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**Research article****Uniqueness and exponential instability in a new two-temperature thermoelastic theory****José R. Fernández<sup>1</sup> and Ramón Quintanilla<sup>2,\*</sup>**<sup>1</sup> Departamento de Matemática Aplicada I, Universidade de Vigo, Escola de Enxeñaría de Telecomunicación, Campus As Lagoas Marcosende s/n, 36310 Vigo, Spain<sup>2</sup> Departamento de Matemáticas, E.S.E.I.A.A.T.-U.P.C., Colom 11, 08222 Terrassa, Barcelona, Spain**\* Correspondence:** Email: ramon.quintanilla@upc.edu; Tel: +34937398162; Fax: +34937398101.

**Abstract:** In this work we consider the temperature-rate dependent two temperatures thermoelastic theory. It has been proposed very recently. We study the case in which the elasticity tensor may not be positive definite. Thus, the problem can be ill posed in the sense of Hadamard. We adapt the logarithmic convexity argument to the specific situation proposed by this theory. That is, we define a suitable function on the solutions satisfying that the logarithm is convex. Uniqueness and instability of the solutions under suitable conditions on the constitutive tensors are proved.

**Keywords:** temperature-rate dependent two-temperature thermoelasticity; uniqueness; logarithmic convexity; exponential instability

**Mathematics Subject Classification:** 35Q74, 74H10, 74H25, 80A19, 74F05

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**1. Introduction**

The models to describe the thermomechanical situations propose (mainly) a heat conduction equation based on the Fourier law. This implies that the heat flux vector is proportional to the gradient of temperature. However, one of the mathematical consequences of this assumption (jointly with the classical energy equation) is that the thermal waves propagate instantaneously. This is not well accepted from the physical point of view because it violates the causality principle. To overcome this difficulty, different alternative heat conduction theories have been suggested in the second part of the last century (see [1–3]). Several recent books [4–6] develop mathematical studies concerning the applicability of the different alternative thermoelastic theories. Models like Lord and Shulman [7], Green and Lindsay [8], phase-lag [9], Green and Naghdi [10–12] or Moore-Gibson-Thompson [13, 14] are currently considered in the literature.

In 1968, Chen and Gurtin (see [15–17]) introduced a theory modifying the Fourier law. In this case,

two temperatures, the thermodynamic and the conductive temperatures which were related by a certain elliptic equation, were considered. In 2006, Youssef [18] proposed to make a similar modification to the hyperbolic theories proposed by Lord and Shulman [7] or the Green and Lindsay [8]. More recently, Quintanilla proposed to modify the phase-lag theories in a similar way [19, 20]. These new theories based on two temperatures have deserved much attention in the last years (see, for instance, [21–26]). Last year, Shivay and Mukhopadhyay [27] proposed another one related with the Green and Lindsay theory, but in such a way that we have a temperature rate dependent for the two temperatures. In this new theory, the relation between the two temperatures is modified. Therefore, it proposes a new mathematical problem to clarify its applicability. We believe that the mathematical and physical study of this theory will be relevant to see this issue. Our contribution is addressed in this direction.

In this paper, we will obtain uniqueness and instability results for the solutions of these new thermoelastic theories. We recall that a uniqueness result for one of the cases considered here was obtained in [27]. However, in this case the authors assumed that the elasticity tensor is positive definite.\* In the present work, we do not impose this requirement. We believe that it is relevant to obtain qualitative results without this condition because, on one hand, the axioms of thermomechanics do not imply this condition and, on the other hand, we recall that when we consider prestressed thermoelastic situations [28–31], we do not have any guarantee that the elasticity tensor would be positive definite. In fact, in the case that the material is initially stressed, the elasticity tensor can be written as

$$C_{ijrs} = C_{ijrs}^* + t_{ij}\delta_{rs},$$

where  $C_{ijrs}^*$  is the elasticity tensor when the Lagrangian strain vanishes,  $t_{ij}$  is the initial stress and  $\delta_{ij}$  is the Kronecker delta. We also note that the fact that the elasticity tensor is not positive definite implies that, for a suitable tensor  $\xi_{ij}$ , the following inequality holds:

$$C_{ijkl}\xi_{ij}\xi_{kl} \leq 0.$$

Thus, the elastic energy can be less or equal to zero for suitable deformations.

Therefore, to clarify what happens in this case is a relevant issue to be studied. It is worth recalling that results of this kind have been obtained for several classical and nonclassical thermoelastic theories (see, among others, [24, 32–35]). So, we adapt the analysis to the specific situation determined by the theory considered here.

We note that the instability results obtained in this paper have a relevant thermomechanical interpretation. We will prove that, for certain initially stressed thermoelastic solids, the dissipation induced by the heat conduction cannot bring the system to the stability when the elasticity tensor is not positive definite.

In the next section, we recall the basic equations of this new theory as well as the basic assumptions we use in the paper. In the third section, we consider the problem in the case that the theory depends on the rate of the thermodynamic temperature, but not on the rate of the conductive temperature, and we prove the uniqueness and the exponential instability results. In that case, we pay attention to the analysis and we give the details of the proof. Fourth section is devoted to the theory depending on the rate of both temperatures. Since the analysis is very similar, we only give the expressions of the new

\*We recall that this assumption can be interpreted with the help of the theory of elastic stability; however, this is only a mathematical interpretation, but it is not a consequence of the axioms of the thermomechanics.

relevant functions for the study of this problem, but we omit the details of the analysis to shortcut the length of the contribution. We end this note by the conclusions section where we recall the main results of the paper.

## 2. Basic equations

We recall in this section the basic equations and assumptions under we will work in this paper. We will consider a three-dimensional domain  $B$  with a boundary smooth enough to guarantee the applicability of the divergence theorem.

The basic equations for the temperature-rate dependent two-temperature thermoelasticity involving centrosymmetric materials are given by the evolution equations (see [27]):

$$\begin{aligned}\rho \ddot{u}_i &= t_{i,j,j}, \\ \rho \dot{\eta} &= q_{i,i},\end{aligned}$$

the constitutive equations:

$$\begin{aligned}t_{ij} &= C_{ijkl}u_{k,l} - \beta_{ij}(\theta + \alpha\dot{\theta}), \\ \rho\eta &= h\dot{\theta} + d\theta + \beta_{ij}u_{i,j}, \\ q_i &= K_{ij}\phi_{,j},\end{aligned}$$

and the two-temperature relation in the case that we assume that the theory is independent on the rate conductivity temperature:

$$\phi - m(K_{ij}\phi_{,i})_{,j} = \theta + \alpha\dot{\theta}. \quad (2.1)$$

In the general case when the theory also takes into account the rate conductivity temperature, we have

$$\alpha\dot{\phi} + \phi - m(K_{ij}\phi_{,i})_{,j} = \theta + \alpha\dot{\theta}. \quad (2.2)$$

In the above system of equations,  $\rho$  is the mass density,  $u_i$  is the displacement vector,  $t_{ij}$  is the stress tensor,  $\eta$  is the entropy,  $q_i$  is the heat flux,  $\phi$  and  $\theta$  are the conductive and the thermodynamic temperatures,  $C_{ijkl}$  is the elasticity tensor,  $K_{ij}$  is the thermal conductivity tensor,  $\beta_{ij}$  is the coupling tensor,  $\alpha$  is the first relaxation parameter,  $d = \rho c_e \theta_0^{-1}$ , where  $c_e$  is the thermal capacity,  $\theta_0$  is the reference temperature and  $h = dt_2$  where  $t_2$  is the second relaxation parameter. It is worth noting that  $m$  and  $\alpha$  are two constants, but we do not need that  $h$  or  $d$  are also constants.

It is worth recalling that  $d, \alpha$  and  $h$  cannot be selected without any restriction because of the dissipation inequality. To be precise, the next condition (iii) is a consequence of it (see [8]) as well as condition (iv).

In this paper, we will assume that

- (i) The mass density and the constitutive functions  $h$  and  $d$  are strictly positive. That is, they are greater than zero in the whole domain.
- (ii) The parameters  $m$  and  $\alpha$  are positive.
- (iii) We also assume that  $d\alpha > h$ .

(iv) The thermal conductivity tensor is symmetric ( $K_{ij} = K_{ji}$ ) and positive definite. That is, there exists a positive constant  $C^*$  such that

$$K_{ij}\xi_i\xi_j \geq C^*\xi_i\xi_i,$$

for every vector  $(\xi_i)$ .

(v) The elasticity tensor satisfies the major symmetry. That is,  $C_{ijkl} = C_{klji}$ .

These assumptions are natural from the thermomechanical point of view in the context of the theory; however, we do not assume that the elasticity tensor is positive definite. In fact, in the case that we assume that the material is initially stressed it could fail to satisfy this condition on this tensor. When the elasticity tensor is not positive definite, then the problem (usually) is *ill posed* in the sense of Hadamard.

If we substitute the constitutive equations into the evolution equations, we obtain the following linear system of partial differential equations:

$$\rho\ddot{u}_i = (C_{ijkl}u_{k,l} - \beta_{ij}(\theta + \alpha\dot{\theta})),_j, \quad (2.3)$$

$$h\ddot{\theta} + d\dot{\theta} = -\beta_{ij}\dot{u}_{i,j} + (K_{ij}\phi_{,j}),_i. \quad (2.4)$$

To propose the well posed problem we will need to impose the initial conditions:

$$\begin{aligned} u_i(x, 0) &= u_i^0(x), & \dot{u}_i(x, 0) &= v_i^0(x), & \phi(x, 0) &= \phi^0(x), \\ \theta(x, 0) &= \theta^0(x), & \dot{\theta}(x, 0) &= \vartheta^0(x), \end{aligned} \quad (2.5)$$

where  $u_i^0, v_i^0, \phi^0, \theta^0, \vartheta^0$  are given functions. We note that these initial conditions correspond to the case when we assume relation (2.2); however, in the case that we consider relation (2.1), we do not impose any initial data on  $\phi$  at time  $t = 0$ . It should be noted that, in this case, we can obtain the initial data for  $\phi^0$  in terms of the initial data  $\theta^0$  and  $\vartheta^0$ . To this end, we recall that the map  $\phi \rightarrow \phi - m(K_{ij}\phi_{,i}),_j$  defines an isomorphism between  $W_0^{1,2} \cap W^{2,2}$  and  $L^2$ . If we denote by  $\Phi$  the inverse operator, we see that we can find  $\phi^0 = \Phi(\theta^0 + \alpha\vartheta^0)$ .

Since we assume homogeneous Dirichlet boundary conditions, it follows that

$$u(x, t) = \phi(x, t) = 0, \quad t \in [0, \infty), \quad x \in \partial B. \quad (2.6)$$

We note that, in the case of relation (2.1), the energy equality can be obtained after multiplication of the displacement equation by  $\dot{u}_i$ , the heat equation by  $\theta + \alpha\dot{\theta}$ , the use of equality (2.1), and taking into account the boundary conditions. It follows that

$$\begin{aligned} E(t) &= \frac{1}{2} \int_B \left( \rho\dot{u}_i\dot{u}_i + C_{ijkl}u_{i,j}u_{k,l} + d(\theta + \frac{h}{d}\dot{\theta})^2 + h(\alpha - \frac{h}{d})|\dot{\theta}|^2 \right) dv \\ &+ \int_0^t \int_B \left( (d\alpha - h)|\dot{\theta}|^2 + K_{ij}\phi_{,i}\phi_{,j} + m((K_{ij}\phi_{,i}),_j)^2 \right) dv ds = E(0) \end{aligned}$$

holds.

In the case that we consider relation (2.2) the energy equality is

$$\begin{aligned} E(t) &= \frac{1}{2} \int_B \left( \rho\dot{u}_i\dot{u}_i + C_{ijkl}u_{i,j}u_{k,l} + d(\theta + \frac{h}{d}\dot{\theta})^2 + h(\alpha - \frac{h}{d})|\dot{\theta}|^2 + \alpha K_{ij}\phi_{,i}\phi_{,j} \right) dv \\ &+ \int_0^t \int_B \left( (d\alpha - h)|\dot{\theta}|^2 + K_{ij}\phi_{,i}\phi_{,j} + m((K_{ij}\phi_{,i}),_j)^2 \right) dv ds = E(0). \end{aligned}$$

### 3. The case (2.1)

This section is devoted to prove the uniqueness and the exponential instability of the solutions in the case that we assume that the two temperatures satisfy relation (2.1).

#### 3.1. Uniqueness

In order to obtain a uniqueness result, it will be sufficient to show that the only solution for the problem determined by the null initial conditions is the null solution. Therefore, we assume null initial conditions and so we have that  $E(0) = 0$ .

As we will use the logarithmic convexity argument, it will be convenient to define a suitable function. That is, a function with convex logarithm. In the case (2.1) we consider

$$F(t) = \frac{1}{2} \int_B \rho u_i u_i dv + \frac{1}{2} \int_0^t \int_B ((d\alpha - h)\theta^2 + K_{ij}\eta_{,i}\eta_{,j} + m((K_{ij}\eta_{,i}),_j)^2) dv ds,$$

where

$$\eta(x, t) = \int_0^t \phi(x, s) ds.$$

We have

$$\dot{F}(t) = \int_B \rho u_i \dot{u}_i dv + \frac{1}{2} \int_B ((d\alpha - h)\theta^2 + K_{ij}\eta_{,i}\eta_{,j} + m((K_{ij}\eta_{,i}),_j)^2) dv,$$

and

$$\begin{aligned} \dot{F}(t) &= \int_B (\rho \dot{u}_i \dot{u}_i + \rho u_i \ddot{u}_i) dv \\ &+ \int_B ((d\alpha - h)\theta \dot{\theta} + K_{ij}\eta_{,i}\phi_{,j} + m((K_{ij}\phi_{,i}),_j)((K_{lk}\eta_{,l}),_k)) dv. \end{aligned}$$

It is worth noting that the following equalities

$$\int_B (\rho u_i \ddot{u}_i + C_{ijkl}u_{i,j}u_{k,l}) dv = \int_B \beta_{ij}u_{i,j}(\theta + \alpha\dot{\theta}) dv,$$

and

$$\begin{aligned} \int_B ((h\dot{\theta} + d\theta)(\theta + \alpha\dot{\theta}) + K_{ij}\eta_{,i}\phi_{,j} + m((K_{ij}\phi_{,i}),_j)((K_{lk}\eta_{,l}),_k)) dv \\ = - \int_B \beta_{ij}u_{i,j}(\theta + \alpha\dot{\theta}) dv, \end{aligned}$$

hold.

It then follows that

$$\begin{aligned} \int_B (\rho u_i \ddot{u}_i + C_{ijkl}u_{i,j}u_{k,l}) dv \\ + \int_B ((h\dot{\theta} + d\theta)(\theta + \alpha\dot{\theta}) + K_{ij}\eta_{,i}\phi_{,j} + m((K_{ij}\phi_{,i}),_j)((K_{lk}\eta_{,l}),_k)) dv = 0. \end{aligned}$$

Since we have that

$$(h\dot{\theta} + d\theta)(\theta + \alpha\dot{\theta}) = d(\theta + \frac{h}{d}\dot{\theta})^2 + h(\alpha - \frac{h}{d})|\dot{\theta}|^2 + (d\alpha - h)\theta\dot{\theta},$$

we can write

$$\int_B (\rho u_i \ddot{u}_i + C_{ijkl} u_{i,j} u_{k,l}) dv + \int_B \left( d(\theta + \frac{h}{d} \dot{\theta})^2 + h(\alpha - \frac{h}{d}) |\dot{\theta}|^2 + (d\alpha - h) \theta \dot{\theta} + K_{ij} \eta_{,i} \phi_{,j} + m((K_{ij} \phi_{,i}),_j) ((K_{lk} \eta_{,l}),_k) \right) dv = 0.$$

This is equivalent to

$$\begin{aligned} & \int_B \left( \rho u_i \ddot{u}_i + (d\alpha - h) \theta \dot{\theta} + K_{ij} \eta_{,i} \phi_{,j} + m((K_{ij} \phi_{,i}),_j) ((K_{lk} \eta_{,l}),_k) \right) dv \\ &= - \int_B \left( C_{ijkl} u_{i,j} u_{k,l} + d(\theta + \frac{h}{d} \dot{\theta})^2 + h(\alpha - \frac{h}{d}) |\dot{\theta}|^2 \right) dv. \end{aligned}$$

Then, we obtain that

$$\ddot{F}(t) = \int_B \rho \dot{u}_i \dot{u}_i dv - \int_B (C_{ijkl} u_{i,j} u_{k,l} + d(\theta + \frac{h}{d} \dot{\theta})^2 + h(\alpha - \frac{h}{d}) |\dot{\theta}|^2) dv,$$

and, in view of the energy equality, we find that

$$\ddot{F}(t) = 2 \int_B \rho \dot{u}_i \dot{u}_i dv + 2 \int_0^t \int_B \left( (d\alpha - h) |\dot{\theta}|^2 + K_{ij} \phi_{,i} \phi_{,j} + m((K_{ij} \phi_{,i}),_j)^2 \right) dv ds.$$

Thus, we have

$$F(t) \ddot{F}(t) - (\dot{F}(t))^2 \geq 0 \quad \text{for every } t \geq 0.$$

This inequality implies that

$$\frac{d^2}{dt^2} (\ln F(t)) \geq 0.$$

That is,  $\ln F(t)$  is a convex function with respect to the time variable. It then follows that

$$F(t) \leq F(0)^{1-t/T} F(T)^{t/T},$$

whenever the solution exists until time  $T$ . Since we assume null initial data, we see that  $F(0) = 0$  and, therefore,  $F(t) = 0$  for every  $t$ . This implies that the solution must satisfy that  $u_i = \theta = \phi = 0$ . That is, the solution is the null solution and the uniqueness of solutions is concluded.

We have proved the following.

**Theorem 3.1.** *Let us assume that conditions (i)-(v) hold. Then, the initial boundary value problem determined by system (2.3)-(2.4) with homogeneous Dirichlet conditions (2.6) and initial conditions (2.5) admits at most one solution.*

### 3.2. Exponential instability

Now, we will obtain the instability of the solutions whenever the initial energy of the system can be negative or null. We note that, since we do not assume that the elasticity tensor is positive definite, the initial energy of the system can satisfy this condition. To this end, we first note that, after a time integration of the energy equation, we can see that

$$h \dot{\theta} + d\theta - \beta_{ij} u_{i,j} = h \theta^0 + d\theta - \beta_{ij} u_{i,j}^0 + (K_{ij} \eta_{,i}),_j$$

where  $\eta(x, t)$  has been defined in the previous subsection.

If we define the function  $P(x)$  such that it satisfies

$$(K_{ij}P_{,i})_{,j} = h\theta^0 + d\theta^0 - \beta_{ij}u_{i,j}^0,$$

and  $P(x)$  vanishes at the boundary of  $B$ , we see that

$$h\dot{\theta} + d\theta^0 - \beta_{ij}u_{i,j} = (K_{ij}\beta_{,i})_{,j},$$

where

$$\beta(x, t) = P(x) + \eta(x, t).$$

We note that the existence of function  $P(x)$  is guaranteed by the usual known properties of the elliptic equations.

We may define the function:<sup>†</sup>

$$\begin{aligned} G(t, \omega, t_0) &= \frac{1}{2} \int_B \rho u_i u_i dv \\ &+ \frac{1}{2} \int_0^t \int_B \left( (d\alpha - h)\theta^2 + K_{ij}\beta_{,i}\eta_{,j} + m((K_{ij}\beta_{,i})_{,j})^2 \right) dv ds + \omega(t + t_0)^2, \end{aligned}$$

where  $\omega$  and  $t_0$  are two positive parameters to be selected later.

We have that

$$\begin{aligned} \dot{G}(t, \omega, t_0) &= \int_B \rho u_i \dot{u}_i dv \\ &+ \int_0^t \int_B \left( (d\alpha - h)\theta \dot{\theta} + K_{ij}\beta_{,i}\phi_{,j} + m((K_{ij}\phi_{,i})_{,j})((K_{lk}\beta_{,l})_{,k}) \right) dv ds \\ &+ \frac{1}{2} \int_B \left( (d\alpha - h)|\theta^0|^2 + K_{ij}\phi_{,i}^0\phi_{,j}^0 + m((K_{ij}\phi_{,i}^0)_{,j})^2 \right) dv + 2\omega(t + t_0). \end{aligned}$$

We also find that

$$\begin{aligned} \ddot{G}(t, \omega, t_0) &= \int_B (\rho u_i \ddot{u}_i + \dot{u}_i \dot{u}_i) dv \\ &+ \int_B \left( (d\alpha - h)\theta \dot{\theta} + K_{ij}\beta_{,i}\phi_{,j} + m((K_{ij}\phi_{,i})_{,j})((K_{lk}\beta_{,l})_{,k}) \right) dv + 2\omega. \end{aligned}$$

If we use several similar arguments to the ones proposed in the previous subsection, we can obtain

$$\begin{aligned} \ddot{G}(t, \omega, t_0) &= 2 \int_B \dot{u}_i \dot{u}_i dv \\ &+ 2 \int_0^t \int_B \left( (d\alpha - h)|\dot{\theta}|^2 + K_{ij}\phi_{,i}\phi_{,j} + m((K_{ij}\phi_{,i})_{,j})^2 \right) dv ds + 2(\omega - E(0)). \end{aligned}$$

If we denote

$$\nu = \int_B \left( (d\alpha - h)|\theta^0|^2 + K_{ij}\phi_{,i}^0\phi_{,j}^0 + m((K_{ij}\phi_{,i}^0)_{,j})^2 \right) dv,$$

<sup>†</sup>This kind of function with the terms  $\omega(t + t_0)^2$  are usually inspired in the work of Knops and Payne [36].

we obtain the inequality

$$G(t, \omega, t_0) \ddot{G}(t, \omega, t_0) - \left( \dot{G}(t, \omega, t_0) - \frac{\nu}{2} \right)^2 \geq 2(\omega + E(0))G(t, \omega, t_0).$$

In the case that  $E(0) < 0$ , we can always take  $\omega = -E(0)$  and  $t_0$  large enough to guarantee that  $\dot{G}(0, \omega, t_0) > \nu$ , and then we obtain that

$$G(t, \omega, t_0) \ddot{G}(t, \omega, t_0) - \dot{G}(t, \omega, t_0)(\dot{G}(t, \omega, t_0) - \nu) \geq 0.$$

If we divide by the square of  $G(t, \omega, t_0)$ , we see that

$$\frac{\dot{G}(t, \omega, t_0) - \nu}{G(t, \omega, t_0)}$$

is an increasing function with respect to the time, and then

$$\frac{\dot{G}(t, \omega, t_0) - \nu}{G(t, \omega, t_0)} \geq \frac{\dot{G}(0, \omega, t_0) - \nu}{G(0, \omega, t_0)} \text{ for every } t \geq 0.$$

This inequality implies that

$$\dot{G}(t, \omega, t_0) \geq \frac{\dot{G}(0, \omega, t_0) - \nu}{G(0, \omega, t_0)} G(t, \omega, t_0) + \nu.$$

Therefore, we can write

$$\frac{dG(t, \omega, t_0)}{G(t, \omega, t_0) + \frac{\nu}{\frac{\dot{G}(0, \omega, t_0) - \nu}{G(0, \omega, t_0)}}} \geq \frac{\dot{G}(0, \omega, t_0) - \nu}{G(0, \omega, t_0)} dt.$$

After integrating this inequality, we obtain

$$G(t, \omega, t_0) \geq \frac{G(0, \omega, t_0) \dot{G}(0, \omega, t_0)}{\dot{G}(0, \omega, t_0) - \nu} \exp\left(\frac{\dot{G}(0, \omega, t_0) - \nu}{G(0, \omega, t_0)} t\right) - \frac{\nu G(0, \omega, t_0)}{\dot{G}(0, \omega, t_0) - \nu}. \quad (3.1)$$

This inequality gives the exponential growth of the solutions.

We also note that, in the case that  $E(0) = 0$  and  $\dot{G}(0, 0, t_0) > \nu$ , the previous argument also gives the exponential growth of the solutions.

Therefore, we have proved the following.

**Theorem 3.2.** *Let us assume that conditions (i)-(v) hold. Let us also assume that we have a solution of the initial boundary value problem determined by system (2.3)-(2.4) with homogeneous Dirichlet conditions (2.6) and initial conditions (2.5) such that (i)  $E(0) < 0$ , or (ii)  $E(0) = 0$  and  $\dot{G}(0, 0, t_0) > \nu$ . Then, the solution is exponentially unstable.*

The results are still valid in the case that the elasticity tensor is positive definite. We mean that the uniqueness (but it was known) and the instability results also hold; however, to prove the instability we need to assume that the initial energy must be less or equal to zero, which is only possible in the case that the initial data is zero. Therefore, the result of instability is (of course) still valid, but it is obtained without any consequence because the necessary assumptions cannot be satisfied for the blow-up.

We also remark that the logarithmic convexity argument could be used to show the impossibility of existence of a semigroup when the variable for the displacement is contained in  $\mathbf{L}^2(B)$  (see [37] or [38, page 345]).

From a thermomechanical point of view, we have seen that, for certain prestressed elastic materials, the thermal effects considered in this section do not control the instability given by the lack of the positivity of the elastic tensor.

#### 4. The case (2.2)

This section is devoted to sketch the proof of the uniqueness and the exponential instability of the solutions in the case that we assume that the two temperatures (conductive and thermodynamic) satisfy relation (2.2).

##### 4.1. Uniqueness

Again, in order to prove the uniqueness, it will be sufficient to show that the only solution for the problem determined by the null initial conditions is the null solution.

In the case that we consider system (2.2), we can reproduce the arguments of the previous section, but now we define the function:

$$F(t) = \frac{1}{2} \int_B (\rho u_i u_i + \alpha K_{ij} \eta_{,i} \eta_{,j}) dv + \frac{1}{2} \int_0^t \int_B ((d\alpha - h)\theta^2 + K_{ij} \eta_{,i} \eta_{,j} + m((K_{ij} \eta_{,i})_{,j})^2) dv ds,$$

where  $\eta(x, t)$  is defined as in the previous section.

We have

$$\dot{F}(t) = \int_B (\rho u_i \dot{u}_i + \alpha K_{ij} \eta_{,i} \phi_{,j}) dv + \frac{1}{2} \int_B ((d\alpha - h)\theta^2 + K_{ij} \eta_{,i} \eta_{,j} + m((K_{ij} \eta_{,i})_{,j})^2) dv.$$

After a second derivation and following the same ideas to the ones proposed in the previous section, we can obtain

$$\ddot{F}(t) = 2 \int_B (\rho \ddot{u}_i \dot{u}_i + \alpha K_{ij} \phi_{,i} \phi_{,j}) dv + 2 \int_0^t \int_B ((d\alpha - h)|\dot{\theta}|^2 + K_{ij} \phi_{,i} \phi_{,j} + m((K_{ij} \phi_{,i})_{,j})^2) dv ds.$$

Therefore, the estimate

$$F(t) \leq F(0)^{1-t/T} F(T)^{t/T}$$

holds again whenever the solution exists until time  $T$ . As the initial data is null, we see that  $F(0) = 0$  and so,  $F(t) = 0$  for every  $t$ . Perhaps, the most relevant issue in the analysis of this situation is that we obtain that  $u_i = \theta = \phi = 0$  for every time, but, in this case, we show that  $\phi$  vanishes due to the definition of function  $F$ . The uniqueness of solutions is then concluded.

#### 4.2. Exponential instability

Again, we can prove the instability of the solutions for certain initial conditions. The way to prove this claim is to follow the arguments used in the second subsection of the previous section. The only relevant point is that we shall change a little bit the function to use. In the present case, we shall take the function:

$$G(t, \omega, t_0) = \frac{1}{2} \int_B (\rho u_i u_i + \alpha K_{ij} \beta_{,i} \beta_{,j}) dv + \frac{1}{2} \int_0^t \int_B ((d\alpha - h)\theta^2 + K_{ij} \beta_{,i} \beta_{,j} + m((K_{ij} \beta_{,i})_{,j})^2) dv ds + \omega(t + t_0)^2,$$

where  $\beta(x, t)$  is defined as in the previous section.

We can use the same arguments to the ones proposed previously. The only difference is that, in this case, the initial data for  $\phi^0$  is determined by the data of the problem. Exponential instability of the solutions for the measure determined in this section can be proved. To be precise, estimate (3.1) can be also obtained in this situation, whenever we assume  $E(0) < 0$  or  $E(0) = 0$  and  $\dot{G}(0, 0, t_0) > v$ . Therefore, it follows the impossibility of the existence of a semigroup when the elasticity tensor fails to satisfy the strong ellipticity condition.

We also remark that the physical interpretation of the result at the end of the previous section has a direct counterpart for the theory studied in this section.

## 5. Conclusions

In this paper, we have analysed the system of equations governing the temperature-rate dependent two temperature thermoelastic theory, in the case that we do not impose that the elasticity tensor is positive definite. It is remarkable this case because, on one hand, the axioms of the thermomechanics do not imply that this elasticity tensor should be positive definite and, on the other hand, in the case of initially stressed solids it is natural to expect that this condition is not satisfied. It is also worth recalling that we are dealing with an *ill posed problem* in the sense of Hadamard. Thought this difficulty, in this paper we have proved the uniqueness of solutions for the initial boundary value problem as well as the instability of solutions when the initial energy is not positive. This last result states that the dissipation of the energy determined by the heat conduction cannot bring the system to the stability whenever the elasticity tensor is not positive definite.

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## Conflict of interest

The authors declare that they have no conflict of interest.

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