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Research article

Decay rate for systems of m-nonlinear wave equations with new viscoelastic structures

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Abstract: The article discusses the effect of weak and strong damping terms on decay rate for systems of nonlinear m- wave equations with new viscoelastic structures. The factors that allowed system (1.1) to coexist for a long time are the strong nonlinearities in the sources. We showed, under a novel condition on the kernel function in (2.4), a new scenario for energy decay in (3.7) by using an appropriate energy estimates. This result extend the results in [18, 27] for system of m-equations inspired from the paper [1].

Keywords: wave equation; strong nonlinear system; global solution; exponential decay rate
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1. Introduction and position of problem

We consider, for $x \in \mathbb{R}^n,\ t > 0,\ j = 1, 2, \ldots, m$, the following system of m equations

$$\begin{cases} u_{jtt} + au_{jt} - \Theta(x)\Delta\left(u_j + \omega u_{jt} - \int_0^t \varpi_j(t-s)u_j(s)\,ds\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad = f_j(u_1, u_2, \ldots, u_m) \\ u_j(x, 0) = u_{j0}(x) \\ u_{jt}(x, 0) = u_{j1}(x), \end{cases} \tag{1.1}$$

where $a \in \mathbb{R}$, $\omega > 0$, $n \geq 3$.

Various non-linear sources have been combined as follows, we combine all two consecutive equations together and of course the last equation with the first one, which get the whole system closely linked by the strong nonlinear sources. The functions $f_j(u_1, u_2, \ldots, u_m) \in (\mathbb{R}^m, \mathbb{R})$ are given for

$j = 1, 2, \ldots, m-1$, by

$$f_j(u_1, u_2, \ldots, u_m) = (p+1)\Big[d\Big|\sum_{i=1}^{m} u_i\Big|^{(p-1)} \sum_{i=1}^{m} u_i + e|u_j|^{(p-3)/2} u_j |u_{j+1}|^{(p+1)/2}\Big],$$

and

$$f_m(u_1, u_2, \ldots, u_m) = (p+1)\Big[d\Big|\sum_{i=1}^{m} u_i\Big|^{(p-1)} \sum_{i=1}^{m} u_i + e|u_m|^{(p-3)/2} u_m |u_1|^{(p+1)/2}\Big],$$

with $d, e > 0,\ p > 3$. For simplicity reason, we take $d = e = 1$.
There exists a function $\mathcal{F} \in C^1(\mathbb{R}^3, \mathbb{R})$ such that

$$\sum_{j=1}^{m} u_j f_j(u_1, u_2, \ldots, u_m) = (p+1)\mathcal{F}(u_1, u_2, \ldots, u_m),\ \forall (u_1, u_2, \ldots, u_m) \in \mathbb{R}^m. \tag{1.2}$$

such that

$$(p+1)\mathcal{F}(u_1, u_2, \ldots, u_m) = \Big|\sum_{j=1}^{m} u_j\Big|^{p+1} + 2\Big|\sum_{j=1}^{m-1} u_j u_{j+1}\Big|^{(p+1)/2} + 2|u_m u_1|^{(p+1)/2}. \tag{1.3}$$

In order to use Poincare's inequality which is a key in calculus for the PDEs, we will study the problem (1.1) in the presence of a density function θ to find a generalized formula for Poincare's inequality that can be used in unbounded domain $\mathbb{R}^n$. The function $\Theta(x) > 0$ for all $x \in \mathbb{R}^n$ is a density and $(\Theta)^{-1}(x) = 1/\Theta(x) \equiv \theta(x)$ such that

$$\theta \in L^{\tau}(\mathbb{R}^n) \quad \text{with} \quad \tau = \frac{2n}{2n - rn + 2r} \quad \text{for} \quad 2 \leq r \leq \frac{2n}{n-2}. \tag{1.4}$$

We define a new space related to the nature of our system, taking into account the boundless of space $\mathbb{R}^n$. The function spaces $\mathcal{H}$ is defined as the closure of $C_0^{\infty}(\mathbb{R}^n)$, as in [20], we have

$$\mathcal{H} = \{v \in L^{\frac{2n}{n-2}}(\mathbb{R}^n) \mid \nabla v \in L^2(\mathbb{R}^n)^n\}.$$

with respect to the norm $\|v\|_{\mathcal{H}} = (v, v)_{\mathcal{H}}^{1/2}$ for the inner product

$$(v, w)_{\mathcal{H}} = \int_{\mathbb{R}^n} \nabla v \cdot \nabla w \, dx,$$

and $L^2_{\theta}(\mathbb{R}^n)$ as that to the norm $\|v\|_{L^2_{\theta}} = (v, v)_{L^2_{\theta}}^{1/2}$ for

$$(v, w)_{L^2_{\theta}} = \int_{\mathbb{R}^n} \theta v w \, dx.$$

For general $r \in [1, +\infty)$

$$\|v\|_{L^r_{\theta}} = \left(\int_{\mathbb{R}^n} \theta \, |v|^r \, dx\right)^{\frac{1}{r}}.$$

is the norm of the weighted space $L^r_\theta(\mathbb{R}^n)$.

The following references in connection to our system for a single equation [6] and [7]. The work [6] was the pioneer in the literature for the single equation, source of inspiration of several works, while the work [7] is a recent generalization of [6] by introducing less dissipative effects (See [8, 9, 19, 24, 26]). With regard to the study of this type of systems without viscoelasticity, with the existence of both weak damping u_t and strong damping Δu_t, under condition (3.2), we mention the work recently published in one equation in [14]

$$\begin{cases} u_{tt} + \mu u_t - \Delta u - \omega \Delta u_t = u \ln |u|, \ (x,t) \in \Omega \times (0,\infty) \\ u(x,t) = 0, x \in \partial\Omega, t \geq 0 \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \end{cases} \tag{1.5}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 1$ with a smooth boundary $\partial\Omega$. The aim goal was mainly on the local existence of weak solution by using contraction mapping principle and of course the authors showed the global existence, decay rate and infinite time blow up of the solution with certain conditions on initial energy.

In the case of non-bounded domain $\mathbb{R}^n$, we mention the paper recently published by T. Miyasita and Kh. Zennir in [18], where they considered equation as follows

$$u_{tt} + au_t - \phi(x)\Delta\left(u + \omega u_t - \int_0^t g(t-s)u(s)\,ds\right) = u|u|^{p-1}, \tag{1.6}$$

with initial data

$$\begin{cases} u(x,0) = u_0(x) \\ u_t(x,0) = u_1(x). \end{cases} \tag{1.7}$$

The authors succeeded in highlighting the existence of unique local solution and they continued to expand it to be global in time. The rate of the decay for solution was the main result, for more results related to decay rate of solution of this type of problems, please see [15, 23, 25, 28].

Regarding the study of the coupled system of two nonlinear wave equations, we mention the work done by Baowei Feng *et al.* which was considered in [12], a coupled system for viscoelastic wave equations with nonlinear sources in bounded domain with smooth boundary as follows

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + u_t = f_1(u,v) \\ \\ v_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(s)\,ds + v_t = f_2(u,v). \end{cases} \tag{1.8}$$

Under appropriate hypotheses, they established a general decay result by multiplication techniques to extend some existing results for a single equation to the case of a coupled system.

There are several results in this direction, notably the generalization made by Shun in a complicate nonlinear case with degenerate damping term in [21]. The IBVP for a system of nonlinear viscoelastic wave equations in a bounded domain was considered in the problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + (|u|^k + |v|^q)|u_t|^{m-1}u_t = f_1(u,v) \\ v_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(s)\,ds + (|v|^\theta + |u|^\rho)|v_t|^{r-1}v_t = f_2(u,v) \\ u(x,t) = v(x,t) = 0, x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x), v(x,0) = v_0(x) \\ u_t(x,0) = u_1(x), v_t(x,0) = v_1(x), \end{cases} \tag{1.9}$$

where Ω is a bounded domain with a smooth boundary. Given certain conditions on the kernel functions, degenerate damping and nonlinear source terms, they got a decay rate of the energy function for some initial data.

In $n-$equations, paper in [1] considered a system

$$u_{itt} + \gamma u_{it} - \Delta u_i + u_i = \sum_{i,j=1,i\neq j}^{m} |u_j|^{p_j}|u_i|^{p_i}u_i,\ i = 1, 2, \ldots, m, \tag{1.10}$$

where the absence of global solutions with positive initial energy was investigated. Next, a nonexistence of global solutions for system of three semilinear hyperbolic equations was introduced in [3]. A coupled system of semilinear hyperbolic equations was investigated by many authors and many results were obtained with the nonlinearities in the form $f_1 = |u|^{p-1}|v|^{q+1}u, f_2 = |v|^{p-1}|u|^{q+1}v$. (Please, see [2, 16, 22])

2. Preliminaries

We introduce the Sobolev embedding and generalized Poincaré inequalities.

Lemma 2.1. *[13, 18] Let θ satisfy (1.4). For positive constants $C_\tau > 0$ and $C_P > 0$ depending only on θ and n, we have*

$$\|v\|_{\frac{2n}{n-2}} \leq C_\tau \|v\|_{\mathcal{H}},\ \ \|v\|_{L^2_\theta} \leq C_P \|v\|_{\mathcal{H}},$$

and

$$\|v\|_{L^r_\theta} \leq C_r \|v\|_{\mathcal{H}},\ \ C_r = C_\tau \|\theta\|_\tau^{\frac{1}{r}},$$

hold for $v \in \mathcal{H}$. Here $\tau = 2n/(2n - rn + 2r)$ for $1 \leq r \leq 2n/(n - 2)$.

In the 1950s and 1970s, the linear theory of viscoelasticity was extensively developed and now, it becomes widely used to represent this term using several improvements to the nature of decreasing the kernel function. We assume that the kernel functions $\varpi_j \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfying

$$1 - \overline{\varpi_j} = \rho_j > 0 \quad \text{for} \quad \overline{\varpi_j} = \int_0^{+\infty} \varpi_j(s)\, ds,\ \varpi_j'(t) \leq 0,\ \varpi_j(0) > 0. \tag{2.1}$$

We mean by $\mathbb{R}^+$ the set $\{\tau \mid \tau \geq 0\}$. Noting by

$$\mu(t) = \max_{t\geq 0} \Big\{\varpi_1(t), \varpi_2(t), \ldots, \varpi_m(t)\Big\}, \tag{2.2}$$

and

$$\mu_0(t) = \min_{t\geq 0} \Big\{ \int_0^t \varpi_1(s)ds, \int_0^t \varpi_2(s)ds, \ldots, \int_0^t \varpi_m(s)ds\Big\}. \tag{2.3}$$

We assume that there is a function $\chi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ which is linear or is strictly convex C^2 function on $(0, \varepsilon_0), \varepsilon_0 \leq \varpi_j(0)$, with $\chi(0) = \chi'(0) = 0$ and a positive nonincreasing differentiable function $\xi : [0, \infty) \to [0, \infty)$, such that the novel properties

$$\varpi_j'(t) + \xi(t)\chi(\varpi_j(t)) \leq 0, \quad \chi(0) = 0, \quad \chi'(0) > 0 \quad \text{and} \quad \chi''(\varrho) \geq 0,\ i = 1, 2, \ldots, m, \tag{2.4}$$

satisfied for any $\varrho \geq 0$.
We note that, if χ is a strictly increasing convex C^2- function on $(0,\tau]$ with $\chi(0)=\chi'(0)=0$, then χ has an extension $\bar{\chi}$, which is strictly increasing and strictly convex C^2-function on $(0,\infty)$. For example, $\bar{\chi}$ can be given by

$$\bar{\chi}(t)=\frac{1}{2}\chi''(\tau)t^2+[\chi'(\tau)-\chi''(\tau)\tau]t+\chi(\tau)-\chi'(\tau)\tau+\frac{1}{2}\chi''(\tau)\tau^2,\ t>\tau.$$

Hölder and Young's inequalities give

$$\begin{aligned}\left\|u_iu_j\right\|^{(p+1)/2}_{L^{(p+1)/2}_\theta} &\leq \left(\|u_i\|^2_{L^{(p+1)}_\theta}+\|u_j\|^2_{L^{(p+1)}_\theta}\right)^{(p+1)/2}\\ &\leq \left(\rho_i\|u_i\|^2_{\mathcal{H}}+\rho_j\|u_j\|^2_{\mathcal{H}}\right)^{(p+1)/2},\end{aligned} \tag{2.5}$$

Thanks to Minkowski's inequality to give

$$\begin{aligned}\left\|\sum_{j=1}^m u_j\right\|^{(p+1)}_{L^{(p+1)}_\theta} &\leq c\left(\sum_{j=1}^m\|u_j\|^2_{L^{(p+1)}_\theta}\right)^{(p+1)/2}\\ &\leq c\left(\sum_{j=1}^m\|u_j\|^2_{\mathcal{H}}\right)^{(p+1)/2}.\end{aligned}$$

Then, there exist $\eta>0$ such that

$$\begin{aligned}&\left\|\sum_{j=1}^m u_j\right\|^{(p+1)}_{L^{(p+1)}_\theta}+2\left\|\sum_{j=1}^{m-1}u_ju_{j+1}\right\|^{(p+1)/2}_{L^{(p+1)/2}_\theta}+2\|u_mu_1\|^{(p+1)/2}_{L^{(p+1)/2}_\theta}\\ &\leq \eta\left(\sum_{j=1}^m\rho_j\|u_j\|^2_{\mathcal{H}}\right)^{(p+1)/2}.\end{aligned} \tag{2.6}$$

We need to define positive constants λ_0 and $\mathcal{E}_0$ by

$$\lambda_0\equiv\eta^{-1/(p-1)}\quad\text{and}\quad\mathcal{E}_0=\Big(\frac{1}{2}-\frac{1}{p+1}\Big)\eta^{-2/(p-1)}. \tag{2.7}$$

The mainly aim of the present paper is to obtain a novel decay rate of solution from the convexity property of the function χ given in Theorem 3.4.
We denote, as in [18], an eigenpair $\{(\lambda_i,e_i)\}_{i\in\mathbb{N}}\subset\mathbb{R}\times\mathcal{H}$ of

$$-\Theta(x)\Delta e_i=\lambda_ie_i\quad x\in\mathbb{R}^n,$$

for any $i\in\mathbb{N}$. Then

$$0<\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_i\leq\cdots\uparrow+\infty,$$

holds and $\{e_i\}$ is a complete orthonormal system in $\mathcal{H}$.

Definition 2.2. *The vectors* $(u_1, u_2, \ldots, u_m)$ *is said a weak solution to (1.1) on* $[0, T]$ *if satisfies for* $x \in \mathbb{R}^n$

$$\begin{aligned} & \int_{\mathbb{R}^n} u_{jtt}\varphi_j dx + a\int_{\mathbb{R}^n} u_{jt}\varphi_j dx \\ + & \int_{\mathbb{R}^n} \Theta(x)\nabla\left(u_j + \omega u_{jt} - \int_0^t \varpi_j(t-s)u_j(s)\,ds\right)\nabla\varphi_j dx \\ = & \int_{\mathbb{R}^n} f_j(u_1, u_2, \ldots, u_m)\varphi_j dx, \end{aligned} \tag{2.8}$$

for all test functions $\varphi_j \in \mathcal{H}$, $j = 1, 2, \ldots, m$ for almost all $t \in [0, T]$.

3. Statement of main results

The local solution (in time $[0, T]$) is given in next Theorem.

Theorem 3.1. *(Local existence) Assume that*

$$1 < p \leq \frac{n+2}{n-2} \quad \text{and} \quad n \geq 3. \tag{3.1}$$

Let $(u_{10}, u_{20}, \ldots u_{m0}) \in \mathcal{H}^m$ *and* $(u_{11}, u_{21}, \ldots, u_{m1}) \in [L^2_\theta(\mathbb{R}^n)]^m$*. Under the assumptions (1.4)-(1.3) and (2.1)-(2.4), suppose that*

$$a + \lambda_1\omega > 0. \tag{3.2}$$

Then (1.1) admits a unique local solution $(u_1, u_2, \ldots, u_m)$ *such that*

$$(u_1, u_2, \ldots, u_m) \in \mathcal{X}_T^m,\ \mathcal{X}_T \equiv C([0, T]; \mathcal{H}) \cap C^1([0, T]; L^2_\theta(\mathbb{R}^n)),$$

for sufficiently small $T > 0$.

Remark 3.2. *The constant* λ_1 *introduced in (3.2) being the first eigenvalue of the operator* $-\Delta$.

We will show now the global solution in time established in Theorem 3.3. Let us introduce the potential energy $J : \mathcal{H}^m \to \mathbb{R}$ defined by

$$J(u_1, u_2, \ldots, u_m) = \sum_{j=1}^m \left(1 - \int_0^t \varpi_j(s)\,ds\right)\left\|u_j\right\|^2_{\mathcal{H}} + \sum_{j=1}^m \left(\varpi_j \circ u_j\right). \tag{3.3}$$

The modified energy is defined by

$$\mathcal{E}(t) = \frac{1}{2}\sum_{j=1}^m \left\|u_{jt}\right\|^2_{L^2_\theta} + \frac{1}{2}J(u_1, u_2, \ldots, u_m) - \int_{\mathbb{R}^n} \theta(x)\mathcal{F}(u_1, u_2, \ldots, u_m)dx, \tag{3.4}$$

here

$$\left(\varpi_j \circ w\right)(t) = \int_0^t \varpi_j(t-s)\left\|w(t) - w(s)\right\|^2_{\mathcal{H}}\,ds,$$

for any $w \in L^2(\mathbb{R}^n)$, $j = 1, 2, \ldots, m$.

Theorem 3.3. *(Global existence) Let (1.4)-(1.3) and (2.1)-(2.4) hold. Under (3.1), (3.2) and for sufficiently small* $(u_{10}, u_{11}), (u_{20}, u_{21}), \ldots, (u_{m0}, u_{m1}) \in \mathcal{H} \times L^2_\theta(\mathbb{R}^n)$*, problem (1.1) admits a unique global solution* $(u_1, u_2, \ldots, u_m)$ *such that*

$$(u_1, u_2, \ldots, u_m) \in \mathcal{X}^m, \ \mathcal{X} \equiv C([0, +\infty); \mathcal{H}) \cap C^1([0, +\infty); L^2_\theta(\mathbb{R}^n)). \tag{3.5}$$

The decay rate for solution is given in the next Theorem.

Theorem 3.4. *(Decay of solution) Let (1.4)-(1.3) and (2.1)-(2.4) hold. Under conditions (3.1), (3.2) and*

$$\gamma = \eta\Big(\frac{2(p+1)}{p-1}\mathcal{E}(0)\Big)^{(p-1)/2} < 1, \tag{3.6}$$

there exists $t_0 > 0$ *depending only on* ϖ_j*,* a*,* ω*,* λ_1 *and* $\chi'(0)$ *such that*

$$0 \leq \mathcal{E}(t) < \mathcal{E}(t_0) \exp\left(-\int_{t_0}^{t} \frac{\mu(s)}{1 - \mu_0(t)}\right), \tag{3.7}$$

holds for all $t \geq t_0$*.*

In particular, by the positivity of μ in (2.2), we have, as in [17],

$$0 \leq \mathcal{E}(t) < \mathcal{E}(t_0) \exp\left(-\int_{t_0}^{t} \mu(s)\, ds\right),$$

for a single wave equation.

Lemma 3.5. *For* $(u_1, u_2, \ldots, u_m) \in \mathcal{X}^m_T$*, the functional* $\mathcal{E}(t)$ *associated with problem (1.1) is a decreasing energy.*

Proof. For $0 \leq t_1 < t_2 \leq T$, we have

$$\begin{aligned} & \mathcal{E}(t_2) - \mathcal{E}(t_1) \\ = & \int_{t_1}^{t_2} \frac{d}{dt}\mathcal{E}(t)\, dt \\ = & -\sum_{j=1}^{m} \int_{t_1}^{t_2} \left(a \left\| u_{jt} \right\|^2_{L^2_\theta} + \omega \left\| u_{jt} \right\|^2_{\mathcal{H}} + \frac{1}{2}\varpi_j(t) \left\| u_j \right\|^2_{\mathcal{H}} - \frac{1}{2}(\varpi'_j \circ u_j) \right) dt \\ \leq & \ 0, \end{aligned}$$

owing to (2.1)-(2.4). □

We define an inner product as

$$(v, w)_* = \omega \int_{\mathbb{R}^n} \nabla v \cdot \nabla w\, dx + a \int_{\mathbb{R}^n} \theta v w\, dx,$$

and the associated norm is given by

$$\|v\|_* = \sqrt{(v, v)_*}.$$

$\forall v, w \in \mathcal{H}$. By (3.2), we get

$$(v, v)_* = \omega \int_{\mathbb{R}^n} |\nabla v|^2\, dx + a \int_{\mathbb{R}^n} \theta v^2\, dx \geq (\omega\lambda_1 + a) \int_{\mathbb{R}^n} \theta v^2\, dx \geq 0.$$

The following Lemma yields.

Lemma 3.6. *Let θ satisfy (1.4). Under condition (3.2), we get*

$$\sqrt{\omega}\,\|v\|_{\mathcal{H}} \leq \|v\|_{*} \leq \sqrt{\omega + C_P^2}\,\|v\|_{\mathcal{H}}\,,$$

for $v \in \mathcal{H}$.

4. Proof of existence results

We give here the outline of the proof for local solution by a standard procedure (See [15, 28]).

Proof. (Of Theorem 3.1.) Let $(u_{10}, u_{11}), (u_{20}, u_{21}), \ldots, (u_{m0}, u_{m1}) \in \mathcal{H} \times L^2_\theta(\mathbb{R}^n)$. The presence of the nonlinear terms in the right hand side of our problem (1.1) gives us negative values of the energy. For this purpose, for any fixed $(u_1, u_2, \ldots, u_m) \in \mathcal{X}_T^m$, we can obtain first, a weak solution of the related system

$$\begin{cases} z_{jtt} + az_{jt} - \Theta(x)\Delta\left(z_j + \omega z_{jt}\right) & +\Theta(x)\Delta\int_0^t \varpi_j(t-s)z_j(s)\,ds \\ & = f_j(u_1, u_2, \ldots, u_m) \\ z_j(x,0) = u_{j0}(x) \\ z_{jt}(x,0) = u_{j1}(x). \end{cases} \tag{4.1}$$

The Faedo-Galerkin's method consist to construct approximations of solutions $(z_{1n}, z_{2n}, \ldots, z_{mn},)$ for (4.1), then we obtain a prior estimates necessary to guarantee the convergence of approximations. In the last step we pass to the limit of the approximations by using the compactness of some embedding in the Sobolev spaces. The uniqueness is obtain by letting two solutions for (4.1) and then, after ordinary calculations, we find that the solutions are equal.
Some details regarding the transition to ODE systems are given, for this end let $\{e_i\}$ be the Galerkin basis and let

$$W_{jn} = span\{e_{j1}, e_{j2}, \ldots., e_{jn}\},\ j = 1, \ldots, m.$$

Given initial data $u_{j0} \in \mathcal{H}$, $u_{j1} \in L^2_\theta(\mathbb{R}^n)$, we define the approximations

$$z_{jn} = \sum_{i=1}^{n} g_{jin}(t)e_{ji}(x), \tag{4.2}$$

which satisfy the following approximate problem

$$\begin{aligned} &\left(z_{jntt}, e_{ji}\right) + \left(az_{jnt}, e_{ji}\right) - \left(\Theta(x)\Delta\left(z_{jn} + \omega z_{jnt}\right), e_{ji}\right) \\ &= -\left(\Theta(x)\Delta\int_0^t \varpi_j(t-s)z_{jn}(s)\,ds, e_{ji}\right) + \left(f_j(u_1, u_2, \ldots, u_m), e_{ji}\right), \end{aligned} \tag{4.3}$$

with initial conditions

$$z_{jn}(x,0) = u^n_{j0}(x),\ z_{jnt}(x,0) = u^n_{j1}(x), \tag{4.4}$$

which satisfies

$$u^n_{j0} \to u_{j0},\ \ strongly\ \ in\ \ \mathcal{H}$$

$$u_1^{jn} \to u_{j1}, \;\; strongly \;\; in \;\; L^2_\theta(\mathbb{R}^n). \tag{4.5}$$

Taking $e_{ji} = g_{ji}$ in (4.3) yields the following Cauchy problem for a ordinary differential equation with unknown g^n_{ji}.

$$\begin{aligned} & g^n_{jitt}(t) + ag^n_{jit}(t) + \lambda_i \left(g^n_{ji}(t) + \omega g^n_{jit}(t) \right) \\ & = \lambda_i \int_0^t \varpi_j(t-s) g^n_{ji}(s)\, ds + \left(f_j(u_1, u_2, \ldots, u_m), g_{ji} \right), \end{aligned} \tag{4.6}$$

By using the Caratheodory Theorem for standard ordinary differential equations theory, the problem (4.3)-(4.4) has a solutions $(g_{1in}, g_{2in}, \ldots, g_{min})_{i=1,n} \in (H^3[0,T])^m$ and by using the embedding $H^m[0,T] \to C^m[0,T]$, we deduce that the solution $(g_{1in}, g_{2in}, \ldots, g_{min})_{i=1,n} \in (C^2[0,T])^4$. In turn, this gives a unique $(z_{1n}, z_{2n}, \ldots, z_{mn})$ defined by (4.2) and satisfying (4.3).
To return to the problem (1.1), we should find a solution map

$$\top : (u_1, u_2, \ldots, u_m) \mapsto (z_1, z_2, \ldots, z_m)$$

from $\mathcal{X}^m_T$ to $\mathcal{X}^m_T$. We are now ready to show that $\top$ is a contraction mapping in an appropriate subset of $\mathcal{X}^m_T$ for a small $T > 0$. Hence $\top$ has a fixed point

$$\top(u_1, u_2, \ldots, u_m) = (u_1, u_2, \ldots, u_m),$$

which gives a unique solution in $\mathcal{X}^m_T$. $\square$

We will show the global solution. For this end, by using conditions on functions ϖ_j, we have

$$\begin{aligned} \mathcal{E}(t) &\geq \frac{1}{2} J(u_1, u_2, \ldots, u_m) - \int_{\mathbb{R}^n} \theta(x) \mathcal{F}(u_1, u_2, \ldots, u_m) dx \\ &\geq \frac{1}{2} J(u_1, u_2, \ldots, u_m) - \frac{1}{p+1} \left\| \sum_{j=1}^m u_j \right\|^{(p+1)}_{L^{(p+1)}_\theta} \\ &- \frac{2}{p+1} \Big(\Big\| \sum_{j=1}^{m-1} u_j u_{j+1} \Big\|^{(p+1)/2}_{L^{(p+1)/2}_\theta} + \| u_m u_1 \|^{(p+1)/2}_{L^{(p+1)/2}_\theta} \Big) \\ &\geq \frac{1}{2} J(u_1, u_2, \ldots, u_m) - \frac{\eta}{p+1} \Big[\sum_{j=1}^m \rho_j \left\| u_j \right\|^2_{\mathcal{H}} \Big]^{(p+1)/2} \\ &\geq \frac{1}{2} J(u_1, u_2, \ldots, u_m) - \frac{\eta}{p+1} \Big(J(u_1, u_2, \ldots, u_m) \Big)^{(p+1)/2} \\ &= G(\beta), \end{aligned} \tag{4.7}$$

here $\beta^2 = J(u_1, u_2, \ldots, u_m)$, for $t \in [0,T)$, where

$$G(\xi) = \frac{1}{2}\xi^2 - \frac{\eta}{p+1} \xi^{(p+1)}.$$

Noting that $\mathcal{E}_0 = G(\lambda_0)$, given in (2.7). Then

$$\begin{cases} G(\xi) \geq 0 \quad in \;\; \xi \in [0, \lambda_0] \\ \\ G(\xi) < 0 \quad in \;\; \xi > \lambda_0. \end{cases} \tag{4.8}$$

Moreover, $\lim\limits_{\xi \to +\infty} G(\xi) \to -\infty$. Then, we have the following Lemma

Lemma 4.1. *Let* $0 \leq \mathcal{E}(0) < \mathcal{E}_0$.
(i) *If* $\sum\limits_{j=1}^{m} \left\|u_{j0}\right\|_{\mathcal{H}}^2 < \lambda_0^2$, *then local solution of (1.1) satisfies*

$$J(u_1, u_2, \ldots, u_m) < \lambda_0^2, \;\; \forall t \in [0, T).$$

(ii) *If* $\sum\limits_{j=1}^{m} \left\|u_{j0}\right\|_{\mathcal{H}}^2 > \lambda_0^2$, *then, local solution of (1.1) satisfies*

$$\sum_{j=1}^{m} \left\|u_j\right\|_{\mathcal{H}}^2 > \lambda_1^2, \;\; \forall t \in [0, T), \lambda_1 > \lambda_0.$$

Proof. Since $0 \leq \mathcal{E}(0) < \mathcal{E}_0 = G(\lambda_0)$, there exist ξ_1 and ξ_2 such that $G(\xi_1) = G(\xi_2) = \mathcal{E}(0)$ with $0 < \xi_1 < \lambda_0 < \xi_2$.
The case (i)**.** By (4.7), we have

$$G(J(u_{10}, u_{20}, \ldots u_{m0})) \leq \mathcal{E}(0) = G(\xi_1),$$

which implies that $J(u_{10}, u_{20}, \ldots u_{m0}) \leq \xi_1^2$. Then, we claim that $J(u_1, u_2, \ldots, u_m) \leq \xi_1^2, \;\; \forall t \in [0, T)$. Moreover, there exists $t_0 \in (0, T)$ such that

$$\xi_1^2 < J(u_1(t_0), u_2(t_0), \ldots, u_m(t_0)) < \xi_2^2.$$

Then

$$G(J(u_1(t_0), u_2(t_0), \ldots, u_m(t_0)) > \mathcal{E}(0) \geq \mathcal{E}(t_0),$$

by Lemma 3.5, which contradicts (4.7). Hence we have

$$J(u_1, u_2, \ldots, u_m) \leq \xi_1^2 < \lambda_0^2, \;\; \forall t \in [0, T).$$

The case (ii)**.** We can now show that $\sum\limits_{j=1}^{m} \left\|u_{j0}\right\|_{\mathcal{H}}^2 \geq \xi_2^2$ and that $\sum\limits_{j=1}^{m} \left\|u_j\right\|_{\mathcal{H}}^2 \geq \xi_2^2 > \lambda_0^2$ in the same way as (i). □

Proof. (*Of Theorem 3.3.*) Let $(u_{10}, u_{11}), (u_{20}, u_{21}), \ldots, (u_{m0}, u_{m1}) \in \mathcal{H} \times L_\theta^2(\mathbb{R}^n)$ satisfy both $0 \leq \mathcal{E}(0) < \mathcal{E}_0$ and $\sum\limits_{j=1}^{m} \left\|u_{j0}\right\|_{\mathcal{H}}^2 < \lambda_0^2$. By Lemma 3.5 and Lemma 4.1, we have

$$\sum_{j=1}^{m} \left\|u_{jt}\right\|_{L_\theta^2}^2 + \sum_{j=1}^{m} \rho_j \left\|u_j\right\|_{\mathcal{H}}^2$$

$$
\begin{aligned}
&\leq \sum_{j=1}^{m} \left\|u_{jt}\right\|_{L_\theta^2}^2 + \sum_{j=1}^{m} \Big[\Big(1 - \int_0^t \varpi_j(s)\, ds\Big) \left\|u_j\right\|_{\mathcal{H}}^2 + \big(\varpi_j \circ u_j\big) \Big] \\
&\leq 2\mathcal{E}(t) + \frac{2\eta}{p+1} \Big(\sum_{j=1}^{m} \rho_j \left\|u_j\right\|_{\mathcal{H}}^2 \Big)^{(p+1)/2} \\
&\leq 2\mathcal{E}(0) + \frac{2\eta}{p+1} \Big(J(u_1, u_2, \ldots, u_m) \Big)^{(p+1)/2} \\
&\leq 2\mathcal{E}_0 + \frac{2\eta}{p+1} \lambda_0^{p+1} \\
&= \eta^{-2/(p-1)}.
\end{aligned} \tag{4.9}
$$

This completes the proof. □

5. Proof of general decay result

Let

$$
\begin{aligned}
\Lambda(u_1, u_2, \ldots, u_m) &= \frac{1}{2} \sum_{j=1}^{m} \Big[\Big(1 - \int_0^t \varpi_j(s)\, ds\Big) \left\|u_j\right\|_{\mathcal{H}}^2 + \big(\varpi_j \circ u_j\big) \Big] \\
&- \int_{\mathbb{R}^n} \theta(x) \mathcal{F}(u_1, u_2, \ldots, u_m) dx,
\end{aligned}
$$

$$
\begin{aligned}
\Pi(u_1, u_2, \ldots, u_m) &= \sum_{j=1}^{m} \Big[\Big(1 - \int_0^t \varpi_j(s)\, ds\Big) \left\|u_j\right\|_{\mathcal{H}}^2 + \big(\varpi_j \circ u_j\big) \Big] \\
&- (p+1) \int_{\mathbb{R}^n} \theta(x) \mathcal{F}(u_1, u_2, \ldots, u_m) dx.
\end{aligned}
$$

Lemma 5.1. *Let $(u_1, u_2, \ldots, u_m)$ be the solution of problem (1.1). If*

$$
\sum_{j=1}^{m} \left\|u_{j0}\right\|_{\mathcal{H}}^2 - (p+1) \int_{\mathbb{R}^n} \theta(x) \mathcal{F}(u_1, u_2, \ldots, u_m) dx > 0. \tag{5.1}
$$

Then, under condition (3.6), the functional $\Pi(u_1, u_2, \ldots, u_m) > 0,\ \forall t > 0$.

Proof. By (5.1) and continuity, there exists a time $t_1 > 0$ such that

$$
\Pi(u_1, u_2, \ldots, u_m) \geq 0, \forall t < t_1.
$$

Let

$$
Y = \{(u_1, u_2, \ldots, u_m) \mid \Pi(u_1(t_0), u_2(t_0), \ldots, u_m(t_0)) = 0,\ \Pi(u_1, u_2, \ldots, u_m) > 0, \forall t \in [0, t_0)\}. \tag{5.2}
$$

Then, by (5.1), we have for all $(u_1, u_2, \ldots, u_m) \in Y$,

$$
\Lambda(u_1, u_2, \ldots, u_m)
$$

$$
\begin{aligned}
&= \frac{p-1}{2(p+1)} \sum_{j=1}^{m} \left(1 - \int_0^t \varpi_j(s)\, ds\right) \left\|u_j\right\|_{\mathcal{H}}^2 \\
&+ \frac{p-1}{2(p+1)} \sum_{j=1}^{m} \left(\varpi_j \circ u_j\right) + \frac{1}{p+1} \Pi(u_1, u_2, \ldots, u_m) \\
&\geq \frac{p-1}{2(p+1)} \sum_{j=1}^{m} \left[\rho_j \left\|u_j\right\|_{\mathcal{H}}^2 + \left(\varpi_j \circ u_j\right)\right].
\end{aligned}
$$

Owing to (3.4), it follows for $(u_1, u_2, \ldots, u_m) \in Y$

$$
\rho_j \left\|u_j\right\|_{\mathcal{H}}^2 \leq \frac{2(p+1)}{p-1} \Lambda(u_1, u_2, \ldots, u_m) \leq \frac{2(p+1)}{p-1} \mathcal{E}(t) \leq \frac{2(p+1)}{p-1} \mathcal{E}(0). \tag{5.3}
$$

By (2.6), (3.6) we have

$$
\begin{aligned}
& (p+1) \int_{\mathbb{R}^n} \mathcal{F}(u_1(t_0), u_2(t_0), \ldots, u_m(t_0)) \\
\leq\ & \eta \sum_{j=1}^{m} \left(\rho_j \left\|u_j(t_0)\right\|_{\mathcal{H}}^2\right)^{(p+1)/2} \\
\leq\ & \eta \Big(\frac{2(p+1)}{p-1} E(0)\Big)^{(p-1)/2} \sum_{j=1}^{m} \rho_j \left\|u_j(t_0)\right\|_{\mathcal{H}}^2 \\
\leq\ & \gamma \sum_{j=1}^{m} \rho_j \left\|u_j(t_0)\right\|_{\mathcal{H}}^2 \\
<\ & \sum_{j=1}^{m} \Big(1 - \int_0^{t_0} \varpi_j(s) ds\Big) \left\|u_j(t_0)\right\|_{\mathcal{H}}^2 \\
<\ & \sum_{j=1}^{m} \Big(1 - \int_0^{t_0} \varpi_j(s) ds\Big) \left\|u_j(t_0)\right\|_{\mathcal{H}}^2 \\
+\ & \sum_{j=1}^{m} (\varpi_j \circ u_j(t_0)),
\end{aligned} \tag{5.4}
$$

hence $\Pi(u_1(t_0), u_2(t_0), \ldots, u_m(t_0)) > 0$ on Y, which contradicts the definition of Y since $\Pi(u_1(t_0), u_2(t_0), \ldots, u_m(t_0)) = 0$. Thus $\Pi(u_1, u_2, \ldots, u_m) > 0,\ \forall t > 0$. □

We are ready to prove the decay rate.

Proof. (*Of Theorem 3.4.*) By (2.6) and (5.3), we have for $t \geq 0$

$$
0 < \sum_{j=1}^{m} \rho_j \left\|u_j\right\|_{\mathcal{H}}^2 \leq \frac{2(p+1)}{p-1} \mathcal{E}(t). \tag{5.5}
$$

Let

$$
I(t) = \frac{\mu(t)}{1 - \mu_0(t)},
$$

where μ and μ_0 defined in (2.2) and (2.3).
Noting that $\lim\limits_{t\to+\infty}\mu(t)=0$ by (2.1)-(2.3), we have

$$\lim_{t\to+\infty} I(t)=0,\ \ I(t)>0,\ \forall t\geq 0.$$

Then, we take $t_0>0$ such that

$$0<I(t)<\min_{t\geq 0}\left\{2\left(\omega\lambda_1+a\right),\xi(t)\chi'(0)\right\}, \tag{5.6}$$

with (2.4) for all $t>t_0$. Due to (3.4), we have

$$\begin{aligned}\mathcal{E}(t) &\leq \frac{1}{2}\sum_{j=1}^{m}\left\|u_{jt}\right\|_{L^2_\theta}^2+\frac{1}{2}\sum_{j=1}^{m}\left(\varpi_j\circ u_j\right)+\frac{1}{2}\sum_{j=1}^{m}\left(1-\int_0^t\varpi_j(s)\,ds\right)\left\|u_j\right\|_{\mathcal{H}}^2\\ &\leq \frac{1}{2}\sum_{j=1}^{m}\left\|u_{jt}\right\|_{L^2_\theta}^2+\frac{1}{2}\sum_{j=1}^{m}\left(\varpi_j\circ u_j\right)+\frac{1}{2}(1-\mu_0(t))\sum_{j=1}^{m}\left\|u_j\right\|_{\mathcal{H}}^2.\end{aligned}$$

Then, by definition of $I(t)$, we have

$$I(t)\mathcal{E}(t)\leq\frac{1}{2}I(t)\sum_{j=1}^{m}\left\|u_{jt}\right\|_{L^2_\theta}^2+\frac{1}{2}\mu(t)\sum_{j=1}^{m}\left\|u_j\right\|_{\mathcal{H}}^2+\frac{1}{2}I(t)\sum_{j=1}^{m}\left(\varpi_j\circ u_j\right), \tag{5.7}$$

and Lemma 3.5, we have for all $t_1,t_2\geq 0$

$$\begin{aligned}&\mathcal{E}(t_2)-\mathcal{E}(t_1)\\ &\quad\leq -\int_{t_1}^{t_2}\left(a\sum_{j=1}^{m}\left\|u_{jt}\right\|_{L^2_\theta}^2+\omega\sum_{j=1}^{m}\left\|u_{jt}\right\|_{\mathcal{H}}^2+\frac{1}{2}\mu(t)\sum_{j=1}^{m}\left\|u_j\right\|_{\mathcal{H}}^2\right)dt\\ &\quad+\int_{t_1}^{t_2}\frac{1}{2}\sum_{j=1}^{m}(\varpi'_j\circ u_j)\,dt\end{aligned}$$

then, by generalized Poincaré's inequalities, we get

$$\mathcal{E}'(t)\ \leq\ -(\omega\lambda_1+a)\sum_{j=1}^{m}\left\|u_{jt}\right\|_{L^2_\theta}^2-\frac{1}{2}\mu(t)\sum_{j=1}^{m}\left\|u_j\right\|_{\mathcal{H}}^2+\frac{1}{2}\sum_{j=1}^{m}(\varpi'_j\circ u_j),$$

Finally, by (5.6), $\forall t\geq t_0$, we have

$$\begin{aligned}&\mathcal{E}'(t)+I(t)\mathcal{E}(t)\\ &\leq \left\{\frac{1}{2}I(t)-(\omega\lambda_1+a)\right\}\sum_{j=1}^{m}\left\|u_{jt}\right\|_{L^2_\theta}^2\\ &+\frac{1}{2}\sum_{j=1}^{m}(\varpi'_j\circ u_j)+\frac{1}{2}I(t)\sum_{j=1}^{m}(\varpi_j\circ u_j)\\ &\leq \frac{1}{2}\sum_{j=1}^{m}\int_0^t\left\{\varpi'_j(t-\tau)+I(t)\varpi_j(t-\tau)\right\}\left\|u_j(t)-u_j(\tau)\right\|_{\mathcal{H}}^2\,d\tau\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2}\sum_{j=1}^{m}\int_0^t \Big\{\varpi_j'(\tau) + I(t)\varpi_j(\tau)\Big\} \left\|u_j(t) - u_j(t-\tau)\right\|_{\mathcal{H}}^2 \, d\tau \\
&\leq \frac{1}{2}\sum_{j=1}^{m}\int_0^t \Big\{-\xi(\tau)\chi\Big(\varpi_j(\tau)\Big) + \xi(\tau)\chi'(0)\varpi_j(\tau)\Big\} \left\|u_j(t) - u_j(t-\tau)\right\|_{\mathcal{H}}^2 \, d\tau \\
&\leq 0,
\end{aligned}$$

by the convexity of χ and (2.4), we have

$$\xi(t)\chi(\varrho) \geq \xi(t)\chi(0) + \xi(t)\chi'(0)\varrho \geq \xi(t)\chi'(0)\varrho.$$

Then

$$\mathcal{E}(t) \leq \mathcal{E}(t_0)\exp\left(-\int_{t_0}^t I(s)ds\right),$$

which completes the proof. □

6. Conclusions

The paper deals with a kind of $m-$nonlinear wave equations with viscoelastic structures. We considered the local existence, global existence and exponential decay rate of solution. We discussed the effects of weak and strong damping terms on decay rate. The methods are standard for local existence and we extended the local solution to a global one by appropriate energy estimates. At last, We obtained a novel decay rate of solution from the convexity property of the function which extends the results in [Math. Meth. Appl. Sci., 43(3), 1138 (2020); Mathematics 8(2), 203 (2020)]. The treatment of Cauchy problem for a family of effectively damped single wave models with a nonlinear memory on the righthand side, that is for $x \in \mathbb{R}^n$

$$u_{tt} + (1+t)^r u_t - \Delta(u + \omega u_t) = \int_0^t (t-s)^{-\gamma}\|u(s,.)\|^p ds \tag{6.1}$$

where $\omega > 0, p > 1, r \in (-1, 1)$ and $\gamma \in (0, 1)$, remains as an open problem, which will be our next work, based on [4, 5, 10, 11].

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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