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Research article

Hankel and Toeplitz determinant for a subclass of multivalent q-starlike functions of order α

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Abstract: In this paper our aim is to study some valuable problems dealing with newly defined subclass of multivalent q-starlike functions. These problems include the initial coefficient estimates, Toeplitz matrices, Hankel determinant, Fekete-Szego problem, upper bounds of the functional $\left|a_{p+1} - \mu a_{p+1}^2\right|$ for the subclass of multivalent q-starlike functions. As applications we study a q-Bernardi integral operator for a subclass of multivalent q-starlike functions. Furthermore, we also highlight some known consequence of our main results.

Keywords: quantum (or q-) calculus; q-derivative operator; multivalent functions; q-starlike functions; Hankel determinant; Toeplitz matrices
Mathematics Subject Classification: Primary: 05A30, 30C45; Secondary: 11B65, 47B38

1. Introduction and definitions

The function class $\mathcal{H}(E)$ is a collection of the function f which are holomorphic in the open unit disc

$$E = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Let $\mathcal{A}_p$ denote the class of all functions f which are analytic and p-valent in the open unit disk E and has the Taylor series expansion of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (p \in \mathbb{N} = \{1, 2, ...\}). \tag{1.1}$$

For briefly, we write as:

$$\mathcal{A}_1 = \mathcal{A}.$$

Moreover, $\mathcal{S}$ represents the subclass of $\mathcal{A}$ which is univalent in open unit disk E. Further in area of Geometric Function Theory, numerous researchers offered their studies for the class of analytic function and its subclasses as well. The role of geometric properties is remarkable in the study of analytic functions, for instance convexity, starlikeness, close-to-convexity. A function $f \in \mathcal{A}_p$ is known as p-valently starlike $\left(\mathcal{S}_p^*\right)$ and convex $\left(\mathcal{K}_p\right)$, whenever it satisfies the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \ (z \in E)$$

and

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \ (z \in E).$$

Moreover, a function $f(z) \in \mathcal{A}_p$, is said to be p-valently starlike function of order α, written as $f(z) \in \mathcal{S}_p^*(\alpha)$, if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \ (z \in E).$$

Similarly, a function $f(z) \in \mathcal{A}_p$, is said to be p-valently convex functions of order α, written as $f(z) \in \mathcal{K}_p(\alpha)$, if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \ (z \in E),$$

for some $0 \leq \alpha < p$. In particular, we have

$$\mathcal{S}_p^*(0) = \mathcal{S}_p^*$$

and

$$\mathcal{K}_p(0) = \mathcal{K}_p.$$

The convolution (Hadamard product) of $f(z)$ and $g(z)$ is defined as:

$$f(z) * g(z) = \sum_{n=0}^{\infty} a_n b_n z^n = g(z) * f(z),$$

where

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n, \quad (z \in E).$$

Let $\mathcal{P}$ denote the well-known Carathéodory class of functions m, analytic in the open unit disk E of the form

$$m(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{1.2}$$

and satisfy

$$\Re\,(m(z)) > 0.$$

The quantum (or q-) calculus has a great important because of its applications in several fields of mathematics, physics and some related areas. The importance of q-derivative operator (D_q) is pretty recongnizable by its applications in the study of numerous subclasses of analytic functions. Initially in 1908, Jackson [14] defined the q-analogue of derivative and integral operator as well as provided some of their applications. Further in [11] Ismail et al. gave the idea of q-extension of class of q-starlike functions after that Srivastava [37] studied q-calculus in the context of univalent functions theory, also numerous mathematician studied q-calculus in the context of geometric functions theory. Kanas and Raducanu [17] introduced the q-analogue of Ruscheweyh differential operator and Arif et al. [3,4] discussed some of its applications for multivalent functions while Zhang et al. in [50] studied q-starlike functions related with generalized conic domain $\Omega_{k,\alpha}$. By using the concept of convolution Srivastava et al. [40] introduced q-Noor integral operator and studied some of its applications, also Srivastava et al. published set of articles in which they concentrated class of q-starlike functions from different aspects (see [24, 41, 42, 44, 46, 47]). Additionally, a recently published survey-cum-expository review article by Srivastava [38] is potentially useful for researchers and scholars working on these topics. For some more recent investigation about q-calculus we may refer to [1, 18–23, 25, 31–34, 38, 39, 45].

For better understanding of the article we recall some concept details and definitions of the q-difference calculus. Throughout the article we presume that

$$0 < q < 1 \quad \text{and} \quad p \in \mathbb{N} = \{1, 2, 3...\}.$$

Definition 1. ([10]) The q-number $[t]_q$ for $q \in (0,1)$ is defined as:

$$[t]_q = \begin{cases} \frac{1-q^t}{1-q}, & (t \in \mathbb{C}), \\ \\ \sum\limits_{k=0}^{n-1} q^k, & (t = n \in \mathbb{N}). \end{cases}$$

Definition 2. The q-factorial $[n]_q!$ for $q \in (0,1)$ is defined as:

$$[n]_q! = \begin{cases} 1, & (n = 0), \\ \\ \prod\limits_{k=1}^{n} [k]_q, & (n \in \mathbb{N}). \end{cases}$$

Definition 3. The q-generalized Pochhammer symbol $[t]_{n,q}$, $t \in \mathbb{C}$, is defined as:

$$[t]_{n,q} = \begin{cases} 1, & (n = 0), \\ \\ [t]_q[t+1]_q[t+2]_q...[t+n-1]_q, & (n \in \mathbb{N}). \end{cases}$$

And the q-Gamma function be defined as:

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t) \quad \text{and} \quad \Gamma_q(1) = 1.$$

Definition 4. The q-integral of any function $f(z)$ was defined be Jackson [15] as follows:

$$\int f(z) d_q z = (1-q)\, z \sum_{n=0}^{\infty} f\left(q^n z\right) q^n$$

provided that the series on right hand side converges absolutely.

Definition 5. ([14]) For given $q \in (0,1)$, the q-derivative operator or q-difference operator of f is defined by:

$$\begin{aligned} D_q f(z) &= \frac{f(z) - f(qz)}{(1-q)z}, \quad z \neq 0, q \neq 1, \qquad (1.3) \\ &= 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}. \end{aligned}$$

Now we extend the idea of q-difference operator to a function f given by (1.1) from the class $\mathcal{A}_p$ as:

Definition 6. For $f \in \mathcal{A}_p$, let the q-derivative operator (or q-difference operator) be defined as:

$$\begin{aligned} D_q f(z) &= \frac{f(z) - f(qz)}{(1-q)z}, \quad z \neq 0, q \neq 1, \\ &= [p]_q z^{p-1} + \sum_{n=p+1}^{\infty} [n]_q a_n z^{n-1}. \qquad (1.4) \end{aligned}$$

We can observe that for $p = 1$, and $q \to 1-$ in (1.4) we have

$$\lim_{q \to 1-} D_q f(z) = f'(z).$$

Definition 7. An analytic function $f(z) \in \mathcal{S}_p^*(\alpha, q)$ of p-valent q-starlike functions of order α in E, if $f(z) \in \mathcal{A}_p$, satisfies the condition

$$\Re\left(\frac{z D_q f(z)}{f(z)}\right) > \alpha, \ (z \in E),$$

for some $0 \leq \alpha < p$.

Definition 8. An analytic function $f(z) \in \mathcal{K}_p(\alpha, q)$ of p-valent q-convex functions of order α in E, if $f(z) \in \mathcal{A}_p$, satisfies the condition

$$\Re\left(\frac{D_q\left(z D_q f(z)\right)}{D_q f(z)}\right) > \alpha, \ (z \in E),$$

for some $0 \leq \alpha < p$.

Remark 1. Let $f(z) \in \mathcal{A}_p$, it follows that

$$f(z) \in \mathcal{K}_p(\alpha, q) \text{ if and only if } \frac{zD_q f(z)}{[p]_q} \in \mathcal{S}_p^*(\alpha, q)$$

and

$$f(z) \in \mathcal{S}_p^*(\alpha, q) \text{ if and only if } \int_0^z \frac{[p]_q f(\zeta)}{\zeta} d_q\zeta \in \mathcal{K}_p(\alpha, q).$$

Remark 2. By putting value of parameters α and p we can get some new subclasses of analytic functions:

$$\mathcal{S}_p^*(q) = \mathcal{S}_p^*(0, q), \quad \mathcal{S}^*(\alpha, q) = \mathcal{S}_1^*(\alpha, q), \quad \mathcal{K}_p(q) = \mathcal{K}_p(0, q) \text{ and } \mathcal{K}(\alpha, q) = \mathcal{K}_1(\alpha, q).$$

Remark 3. By taking $q \to 1-$, then we obtain two known subclasses $\mathcal{S}_p^*(\alpha)$ and $\mathcal{K}_p(\alpha)$ of p-valently starlike and convex functions of order α, introduced by Hayami and Owa in [12].

Let $n \in \mathbb{N}_0$ and $j \in \mathbb{N}$. The jth Hankel determinant was introduced and studied in [29]:

$$H_j(n) = \begin{vmatrix} a_n & a_{n+1} \dots & a_{n+j-1} \\ a_{n+1} & a_{n+2} \dots & a_{n+j-2} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ a_{n+j-1} & a_{n+j-2} \dots & a_{n+2j-2} \end{vmatrix},$$

where $a_1 = 1$. The Hankel determinant $H_2(1)$ represents a Fekete-Szegö functional $\left|a_3 - a_2^2\right|$. This functional has been further generalized as $\left|a_3 - \mu a_2^2\right|$ for some real or complex number μ and also the functional $\left|a_2 a_4 - a_3^2\right|$ is equivalent to $H_2(2)$ (see [16]). Babalola [5] studied the Hankel determinant $H_3(1)$ (see also [43]). The symmetric Toeplitz determinant $T_j(n)$ is defined as follows:

$$T_j(n) = \begin{vmatrix} a_n & a_{n+1} \dots & a_{n+j-1} \\ a_{n+1} & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ a_{n+j-1} & \dots & a_n \end{vmatrix}, \tag{1.5}$$

so that

$$T_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}, \quad T_2(3) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_3 \end{vmatrix}, \quad T_3(2) = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_3 & a_2 \end{vmatrix}$$

and so on. The problem of finding the best possible bounds for $||a_{n+1}| - |a_n||$ has a long history (see [8]). In particular, several authors [13, 44] have studied $T_j(n)$ for several classes.

For our simplicity, we replace $n = n + p - 1$, into (1.5), then the symmetric Toeplitz determinant $T_j(n)$ can be written as:

$$T_j(n+p-1) = \begin{vmatrix} a_{n+p-1} & a_{n+p} & \dots a_{n+p+j-2} \\ a_{n+p} & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ a_{n+p+j-2} & \dots & a_{n+p-1} \end{vmatrix},$$

so that

$$T_2(p+1) = \begin{vmatrix} a_{p+1} & a_{p+2} \\ a_{p+2} & a_{p+1} \end{vmatrix}, \quad T_2(p+2) = \begin{vmatrix} a_{p+2} & a_{p+3} \\ a_{p+3} & a_{p+2} \end{vmatrix},$$

$$T_3(p+1) = \begin{vmatrix} a_{p+1} & a_{p+2} & a_{p+3} \\ a_{p+2} & a_{p+1} & a_{p+2} \\ a_{p+3} & a_{p+2} & a_{p+1} \end{vmatrix}.$$

Hankel determinants generated by perturbed Gaussian, Laguerre and Jacobi weights play an important role in Random Matrix Theory, since they represent the partition functions for the perturbed Gaussian, Laguerre and Jacobi unitary ensembles, see for example [7,26–28,49].

2. A set of Lemmas

In order to discuss our problems, we need some lemmas.

Lemma 1. *(see [8]). If a function* $m(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{P}$*, then*

$$|c_n| \leq 2, \quad n \geq 1.$$

The inequality is sharp for

$$f(z) = \frac{1+z}{1-z}.$$

Lemma 2. *If a function* $m(z) = [p]_q + \sum_{n=1}^{\infty} c_n z^n$ *satisfies the following inequality*

$$\Re\,(m(z)) \geq \alpha$$

for some α, $(0 \leq \alpha < p)$*, then*

$$|c_n| \leq 2\left([p]_q - \alpha\right), \; n \geq 1.$$

The result is sharp for

$$m(z) = \frac{[p]_q + \left([p]_q - 2\alpha\right)z}{1-z} = [p]_q + \sum_{n=1}^{\infty} 2\left([p]_q - \alpha\right)z^n.$$

Proof. Let

$$l(z) = \frac{m(z)-\alpha}{[p]_q - \alpha} = 1 + \sum_{n=1}^{\infty} \frac{c_n}{[p]_q - \alpha} z^n.$$

Noting that $l(z) \in \mathcal{P}$ and using Lemma 1, we see that

$$\left|\frac{c_n}{[p]_q - \alpha}\right| \leq 2, \quad n \geq 1,$$

which implies

$$|c_n| \leq 2\left([p]_q - \alpha\right), \quad n \geq 1.$$

□

Remark 4. When $q \to 1-$, then Lemma 2, reduces to the lemma which was introduced by Hayami et al. [12].

Lemma 3. *([36]) If m is analytic in E and of the form (1.2), then*

$$2c_2 = c_1^2 + x(4 - c_1^2)$$

and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - (4 - c_1^2)c_1 x^2 + 2(4 - c_1^2)(1 - \left|x^2\right|)z,$$

for some $x, z \in \mathbb{C}$, with $|z| \le 1$, and $|x| \le 1$.

By virtue of Lemma 3, we have

Lemma 4. *If* $m(z) = [p]_q + \sum_{n=1}^{\infty} c_n z^n$ *satisfy* $\Re(m(z) > \alpha$, *for some* α $(0 \le \alpha < p)$, *then*

$$2\left([p]_q - \alpha\right) c_2 = c_1^2 + \left\{4\left([p]_q - \alpha\right)^2 - c_1^2\right\} x$$

and

$$\begin{aligned} 4\left([p]_q - \alpha\right)^2 c_3 &= c_1^3 + 2\left\{4\left([p]_q - \alpha\right)^2 - c_1^2\right\} c_1 x - \left\{4\left([p]_q - \alpha\right)^2 - c_1^2\right\} c_1 x^2 + \\ & \quad 2\left([p]_q - \alpha\right)\left\{4\left([p]_q - \alpha\right)^2 - c_1^2\right\}(1 - \left|x^2\right|)z, \end{aligned}$$

for some $x, z \in \mathbb{C}$, with $|z| \le 1$, and $|x| \le 1$.

Proof. Since $l(z) = \frac{m(z)-\alpha}{[p]_q-\alpha} = 1 + \sum_{n=1}^{\infty} \frac{c_n}{[p]_q-\alpha} z^n \in \mathcal{P}$, replacing c_2 and c_3 by $\frac{c_2}{[p]_q-\alpha}$and $\frac{c_3}{[p]_q-\alpha}$ in Lemma 3, respectively, we immediately have the relations of Lemma 4. □

Remark 5. When $q \to 1-$, then Lemma 4, reduces to the lemma which was introduced by Hayami et al. [12].

Lemma 5. *([9]) Let the function m(z) given by (1.2) having positive real part in E. Also let* $\mu \in \mathbb{C}$, *then*

$$|c_n - \mu c_k c_{n-k}| \le 2 \max\left(1, |2\mu - 1|\right), \ 1 \le k \le n - k.$$

3. Main results

Theorem 1. *Let the function f given by (1.1) belong to the class* $\mathcal{S}_p^*(\alpha, q)$, *then*

$$\begin{aligned} \left|a_{p+1}\right| &\le \frac{2([p]_q - \alpha)}{[p+1]_q - [p]_q}, \\ \left|a_{p+2}\right| &\le \frac{2([p]_q - \alpha)}{[p+2]_q - [p]_q}\left\{1 + \frac{2([p]_q - \alpha)}{[p+1]_q - [p]_q}\right\}, \\ \left|a_{p+3}\right| &\le \frac{2([p]_q - \alpha)}{[p+3]_q - [p]_q}\left[1 + 2([p]_q - \alpha)\Lambda_2\left\{\rho_3 + 2([p]_q - \alpha)\right\}\right], \end{aligned}$$

where Λ_2 *is given by (3.6).*

Proof. Let $f \in \mathcal{S}^{*}(\alpha, q)$, then their exist a function $\mathcal{P}(z) = [p]_q + \sum_{n=1}^{\infty} c_n z^n$ such that $\Re(m(z)) > \alpha$ and

$$\frac{z\left(D_q f\right)(z)}{f(z)} = m(z),$$

which implies that

$$z\left(D_q f\right)(z) = m(z) f(z).$$

Therefore, we have

$$\left([n]_q - [p]_q\right) a_n = \sum_{l=p}^{n-1} a_l c_{n-l}, \tag{3.1}$$

where $n \geq p+1, a_p = 1, c_0 = [p]_q$. From (3.1), we have

$$\begin{aligned}
a_{p+1} &= \frac{c_1}{[p+1]_q - [p]_q}, &(3.2)\\
a_{p+2} &= \frac{1}{[p+2]_q - [p]_q}\left\{c_2 + \frac{c_1^2}{\left([p+1]_q - [p]_q\right)}\right\}, &(3.3)\\
a_{p+3} &= \frac{1}{[p+3]_q - [p]_q}\left\{c_3 + \Lambda_1 c_1 c_2 + \Lambda_2 c_1^3\right\}, &(3.4)
\end{aligned}$$

where

$$\begin{aligned}
\Lambda_1 &= \Lambda_2 \rho_3, &(3.5)\\
\Lambda_2 &= \frac{1}{\left([p+1]_q - [p]_q\right)\left([p+2]_q - [p]_q\right)}, &(3.6)\\
\rho_3 &= [p+1]_q + [p+2]_q - 2[p]_q. &(3.7)
\end{aligned}$$

By using Lemma 2, we obtain the required result. □

Theorem 2. *Let an analytic function f given by (1.1) be in the class $\mathcal{S}_p^{*}(\alpha, q)$, then*

$$T_3((p+1) \leq \Lambda_3 \left[\Omega_4 + 4\left([p]_q - \alpha\right)^2 \Omega_5 + \Omega_7 + \Omega_8 \left|1 - \frac{2\left([p]_q - \alpha\right)\Omega_6}{\Omega_8}\right|\right],$$

where

$$\Lambda_3 = 4\left([p]_q - \alpha\right)^2 \left[\Omega_1 + \Omega_2\left(1 + \Omega_3\right)\right], \; \Omega_1 = \frac{2\left([p]_q - \alpha\right)}{[p+1]_q - [p]_q}, \tag{3.8}$$

$$\Omega_2 = \frac{2\left([p]_q - \alpha\right)}{[p+3]_q - [p]_q}, \tag{3.9}$$

$$\Omega_3 = 2\left([p]_q - \alpha\right)\Lambda_2\left\{\rho_3 + 2\left([p]_q - \alpha\right)\right\}, \tag{3.10}$$

$$\Omega_4 = \frac{1}{\left([p+1]_q - [p]_q\right)^2}, \quad \Omega_5 = 2\Lambda_2\Lambda_2 - \Lambda_2\rho_4, \tag{3.11}$$

$$\Omega_6 = \frac{4\Lambda_2}{\left([p+2]_q - [p]_q\right)} - \Lambda_2\rho_3\rho_4,$$

$$\Omega_7 = \frac{2}{\left([p+2]_q - [p]_q\right)^2}, \tag{3.12}$$

$$\Omega_8 = \rho_4 = \frac{1}{\left([p+1]_q - [p]_q\right)\left([p+3]_q - [p]_q\right)}. \tag{3.13}$$

Proof. A detailed calculation of $T_3(p+1)$ is in order.

$$T_3(p+1) = \left(a_{p+1} - a_{p+3}\right)\left(a_{p+1}^2 - 2a_{p+2}^2 + a_{p+1}a_{p+3}\right),$$

where a_{p+1}, a_{p+2}, and a_{p+3} is given by (3.2), (3.3) and (3.4).

Now if $f \in \mathcal{S}^*(\alpha, q)$, then we have

$$\begin{aligned} \left|a_{p+1} - a_{p+3}\right| &\leq \left|a_{p+1}\right| + \left|a_{p+3}\right|, \\ &\leq \Omega_1 + \Omega_2\left(1 + \Omega_3\right), \end{aligned} \tag{3.14}$$

where Ω_1, Ω_2, Ω_3, is given by (3.9), (3.10) and (3.11).

We need to maximize $\left|a_{p+1}^2 - 2a_{p+2}^2 + a_{p+1}a_{p+3}\right|$ for $f \in \mathcal{S}^*(\alpha, q)$, so by writing a_{p+1}, a_{p+2}, a_{p+3} in terms of c_1, c_2, c_3, with the help of (3.2), (3.3) and (3.4), we get

$$\begin{aligned} & \left|a_{p+1}^2 - 2a_{p+2}^2 + a_{p+1}a_{p+3}\right| \\ \leq\ & \left|\Omega_4 c_1^2 - \Omega_5 c_1^4 - \Omega_6 c_1^2 c_2 - \Omega_7 c_2^2 + \Omega_8 c_1 c_3\right|, \\ \leq\ & \Omega_4 c_1^2 + \Omega_5 c_1^4 + \Omega_7 c_2^2 + \Omega_8 c_1 \left|c_3 - \frac{\Omega_6 c_1 c_2}{\Omega_8}\right|. \end{aligned} \tag{3.15}$$

Finally applying Lemmas 2 and 5 along with (3.14) and (3.15), we obtained the required result. □

For $q \to 1-$, $p = 1$ and $\alpha = 0$, we have following known corollary.

Corollary 1. *([2]). Let an analytic function f be in the class $\mathcal{S}^*$, then*

$$T_3(2) \leq 84.$$

Theorem 3. *If an analytic function f given by (1.1) belongs to the class $\mathcal{S}_p^*(\alpha, q)$, then*

$$\left|a_{p+1}a_{p+3} - a_{p+2}^2\right| \leq \frac{4}{([p]_q - \alpha)^2([p+2]_q - [p]_q)^2}.$$

Proof. Making use of (3.2), (3.3) and (3.4), we have

$$\begin{aligned} & a_{p+1}a_{p+3} - a_{p+2}^2 \\ = \ & \rho_4 c_1 c_3 + (\Lambda_2\rho_3 - \mathfrak{B})\, c_1^2 c_2 - \mathcal{D} c_2^2 + (\Lambda_2\rho_4 - \Lambda_2\Lambda_2)\, c_1^4, \end{aligned}$$

where

$$\mathcal{D} = \frac{1}{\left([p+2]_q - [p]_q\right)^2}, \quad \mathfrak{B} = \frac{2\Lambda_2}{[p+2]_q - [p]_q}.$$

By using Lemma 3 and we take $\Upsilon = 4\left([p]_q - \alpha\right)^2 - c_1^2$ and $\mathcal{Z} = \left(1 - |x|^2\right) z$. Without loss of generality we assume that $c = c_1$, $\left(0 \leq c \leq 2\left([p]_q - \alpha\right)\right)$, so that

$$a_{p+1}a_{p+3} - a_{p+2}^2 = \lambda_1 c^4 + \lambda_2 \Upsilon c^2 x - \lambda_3 \Upsilon c^2 x^2 - \lambda_4 \Upsilon^2 x^2 + \lambda_5 \Upsilon c \mathcal{Z}, \tag{3.16}$$

where

$$\begin{aligned} \lambda_1 &= \frac{\rho_4}{4([p]_q - \alpha)^2} + \frac{\Lambda_2\rho_3 - \mathfrak{B}}{2([p]_q - \alpha)} - \frac{\mathcal{D}}{4([p]_q - \alpha)^2} - \frac{\mathcal{D}\,(\Lambda_2\rho_4 - \Lambda_2\Lambda_2)}{4([p]_q - \alpha)^2}, \\ \lambda_2 &= \frac{\rho_4}{2([p]_q - \alpha)^2} + \frac{\Lambda_2\rho_3 - \mathfrak{B}}{2([p]_q - \alpha)} - \frac{\mathcal{D}}{2([p]_q - \alpha)^2}, \\ \lambda_3 &= \frac{\rho_4}{4([p]_q - \alpha)^2}, \ \lambda_4 = \frac{\mathcal{D}}{4([p]_q - \alpha)^2}, \ \lambda_5 = \frac{\rho_4}{2([p]_q - \alpha)}. \end{aligned}$$

Taking the modulus on (3.16) and using triangle inequality, we find that

$$\begin{aligned} & \left|a_{p+1}a_{p+3} - a_{p+2}^2\right| \\ \leq \ & |\lambda_1|\, c^4 + |\lambda_2|\, \Upsilon c^2\, |x| + |\lambda_3|\, \Upsilon c^2\, |x|^2 + |\lambda_4|\, \Upsilon^2\, |x|^2 + |\lambda_5| \left(1 - |x|^2\right) c\Upsilon \\ = \ & \mathcal{G}(c, |x|). \end{aligned}$$

Now, trivially we have

$$\mathcal{G}'(c, |x|) > 0$$

on $[0, 1]$, which shows that $\mathcal{G}(c, |x|)$ is an increasing function in an interval $[0, 1]$, therefore maximum value occurs at $x = 1$ and Max $\mathcal{G}(c, |1|) = \mathcal{G}(c)$.

$$\mathcal{G}(c, |1|) = |\lambda_1|\, c^4 + |\lambda_2|\, \Upsilon c^2 + |\lambda_3|\, \Upsilon c^2 + |\lambda_4|\, \Upsilon^2$$

and

$$\mathcal{G}(c) = |\lambda_1|\, c^4 + |\lambda_2|\, \Upsilon c^2 + |\lambda_3|\, \Upsilon c^2 + |\lambda_4|\, \Upsilon^2.$$

Hence, by putting $\Upsilon = 4 - c_1^2$ and after some simplification, we have

$$\mathcal{G}(c) = (|\lambda_1| - |\lambda_2| - |\lambda_3| + |\lambda_4|)\, c^4 + 4\,(|\lambda_2| + |\lambda_3| - 2\,|\lambda_4|)\, c^2 + 16\,|\lambda_4|\,.$$

We consider $\mathcal{G}'(c) = 0$, for optimum value of $\mathcal{G}(c)$, which implies that $c = 0$. So $\mathcal{G}(c)$ has a maximum value at $c = 0$. Hence the maximum value of $\mathcal{G}(c)$ is given by

$$16\,|\lambda_4|\,. \tag{3.17}$$

Which occurs at $c = 0$ or

$$c^2 = \frac{4\left(|\lambda_2| + |\lambda_3| - 2\,|\lambda_4|\right)}{|\lambda_1| - |\lambda_2| - |\lambda_3| + |\lambda_4|}.$$

Hence, by putting $\lambda_4 = \frac{\mathcal{D}}{4([p]_q-\alpha)^2}$ and $\mathcal{D} = \frac{1}{\left([p+2]_q-[p]_q\right)^2}$ in (3.17) and after some simplification, we obtained the desired result. □

For $q \to 1-$, $p = 1$ and $\alpha = 0$, we have following known corollary.

Corollary 2. *([16]). If an analytic function f belongs to the class $\mathcal{S}^*$, then*

$$\left|a_2a_4 - a_3^2\right| \le 1.$$

3.1. Fekete–Szegö problem

Theorem 4. *Let f be the function given by (1.1) belongs to the class $\mathcal{S}_p^*(\alpha, q)$, $0 \le \alpha < p$, then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \le \begin{cases} \dfrac{2\left([p]_q-\alpha\right)}{\left([p+2]_q-[p]_q\right)}\left\{\rho_1 - \rho_2\mu\right\}, & \text{if } \mu \le \rho_5, \\[2ex] \dfrac{2\left([p]_q-\alpha\right)}{\left([p+2]_q-[p]_q\right)}, & \text{if } \rho_5 \le \mu \le \rho_6, \\[2ex] \dfrac{2\left([p]_q-\alpha\right)}{\left([p+1]_q-[p]_q\right)^2\left([p+2]_q-[p]_q\right)}\left\{\rho_2\mu - \rho_1\right\}, & \text{if } \mu \ge \rho_6, \end{cases}$$

where

$$\begin{aligned}
\rho_1 &= \left\{2\left([p]_q - \alpha\right)\left([p+1]_q - [p]_q\right) + \left([p+1]_q - [p]_q\right)^2\right\}, \\
\rho_2 &= 2\left([p]_q - \alpha\right)\left([p+2]_q - [p]_q\right), \\
\rho_5 &= \frac{\left([p+1]_q - [p]_q\right)\left\{2\left([p]_q - \alpha\right) + \left([p+1]_q - [p]_q\right)\right\} - 1}{2\left([p]_q - \alpha\right)\left([p+2]_q - [p]_q\right)}, \\
\rho_6 &= \frac{\left([p+1]_q - [p]_q\right)\left([p]_q - \alpha + \left([p+1]_q - [p]_q\right)\right)}{\left([p]_q - \alpha\right)\left([p+2]_q - [p]_q\right)}.
\end{aligned}$$

Proof. From (3.2) and (3.3) and we can suppose that $c_1 = c\left(0 \le c \le 2([p]_q - \alpha)\right)$, without loss of generality we derive

$$\left|a_{p+2} - \mu a_{p+1}^2\right| = \frac{1}{\rho_7}\left|\left\{\rho_1 - \rho_2\mu\right\}c^2 + \left([p+1]_q - [p]_q\right)^2\left\{4\left([p]_q - \alpha\right)^2 - c^2\right\}\rho\right|$$

$$= \quad A(\rho),$$

where

$$\rho_7 = 2\left([p]_q - \alpha\right)\left([p+1]_q - [p]_q\right)^2\left([p+2]_q - [p]_q\right).$$

Applying the triangle inequality, we deduce

$$\begin{aligned} A(\rho) &\leq \frac{1}{\rho_7}\left|\{\rho_1 - \rho_2\mu\}\right| c^2 + \left([p+1]_q - [p]_q\right)^2\left\{4\left([p]_q - \alpha\right)^2 - c^2\right\} \\ &= \begin{cases} \frac{1}{\rho_7}\left[\left\{2\left([p]_q - \alpha\right)\{\rho_{11} - \rho_{12}\mu\}\right\} c^2 + \rho_9\right], & \text{if } \mu \leq \rho_8, \\ \\ \frac{1}{\rho_7}\left[2\left\{\left([p]_q - \alpha\right)\left([p+2]_q - [p]_q\right)\mu - \rho_{10}\right\} c^2 + \rho_9\right], & \text{if } \mu \geq \rho_8, \end{cases} \end{aligned}$$

where

$$\begin{aligned} \rho_8 &= \frac{2\left([p]_q - \alpha\right)\left([p+1]_q - [p]_q\right) + \left([p+1]_q - [p]_q\right)^2}{2\left([p]_q - \alpha\right)\left([p+2]_q - [p]_q\right)}, \\ \rho_9 &= 4\left([p]_q - \alpha\right)^2\left([p+1]_q - [p]_q\right)^2, \\ \rho_{10} &= \left([p+1]_q - [p]_q\right)\left\{\left([p]_q - \alpha\right) + \left([p+1]_q - [p]_q\right)\right\}, \\ \rho_{11} &= \left([p+1]_q - [p]_q\right), \ \rho_{12} = \left([p+2]_q - [p]_q\right), \\ \rho_{13} &= \frac{2\left([p]_q - \alpha\right)}{\left([p+1]_q - [p]_q\right)^2\left([p+2]_q - [p]_q\right)}. \end{aligned}$$

$$\left|a_{p+2} - \mu a_{p+1}^2\right|$$

$$\leq \begin{cases} \frac{2\left([p]_q-\alpha\right)}{\left([p+2]_q-[p]_q\right)}\{\rho_1 - \rho_2\mu\}, & \text{if } \mu \leq \rho_5, \ c = 2\left([p]_q - \alpha\right), \\ \\ \frac{2\left([p]_q-\alpha\right)}{\left([p+2]_q-[p]_q\right)}, & \text{if } \rho_5 \leq \mu \leq \rho_8, \ c = 0, \\ \\ \frac{2\left([p]_q-\alpha\right)}{\left([p+2]_q-[p]_q\right)}, & \text{if } \rho_8 \leq \mu \leq \rho_6, \ c = 0, \\ \\ \rho_{13}\left\{\rho_2\mu - \left\{2\left([p]_q - \alpha\right)\rho_{11} + \rho_{11}^2\right\}\right\}, & \text{if } \mu \geq \rho_6, \ c = 2\left([p]_q - \alpha\right). \end{cases}$$

□

If $q \to 1-$ in Theorem 4, we thus obtain the following known result.

Corollary 3. *([12]). Let f be the function given by (1.1) belongs to the class $\mathcal{S}_p^*(\alpha)$, $0 \leq \alpha < p$, then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \begin{cases} (p-\alpha)\left\{\{2(p-\alpha)+1\} - 4(p-\alpha)\mu\right\}, & \text{if } \mu \leq \frac{1}{2}, \\ (p-\alpha), & \text{if } \frac{1}{2} \leq \mu \leq \frac{p-\alpha+1}{2(p-\alpha)}, \\ (p-\alpha)\left\{4(p-\alpha)\mu - \{2(p-\alpha)+1\}\right\}, & \text{if } \mu \geq \frac{p-\alpha+1}{2(p-\alpha)}. \end{cases}$$

3.2. Applications of our main results

In this section, firstly we recall that the q-Bernardi integral operator for multivalent functions $\mathcal{L}(f) = \mathcal{B}_{p,\beta}^q$ given in [35] as:

Let $f \in \mathcal{A}_p$, then $\mathcal{L}:\mathcal{A}_p \to \mathcal{A}_p$ is called the q-analogue of Benardi integral operator for multivalent functions defined by $\mathcal{L}(f) = \mathcal{B}_{q,\beta}^q$ with $\beta > -p$, where, $\mathcal{B}_{q,\beta}^q$ is given by

$$\begin{aligned} \mathcal{B}_{p,\beta}^q f(z) &= \frac{[p+\beta]_q}{z^\beta} \int_0^z t^{\beta-1} f(t) d_q t \,, && (3.18) \\ &= z^p + \sum_{n=1}^{\infty} \frac{[\beta+p]_q}{[n+\beta+p]_q} a_{n+p} z^{n+p}, \quad z \in E, \\ &= z^p + \sum_{n=1}^{\infty} \mathcal{B}_{n+p} a_{n+p} z^{n+p}. && (3.19) \end{aligned}$$

The series given in (3.19) converges absolutely in E.

Remark 6. For $q \to 1-$, then the operator $\mathcal{B}_{p,\beta}^q$ reduces to the integral operator studied in [48].

Remark 7. For $p = 1$, we obtain the q-Bernardi integral operator introduced in [30].

Remark 8. If $q \to 1-$ and $p = 1$, we obtain the familiar Bernardi integral operator studied in [6].

Theorem 5. *If f is of the form (1.1), belongs to the class $\mathcal{S}_p^*(\alpha, q)$, and*

$$\mathcal{B}_{p,\beta}^q f(z) = z^p + \sum_{n=1}^{\infty} \mathcal{B}_{n+p} a_{n+p} z^{n+p},$$

where $\mathcal{B}_{p,\beta}^q$ is the integral operator given by (3.18), then

$$\begin{aligned} \left|a_{p+1}\right| &\leq \frac{2([p]_q - \alpha)}{\left([p+1]_q - [p]_q\right)\mathcal{B}_{p+1}}, \\ \left|a_{p+2}\right| &\leq \frac{2([p]_q - \alpha)}{\left([p+2]_q - [p]_q\right)\mathcal{B}_{p+2}} \left\{1 + \frac{2([p]_q - \alpha)}{\left([p+1]_q - [p]_q\right)\mathcal{B}_{p+1}}\right\}, \\ \left|a_{p+3}\right| &\leq \frac{2([p]_q - \alpha)}{\left([p+3]_q - [p]_q\right)\mathcal{B}_{p+3}} \left[1 + \frac{2([p]_q - \alpha)\rho_{14}}{\rho_{15}}\right], \end{aligned}$$

where

$$\rho_{14} = \left\{\left(\left([p+1]_q - [p]_q\right)\mathcal{B}_{p+1} + \left([p+2]_q - [p]_q\right)\mathcal{B}_{p+2}\right) + 2([p]_q - \alpha)\right\},$$

$$\rho_{15} = \left([p+1]_q - [p]_q\right)\left([p+2]_q - [p]_q\right)\mathcal{B}_{p+1}\mathcal{B}_{p+2}.$$

Proof. The proof follows easily by using (3.19) and Theorem 1. □

Theorem 6. *Let an analytic function f given by (1.1) be in the class $\mathcal{S}_p^*(\alpha, q)$, in addition $\mathcal{B}_{p,\beta}^q$ is the integral operator defined by (3.18) and is of the form (3.19), then*

$$T_3((p+1) \leq \Upsilon_3 \begin{bmatrix} \frac{\Omega_4}{\mathcal{B}_{p+1}^2} + 4\left([p]_q - \alpha\right)^2 \Omega_{10} + \frac{\Omega_7}{\mathcal{B}_{p+2}^2} \\ + \frac{\Omega_8}{\mathcal{B}_{p+1}\mathcal{B}_{p+3}} \left|1 - \frac{2\left([p]_q - \alpha\right)\mathcal{B}_{p+1}\mathcal{B}_{p+3}\Omega_{11}}{\Omega_8}\right| \end{bmatrix},$$

where

$$\begin{aligned} \Upsilon_3 &= 4\left([p]_q - \alpha\right)^2 \left[\frac{\Omega_1}{\mathcal{B}_{p+1}} + \frac{\Omega_2}{\mathcal{B}_{p+3}}(1 + \Omega_9)\right], \\ \Omega_9 &= \Lambda_p\left(\frac{\rho_{14}}{\mathcal{B}_{p+1}\mathcal{B}_{p+2}}\right), \\ \Omega_{10} &= \Lambda_4 - \Lambda_5, \ \ \Omega_{11} = \Lambda_6 - \Lambda_7, \end{aligned}$$

$$\begin{aligned} \Lambda_4 &= \frac{2\Lambda_2\Lambda_2}{\mathcal{B}_{p+1}^2\mathcal{B}_{p+2}^2}, \ \ \Lambda_5 = \frac{\Lambda_2\rho_4}{\mathcal{B}_{p+1}^2\mathcal{B}_{p+2}\mathcal{B}_{p+3}}, \\ \Lambda_6 &= \frac{4\Lambda_2}{\left([p+2]_q - [p]_q\right)\mathcal{B}_{p+1}\mathcal{B}_{p+2}^2}, \\ \Lambda_7 &= \frac{\Lambda_8\Lambda_2\rho_4}{\mathcal{B}_{p+1}^2\mathcal{B}_{p+2}\mathcal{B}_{p+3}}, \\ \Lambda_8 &= \left([p+1]_q - [p]_q\right)\mathcal{B}_{p+1} + \left([p+2]_q - [p]_q\right)\mathcal{B}_{p+2}. \end{aligned}$$

Proof. The proof follows easily by using (3.19) and Theorem 2. □

Theorem 7. *If an analytic function f given by (1.1) belongs to the class $\mathcal{S}_p^*(\alpha, q)$, in addition $\mathcal{B}_{p,\beta}^q$ is the integral operator is defined by (3.18) and is of the form (3.19), then*

$$\left|a_{p+1}a_{p+3} - a_{p+2}^2\right| \leq \frac{4}{([p]_q - \alpha)^2\left([p+2]_q - [p]_q\right)^2 \mathcal{B}_{p+2}^2}.$$

Theorem 8. *Let f be the function given by (1.1) belongs to the class $\mathcal{S}_p^*(\alpha, q)$, in addition $\mathcal{B}_{p,\beta}^q$ is the integral operator defined by (3.18) and is of the form (3.19), then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \begin{cases} \frac{2\left([p]_q - \alpha\right)}{\left([p+2]_q - [p]_q\right)\mathcal{B}_{p+2}} \left\{\rho_{16} - \rho_2 \mathcal{B}_{p+2}\mu\right\}, & \text{if } \mu \leq \rho_{17}, \\ \frac{2\left([p]_q - \alpha\right)}{\left([p+2]_q - [p]_q\right)\mathcal{B}_{p+2}}, & \text{if } \rho_{17} \leq \mu \leq \rho_{18}, \\ \frac{2\Lambda_2\left([p]_q - \alpha\right)}{\left([p+1]_q - [p]_q\right)\mathcal{B}_{p+1}^2 \mathcal{B}_{p+2}} \left\{\rho_2 \mathcal{B}_{p+2}\mu - \rho_{16}\right\}, & \text{if } \mu \geq \rho_{18}, \end{cases}$$

where

$$\begin{aligned} \rho_{16} &= \left\{2\left([p]_q - \alpha\right)\left([p+1]_q - [p]_q\right)\mathcal{B}_{p+1} + \left([p+1]_q - [p]_q\right)^2 \mathcal{B}_{p+1}^2\right\}, \\ \rho_{17} &= \frac{\left([p+1]_q - [p]_q\right)\mathcal{B}_{p+1}\left\{2\left([p]_q - \alpha\right) + \left([p+1]_q - [p]_q\right)\mathcal{B}_{p+1}\right\} - 1}{2\left([p]_q - \alpha\right)\left([p+2]_q - [p]_q\right)\mathcal{B}_{p+2}}, \\ \rho_{18} &= \frac{\left([p+1]_q - [p]_q\right)\mathcal{B}_{p+1}\left([p]_q - \alpha + \left([p+1]_q - [p]_q\right)\mathcal{B}_{p+1}\right)}{\left([p]_q - \alpha\right)\left([p+2]_q - [p]_q\right)\mathcal{B}_{p+2}}, \end{aligned}$$

and Λ_2 is given by (3.6).

4. Conclusions

Motivated by a number of recent works, we have made use of the quantum (or q-) calculus to define and investigate new subclass of multivalent q-starlike functions in open unit disk E. We have studied about Hankel determinant, Toeplitz matrices, Fekete–Szegö inequalities. Furthermore we discussed applications of our main results by using q-Bernardi integral operator for multivalent functions. All the results that have discussed in this paper can easily investigate for the subclass of meromorphic q-convex functions ($\mathcal{K}_p(\alpha, q)$) of order α in E, respectively.

Basic (or q-) series and basic (or q-) polynomials, especially the basic (or q-) hypergeometric functions and basic (or q-) hypergeometric polynomials, are applicable particularly in several diverse areas (see [38], p328). Moreover, in this recently-published survey-cum expository review article by Srivastava [38], the so called (p, q)-calculus was exposed to be a rather trivial and inconsequential variation of the classical q-calculus (see for details [38], p340).

By this observation of Srivastava in [38], we can make clear link between the q-analysis and (p, q)-analysis and the results for q-analogues which we have included in this paper for $0 < q < 1$, can be easily transformed into the related results for the (p, q)-analogues with $(0 < q < p \leq 1)$.

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Conflict of interest

The authors declare that they have no competing interests.

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