambda\in\mathbb{R}, \\ z(a)=0, z(\eta)=0,\ z(T)=\mu\,{}^{\rho}\mathcal{I}^{\delta}_{a^+}z(\xi),\ a<\eta<\xi<T. \end{cases} \tag{1.1}$$

Seemab et al. [26] investigated the existence and UHR stability results for a nonlinear implicit LE involving a ϑ-Caputo FD

$$\begin{cases} {}_{c}\mathcal{D}^{\alpha_1,\vartheta}_{a^+}\left({}_{c}\mathcal{D}^{\alpha_2,\vartheta}_{a^+}+\lambda\right)z(\varsigma)=\mathcal{F}(\varsigma,z(\varsigma),{}_{c}\mathcal{D}^{\alpha_1,\vartheta}_{a^+}z(\varsigma)), \quad \varsigma\in(a,T),\ \lambda>0, \\ z(a)=0, z(\eta)=0,\ z(T)=\mu\,\mathcal{I}^{\delta;\vartheta}_{a^+}z(\xi),\ 0\le a<\eta<\xi<T<\infty. \end{cases} \tag{1.2}$$

The Hyers-Ulam (U-H) stability and existence for various types of generalized FDEs are established in the papers [31–43].

Motivated by the above works and inspired by novel developments in ϑ-FC, in the reported research, we investigate the existence, uniqueness, and U-H-type stability of the solutions for the nonlinear fractional Langevin differential equation (for short FLDE) described by

$$\begin{cases} \mathcal{D}^{\alpha_2;\vartheta}_{a^+}\left(\mathcal{D}^{\alpha_1;\vartheta}_{a^+}+\lambda\right)z(\varsigma)=\mathcal{F}(\varsigma,z(\varsigma)), \quad \varsigma\in \mathrm{J}=[a,b], \\ z(a)=0, z'(a)=0, z''(a)=0, \\ \mathcal{D}^{\alpha_1;\vartheta}_{a^+}z(a)=\mathcal{I}^{\delta;\vartheta}_{a^+}z(\xi), \\ \mathcal{D}^{\alpha_1;\vartheta}_{a^+}z(b)+\kappa z(b)=0, \end{cases} \tag{1.3}$$

where $\mathcal{D}^{\varepsilon;\vartheta}_{a^+}$ denote the ϑ-Caputo FD of order $\varepsilon\in\{\alpha_1,\alpha_2\}$ such that $\alpha_1\in(2,3]$, $\alpha_2\in(1,2]$, $\xi\in(a,b)$, $\delta>0$, $\mathcal{I}^{\alpha_1;\vartheta}_{a^+}$ is the ϑ-fractional integral of the Riemann-Liouville (RL) type, $\mathcal{F}:\mathrm{J}\times\mathbb{R}\to\mathbb{R}$ is a continuous function, and $\lambda,\kappa\in\mathbb{R}$, $\xi\in(a,b)$.

The considered problem in this work is more general, in other words, when we take certain values of function ϑ, the problem (1.3) is reduced to many problems in the frame of classical fractional operators. Also, the gained results here are novel contributes and an extension of the evolution of FDEs that involving a generalized Caputo operator, especially, the study of stability analysis of Ulam-Hyers type of fractional Langevin equations is a qualitative addition to this work. Besides, analysis of the results was restricted to a minimum of assumptions.

Here is a brief outline of the paper. Section 2 provides the definitions and preliminary results required to prove our main findings. In Section 3, we establish the existence, uniqueness, and stability in the sense of Ulam for the system (1.3). In Section 5, we give some related examples to light the gained results.

2. Preliminaries and lemmas

We start this part by giving some basic definitions and results required for fractional analysis.

Consider the space of real and continuous functions $\mathcal{U} = C(\mathrm{J}, \mathbb{R})$ space with the norm

$$\|z\|_\infty = \sup\{|z(\varsigma)| : \quad \varsigma \in \mathrm{J}\}.$$

Let $\vartheta \in C^1 = C^1(\mathrm{J}, \mathbb{R})$ be an increasing differentiable function such that $\vartheta'(\varsigma) \neq 0$, for all $\varsigma \in \mathrm{J}$.

Now, we start by defining ϑ-fractionals operators as follows:

Definition 2.1. [1] The ϑ-RL fractional integral of order $\alpha_1 > 0$ for an integrable function $\omega \colon \mathrm{J} \longrightarrow \mathbb{R}$ is given by

$$\mathcal{I}_{a^+}^{\alpha_1;\vartheta}\omega(\varsigma) = \frac{1}{\Gamma(\alpha_1)} \int_a^\varsigma \vartheta'(s)(\vartheta(\varsigma) - \vartheta(s))^{\alpha_1 - 1}\omega(s)\mathrm{ds}. \tag{2.1}$$

Definition 2.2. [1] Let $\alpha_1 \in (n-1, n)$, $n \in \mathbb{N}$, $\omega : \mathrm{J} \to \mathbb{R}$ is an integrable function, and $\vartheta \in C^n(\mathrm{J}, \mathbb{R})$, the ϑ-RL FD of a function ω of order α_1 is given by

$$\mathcal{D}_{a^+}^{\alpha_1;\vartheta}\omega(\varsigma) = \left(\frac{D_\varsigma}{\vartheta'(\varsigma)}\right)^n \ \mathcal{I}_{a^+}^{n-\alpha_1;\vartheta}\omega(\varsigma),$$

where $n = [\alpha_1] + 1$ and $D_\varsigma = \frac{d}{dt}$.

Definition 2.3. [9] For $\alpha_1 \in (n-1, n)$, and $\omega, \vartheta \in C^n(\mathrm{J}, \mathbb{R})$, the ϑ-Caputo FD of a function ω of order α_1 is given by

$${}^c\mathcal{D}_{a^+}^{\alpha_1;\vartheta}\omega(\varsigma) = \ \mathbb{I}_{a^+}^{n-\alpha_1;\vartheta}\omega_\vartheta^{[n]}(\varsigma),$$

where $n = [\alpha_1] + 1$ for $\alpha_1 \notin \mathbb{N}$, $n = \alpha_1$ for $\alpha_1 \in \mathbb{N}$, and $\omega_\vartheta^{[n]}(\varsigma) = \left(\frac{D_\varsigma}{\vartheta'(\varsigma)}\right)^n \omega(\varsigma)$.

From the above definition, we can express ϑ-Caputo FD by formula

$${}^c\mathcal{D}_{a^+}^{\alpha_1;\vartheta}\omega(\varsigma) = \begin{cases} \int_a^\varsigma \frac{\vartheta'(s)(\vartheta(\varsigma)-\vartheta(s))^{n-\alpha_1-1}}{\Gamma(n-\alpha_1)}\omega_\vartheta^{[n]}(s)\mathrm{ds} & , \quad \text{if } \alpha_1 \notin \mathbb{N}, \\ \omega_\vartheta^{[n]}(\varsigma) & , \quad \text{if } \alpha_1 \in \mathbb{N}. \end{cases} \tag{2.2}$$

Also, the ϑ-Caputo FD of order α_1 of ω is defined as

$${}^c\mathcal{D}_{a^+}^{\alpha_1;\vartheta}\omega(\varsigma) = \mathcal{D}_{a^+}^{\alpha_1;\vartheta}\left[\omega(\varsigma) - \sum_{k=0}^{n-1}\frac{\omega_\vartheta^{[k]}(a)}{k!}(\vartheta(\varsigma) - \vartheta(a))^k\right].$$

For more details see [9, Theorem 3].

Lemma 2.4. *[1] For $\alpha_1, \alpha_2 > 0$, and $\omega \in C(\mathrm{J}, \mathbb{R})$, we have*

$$\mathcal{I}_{a^+}^{\alpha_1;\vartheta} \mathcal{I}_{a^+}^{\alpha_2;\vartheta} \omega(\varsigma) = \mathcal{I}_{a^+}^{\alpha_1+\alpha_2;\vartheta} \omega(\varsigma), \ a.e. \ \varsigma \in \mathrm{J}.$$

Lemma 2.5. *[44] Let $\alpha_1 > 0$.*
If $\omega \in C(\mathrm{J}, \mathbb{R})$, then

$$ {}^c\mathcal{D}_{a^+}^{\alpha_1;\vartheta} \mathcal{I}_{a^+}^{\alpha_1;\vartheta} \omega(\varsigma) = \omega(\varsigma), \ \varsigma \in \mathrm{J},$$

and if $\omega \in C^{n-1}(\mathrm{J}, \mathbb{R})$, then

$$\mathcal{I}_{a^+}^{\alpha_1;\vartheta} \, {}^c\mathcal{D}_{a^+}^{\alpha_1;\vartheta} \omega(\varsigma) = \omega(\varsigma) - \sum_{k=0}^{n-1} \frac{\omega_{\vartheta}^{[k]}(a)}{k!} \left[\vartheta(\varsigma) - \vartheta(a)\right]^k, \quad \varsigma \in \mathrm{J}.$$

for all $\varsigma \in \mathrm{J}$. Moreover, if $m \in \mathrm{N}$ be an integer and $\omega \in \mathrm{C}^{n+m}(\mathrm{J}, \mathbb{R})$ a function. Then, the following holds:

$$\left(\frac{1}{\vartheta'(\varsigma)} \frac{\mathrm{d}}{\mathrm{dt}}\right)^m \cdot \mathcal{D}_{a^+}^{\alpha_1;\phi} \omega(\varsigma) = {}^c\mathcal{D}_{a^+}^{a+m;\phi} \omega(\varsigma) + \sum_{k=0}^{m-1} \frac{(\vartheta(\varsigma) - \vartheta(a))^{k+n-\hat{\alpha_1}-m}}{\Gamma(k+n-\alpha_1-m+1)} \omega_{\vartheta}^{[k+n]}(a) \tag{2.3}$$

Observe that from Eq (2.3), if $\omega_{\vartheta}^{[k]}(a) = 0$, for all $k = n, n+1, \ldots, n+m-1$ we can get the following relation

$$z_{\vartheta}^{[m]} \cdot \mathcal{D}_{a^+}^{a,\vartheta} z(\varsigma) = {}^c\mathcal{D}_{a^+}^{\alpha_1+m;\vartheta} z(\varsigma), \quad \varsigma \in \mathrm{J}$$

Lemma 2.6. *[1, 9] For $\varsigma > a, \ \alpha_1 \geq 0, \alpha_2 > 0$, we have*

- $\mathcal{I}_{a^+}^{\alpha_1;\vartheta} (\vartheta(\varsigma) - \vartheta(a))^{\alpha_2-1} = \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2+\alpha_1)} (\vartheta(\varsigma) - \vartheta(a))^{\alpha_2+\alpha_1-1}$,
- ${}^c\mathcal{D}_{a^+}^{\alpha_1;\vartheta} (\vartheta(\varsigma) - \vartheta(a))^{\alpha_2-1} = \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2-\alpha_1)} (\vartheta(\varsigma) - \vartheta(a))^{\alpha_2-\alpha_1-1}$,
- ${}^c\mathcal{D}_{a^+}^{\alpha_1;\vartheta} (\vartheta(\varsigma) - \vartheta(a))^k = 0, \forall k \in \{0, \ldots, n-1\}, n \in \mathbb{N}$.

Theorem 2.7. *(Banach's FPT [45]). Let (R, d) be a nonempty complete metric space with a contraction mapping $\mathcal{G} : R \to R$ i.e., $\mathrm{d}(\mathcal{G}z, \mathcal{G}\varkappa) \leq \mathrm{L} \, \mathrm{d}(z, \varkappa)$ for all $z, \varkappa \in R$, where $\mathrm{L} \in (0, 1)$ is a constant. Then $\mathcal{G}$ possesses a unique fixed point.*

Theorem 2.8. *(Kransnoselskii's FPT [46]). Let E be a Banach space. Let S is a nonempty convex, closed and bounded subset of E and let A_1, A_2 be mapping from S to E such that:*
(i) $A_1 z + A_2 \varkappa \in S$ whenever $z, \varkappa \in S$
(ii) A_1 is continuous and compact;
(iii) A_2 is a contraction.
Then there exists $z \in S$ such that $z = A_1 z + A_2 z$.

3. Main results

This portion interests in the existence, uniqueness, and Ulam stability of solutions to the suggested problem (1.3).

The next auxiliary lemma, which attentions the linear term of a problem (1.3), plays a central role in the afterward findings.

Lemma 3.1. *Let* $\alpha_1 \in (2,3]$, $\alpha_2 \in (1,2]$. *Then the linear BVP*

$$\begin{cases} \mathcal{D}_{a^+}^{\alpha_2;\vartheta}\left(\mathcal{D}^{\alpha_1;\vartheta}+\lambda\right)z(\varsigma)=\sigma(\varsigma), \quad \varsigma\in \mathrm{J}=[a,b], \\ z(a)=0, z'(a)=0, z''(a)=0, \\ \mathcal{D}_{a^+}^{\alpha_1;\vartheta}z(a)=\mathcal{I}_{a^+}^{\delta;\vartheta}z(\gamma), \\ \mathcal{D}_{a^+}^{\alpha_1;\vartheta}z(b)+\kappa z(b)=0, \end{cases} \tag{3.1}$$

has a unique solution defined by

$$\begin{aligned} z(\varsigma)=&\ \mathcal{I}_{a^+}^{\alpha_1+\alpha_2;\vartheta}\sigma(\varsigma)-\lambda\mathcal{I}_{a^+}^{\alpha_1;\vartheta}z(\varsigma)+\mu(\varsigma)\mathcal{I}_{a^+}^{\delta;\vartheta}\sigma(\xi) \\ &+\nu(\varsigma)\left\{\lambda(\kappa-\lambda)\mathcal{I}_{a^+}^{\alpha_1;\vartheta}z(b)-(\kappa-\lambda)\mathcal{I}_{a^+}^{\alpha_1+\alpha_2;\vartheta}\sigma(b)-\mathcal{I}_{a^+}^{\alpha_2;\vartheta}\sigma(b)-\varpi\mathcal{I}_{a^+}^{\delta;\vartheta}z(\xi)\right\}, \end{aligned} \tag{3.2}$$

where

$$\mu(\varsigma)=\frac{(\vartheta(\varsigma)-\vartheta(a))^{\alpha_1}}{\Gamma(\alpha_1+1)}, \tag{3.3}$$

and

$$\nu(\varsigma)=\frac{(\vartheta(\varsigma)-\vartheta(a))^{\alpha_1+1}}{(\vartheta(b)-\vartheta(a))\,\Gamma(\alpha_1+2)+(\kappa-\lambda)}, \tag{3.4}$$

with

$$\varpi \ = \ \left(1+\frac{(\kappa-\lambda)}{\Gamma(\alpha_1+1)}\right), \tag{3.5}$$

Proof. Applying the RL operator $\mathcal{I}_{a^+}^{\alpha_2;\vartheta}$ to (3.1) it follows from Lemma 2.5 that

$$\left(\mathcal{D}_{a^+}^{\alpha_1}+\lambda\right)z(\varsigma)=c_0+c_1\,(\vartheta(\varsigma)-\vartheta(a))+\mathcal{I}_{a^+}^{\alpha_2;\vartheta}\sigma(\varsigma), \quad \varsigma\in(a,b]. \tag{3.6}$$

Again, we apply the RL operator $\mathcal{I}_{a^+}^{\alpha_1;\vartheta}$ and use the results of Lemma 2.5 to get

$$\begin{aligned} z(\varsigma)=&\mathcal{I}_{a^+}^{\alpha_1+\alpha_2;\vartheta}\sigma(\varsigma)-\lambda\mathcal{I}_{a^+}^{\alpha_1}z(\varsigma) \\ &+c_0\frac{(\vartheta(\varsigma)-\vartheta(a))^{\alpha_1+1}}{\Gamma(\alpha_1+2)}+c_1\frac{(\vartheta(\varsigma)-\vartheta(a))^{\alpha_1}}{\Gamma(\alpha_1+1)}+c_2\,(\vartheta(\varsigma)-\vartheta(a))^2+c_3\,(\vartheta(\varsigma)-\vartheta(a))+c_4, \end{aligned} \tag{3.7}$$

where $c_0, c_1, c_2, c_3, c_4 \in \mathbb{R}$. By utilizing the boundary conditions in (3.1) and (3.7), we obtain $c_2 = 0, c_3 = 0, c_4 = 0$.
Hence,

$$z(\varsigma)=\mathcal{I}_{a^+}^{\alpha_1+\alpha_2;\vartheta}\sigma(\varsigma)-\lambda\mathcal{I}_{a^+}^{\alpha_1}z(\varsigma)+c_0\frac{(\vartheta(\varsigma)-\vartheta(a))^{\alpha_1+1}}{\Gamma(\alpha_1+2)}+c_1\frac{(\vartheta(\varsigma)-\vartheta(a))^{\alpha_1}}{\Gamma(\alpha_1+1)}, \tag{3.8}$$

Now, by using the conditions $\mathcal{D}_{a^+}^{\alpha_1;\vartheta}z(a)=\mathcal{I}_{a^+}^{\delta;\vartheta}z(\gamma)$ and $\mathcal{D}_{a^+}^{\alpha_1;\vartheta}z(b)+\kappa z(b)=0$, we get

$$c_1 = I_{a^+}^{\delta;\vartheta}z(\xi), \tag{3.9}$$

$$\begin{aligned} c_0=&\left(\frac{(\vartheta(\varsigma)-\vartheta(a))^{\alpha_1+1}}{(\vartheta(b)-\vartheta(a))\,\Gamma(\alpha_1+2)+(\kappa-\lambda)}\right) \\ &\times\left\{\lambda(\kappa-\lambda)\mathcal{I}_{a^+}^{\alpha_1;\vartheta}z(b)-(\kappa-\lambda)\mathcal{I}_{a^+}^{\alpha_1+\alpha_2;\vartheta}\sigma(b)-\mathcal{I}_{a^+}^{\alpha_2;\vartheta}\sigma(b)-\left(1+\frac{(\kappa-\lambda)}{\Gamma(\alpha_1+1)}\right)\mathcal{I}_{a^+}^{\delta;\vartheta}z(\xi)\right\}. \end{aligned} \tag{3.10}$$

Substituting c_0 and c_1 in (3.8), we finish with (3.2).
The reverse direction can be shown easily with the help of results in Lemmas 2.5 and 2.6, i.e. Eq (3.2) solves problem (3.1). This ends the proof. □

Now, we shall need to the following lemma:

Lemma 3.2. *The functions μ and ν are continuous functions on* J *and satisfy the following properties:*
(1) $\mu^* = \max_{0\leq\varsigma\leq b}|\mu(\varsigma)|$,
(2) $\nu^* = \max_{0<\varsigma<b}|\nu(\varsigma)|$,
where μ and ν are defined by Lemma 3.1.

Here, we give the following hypotheses:
(H1) The function $\mathcal{F} : \mathrm{J}\times\mathbb{R}\to\mathbb{R}$ is continuous.
(H2) There exists a constant $\mathcal{L} > 0$ such that

$$|\mathcal{F}(\varsigma, z) - \mathcal{F}(\varsigma, \varkappa)| \leq \mathcal{L}|z - \varkappa|, \quad \varsigma\in\mathrm{J}, \quad z, \varkappa\in\mathbb{R}.$$

(H3) There exist positive functions $h(\varsigma)\in C(\mathrm{J}, \mathbb{R}^+)$ with bounds $\|h\|$ such that

$$|\mathcal{F}(\varsigma, z(\varsigma))| \leq h(\varsigma), \quad \text{for all } (\varsigma, z)\in\mathrm{J}\times\mathbb{R}.$$

For simplicity, we denote

$$\mathcal{M} := \sup_{\varsigma\in[a,b]}|\mathcal{F}(\varsigma, 0)|.$$

$$\begin{aligned}\Delta : = &\left\{\left(\mathcal{L}\frac{(\vartheta(b)-\vartheta(a))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}(1+\nu^*|\kappa-\lambda|) + |\lambda|(1+\nu^*|\kappa-\lambda|)\frac{(\vartheta(b)-\vartheta(a))^{\alpha_1}}{\Gamma(\alpha_1+1)}\right)\right.\\ &\left.+\left(\mathcal{L}\nu^*\frac{(\vartheta(b)-\vartheta(a))^{\alpha_2}}{\Gamma(\alpha_2+1)} + (\mu^*+\nu^*|\varpi|)\frac{(\vartheta(\xi)-\vartheta(a))^{\delta}}{\Gamma(\delta+1)}\right)\right\},\end{aligned} \tag{3.11}$$

$$\mathcal{G}^{\chi}_{\vartheta}(\varsigma, s) = \frac{\vartheta'(s)(\vartheta(\varsigma)-\vartheta(s))^{\chi-1}}{\Gamma(\chi)}, \quad \chi > 0. \tag{3.12}$$

As a result of Lemma 3.1, we have the subsequent lemma:

Lemma 3.3. *Suppose that $\mathcal{F} : \mathrm{J}\times\mathbb{R}\to\mathbb{R}$ is continuous. A function $z(\varsigma)$ solves* (1.3) *if and only if it is a fixed-point of the operator $\mathcal{G} : \mathcal{U}\to\mathcal{U}$ defined by*

$$\begin{aligned}\mathcal{G}z(\varsigma) = &\int_a^{\varsigma}\mathcal{G}^{\alpha_1+\alpha_2}_{\vartheta}(\varsigma, s)\mathcal{F}(s, z(s))\,\mathrm{ds} + \lambda\int_a^{\varsigma}\mathcal{G}^{\alpha_1}_{\vartheta}(\varsigma, s)z(s)\mathrm{ds}\\ &+\mu(\varsigma)\int_a^{\xi}\mathcal{G}^{\delta}_{\vartheta}(\xi, s)\mathcal{F}(s, z(s))\,\mathrm{ds}\\ &+\nu(\varsigma)\left\{\lambda(\kappa-\lambda)\int_a^b\mathcal{G}^{\alpha_1}_{\vartheta}(b, s)z(s)\mathrm{ds} - (\kappa-\lambda)\int_a^b\mathcal{G}^{\alpha_1+\alpha_2}_{\vartheta}(b, s)\mathcal{F}(s, z(s))\,\mathrm{ds}\right.\\ &\left.-\int_a^b\mathcal{G}^{\alpha_2}_{\vartheta}(b, s)\mathcal{F}(s, z(s))\,\mathrm{ds} - \varpi\int_a^{\xi}\mathcal{G}^{\delta}_{\vartheta}(\xi, s)z(s)\mathrm{ds}\right\}.\end{aligned} \tag{3.13}$$

Now, we are willing to give our first result which based on Theorem 2.7.

Theorem 3.4. *Suppose that (H1) and (H2) hold. If $\Delta < 1$, where Δ is given by (3.11), then there exists a unique solution for (1.3) on the interval* J.

Proof. Thanks to Lemma 3.1, we consider the operator $\mathcal{G} : \mathcal{U} \to \mathcal{U}$ defined by (3.13). Thus, $\mathcal{G}$ is well defined as $\mathcal{F}$ is a continuous. Then the fixed point of $\mathcal{G}$ coincides with the solution of FLDE (1.3). Next, the Theorem 2.7 will be used to prove that $\mathcal{G}$ has a fixed point. For this end, we show that $\mathcal{G}$ is a contraction.

Let $\mathcal{B}_{\mathcal{R}} = \{z \in \mathcal{U} : \|z\| \leq \mathcal{R}\}$, where $\mathcal{R} > \frac{\mathcal{M}\Lambda_1}{1-\mathcal{L}\Lambda_1-\Lambda_2}$. Since

$$\begin{aligned}|\mathcal{F}(s,z(s)| &= |\mathcal{F}(s,z(s)) - \mathcal{F}(s,0) + \mathcal{F}(s,0)|\\ &\leq |\mathcal{F}(s,z(s)) - \mathcal{F}(s,0)| + |\mathcal{F}(s,0)|\\ &\leq (\mathcal{L}|z(s)| + |\mathcal{F}(s,0)|)\\ &\leq \mathcal{L}\mathcal{R} + \mathcal{M}\end{aligned}$$

we obtain

$$\begin{aligned}|\mathcal{G}z(\varsigma)| = &\int_a^\varsigma \mathcal{G}_\vartheta^{\alpha_1+\alpha_2}(\varsigma,s)|\mathcal{F}(s,z(s))|\mathbf{ds} + |\lambda|\int_a^\varsigma \mathcal{G}_\vartheta^{\alpha_1}(\varsigma,s)|z(s)|\mathbf{ds}\\ &+ \mu^*\int_a^\xi \mathcal{G}_\vartheta^{\delta}(\xi,s)|z(s)|\mathbf{ds} + \nu(\varsigma)\left\{|\lambda(\kappa-\lambda)|\int_a^b \mathcal{G}_\vartheta^{\alpha_1}(b,s)|z(s)|\mathrm{ds}\right.\\ &+ (\kappa-\lambda)\int_a^b \mathcal{G}_\vartheta^{\alpha_1+\alpha_2}(b,s)|\mathcal{F}(s,z(s))|\mathrm{ds} + \int_a^b \mathcal{G}_\vartheta^{\alpha_2}(b,s)|\mathcal{F}(s,z(s))|\mathrm{ds}\\ &\left.+\varpi\int_a^\xi \mathcal{G}_\vartheta^{\delta}(\xi,s)|z(s)|\mathrm{ds}\right\}\\ &\leq \mathcal{L}\mathcal{R} + \mathcal{M}\left\{\int_a^\varsigma \mathcal{G}_\vartheta^{\alpha_1+\alpha_2}(\varsigma,s)|\mathcal{F}(s,z(s))|\mathbf{ds} + \nu(\varsigma)\left\{|\kappa-\lambda|\int_a^b \mathcal{G}_\vartheta^{\alpha_1+\alpha_2}(b,s)|\mathcal{F}(s,z(s))|\mathrm{ds}\right.\right.\\ &\left.\left.+\nu^*\int_a^b \mathcal{G}_\vartheta^{\alpha_2}(b,s)|\mathcal{F}(s,z(s))|\mathrm{ds}\right\}\right.\\ &+ \mathcal{R}\left\{|\lambda|\int_a^\varsigma \mathcal{G}_\vartheta^{\alpha_1}(\varsigma,s)|z(s)|\mathbf{ds} + \mu^*\int_a^\xi \mathcal{G}_\vartheta^{\delta}(\xi,s)|z(s)|\mathbf{ds} + \nu(\varsigma)\left\{|\lambda(\kappa-\lambda)|\int_a^b \mathcal{G}_\vartheta^{\alpha_1}(b,s)|z(s)|\mathrm{ds}\right.\right.\\ &\left.\left.+\varpi\int_a^\xi \mathcal{G}_\vartheta^{\delta}(\xi,s)|z(s)|\mathrm{ds}\right\}\right\}\\ &\leq \mathcal{L}\mathcal{R} + \mathcal{M}\left\{\frac{(\vartheta(b)-\vartheta(a))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}(1+\nu^*|\kappa-\lambda|) + \nu^*\frac{(\vartheta(b)-\vartheta(a))^{\alpha_2}}{\Gamma(\alpha_2+1)}\right\}\\ &+ \mathcal{R}\left\{|\lambda|(1+\nu^*|\kappa-\lambda|)\frac{(\vartheta(b)-\vartheta(a))^{\alpha_1}}{\Gamma(\alpha_1+1)} + (\mu^*+\nu^*|\varpi|)\frac{(\vartheta(\xi)-\vartheta(a))^{\delta}}{\Gamma(\delta+1)}\right\}\\ &\leq (\mathcal{L}\mathcal{R}+\mathcal{M})\Lambda_1 + \Lambda_2\mathcal{R} \leq \mathcal{R}.\end{aligned} \tag{3.14}$$

which implies that $\|\mathcal{G}z\| \leq \mathcal{R}$, i.e.,

$$\mathcal{G}\mathcal{B}_{\mathcal{R}} \subseteq \mathcal{B}_{\mathcal{R}}.$$

Now, let $z, \varkappa \in \mathcal{U}$. Then, for every $\varsigma \in \mathrm{J}$, using (H2), we can get

$$
\begin{aligned}
|\mathcal{G}\varkappa(\varsigma)-\mathcal{G}z(\varsigma)| \leq & \int_a^{\varsigma} \mathcal{G}_{\vartheta}^{\alpha_1+\alpha_2}(\varsigma,s)|\mathcal{F}(s,\varkappa(s))-\mathcal{F}(s,z(s))|\mathbf{ds}+|\lambda|\int_a^{\varsigma} \mathcal{G}_{\vartheta}^{\alpha_1}(\varsigma,s)|\varkappa(s)-z(s)|\mathbf{ds} \\
& +\mu^*\int_a^{\xi} \mathcal{G}_{\vartheta}^{\delta}(\xi,s)|\varkappa(s)-z(s)|\mathbf{ds}+\nu(\varsigma)\bigg\{|\lambda(\kappa-\lambda)|\int_a^b \mathcal{G}_{\vartheta}^{\alpha_1}(b,s)|\varkappa(s)-z(s)|\mathrm{ds} \\
& +|\kappa-\lambda|\int_a^b \mathcal{G}_{\vartheta}^{\alpha_1+\alpha_2}(b,s)|\mathcal{F}(s,\varkappa(s))-\mathcal{F}(s,z(s))|\mathrm{ds} \\
& +\int_a^b \mathcal{G}_{\vartheta}^{\alpha_2}(b,s)|\mathcal{F}(s,\varkappa(s))-\mathcal{F}(s,z(s))|\mathrm{ds}+\varpi\int_a^{\xi} \mathcal{G}_{\vartheta}^{\delta}(\xi,s)|\varkappa(s)-z(s)|\mathrm{ds}\bigg\} \\
& \leq \int_a^{\varsigma} \mathcal{L}\mathcal{G}_{\vartheta}^{\alpha+\alpha_2}(\varsigma,s)|\varkappa(s)-z(s)|\mathbf{ds}+|\lambda|\int_a^{\varsigma} \mathcal{G}_{\vartheta}^{\alpha_1}(\varsigma,s)|\varkappa(s)-z(s)|\mathbf{ds} \\
& +\mu^*\int_a^{\xi} \mathcal{G}_{\vartheta}^{\delta}(\xi,s)|\varkappa(s)-z(s)|\mathbf{ds}+\nu(\varsigma)\bigg\{|\lambda(\kappa-\lambda)|\int_a^b \mathcal{G}_{\vartheta}^{\alpha_1}(b,s)|\varkappa(s)-z(s)|\mathrm{ds} \\
& +(\kappa-\lambda)\int_a^b \mathcal{G}_{\vartheta}^{\alpha_1+\alpha_2}(b,s)\mathcal{L}|\varkappa(s)-z(s)|\mathrm{ds} \\
& +\int_a^b \mathcal{G}_{\vartheta}^{\alpha_2}(b,s)\mathcal{L}|\varkappa(s)-z(s)|\mathrm{ds}+\varpi\int_a^{\xi} \mathcal{G}_{\vartheta}^{\delta}(\xi,s)|\varkappa(s)-z(s)|\mathrm{ds}\bigg\} \\
& =\int_a^{\varsigma}\left(\mathcal{L}\mathcal{G}_{\vartheta}^{\alpha_1+\alpha_2}(\varsigma,s)+|\lambda|\mathcal{G}_{\vartheta}^{\alpha_1}(\varsigma,s)\right)|\varkappa(s)-z(s)|\mathbf{ds} \\
& +(\mu^*+\nu^*|\varpi|)\int_a^{\xi} \mathcal{G}_{\vartheta}^{\delta}(\xi,s)|\varkappa(s)-z(s)|\mathbf{ds} \\
& +\nu^*\int_a^b\left(|\kappa-\lambda|\left(\lambda\mathcal{G}_{\vartheta}^{\alpha_1}(b,s)+\mathcal{L}\mathcal{G}_{\vartheta}^{\alpha_1+\alpha_2}(\varsigma,s)\right)+\mathcal{L}\mathcal{G}_{\vartheta}^{\alpha_2}(b,s)\right)|\varkappa(s)-z(s)|\mathbf{ds} \\
& \leq \|\varkappa-z\|_{\infty}\bigg\{\int_a^{\varsigma}\left(\mathcal{L}\mathcal{G}_{\vartheta}^{\alpha_1+\alpha_2}(\varsigma,s)+|\lambda|\mathcal{G}_{\vartheta}^{\alpha_1}(\varsigma,s)\right)\mathrm{ds}+(\mu^*+\nu^*|\varpi|)\int_a^{\xi} \mathcal{G}_{\vartheta}^{\delta}(\xi,s)\mathbf{ds} \\
& +\nu^*\int_a^b\left(|\kappa-\lambda|\left(\lambda\mathcal{G}_{\vartheta}^{\alpha_1}(b,s)+\mathcal{L}\mathcal{G}_{\vartheta}^{\alpha_1+\alpha_2}(\varsigma,s)\right)+\mathcal{L}\mathcal{G}_{\vartheta}^{\alpha_2}(b,s)\right)\mathbf{ds}\bigg\}
\end{aligned}
\tag{3.15}
$$

Also note that

$$
\int_a^{\varsigma} \mathcal{G}_{\vartheta}^{\chi}(\varsigma,s)\mathrm{d}s \leq \frac{(\vartheta(b)-\vartheta(a))^{\chi}}{\Gamma(\chi+1)}, \quad \chi>0.
$$

Using the above arguments, we get

$$
\begin{aligned}
\|\mathcal{G}\varkappa-\mathcal{G}z\|_{\infty} \leq & \Bigg\{\left(\mathcal{L}\frac{(\vartheta(b)-\vartheta(a))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}(1+\nu^*|\kappa-\lambda|)+|\lambda|(1+\nu^*|\kappa-\lambda|)\frac{(\vartheta(b)-\vartheta(a))^{\alpha_1}}{\Gamma(\alpha_1+1)}\right) \\
& +\left(\mathcal{L}\nu^*\frac{(\vartheta(b)-\vartheta(a))^{\alpha_2}}{\Gamma(\alpha_2+1)}+(\mu^*+\nu^*|\varpi|)\frac{(\vartheta(\xi)-\vartheta(a))^{\delta}}{\Gamma(\delta+1)}\right)\Bigg\}\|\varkappa-z\|_{\infty} \\
& =\Delta\|\varkappa-z\|_{\infty}.
\end{aligned}
$$

As $\Delta<1$, we derive that $\mathcal{G}$ is a contraction. Hence, by Theorem 2.7, $\mathcal{G}$ has a unique fixed point which is a unique solution of FLDE (1.3). This ends the proof. □

Now, we apply the Theorem 2.8 to obtain the existence result.

Theorem 3.5. *Let us assume (H1)–(H3) hold. Then FLDE (1.3) has at least one solution on* J *if* $\Lambda_3 < 1$, *where it is supposed that*

$$\begin{aligned}\Lambda_3 := \Bigg\{ & \left(\mathcal{L} \frac{(\vartheta(b)-\vartheta(a))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} (\nu^*|\kappa-\lambda|) + |\lambda| (\nu^*|\kappa-\lambda|) \frac{(\vartheta(b)-\vartheta(a))^{\alpha_1}}{\Gamma(\alpha_1+1)} \right) \\ & + \left(\mathcal{L}\nu^* \frac{(\vartheta(b)-\vartheta(a))^{\alpha_2}}{\Gamma(\alpha_2+1)} + (\mu^* + \nu^*|\varpi|) \frac{(\vartheta(\xi)-\vartheta(a))^{\delta}}{\Gamma(\delta+1)} \right) \Bigg\}.\end{aligned}$$

Proof. By the assumption (H3), we can fix

$$\rho \geq \frac{\lambda_1 \|h\|}{(1-\lambda_2)},$$

where $B_\rho = \{z \in \mathcal{U} : \|z\| \leq \rho\}$. Let us split the operator $\mathcal{G} : \mathcal{U} \to \mathcal{U}$ defined by (3.13) as $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$, where $\mathcal{G}_1$ and $\mathcal{G}_2$ are given by

$$\mathcal{G}_1 z(\varsigma) = \int_a^{\varsigma} \mathcal{G}_\vartheta^{\alpha_1+\alpha_2}(\varsigma, s) \mathcal{F}(s, z(s))\,ds + \lambda \int_a^{\varsigma} \mathcal{G}_\vartheta^{\alpha_1}(\varsigma, s) z(s) ds, \tag{3.16}$$

and

$$\begin{aligned}\mathcal{G}_2 z(\varsigma) = & \mu(\varsigma) \int_a^{\xi} \mathcal{G}_\vartheta^{\delta}(\xi, s) \mathcal{F}(s, z(s))\,ds \\ & + \nu(\varsigma) \Bigg\{ \lambda(\kappa-\lambda) \int_a^b \mathcal{G}_\vartheta^{\alpha_1}(b, s) z(s) ds - (\kappa-\lambda) \int_a^b \mathcal{G}_\vartheta^{\alpha_1+\alpha_2}(b, s) \mathcal{F}(s, z(s))\,ds \\ & - \int_a^b \mathcal{G}_\vartheta^{\alpha_2}(b, s) \mathcal{F}(s, z(s))\,ds - \varpi \int_a^{\xi} \mathcal{G}_\vartheta^{\delta}(\xi, s) z(s) ds \Bigg\}.\end{aligned} \tag{3.17}$$

The proof will be split into numerous steps:

Step 1: $\mathcal{G}_1(z) + \mathcal{G}_2(z) \in B_\rho$.

$$\begin{aligned}\|\mathcal{G}_1 z + \mathcal{G}_2 z_1\| = & \sup_{\varsigma \in J} |\mathcal{G}_1 z(\varsigma) + \mathcal{G}_2 z_1(\varsigma)| \\ \leq & \int_a^{\varsigma} \mathcal{G}_\vartheta^{\alpha_1+\alpha_2}(\varsigma, s) |\mathcal{F}(s, z(s))| d\mathbf{s} + |\lambda| \int_a^{\varsigma} \mathcal{G}_\vartheta^{\alpha_1}(\varsigma, s) |z(s)| d\mathbf{s} \\ & + \mu^* \int_a^{\xi} \mathcal{G}_\vartheta^{\delta}(\xi, s) |z(s)| d\mathbf{s} + \nu(\varsigma) \Bigg\{ |\lambda(\kappa-\lambda)| \int_a^b \mathcal{G}_\vartheta^{\alpha_1}(b, s) |z(s)| ds \\ & + |\kappa-\lambda| \int_a^b \mathcal{G}_\vartheta^{\alpha_1+\alpha_2}(b, s) |\mathcal{F}(s, z(s))| ds \\ & + \int_a^b \mathcal{G}_\vartheta^{\alpha_2}(b, s) |\mathcal{F}(s, z(s))| ds + \varpi \int_a^{\xi} \mathcal{G}_\vartheta^{\delta}(\xi, s) |z(s)| ds \Bigg\}\end{aligned} \tag{3.18}$$

$$
\begin{aligned}
&\le \|h\|\left\{\int_a^{\varsigma} \mathcal{G}_{\vartheta}^{\alpha_1+\alpha_2}(\varsigma, s)|\mathcal{F}(s, z(s))|\mathrm{ds} + \nu(\varsigma)\left\{|\kappa-\lambda| \int_a^b \mathcal{G}_{\vartheta}^{\alpha_1+\alpha_2}(b, s)|\mathcal{F}(s, z(s))|\mathrm{ds}\right.\right.\\
&\left.\left.+\nu^* \int_a^b \mathcal{G}_{\vartheta}^{\alpha_2}(b, s)|\mathcal{F}(s, z(s))|\mathrm{ds}\right\}\right.\\
&+\rho\left\{|\lambda| \int_a^{\varsigma} \mathcal{G}_{\vartheta}^{\alpha_1}(\varsigma, s)|z(s)|\mathrm{ds} + \mu^* \int_a^{\xi} \mathcal{G}_{\vartheta}^{\delta}(\xi, s)|z(s)|\mathrm{ds} + \nu(\varsigma)\left\{|\lambda(\kappa-\lambda)| \int_a^b \mathcal{G}_{\vartheta}^{\alpha_1}(b, s)|z(s)|\mathrm{ds}\right.\right.\\
&\left.\left.+\varpi \int_a^{\xi} \mathcal{G}_{\vartheta}^{\delta}(\xi, s)|z(s)|\mathrm{ds}\right\}\right\}\\
&\le \|h\|\left\{\frac{(\vartheta(b)-\vartheta(a))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}(1+\nu^*|\kappa-\lambda|) + \nu^* \frac{(\vartheta(b)-\vartheta(a))^{\alpha_2}}{\Gamma(\alpha_2+1)}\right\}\\
&+\rho\left\{|\lambda|(1+\nu^*|\kappa-\lambda|)\frac{(\vartheta(b)-\vartheta(a))^{\alpha_1}}{\Gamma(\alpha_1+1)} + (\mu^*+\nu^*|\varpi|)\frac{(\vartheta(\xi)-\vartheta(a))^{\delta}}{\Gamma(\delta+1)}\right\}\\
&\le \|h\|\,\Lambda_1 + \Lambda_2\rho \le \rho.
\end{aligned} \tag{3.19}
$$

Hence

$$\|\mathcal{G}_1(z) + \mathcal{G}_2(z_1)\| \le \rho,$$

which shows that $\mathcal{G}_1 z + \mathcal{G}_2 z_1 \in B_\rho$.

Step 2: $\mathcal{G}_2$ is a contraction map on B_ρ.

Due to the contractitility of $\mathcal{G}$ as in Theorem 3.4, then $\mathcal{G}_2$ is a contraction map too.

Step 3: $\mathcal{G}_1$ is completely continuous on B_ρ.

From the continuity of $\mathcal{F}(\cdot, z(\cdot))$, it follows that $\mathcal{G}_1$ is continuous.

Since

$$
\begin{aligned}
\|\mathcal{G}_1 z\| = \sup_{\varsigma\in \mathrm{J}} |\mathcal{G}_1 z(\varsigma)| &\le \int_a^{\varsigma} \mathcal{G}_{\vartheta}^{\alpha_1+\alpha_2}(\varsigma, s)|\mathcal{F}(s, z(s))|\mathrm{ds} + |\lambda| \int_a^{\varsigma} \mathcal{G}_{\vartheta}^{\alpha_1}(\varsigma, s)|z(s)|\mathrm{ds}\\
&\le \|h\|\frac{(\vartheta(b)-\vartheta(a))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + |\lambda|\frac{(\vartheta(b)-\vartheta(a))^{\alpha_1}}{\Gamma(\alpha_1+1)}\rho := \mathcal{N},\ z \in B_\rho,
\end{aligned}
$$

we get $\|\mathcal{G}_1 z\| \le \mathcal{N}$ which emphasize that $\mathcal{G}_1$ uniformly bounded on B_ρ.

Finally, we prove the compactness of $\mathcal{G}_1$.

For $z \in B_\rho$ and $\varsigma \in \mathrm{J}$, we can estimate the operator derivative as follows:

$$
\begin{aligned}
\left|(\mathcal{G}_1 z)_{\vartheta}^{(1)}(\varsigma)\right| &\le \int_a^{\varsigma} \mathcal{G}_{\vartheta}^{\alpha_1+\alpha_2-1}(\varsigma, s)|\mathcal{F}(s, z(s))|\mathrm{ds} + |\lambda| \int_a^{\varsigma} \mathcal{G}_{\vartheta}^{\alpha_1-1}(\varsigma, s)|z(s)|\mathrm{ds}\\
&\le \|h\|\frac{(\vartheta(b)-\vartheta(a))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + |\lambda|\frac{(\vartheta(b)-\vartheta(a))^{\alpha_1}}{\Gamma(\alpha_1+1)}\rho := \ell,
\end{aligned}
$$

where we used the fact

$$D_{\vartheta}^{k}\, \mathcal{I}_{a^+}^{\alpha_1,\vartheta} = \mathcal{I}_{a^+}^{\alpha_1-k,\vartheta},\ \ \omega_{\vartheta}^{(k)}(\varsigma) = \left(\frac{1}{\vartheta'(\varsigma)}\frac{d}{d\varsigma}\right)^{k}\omega(\varsigma) \text{ for } k = 0, 1, ..., n-1.$$

Hence, for each $\varsigma_1, \varsigma_2 \in \mathrm{J}$ with $a < \varsigma_1 < \varsigma_2 < b$ and for $z \in B_\rho$, we get

$$|(\mathcal{G}_1 z)(\varsigma_2) - (\mathcal{G}_1 z)(\varsigma_1)| = \int_{\varsigma_1}^{\varsigma_2} |(\mathcal{G}_1 z)'(s)|\, ds \le \ell(\varsigma_2 - \varsigma_1),$$

which as $(\varsigma_2 - \varsigma_1)$ tends to zero independent of z. So, $\mathcal{G}_1$ is equicontinuous. In light of the foregoing arguments along with Arzela–Ascoli theorem, we derive that $\mathcal{G}_1$ is compact on B_ρ. Thus, the hypotheses of Theorem 2.8 holds, so there exists at least one solution of (1.3) on J. □

4. Ulam-Hyers stability analysis for the ϑ-Caputo FLDE (1.3)

In the current section, we are interested in studying Ulam-Hyers (U-H) and the generalized Ulam-Hyers stability types of the problem (1.3).

Let $\varepsilon > 0$. We consider the next inequality:

$$\left| \mathcal{D}_{a^+}^{\alpha_2;\vartheta} \left(\mathcal{D}_{a^+}^{\alpha;\vartheta} - \lambda \right) \tilde{z}(\varsigma) - \mathcal{F}(\varsigma, \tilde{z}(\varsigma)) \right| \le \varepsilon, \quad \varsigma \in \mathrm{J}. \tag{4.1}$$

Definition 4.1. FLDE (1.3) is stable in the frame of U-H type if there exists $c_{\mathcal{F}} \in \mathbb{R}^+$ such that for every $\varepsilon > 0$ and for each solution $\tilde{z} \in \mathcal{U}$ of the inequality (4.1) there exists a solution $z \in \mathcal{U}$ of (1.3) with

$$|\tilde{z}(\varsigma) - z(\varsigma)| \le \varepsilon c_{\mathcal{F}}, \quad \varsigma \in \mathrm{J}.$$

Definition 4.2. FLDE (1.3) has the generalized U-H stability if there exists $C_{\mathcal{F}} : \mathrm{C}(\mathbb{R}_+, \mathbb{R}_+)$ along with $C_{\mathcal{F}}(0) = 0$ such that for every $\varepsilon > 0$ and for each solution $\tilde{z} \in \mathcal{U}$ of the inequality (4.1), a solution $z \in C(\mathrm{J}, \mathbb{R})$ of (1.3) exists uniquely for which

$$|\tilde{z}(\varsigma) - z(\varsigma)| \le C_{\mathcal{F}}(\varepsilon), \quad \varsigma \in \mathrm{J}.$$

Remark 4.3. A function $\tilde{z} \in \mathcal{U}$ is a solution of the inequality (4.1) if and only if there exists a function $\varrho \in \mathcal{U}$ (which depends on solution $\tilde{z}$) such that

1.$|\varrho(\varsigma)| \le \varepsilon, \varsigma \in \mathrm{J}$.

2.$\mathcal{D}_{a^+}^{\alpha_2;\vartheta} \left(\mathcal{D}_{a^+}^{a,\vartheta} - \lambda \right) \tilde{z}(\varsigma) = \mathcal{F}(\varsigma, \tilde{z}(\varsigma)) + \varrho(\varsigma), \quad \varsigma \in \mathrm{J}$.

Theorem 4.4. *Let $\Delta < 1$, (H1) and (H2) hold. Then the FLDE (1.3) is U-H stable on* J *and consequently generalized U-H stable.*

Proof. For $\varepsilon > 0$ and $\tilde{z} \in C(\mathrm{J}, \mathbb{R})$ be a function which fulfills the inequality (4.1). Let $z \in \mathcal{U}$ the unique solution of

$$\begin{cases} \mathcal{D}_{a^+}^{\alpha_2;\vartheta} \left(\mathcal{D}^{\alpha_1;\vartheta} + \lambda \right) z(\varsigma) = \mathcal{F}(\varsigma, z(\varsigma)), \quad \varsigma \in \mathrm{J} = [a, b], \\ z(a) = 0, z'(a) = 0, z''(a) = 0, \\ \mathcal{D}_{a^+}^{\alpha_1;\vartheta} z(a) = \mathcal{I}_{a^+}^{\delta;\vartheta} z(\gamma), \\ \mathcal{D}_{a^+}^{\alpha_1;\vartheta} z(b) + \kappa z(b) = 0. \end{cases} \tag{4.2}$$

By Lemma 3.1, we have

$$\begin{aligned} z(\varsigma) = &\int_a^\varsigma \mathcal{G}_\vartheta^{\alpha_1+\alpha_2}(\varsigma, s) \mathcal{F}(s, z(s))\, \mathrm{ds} + \lambda \int_a^\varsigma \mathcal{G}_\vartheta^{\alpha_1}(\varsigma, s) z(s) \mathrm{ds} \\ &+ \mu(\varsigma) \int_a^\xi \mathcal{G}_\vartheta^{\delta}(\xi, s) \mathcal{F}(s, z(s))\, \mathrm{ds} \\ &+ \nu(\varsigma) \left\{ \lambda(\kappa - \lambda) \int_a^b \mathcal{G}_\vartheta^{\alpha_1}(b, s) z(s) \mathrm{ds} - (\kappa - \lambda) \int_a^b \mathcal{G}_\vartheta^{\alpha_1+\alpha_2}(b, s) \mathcal{F}(s, z(s))\, \mathrm{ds} \right. \\ &\left. - \int_a^b \mathcal{G}_\vartheta^{\alpha_2}(b, s) \mathcal{F}(s, z(s))\, \mathrm{ds} - \varpi \int_a^\xi \mathcal{G}_\vartheta^{\delta}(\xi, s) z(s) \mathrm{ds} \right\}. \end{aligned} \tag{4.3}$$

Since we have assumed that $\widetilde{z}$ is a solution of (4.1), hence by Remark 4.3

$$\begin{cases} \mathcal{D}_{a^+}^{\alpha_2;\vartheta}(\mathcal{D}_{a^+}^{\alpha_1;\vartheta}+\lambda)\widetilde{z}(\varsigma)=\mathcal{F}(\varsigma,\widetilde{z}(\varsigma))+\varrho(\varsigma), \quad \varsigma\in \mathrm{J}=[0,b],\\ z(a)=0, z'(a)=0, z''(a)=0,\\ \mathcal{D}_{a^+}^{\alpha_1;\vartheta}z(a)=\mathcal{I}_{a^+}^{\delta;\vartheta}z(\gamma),\\ \mathcal{D}_{a^+}^{\alpha_1;\vartheta}z(b)+\kappa z(b)=0. \end{cases} \tag{4.4}$$

Again by Lemma 3.1, we have

$$\begin{aligned} \widetilde{z}(\varsigma) = &\int_a^{\varsigma} \mathcal{G}_{\vartheta}^{\alpha_1+\alpha_2}(\varsigma,s)\mathcal{F}(s,\widetilde{z}(s))\,\mathrm{ds}+\lambda\int_a^{\varsigma}\mathcal{G}_{\vartheta}^{\alpha_1}(\varsigma,s)\widetilde{z}(s)\mathrm{ds}\\ &+\mu(\varsigma)\int_a^{\xi}\mathcal{G}_{\vartheta}^{\delta}(\xi,s)\mathcal{F}(s,\widetilde{z}(s))\,\mathrm{ds}\\ &+\nu(\varsigma)\left\{\lambda(\kappa-\lambda)\int_a^b\mathcal{G}_{\vartheta}^{\alpha_1}(b,s)\widetilde{z}(s)\mathrm{ds}-(\kappa-\lambda)\int_a^b\mathcal{G}_{\vartheta}^{\alpha_1+\alpha_2}(b,s)\mathcal{F}(s,\widetilde{z}(s))\,\mathrm{ds}\right.\\ &\left.-\int_a^b\mathcal{G}_{\vartheta}^{\alpha_2}(b,s)\mathcal{F}(s,\widetilde{z}(s))\,\mathrm{ds}-\varpi\int_a^{\xi}\mathcal{G}_{\vartheta}^{\delta}(\xi,s)\widetilde{z}(s)\mathrm{ds}\right\}\\ &+\int_a^{\varsigma}\mathcal{G}_{\vartheta}^{\alpha_1+\alpha_2}(\varsigma,s)\varrho(s)\,\mathrm{ds}+\mu(\varsigma)\int_a^{\xi}\mathcal{G}_{\vartheta}^{\delta}(\xi,s)\varrho(s)\,\mathrm{ds}\\ &+\nu(\varsigma)\left\{(\lambda-\kappa)\int_a^b\mathcal{G}_{\vartheta}^{\alpha_1+\alpha_2}(b,s)\varrho(s)\,\mathrm{ds}-\int_a^b\mathcal{G}_{\vartheta}^{\alpha_2}(b,s)\varrho(s)\,\mathrm{ds}\right\}. \end{aligned} \tag{4.5}$$

On the other hand, for any $\varsigma\in \mathrm{J}$

$$\begin{aligned} |\tilde{z}(\varsigma)-z(\varsigma)|\leq &\int_a^{\varsigma}\mathcal{G}_{\vartheta}^{\alpha_1+\alpha_2}(\varsigma,s)|\varrho(s)|\mathrm{d}s+\nu^*|\kappa-\lambda|\int_a^b\mathcal{G}_{\vartheta}^{\alpha_1+\alpha_2}(b,s)|\varrho(s)|\mathrm{d}s+\nu^*\int_a^b\mathcal{G}_{\vartheta}^{\alpha_2}(b,s)|\varrho(s)|\mathrm{d}s\\ &+\int_a^{\varsigma}\mathcal{G}_{\vartheta}^{\alpha_1+\alpha_2}(\varsigma,s)|\mathcal{F}(s,\varkappa(s))-\mathcal{F}(s,z(s))|\mathrm{ds}+|\lambda|\int_a^{\varsigma}\mathcal{G}_{\vartheta}^{\alpha_1}(\varsigma,s)|\varkappa(s)-z(s)|\mathrm{ds}\\ &+\mu^*\int_a^{\xi}\mathcal{G}_{\vartheta}^{\delta}(\xi,s)|\varkappa(s)-z(s)|\mathrm{ds}+\nu(\varsigma)\left\{|\lambda(\kappa-\lambda)|\int_a^b\mathcal{G}_{\vartheta}^{\alpha_1}(b,s)|\varkappa(s)-z(s)|\mathrm{ds}\right.\\ &+(\kappa-\lambda)\int_a^b\mathcal{G}_{\vartheta}^{\alpha_1+\alpha_2}(b,s)|\mathcal{F}(s,\varkappa(s))-\mathcal{F}(s,z(s))|\mathrm{ds}\\ &\left.+\int_a^b\mathcal{G}_{\vartheta}^{\alpha_2}(b,s)|\mathcal{F}(s,\varkappa(s))-\mathcal{F}(s,z(s))|\mathrm{ds}+\varpi\int_a^{\xi}\mathcal{G}_{\vartheta}^{\delta}(\xi,s)|\varkappa(s)-z(s)|\mathrm{ds}\right\}. \end{aligned}$$

Using part (i) of Remark 4.3 and (H2), we get

$$|\tilde{z}(\varsigma)-z(\varsigma)|\leq\left(\frac{(\vartheta(b)-\vartheta(a))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}[\nu^*|\kappa-\lambda|+1]+\nu^*\frac{(\vartheta(b)-\vartheta(a))^{\alpha_2}}{\Gamma(\alpha_2+1)}\right)\varepsilon+\Delta\|\tilde{z}-z\|,$$

where Δ is defined by (3.11). In consequence, it follows that

$$\|\tilde{z}-z\|_\infty\leq\left(\frac{(\vartheta(b)-\vartheta(a))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}\frac{[\nu^*|\kappa-\lambda|+1]}{(1-\Delta)}+\frac{\nu^*}{(1-\Delta)}\frac{(\vartheta(b)-\vartheta(a))^{\alpha_2}}{\Gamma(\alpha_2+1)}\right)\varepsilon.$$

If we let $c_{\mathcal{F}}=\left(\frac{(\vartheta(b)-\vartheta(a))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}\frac{[\nu^*(\kappa-\lambda)+1]}{(1-\Delta)}+\frac{\nu^*}{(1-\Delta)}\frac{(\vartheta(b)-\vartheta(a))^{\alpha_2}}{\Gamma(\alpha_2+1)}\right)$, then, the U-H stability condition is satisfied. More generally, for $C_{\mathcal{F}}(\varepsilon)=\left(\frac{(\vartheta(b)-\vartheta(a))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}\frac{[\nu^*|\kappa-\lambda|+1]}{(1-\Delta)}+\frac{\nu^*}{(1-\Delta)}\frac{(\vartheta(b)-\vartheta(a))^{\alpha_2}}{\Gamma(\alpha_2+1)}\right)\varepsilon$; $C_{\mathcal{F}}(0)=0$ the generalized U-H stability condition is also fulfilled. □

5. Examples

This section is intended to illustrate the reported results with relevant examples.

Example 5.1. We formulate the system of FLDE in the frame of Caputo type:

$$\begin{cases} {}^{c}\mathcal{D}_{0^+}^{1.2}\left({}^{c}\mathcal{D}_{0^+}^{2.5}+0.4\right)z(\varsigma)=\frac{1}{e^{\varsigma}+9}\left(1+\frac{|z(\varsigma)|}{1+|z(\varsigma)|}\right), \varsigma\in[0,1],\\ z(0)=0, z'(0)=0, z''(0)=0,\\ \mathcal{D}_{0^+}^{1.2;\varsigma}z(0)=\mathcal{I}_{0^+}^{0.5;\vartheta}z(0.1),\\ \mathcal{D}_{0^+}^{1.2;\varsigma}z(1)+0.5z(1)=0. \end{cases} \tag{5.1}$$

In this case we take

$$\alpha_1=2.5, \alpha_2=1.2, \lambda=0.4, \kappa=0.5, \delta=0.5, \xi=0.1, a=0, b=1, \vartheta(\varsigma)=\varsigma$$

$$\text{and}\quad \mathcal{F}(\varsigma,z)=\frac{1}{e^{\varsigma}+9}\left(1+\frac{|z(\varsigma)|}{1+|z(\varsigma)|}\right).$$

Obviously, the hypothesis (H1) of the Theorem 3.4 is fulfilled. On the opposite hand, for each $\varsigma\in[0,1], z,\varkappa\in\mathbb{R}$ we have

$$|\mathcal{F}(\varsigma,z)-\mathcal{F}(\varsigma,\varkappa)|\le\frac{1}{10}|z-\varkappa|.$$

Hence, (H_2) holds with $\mathcal{L}=0.1$. Thus, we find that $\Delta=0.7635<1$. Since all the assumptions in Theorem 3.4 hold, the FLDE (5.1) has a unique solution on $[0,1]$. Moreover, Theorem 4.4 ensures that the FLDE (1.3) is U-H stable and generalized U-H stable.

Example 5.2. We formulate the system of FLDE in the frame of Hadamard type:

$$\begin{cases} {}^{H}\mathcal{D}_{1^+}^{1.5;\ln\varsigma}\left({}^{H}\mathcal{D}_{1^+}^{2.7;\ln\varsigma}+0.18\right)z(\varsigma)=\frac{1}{(\varsigma+1)^2}(1+\sin z(\varsigma)), \varsigma\in[1,e],\\ z(1)=0, z'(1)=0, z''(1)=0,\\ {}^{H}\mathcal{D}_{1^+}^{1.5;\ln\varsigma}z(1)={}^{H}\mathcal{I}_{1^+}^{0.9;\ln\varsigma}z(2),\\ {}^{H}\mathcal{D}_{1^+}^{1.5;\ln\varsigma}z(e)+0.2z(e)=0. \end{cases} \tag{5.2}$$

Here

$$\mathcal{F}(\varsigma,z(\varsigma))=\frac{1}{5(\varsigma+1)^2}(1+\sin z(\varsigma)). \tag{5.3}$$

Obviously, the assumption (H1) of the Theorem 3.4 holds. On the other hand, for any $\varsigma\in[1,e], z,\varkappa\in\mathbb{R}$ we get

$$|\mathcal{F}(\varsigma,z)-\mathcal{F}(\varsigma,\varkappa)|\le\frac{1}{10}|z-\varkappa|.$$

Consequently, (H_2) holds with $\mathcal{L}=0.1$. Besides, by computation directly we find that $\Delta=0.8731<1$.

To illustrate Theorem 3.5, it is clear that the function f satisfies (H1) and (H3) with $\|h\|=0.1$. In addition, $\Lambda_3\approx0.7823<1$. It follows from theorem 3.5 that the FLDE (5.2) has a unique solution on $[1,e]$.

Example 5.3. We formulate the system of FLDE in the frame of ϑ-Caputo type:

$$\begin{cases} {}^{c}\mathcal{D}_{0^+}^{1.9;\vartheta}\left({}^{c}\mathcal{D}_{0^+}^{2.9;\vartheta}+0.25\right)z(\varsigma)=\frac{e^{\varsigma}}{4(e^{\varsigma}+1)}(1+\arctan z(\varsigma)), \varsigma\in[0,1],\\ z(0)=0, z'(0)=0, z''(0)=0,\\ \mathcal{D}_{0^+}^{\alpha_1;\vartheta}z(0)=\mathcal{I}_{a^+}^{5.4;\vartheta}z(0.3),\\ \mathcal{D}_{0^+}^{\alpha_1;\vartheta}z(1)+0.1z(1)=0. \end{cases} \tag{5.4}$$

Take

$$\alpha_1=1.9, \alpha_2=2.5, \lambda=0.25, \delta=5.4, \kappa=0.1, \xi=0.3, a_0=0, b_0=1, \vartheta(\varsigma)=e^{\varsigma}$$

$$\mathcal{F}(\varsigma,z)=\frac{e^{\varsigma}}{4(e^{\varsigma}+1)}(1+\arctan z(\varsigma)). \tag{5.5}$$

For any $\varsigma\in[0,1], z,\varkappa\in\mathbb{R}$ we obtain

$$|\mathcal{F}(\varsigma,z)-\mathcal{F}(\varsigma,\varkappa)|\le\frac{1}{8}|z-\varkappa|.$$

Hence, (H2) holds with $\mathcal{L}=0.125$. Moreover, by computation directly we find that $\Delta=0.7133<1$. It follows from Theorem 3.4 that FLDE (5.4) has a unique solution on $[0,1]$.

To illustrate Theorem 3.5, it is clear that the function $\mathcal{F}(\varsigma,z)$ given by (5.5) satisfies the hypotheses (H1)–(H3) with $\|h\|=0.125.$ and $\Lambda_3\approx0.3247<1$. It follows from Theorem 3.5 that the FLDE (5.2) has a unique solution on $[0,1]$.

6. Conclusions

In this reported article, we have considered a class of nonlinear Langevin equations involving two different fractional orders in the frame of Caputo function-dependent-kernel fractional derivatives with antiperiodic boundary conditions. The existence and uniqueness results are established for the suggested problem. Our perspective is based on properties of ϑ-Caputo's derivatives and applying of Krasnoselskii's and Banach's fixed point theorems. Moreover, we discuss the Ulam-Hyers stability criteria for the at-hand problem. Some related examples illustrating the effectiveness of the theoretical results are presented. The results obtained are recent and provide extensions to some known results in the literature. Furthermore, they cover many fractional Langevin equations that contain classical fractional operators.

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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