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Research article

Optimality conditions for variational problems involving distributed-order fractional derivatives with arbitrary kernels

Fátima Cruz, Ricardo Almeida*and Natália Martins

Center for Research and Development in Mathematics and Applications, Department of Mathematics, University of Aveiro, Aveiro, Portugal

* **Correspondence:** Email: ricardo.almeida@ua.pt.

Abstract: In this work we study necessary and sufficient optimality conditions for variational problems dealing with a new fractional derivative. This fractional derivative combines two known operators: distributed-order derivatives and derivatives with arbitrary kernels. After proving a fractional integration by parts formula, we obtain the Euler–Lagrange equation and natural boundary conditions for the fundamental variational problem. Also, fractional variational problems with integral and holonomic constraints are considered. We end with some examples to exemplify our results.

Keywords: fractional calculus; calculus of variations; Euler-Lagrange equation; isoperimetric problems; holonomic problems
Mathematics Subject Classification: 26A33, 49K05

1. Introduction

The fractional calculus is an old subject and presents an extension of ordinary calculus [12, 14]. It began at the same time with the works of Leibniz on differential calculus, where he questioned what could be a derivative of arbitrary real order $\alpha > 0$. Since then, a large number of definitions of fractional order integral and derivative operators have appeared. Thus, we find in the literature numerous works dealing with similar topics, but for different fractional operators. One possible way to avoid such issue is to consider a more general class of fractional operators, like, for example, fractional integrals and derivatives with arbitrary kernels [3, 12] or other types of general fractional derivatives [16–20].

Another possible approach to fractional calculus is, instead of fixing the fractional order α, the introduction of a new function that acts like a distribution of the orders of differentiation [9, 10]. Our goal in this paper is to combine both ideas into a single operator, in order to obtain new results that will generalize some of the already known.

The main purpose of this paper is to prove optimality conditions for variational problems that depend on distributed-order fractional derivatives with arbitrary kernels. Namely, we will prove the Euler-Lagrange equation and the natural boundary conditions for variational problems with and without integral constraints and also with an holonomic constraint. Moreover, we provide sufficient optimality conditions for all the variational problems studied in this paper. With this work we generalize several existent works on fractional calculus of variations such as [1, 2, 7, 8, 13].

The structure of the paper is as follows. In Section 2, we recall the necessary definitions and results from fractional calculus that are needed to the present work. Our main contributions are presented in Section 3. We finalize the paper with some illustrative examples and concluding remarks.

2. Preliminaries

Throughout the paper, Γ represents the well-known Gamma function and $[\alpha]$ denotes the integer part of $\alpha \in \mathbb{R}$. We begin by recalling the definition of ψ-Riemann–Liouville fractional integrals of a function x of order $\alpha \in \mathbb{R}^+$.

Definition 2.1. [14] Let $\alpha \in \mathbb{R}^+$, $x : [a,b] \to \mathbb{R}$ be an integrable function, and $\psi \in C^1([a,b],\mathbb{R})$ another function with $\psi'(t) > 0$, for all $t \in [a,b]$. The left and right Riemann–Liouville fractional integrals of x with respect to the kernel ψ, of order α, are defined by

$$\mathrm{I}_{a+}^{\alpha,\psi} x(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} x(\tau) d\tau, \quad \text{for } t > a,$$

and

$$\mathrm{I}_{b-}^{\alpha,\psi} x(t) := \frac{1}{\Gamma(\alpha)} \int_t^b \psi'(\tau)(\psi(\tau) - \psi(t))^{\alpha-1} x(\tau) d\tau, \quad \text{for } t < b,$$

respectively.

Definition 2.2. [14] Let $\alpha \in \mathbb{R}^+$, $x : [a,b] \to \mathbb{R}$ an integrable function, and $\psi \in C^n([a,b],\mathbb{R})$ with $\psi'(t) > 0$, for all $t \in [a,b]$. The left and right Riemann–Liouville fractional derivatives of x with respect to the kernel ψ, of order α, are defined by

$$\mathrm{D}_{a+}^{\alpha,\psi} x(t) := \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \mathrm{I}_{a+}^{n-\alpha,\psi} x(t) \quad \text{and} \quad \mathrm{D}_{b-}^{\alpha,\psi} x(t) := \left(-\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \mathrm{I}_{b-}^{n-\alpha,\psi} x(t),$$

respectively, where $n = [\alpha] + 1$.

The operators $\mathrm{D}_{a+}^{\alpha,\psi} x$ and $\mathrm{D}_{b-}^{\alpha,\psi} x$ can be simply called ψ-Riemann–Liouville fractional derivatives of x of order α [5]. If we interchange the order of the ordinary derivative with the fractional integral, we obtain the definition of the ψ-Caputo fractional derivatives of x of order α.

Definition 2.3. [3] Given $\alpha \in \mathbb{R}^+$, let $n \in \mathbb{N}$ be given by $n = [\alpha] + 1$ if $\alpha \notin \mathbb{N}$, and $n = \alpha$ if $\alpha \in \mathbb{N}$. Given two functions $x, \psi \in C^n([a,b],\mathbb{R})$ with $\psi'(t) > 0$, for all $t \in [a,b]$, we define the left and right Caputo fractional derivatives of x with respect to the kernel ψ, of order α, by

$${}^C\mathrm{D}_{a+}^{\alpha,\psi} x(t) := \mathrm{I}_{a+}^{n-\alpha,\psi} \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n x(t) \quad \text{and} \quad {}^C\mathrm{D}_{b-}^{\alpha,\psi} x(t) := \mathrm{I}_{b-}^{n-\alpha,\psi} \left(-\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n x(t),$$

respectively.

Lemma 2.4. *[3] Given $\alpha > 0$, let $n \in \mathbb{N}$ be given by Definition 2.3. For $\beta \in \mathbb{R}$ with $\beta > n$, we have that*

$$^C\mathrm{D}_{a+}^{\alpha,\psi}(\psi(t) - \psi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\psi(t) - \psi(a))^{\beta-\alpha-1}$$

and

$$^C\mathrm{D}_{b-}^{\alpha,\psi}(\psi(b) - \psi(t))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\psi(b) - \psi(t))^{\beta-\alpha-1}.$$

Until the end of the work, the fractional order α belongs to the interval $[0, 1]$ and the kernel ψ is a function on the set $C^1([a, b], \mathbb{R})$, with $\psi'(t) > 0$, for all $t \in [a, b]$.

In order to introduce the new concepts of distributed-order fractional derivatives with respect to another function, in the Riemann–Liouville and in the Caputo sense, we consider a new continuous function $\phi : [0, 1] \to [0, 1]$ that satisfies the condition

$$\int_0^1 \phi(\alpha) d\alpha > 0.$$

Usually, function ϕ is called order-weighting or strength function. For some applications on distributed-order fractional derivatives, we suggest the paper [11].

Definition 2.5. Let $x : [a, b] \to \mathbb{R}$ be an integrable function. The left and right Riemann–Liouville distributed-order fractional derivatives of a function x with respect to the kernel ψ are defined by

$$\mathrm{D}_{a+}^{\phi(\alpha),\psi} x(t) := \int_0^1 \phi(\alpha) \mathrm{D}_{a+}^{\alpha,\psi} x(t) d\alpha \quad \text{and} \quad \mathrm{D}_{b-}^{\phi(\alpha),\psi} x(t) := \int_0^1 \phi(\alpha) \mathrm{D}_{b-}^{\alpha,\psi} x(t) d\alpha,$$

where $\mathrm{D}_{a+}^{\alpha,\psi}$ and $\mathrm{D}_{b-}^{\alpha,\psi}$ are the left and right ψ-Riemann–Liouville fractional derivatives of order α, respectively.

Definition 2.6. The left and right Caputo distributed-order fractional derivatives of a function $x \in C^1([a, b], \mathbb{R})$ with respect to the kernel ψ are defined by

$$^C\mathrm{D}_{a+}^{\phi(\alpha),\psi} x(t) := \int_0^1 \phi(\alpha) {}^C\mathrm{D}_{a+}^{\alpha,\psi} x(t) d\alpha \quad \text{and} \quad {}^C\mathrm{D}_{b-}^{\phi(\alpha),\psi} x(t) := \int_0^1 \phi(\alpha) {}^C\mathrm{D}_{b-}^{\alpha,\psi} x(t) d\alpha,$$

where $^C\mathrm{D}_{a+}^{\alpha,\psi}$ and $^C\mathrm{D}_{b-}^{\alpha,\psi}$ are the left and right ψ-Caputo fractional derivatives of order α, respectively.

For our work we will also need the concepts of distributed-order fractional integrals with respect to the kernel ψ:

$$\mathrm{I}_{a+}^{1-\phi(\alpha),\psi} x(t) := \int_0^1 \phi(\alpha) \mathrm{I}_{a+}^{1-\alpha,\psi} x(t) d\alpha \quad \text{and} \quad \mathrm{I}_{b-}^{1-\phi(\alpha),\psi} x(t) := \int_0^1 \phi(\alpha) \mathrm{I}_{b-}^{1-\alpha,\psi} x(t) d\alpha,$$

where $\mathrm{I}_{a+}^{1-\alpha,\psi}$ and $\mathrm{I}_{b-}^{1-\alpha,\psi}$ are the left and right ψ-Riemann–Liouville fractional integrals of order $1 - \alpha$, respectively.

3. Main results

In the sequel, let us consider two continuous functions $\phi, \varphi : [0,1] \to [0,1]$ satisfying the conditions

$$\int_0^1 \phi(\alpha)d\alpha > 0 \qquad \text{and} \qquad \int_0^1 \varphi(\alpha)d\alpha > 0.$$

The goal of this work is to exhibit necessary and sufficient optimality conditions for the following fractional variational problem:

Problem ($\mathcal{P}$): Find a curve $x \in C^1([a,b],\mathbb{R})$ that minimizes or maximizes the following functional

$$\mathcal{J}(x) := \int_a^b L\left(t, x(t), {}^C\mathrm{D}_{a+}^{\phi(\alpha),\psi} x(t), {}^C\mathrm{D}_{b-}^{\varphi(\alpha),\psi} x(t)\right) dt, \tag{3.1}$$

where $L : [a,b]\times\mathbb{R}^3 \to \mathbb{R}$ is assumed to be continuously differentiable with respect to the second, third and fourth variables. In our study, we will consider the variational problem with and without fixed boundary conditions, and also with an isoperimetric or holonomic constraints.

Before proving our main results, we need to prove the following integration by parts formulae for the left and right Caputo distributed-order fractional derivatives with respect to another function.

Theorem 3.1. *(Integration by parts formulae) Let $x : [a,b] \to \mathbb{R}$ be a continuous function and $y : [a,b] \to \mathbb{R}$ a continuously differentiable function. Then,*

$$\int_a^b x(t)^C\mathrm{D}_{a+}^{\phi(\alpha),\psi} y(t)dt = \int_a^b y(t)\left(\mathrm{D}_{b-}^{\phi(\alpha),\psi}\frac{x(t)}{\psi'(t)}\right)\psi'(t)dt + \left[y(t)\left(\mathrm{I}_{b-}^{1-\phi(\alpha),\psi}\frac{x(t)}{\psi'(t)}\right)\right]_{t=a}^{t=b}$$

and

$$\int_a^b x(t)^C\mathrm{D}_{b-}^{\varphi(\alpha),\psi} y(t)dt = \int_a^b y(t)\left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi}\frac{x(t)}{\psi'(t)}\right)\psi'(t)dt - \left[y(t)\left(\mathrm{I}_{a+}^{1-\varphi(\alpha),\psi}\frac{x(t)}{\psi'(t)}\right)\right]_{t=a}^{t=b}.$$

Proof. By definition of the left ψ-Caputo distributed-order fractional derivative, we have the following:

$$\begin{aligned}\int_a^b x(t)^C\mathrm{D}_{a+}^{\phi(\alpha),\psi} y(t)dt &= \int_a^b x(t)\int_0^1 \phi(\alpha)^C\mathrm{D}_{a+}^{\alpha,\psi} y(t)d\alpha\, dt\\ &= \int_a^b x(t)\int_0^1 \frac{\phi(\alpha)}{\Gamma(1-\alpha)}\int_a^t (\psi(t)-\psi(\tau))^{-\alpha}y'(\tau)d\tau\, d\alpha\, dt\\ &= \int_0^1 \frac{\phi(\alpha)}{\Gamma(1-\alpha)}\int_a^b\int_a^t x(t)(\psi(t)-\psi(\tau))^{-\alpha}y'(\tau)d\tau\, dt\, d\alpha.\end{aligned}$$

Reversing the order of integration, we get

$$\begin{aligned}&\int_0^1 \frac{\phi(\alpha)}{\Gamma(1-\alpha)}\int_a^b\int_a^t x(t)(\psi(t)-\psi(\tau))^{-\alpha}y'(\tau)d\tau\, dt\, d\alpha\\ &= \int_0^1 \frac{\phi(\alpha)}{\Gamma(1-\alpha)}\int_a^b y'(\tau)\int_\tau^b x(t)(\psi(t)-\psi(\tau))^{-\alpha}dt\, d\tau\, d\alpha.\end{aligned}$$

Using the standard integration by parts formula, we have

$$\int_0^1 \frac{\phi(\alpha)}{\Gamma(1-\alpha)}\left[\int_a^b y'(\tau)\int_\tau^b x(t)(\psi(t)-\psi(\tau))^{-\alpha}dt\,d\tau\right]d\alpha$$
$$= \int_0^1 \frac{\phi(\alpha)}{\Gamma(1-\alpha)}\left[\left[y(\tau)\int_\tau^b x(t)(\psi(t)-\psi(\tau))^{-\alpha}dt\right]_{\tau=a}^{\tau=b}\right.$$
$$\left. - \int_a^b y(\tau)\frac{d}{d\tau}\left(\int_\tau^b x(t)(\psi(t)-\psi(\tau))^{-\alpha}dt\right)d\tau\right]d\alpha.$$

Therefore, one gets

$$\begin{aligned}
&\int_a^b x(t)^C\mathrm{D}_{a+}^{\phi(\alpha),\psi}y(t)dt\\
&= \left[y(\tau)\int_0^1 \frac{\phi(\alpha)}{\Gamma(1-\alpha)}\int_\tau^b \psi'(t)(\psi(t)-\psi(\tau))^{-\alpha}\left(\frac{x(t)}{\psi'(t)}\right)dt\,d\alpha\right]_{\tau=a}^{\tau=b}\\
&+ \int_a^b y(\tau)\int_0^1 \frac{\phi(\alpha)}{\Gamma(1-\alpha)}\left(\frac{-1}{\psi'(\tau)}\frac{d}{d\tau}\right)\left(\int_\tau^b \psi'(t)(\psi(t)-\psi(\tau))^{-\alpha}\left(\frac{x(t)}{\psi'(t)}\right)dt\right)\psi'(\tau)\,d\alpha\,d\tau\\
&= \left[y(\tau)\int_0^1 \phi(\alpha)\left(\mathrm{I}_{b-}^{1-\alpha,\psi}\frac{x(\tau)}{\psi'(\tau)}\right)d\alpha\right]_{\tau=a}^{\tau=b} + \int_a^b y(\tau)\int_0^1 \phi(\alpha)\left(\mathrm{D}_{b-}^{\alpha,\psi}\frac{x(\tau)}{\psi'(\tau)}\right)d\alpha\,\psi'(\tau)\,d\tau.
\end{aligned}$$

Hence, we conclude that

$$\int_a^b x(t)^C\mathrm{D}_{a+}^{\phi(\alpha),\psi}y(t)dt = \int_a^b y(t)\left(\mathrm{D}_{b-}^{\phi(\alpha),\psi}\frac{x(t)}{\psi'(t)}\right)\psi'(t)dt + \left[y(t)\left(\mathrm{I}_{b-}^{1-\phi(\alpha),\psi}\frac{x(t)}{\psi'(t)}\right)\right]_{t=a}^{t=b}$$

as desired.

Using similar techniques, we deduce the integration by parts formula involving the operator ${}^C\mathrm{D}_{b-}^{\varphi(\alpha),\psi}$.

□

3.1. Necessary optimality conditions

In what follows, we will denote by $\partial_i L$ the partial derivative of L with respect to its ith-coordinate and use the notation:

$$[x](t) := \left(t, x(t), {}^C\mathrm{D}_{a+}^{\phi(\alpha),\psi}x(t), {}^C\mathrm{D}_{b-}^{\varphi(\alpha),\psi}x(t)\right).$$

We are now in a position to prove our first main result.

Theorem 3.2. *(Fractional Euler–Lagrange equation and natural boundary conditions) Let $x \in C^1([a,b],\mathbb{R})$ be a curve such that functional $\mathcal{J}$ as defined by* (3.1) *attains an extremum. If the maps*

$$t \mapsto \mathrm{D}_{b-}^{\phi(\alpha),\psi}\frac{\partial_3 L[x](t)}{\psi'(t)} \qquad \text{and} \qquad t \mapsto \mathrm{D}_{a+}^{\varphi(\alpha),\psi}\frac{\partial_4 L[x](t)}{\psi'(t)}$$

are continuous on $[a,b]$*, then x satisfies the following Euler-Lagrange equation*

$$\partial_2 L[x](t) + \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi} \frac{\partial_3 L[x](t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi} \frac{\partial_4 L[x](t)}{\psi'(t)}\right)\psi'(t) = 0, \tag{3.2}$$

for all $t \in [a, b]$. Also, if $x(a)$ is free, then

$$\mathrm{I}_{b-}^{1-\phi(\alpha),\psi} \frac{\partial_3 L[x](t)}{\psi'(t)} = \mathrm{I}_{a+}^{1-\varphi(\alpha),\psi} \frac{\partial_4 L[x](t)}{\psi'(t)}, \quad at\ t = a, \tag{3.3}$$

and if $x(b)$ is free, then

$$\mathrm{I}_{b-}^{1-\phi(\alpha),\psi} \frac{\partial_3 L[x](t)}{\psi'(t)} = \mathrm{I}_{a+}^{1-\varphi(\alpha),\psi} \frac{\partial_4 L[x](t)}{\psi'(t)}, \quad at\ t = b. \tag{3.4}$$

Proof. Let $h \in C^1([a, b], \mathbb{R})$ be an arbitrary function and define the function j by $j(\epsilon) := \mathcal{J}(x + \epsilon h)$, $\epsilon \in \mathbb{R}$. Since x is an extremizer of $\mathcal{J}$, $j'(0) = 0$, and, therefore,

$$\int_a^b \left(\partial_2 L[x](t) \cdot h(t) + \partial_3 L[x](t) \cdot {}^C\mathrm{D}_{a+}^{\phi(\alpha),\psi} h(t) + \partial_4 L[x](t) \cdot {}^C\mathrm{D}_{b-}^{\varphi(\alpha),\psi} h(t)\right) dt = 0.$$

Using Theorem 3.1 we obtain

$$\begin{aligned}\int_a^b &\left(\partial_2 L[x](t) + \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi} \frac{\partial_3 L[x](t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi} \frac{\partial_4 L[x](t)}{\psi'(t)}\right)\psi'(t)\right) h(t) dt \\ &+ \left[h(t)\left(\mathrm{I}_{b-}^{1-\phi(\alpha),\psi} \frac{\partial_3 L[x](t)}{\psi'(t)}\right)\right]_{t=a}^{t=b} - \left[h(t)\left(\mathrm{I}_{a+}^{1-\varphi(\alpha),\psi} \frac{\partial_4 L[x](t)}{\psi'(t)}\right)\right]_{t=a}^{t=b} = 0.\end{aligned} \tag{3.5}$$

If we restrict the variations h by considering $h(a) = h(b) = 0$, we have

$$\int_a^b \left(\partial_2 L[x](t) + \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi} \frac{\partial_3 L[x](t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi} \frac{\partial_4 L[x](t)}{\psi'(t)}\right)\psi'(t)\right) h(t) dt = 0.$$

Since h is arbitrary, from the Fundamental Lemma of Calculus of Variations (see [15]), we get

$$\partial_2 L[x](t) + \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi} \frac{\partial_3 L[x](t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi} \frac{\partial_4 L[x](t)}{\psi'(t)}\right)\psi'(t) = 0,$$

for all $t \in [a, b]$, proving the Euler-Lagrange equation (3.2). If $x(a)$ is free, considering $h(a) \neq 0$ and $h(b) = 0$ in (3.5) and using (3.2), we obtain

$$h(a)\left(\mathrm{I}_{a+}^{1-\varphi(\alpha),\psi} \frac{\partial_4 L[x]}{\psi'}(a) - \mathrm{I}_{b-}^{1-\phi(\alpha),\psi} \frac{\partial_3 L[x]}{\psi'}(a)\right) = 0.$$

Since $h(a)$ is arbitrary, we get that, at $t = a$,

$$\mathrm{I}_{b-}^{1-\phi(\alpha),\psi} \frac{\partial_3 L[x](t)}{\psi'(t)} = \mathrm{I}_{a+}^{1-\varphi(\alpha),\psi} \frac{\partial_4 L[x](t)}{\psi'(t)},$$

proving the natural boundary condition (3.3). Similarly, if $x(b)$ is free, considering $h(a) = 0$ and $h(b) \neq 0$ in (3.5) and using (3.2), we deduce the natural boundary condition (3.4). □

Remark 1. It is clear that the variational problem $(\mathcal{P})$ can be easily extended to functionals depending on a vector function $x := (x_1, ..., x_n)$. More precisely, let $L : [a, b] \times \mathbb{R}^{3n} \to \mathbb{R}$ be a continuously differentiable function with respect to the j-th variable, for $j = 2, \ldots, 3n+1$, and consider the functional

$$\mathcal{J}(x) := \int_a^b L[x](t)dt.$$

It follows from the proof of Theorem 3.2 that, if functional $\mathcal{J}$ attains an extremum at $x = (x_1, ..., x_n)$, then, for all $t \in [a, b]$ and $i = 1, ..., n$,

$$\partial_{i+1}L[x](t) + \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi} \frac{\partial_{i+n+1}L[x](t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi} \frac{\partial_{i+2n+1}L[x](t)}{\psi'(t)}\right)\psi'(t) = 0.$$

If the state values $x(a)$ and $x(b)$ are free, then we get the following $2n$ natural boundary conditions:

$$\mathrm{I}_{b-}^{1-\phi(\alpha),\psi} \frac{\partial_{i+n+1}L[x](t)}{\psi'(t)} = \mathrm{I}_{a+}^{1-\varphi(\alpha),\psi} \frac{\partial_{i+2n+1}L[x](t)}{\psi'(t)} \quad \text{at } t = a$$

and

$$\mathrm{I}_{b-}^{1-\phi(\alpha),\psi} \frac{\partial_{i+n+1}L[x](t)}{\psi'(t)} = \mathrm{I}_{a+}^{1-\varphi(\alpha),\psi} \frac{\partial_{i+2n+1}L[x](t)}{\psi'(t)} \quad \text{at } t = b,$$

for all $i = 1, ..., n$.

Next, we consider problem $(\mathcal{P})$ subject to an integral constraint of type

$$\mathcal{I}(x) := \int_a^b G[x](t)dt = k, \tag{3.6}$$

where $k \in \mathbb{R}$ is fixed and $G : [a, b] \times \mathbb{R}^3 \to \mathbb{R}$ is a continuously differentiable function with respect to the second, third and fourth variables. This type of problems are known in the literature as isoperimetric problems.

Theorem 3.3. *(Necessary optimality conditions for isoperimetric problems I) Let x be a curve such that $\mathcal{J}$ attains an extremum at x, when subject to the integral constraint* (3.6)*. Assume that x does not satisfies the equation*

$$\partial_2 G[x](t) + \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi} \frac{\partial_3 G[x](t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi} \frac{\partial_4 G[x](t)}{\psi'(t)}\right)\psi'(t) = 0, \quad t \in [a, b]. \tag{3.7}$$

If the maps

$$t \mapsto \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi} \frac{\partial_3 L[x](t)}{\psi'(t)}\right), \qquad t \mapsto \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi} \frac{\partial_4 L[x](t)}{\psi'(t)}\right),$$
$$t \mapsto \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi} \frac{\partial_3 G[x](t)}{\psi'(t)}\right) \quad \text{and} \quad t \mapsto \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi} \frac{\partial_4 G[x](t)}{\psi'(t)}\right)$$

are continuous on $[a,b]$*, then there exists a real number* λ *such that* x *is a solution of the equation*

$$\partial_2 H[x](t) + \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi} \frac{\partial_3 H[x](t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi} \frac{\partial_4 H[x](t)}{\psi'(t)}\right)\psi'(t) = 0, \tag{3.8}$$

for all $t \in [a,b]$*, where* $H := L + \lambda G$*. Also, if the state variable* $x(a)$ *is free, then*

$$\mathrm{I}_{b-}^{1-\phi(\alpha),\psi} \frac{\partial_3 H[x](t)}{\psi'(t)} = \mathrm{I}_{a+}^{1-\varphi(\alpha),\psi} \frac{\partial_4 H[x](t)}{\psi'(t)} \quad \text{at } t = a, \tag{3.9}$$

and if $x(b)$ *is free, then* x *must satisfy*

$$\mathrm{I}_{b-}^{1-\phi(\alpha),\psi} \frac{\partial_3 H[x](t)}{\psi'(t)} = \mathrm{I}_{a+}^{1-\varphi(\alpha),\psi} \frac{\partial_4 H[x](t)}{\psi'(t)} \quad \text{at } t = b. \tag{3.10}$$

Proof. Suppose that x is an extremizer of functional $\mathcal{J}$ subject to the integral constraint (3.6). Let $h_1, h_2 \in C^1([a,b],\mathbb{R})$ be two functions. First, suppose that $h_i(a) = h_i(b) = 0$, for $i = 1, 2$, and define the two functions i and j in the following way

$$i(\epsilon_1,\epsilon_2) := \mathcal{I}(x + \epsilon_1 h_1 + \epsilon_2 h_2) - k \qquad \text{and} \qquad j(\epsilon_1,\epsilon_2) := \mathcal{J}(x + \epsilon_1 h_1 + \epsilon_2 h_2),$$

for $\epsilon_1, \epsilon_2 \in \mathbb{R}$. Using similar techniques as the ones used in the proof of Theorem 3.2, we get

$$\begin{aligned}\partial_2 i(0,0) = &\int_a^b \left(\partial_2 G[x](t) + \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi} \frac{\partial_3 G[x](t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi} \frac{\partial_4 G[x](t)}{\psi'(t)}\right)\psi'(t)\right) h_2(t)dt \\ &+ \left[\left(I_{b-}^{1-\phi(\alpha),\psi} \frac{\partial_3 G[x](t)}{\psi'(t)}\right) h_2(t)\right]_{t=a}^{t=b} - \left[\left(I_{a+}^{1-\varphi(\alpha),\psi} \frac{\partial_4 G[x](t)}{\psi'(t)}\right) h_2(t)\right]_{t=a}^{t=b}.\end{aligned}$$

Since $h_2(a) = h_2(b) = 0$, we conclude that

$$\partial_2 i(0,0) = \int_a^b \left(\partial_2 G[x](t) + \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi} \frac{\partial_3 G[x](t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi} \frac{\partial_4 G[x](t)}{\psi'(t)}\right)\psi'(t)\right) h_2(t)dt.$$

Since x does not satisfies equation (3.7), one concludes that there exists $t_0 \in [a,b]$ such that,

$$\partial_2 G[x](t_0) + \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi} \frac{\partial_3 G[x](t_0)}{\psi'(t_0)}\right)\psi'(t_0) + \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi} \frac{\partial_4 G[x](t_0)}{\psi'(t_0)}\right)\psi'(t_0) \neq 0.$$

Then, there exists some function h_2 for which $\partial_2 i(0,0) \neq 0$. Also, $i(0,0) = 0$ and so, applying the Implicit Function Theorem, we conclude that there exists a continuously differentiable function g defined on an open set $U \subseteq \mathbb{R}$ containing 0, such that $g(0) = 0$ and $i(\epsilon_1, g(\epsilon_1)) = 0$, for all $\epsilon_1 \in U$. Hence, there exists an infinity subfamily of functions $x + \epsilon_1 h_1 + g(\epsilon_1) h_2$ that satisfies the integral restriction (3.6). From now on we will consider such subfamily of variations. Observe that the vector $(0,0)$ is an extremizer of j, subject to the constraint $i(\cdot,\cdot) = 0$. Since $\nabla i(0,0) \neq (0,0)$, by the Lagrange

Multiplier Rule, there exists a real number λ such that $\nabla(j+\lambda i)(0,0) = (0,0)$. Hence, $\partial_1(j+\lambda i)(0,0)=0$, and, therefore,

$$\begin{aligned}&\int_a^b \Bigg(\partial_2 L[x](t) + \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi}\frac{\partial_3 L[x](t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi}\frac{\partial_4 L[x](t)}{\psi'(t)}\right)\psi'(t)\\&+\lambda\Bigg(\partial_2 G[x](t) + \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi}\frac{\partial_3 G[x](t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi}\frac{\partial_4 G[x](t)}{\psi'(t)}\right)\psi'(t)\Bigg)\Bigg)h_1(t)dt\\&+\left[\left(I_{b-}^{1-\phi(\alpha),\psi}\frac{\partial_3 L[x](t)}{\psi'(t)}\right)h_1(t)\right]_{t=a}^{t=b} - \left[\left(I_{a+}^{1-\varphi(\alpha),\psi}\frac{\partial_4 L[x](t)}{\psi'(t)}\right)h_1(t)\right]_{t=a}^{t=b}\\&+\lambda\left[\left(I_{b-}^{1-\phi(\alpha),\psi}\frac{\partial_3 G[x](t)}{\psi'(t)}\right)h_1(t)\right]_{t=a}^{t=b} - \lambda\left[\left(I_{a+}^{1-\varphi(\alpha),\psi}\frac{\partial_4 G[x](t)}{\psi'(t)}\right)h_1(t)\right]_{t=a}^{t=b} = 0.\end{aligned} \tag{3.11}$$

Since h_1 is an arbitrary function and considering $h_1(a) = h_1(b) = 0$, it follows from the Fundamental Lemma of Calculus of Variations that

$$\begin{aligned}&\partial_2 L[x](t) + \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi}\frac{\partial_3 L[x](t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi}\frac{\partial_4 L[x](t)}{\psi'(t)}\right)\psi'(t)\\&+\lambda\left(\partial_2 G[x](t) + \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi}\frac{\partial_3 G[x](t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi}\frac{\partial_4 G[x](t)}{\psi'(t)}\right)\psi'(t)\right) = 0,\end{aligned}$$

for all $t \in [a,b]$, proving equation (3.8).

Suppose now that $x(a)$ is free and consider variations h_1 with $h_1(a) \neq 0$ and $h_1(b) = 0$. From (3.11) and using (3.8), we conclude that

$$h_1(a)\left(\mathrm{I}_{b-}^{1-\phi(\alpha),\psi}\frac{\partial_3 H[x]}{\psi'}(a) - \mathrm{I}_{a+}^{1-\varphi(\alpha),\psi}\frac{\partial_4 H[x]}{\psi'}(a)\right) = 0,$$

proving (3.9). Similarly, if $x(b)$ is free, then by considering $h_1(a) = 0$ and $h_1(b) \neq 0$ in (3.11) and using (3.8), (3.10) is proved. □

Theorem 3.4. *(Necessary optimality conditions for isoperimetric problems II) Let x be a curve such that $\mathcal{J}$ attains an extremum at x, when subject to the integral constraint* (3.6). *If the maps*

$$\begin{aligned}&t \mapsto \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi}\frac{\partial_3 L[x](t)}{\psi'(t)}\right), \qquad t \mapsto \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi}\frac{\partial_4 L[x](t)}{\psi'(t)}\right),\\&t \mapsto \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi}\frac{\partial_3 G[x](t)}{\psi'(t)}\right) \quad \text{and} \quad t \mapsto \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi}\frac{\partial_4 G[x](t)}{\psi'(t)}\right)\end{aligned}$$

are continuous on $[a,b]$, *then there exists a vector* $(\lambda_0,\lambda) \in \mathbb{R}^2 \setminus \{(0,0)\}$ *such that x is a solution of the equation* (3.8) *for all* $t \in [a,b]$, *with the Hamiltonian H defined as* $H := \lambda_0 L + \lambda G$. *Also, if the state variable* $x(a)$ *is free, then x must satisfy the equation* (3.9) *and if* $x(b)$ *is free, then x must satisfy the equation* (3.10).

Proof. The proof is similar to the one of Theorem 3.3. Since the vector $(0,0)$ is an extremizer of j, subject to the constraint $i(\cdot,\cdot)=0$, the Lagrange Multiplier Rule guarantees the existence of two constants $\lambda_0,\lambda\in\mathbb{R}$, not both zero, such that $\nabla(\lambda_0 j+\lambda i)(0,0)=(0,0)$. Computing $\partial_1(\lambda_0 j+\lambda i)(0,0)=0$, we obtain the desired result. □

Now, we consider problem $(\mathcal{P})$ but in presence of an holonomic restriction. Suppose that the state variable x is a two-dimensional vector function $x=(x_1,x_2)$, where $x_1,x_2\in C^1([a,b],\mathbb{R})$. We impose the following restriction to our variational problem:

$$g(t,x(t))=0,\quad t\in[a,b], \tag{3.12}$$

where $g:[a,b]\times\mathbb{R}^2\to\mathbb{R}$ is a continuously differentiable function. Also, boundary conditions

$$x(a)=x_a \ \text{ and } \ x(b)=x_b,\quad x_a,x_b\in\mathbb{R}^2 \tag{3.13}$$

may be imposed to the variational problem.

Theorem 3.5. *(Necessary optimality conditions for variational problems with an holonomic constraint) Consider the functional*

$$\mathcal{J}(x)=\int_a^b L[x](t)dt, \tag{3.14}$$

defined on $C^1([a,b],\mathbb{R})\times C^1([a,b],\mathbb{R})$ *and subject to the constraint* (3.12)*. If x is an extremizer of functional $\mathcal{J}$, if the maps*

$$t\mapsto\left(\mathrm{D}_{b-}^{\phi(\alpha),\psi}\frac{\partial_{i+3}L[x](t)}{\psi'(t)}\right)\quad \text{and}\quad t\mapsto\left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi}\frac{\partial_{i+5}L[x](t)}{\psi'(t)}\right),\ i=1,2,$$

are continuous, and if

$$\partial_3 g(t,x(t))\neq 0,\quad \forall t\in[a,b],$$

then there exists a continuous function $\lambda:[a,b]\to\mathbb{R}$ such that x is a solution of

$$\begin{aligned}&\partial_{i+1}L[x](t)+\left(\mathrm{D}_{b-}^{\phi(\alpha),\psi}\frac{\partial_{i+3}L[x](t)}{\psi'(t)}\right)\psi'(t)+\left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi}\frac{\partial_{i+5}L[x](t)}{\psi'(t)}\right)\psi'(t)\\&+\lambda(t)\cdot\partial_{i+1}g(t,x(t))=0,\quad\forall t\in[a,b],\quad i=1,2.\end{aligned} \tag{3.15}$$

Also, if $x(a)$ is free, then, for $i=1,2$,

$$\mathrm{I}_{b-}^{1-\phi(\alpha),\psi}\frac{\partial_{i+3}L[x](t)}{\psi'(t)}=\mathrm{I}_{a+}^{1-\varphi(\alpha),\psi}\frac{\partial_{i+5}L[x](t)}{\psi'(t)}\quad \text{at } t=a \tag{3.16}$$

and if $x(b)$ is free, then, for $i=1,2$,

$$\mathrm{I}_{b-}^{1-\phi(\alpha),\psi}\frac{\partial_{i+3}L[x](t)}{\psi'(t)}=\mathrm{I}_{a+}^{1-\varphi(\alpha),\psi}\frac{\partial_{i+5}L[x](t)}{\psi'(t)}\quad \text{at } t=b. \tag{3.17}$$

Proof. Let $h = (h_1, h_2) \in C^1([a,b],\mathbb{R}) \times C^1([a,b],\mathbb{R})$. To prove Eqs (3.15), first assume that $h(a) = (0,0) = h(b)$ and let $\epsilon \in \mathbb{R}$. Since the variations must fulfill the holonomic restriction (3.12), then

$$g(t, x_1(t) + \epsilon h_1(t), x_2(t) + \epsilon h_2(t)) = 0, \quad \forall t \in [a,b]. \tag{3.18}$$

Differentiating (3.18) with respect to ϵ and taking $\epsilon = 0$, we conclude that

$$\partial_3 g(t, x(t)) \cdot h_2(t) = -\partial_2 g(t, x(t)) \cdot h_1(t), \quad \forall t \in [a,b]. \tag{3.19}$$

Define the function $\lambda : [a,b] \to \mathbb{R}$ by the rule

$$\lambda(t) := -\frac{\partial_3 L[x](t) + \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi} \frac{\partial_5 L[x](t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi} \frac{\partial_7 L[x](t)}{\psi'(t)}\right)\psi'(t)}{\partial_3 g(t, x(t))}. \tag{3.20}$$

From the definition of λ, we prove equation (3.15) for $i = 2$. Now, we prove that equation (3.15) holds for $i = 1$. By Eqs (3.19) and (3.20) we obtain

$$\begin{aligned} &\lambda(t) \cdot \partial_2 g(t, x(t)) \cdot h_1(t) \\ &= \left(\partial_3 L[x](t) + \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi} \frac{\partial_5 L[x](t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi} \frac{\partial_7 L[x](t)}{\psi'(t)}\right)\psi'(t)\right) h_2(t). \end{aligned} \tag{3.21}$$

Let us define the new function j by the rule $j(\epsilon) := \mathcal{J}(x_1 + \epsilon h_1, x_2 + \epsilon h_2)$, $\epsilon \in \mathbb{R}$. Since $j'(0) = 0$, we conclude that

$$\begin{aligned} &\int_a^b \left(\partial_2 L[x](t) + \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi} \frac{\partial_4 L[x](t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi} \frac{\partial_6 L[x](t)}{\psi'(t)}\right)\psi'(t)\right) h_1(t) dt \\ &+ \left[\left(\mathrm{I}_{b-}^{1-\phi(\alpha),\psi} \frac{\partial_4 L[x](t)}{\psi'(t)}\right) h_1(t)\right]_{t=a}^{t=b} - \left[\left(\mathrm{I}_{a+}^{1-\varphi(\alpha),\psi} \frac{\partial_6 L[x](t)}{\psi'(t)}\right) h_1(t)\right]_{t=a}^{t=b} \\ &+ \int_a^b \left(\partial_3 L[x](t) + \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi} \frac{\partial_5 L[x](t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi} \frac{\partial_7 L[x](t)}{\psi'(t)}\right)\psi'(t)\right) h_2(t) dt \\ &+ \left[\left(\mathrm{I}_{b-}^{1-\phi(\alpha),\psi} \frac{\partial_5 L[x](t)}{\psi'(t)}\right) h_2(t)\right]_{t=a}^{t=b} - \left[\left(\mathrm{I}_{a+}^{1-\varphi(\alpha),\psi} \frac{\partial_7 L[x](t)}{\psi'(t)}\right) h_2(t)\right]_{t=a}^{t=b} = 0 \end{aligned} \tag{3.22}$$

and by considering $h(a) = h(b) = (0,0)$, we obtain

$$\begin{aligned} &\int_a^b \left(\left(\partial_2 L[x](t) + \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi} \frac{\partial_4 L[x](t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi} \frac{\partial_6 L[x](t)}{\psi'(t)}\right)\psi'(t)\right) h_1(t) \right. \\ &\left. + \left(\partial_3 L[x](t) + \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi} \frac{\partial_5 L[x](t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi} \frac{\partial_7 L[x](t)}{\psi'(t)}\right)\psi'(t)\right) h_2(t)\right) dt = 0. \end{aligned}$$

Using Eq (3.21), we obtain

$$\int_a^b \left(\partial_2 L[x](t) + \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi} \frac{\partial_4 L[x](t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi} \frac{\partial_6 L[x](t)}{\psi'(t)}\right)\psi'(t) + \lambda(t) \cdot \partial_2 g(t, x(t))\right) h_1(t) dt = 0.$$

Since h_1 is arbitrary, from the Fundamental Lemma of Calculus of Variations, we get

$$\partial_2 L[x](t) + \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi} \frac{\partial_4 L[x](t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi} \frac{\partial_6 L[x](t)}{\psi'(t)}\right)\psi'(t) + \lambda(t) \cdot \partial_2 g(t, x(t)) = 0,$$

for all $t \in [a, b]$, proving Eq (3.15) for $i = 1$. We now prove the transversality conditions (3.16) and (3.17). If $x(a)$ is free, then by considering $h(a) \neq (0, 0)$ and $h(b) = (0, 0)$ in (3.22) and using (3.15), (3.16) is proved. If $x(b)$ is free, then consider $h(a) = (0, 0)$ and $h(b) \neq (0, 0)$ to deduce (3.17). □

3.2. Sufficient optimality conditions

Now we will prove sufficient optimality conditions for all the variational problems studied in the last subsection.

Definition 3.6. We say that $f(t, x_2, x_3, ..., x_n)$ is a convex (resp. concave) function in $U \subseteq \mathbb{R}^n$ if $\partial_i f(t, x_2, x_3, ..., x_n)$, $i = 2, \ldots, n$, exist and are continuous, and if

$$f(t, x_2 + \eta_2, x_3 + \eta_3, ..., x_n + \eta_n) - f(t, x_2, x_3, ..., x_n) \geq \text{ (resp. } \leq \text{)} \sum_{i=2}^{n} \partial_i f(t, x_2, x_3, ..., x_n)\eta_i$$

for all $(t, x_2, x_3, ..., x_n), (t, x_2 + \eta_2, x_3 + \eta_3, ..., x_n + \eta_n) \in U$.

Theorem 3.7. *(Sufficient optimality conditions) Suppose that the Lagrangian function L is convex (resp. concave) in* $[a, b] \times \mathbb{R}^3$*. Then, each solution* $\overline{x}$ *of the fractional Euler-Lagrange equation* (3.2) *minimizes (resp. maximizes) the functional* $\mathcal{J}$ *given in* (3.1)*, subject to the boundary conditions* $x(a) = \overline{x}(a)$ *and* $x(b) = \overline{x}(b)$*. Also, if* $x(a)$ *is free, then each solution* $\overline{x}$ *of the equations* (3.2) *and* (3.3) *minimizes (resp. maximizes)* $\mathcal{J}$*. If* $x(b)$ *is free, then each solution* $\overline{x}$ *of the equations* (3.2) *and* (3.4) *minimizes (resp. maximizes)* $\mathcal{J}$*.*

Proof. Let $\eta \in C^1([a, b], \mathbb{R})$ be an arbitrary function. Since L is convex, we have

$$\begin{aligned}
&\mathcal{J}(\overline{x} + \eta) - \mathcal{J}(\overline{x}) \\
&\geq \int_a^b \left(\partial_2 L[\overline{x}](t) \cdot \eta(t) + \partial_3 L[\overline{x}](t) \cdot^C \mathrm{D}_{a+}^{\phi(\alpha),\psi} \eta(t) + \partial_4 L[\overline{x}](t) \cdot^C \mathrm{D}_{b-}^{\varphi(\alpha),\psi} \eta(t)\right) dt.
\end{aligned}$$

Applying Theorem 3.1, we obtain

$$\begin{aligned}
&\mathcal{J}(\overline{x} + \eta) - \mathcal{J}(\overline{x}) \\
&\geq \int_a^b \left(\partial_2 L[\overline{x}](t) + \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi} \frac{\partial_3 L[\overline{x}](t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi} \frac{\partial_4 L[\overline{x}](t)}{\psi'(t)}\right)\psi'(t)\right)\eta(t) dt \\
&+\left[\left(\mathrm{I}_{b-}^{1-\phi(\alpha),\psi} \frac{\partial_3 L[\overline{x}](t)}{\psi'(t)}\right)\eta(t)\right]_{t=a}^{t=b} - \left[\left(\mathrm{I}_{a+}^{1-\varphi(\alpha),\psi} \frac{\partial_4 L[\overline{x}](t)}{\psi'(t)}\right)\eta(t)\right]_{t=a}^{t=b}.
\end{aligned} \tag{3.23}$$

If $x(a)$ and $x(b)$ are fixed, then the admissible variations must fulfill the conditions $\eta(a) = \eta(b) = 0$, and so we get

$$\begin{aligned}
&\mathcal{J}(\overline{x} + \eta) - \mathcal{J}(\overline{x}) \\
&\geq \int_a^b \left(\partial_2 L[\overline{x}](t) + \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi} \frac{\partial_3 L[\overline{x}](t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi} \frac{\partial_4 L[\overline{x}](t)}{\psi'(t)}\right)\psi'(t)\right)\eta(t) dt = 0,
\end{aligned}$$

since $\overline{x}$ is a solution of the fractional Euler-Lagrange equation (3.2). If $x(a)$ is free, then by considering $\eta(a) \neq 0$ and $\eta(b) = 0$ in (3.23), we have

$$\begin{aligned}
&\mathcal{J}(\overline{x}+\eta) - \mathcal{J}(\overline{x}) \\
&\geq \int_a^b \left(\partial_2 L[\overline{x}](t) + \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi} \frac{\partial_3 L[\overline{x}](t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi} \frac{\partial_4 L[\overline{x}](t)}{\psi'(t)}\right)\psi'(t)\right)\eta(t)dt \\
&\qquad + \eta(a)\left(-\mathrm{I}_{b-}^{1-\phi(\alpha),\psi}\frac{\partial_3 L[\overline{x}]}{\psi'}(a) + \mathrm{I}_{a+}^{1-\varphi(\alpha),\psi}\frac{\partial_4 L[\overline{x}]}{\psi'}(a)\right) = 0,
\end{aligned}$$

since $\overline{x}$ is a solution of the fractional equations (3.2) and (3.3). Similarly, if $x(b)$ is free, then by considering $\eta(a) = 0$ and $\eta(b) \neq 0$ in (3.23), since $\overline{x}$ is a solution of the fractional equations (3.2) and (3.4), we conclude that $\mathcal{J}(\overline{x}+\eta) - \mathcal{J}(\overline{x}) \geq 0$. The cases when L is concave are proven in a similar way. □

Theorem 3.8. *(Sufficient optimality conditions for isoperimetric problems) Let us assume that, for some constant λ, the functions L and λG are convex (resp. concave) in $[a,b] \times \mathbb{R}^3$ and define the function H as $H = L + \lambda G$. Then, each solution $\overline{x}$ of the fractional equation* (3.8) *minimizes (resp. maximizes) the functional $\mathcal{J}$ given in* (3.1)*, subject to the restrictions $x(a) = \overline{x}(a)$ and $x(b) = \overline{x}(b)$, and the integral constraint* (3.6)*. Also, if $x(a)$ is free, then each solution $\overline{x}$ of the fractional equations* (3.8) *and* (3.9) *minimizes (resp. maximizes) $\mathcal{J}$ subject to* (3.6)*. If $x(b)$ is free, then each solution $\overline{x}$ of the fractional equations* (3.8) *and* (3.10) *minimizes (resp. maximizes) $\mathcal{J}$ subject to* (3.6).

Proof. First, assume that functions L and λG are convex. It is easy to verify that function H is convex. Let $\eta \in C^1([a,b],\mathbb{R})$ be such that $\eta(a) = \eta(b) = 0$. By Theorem 3.7, $\overline{x}$ minimizes $\widetilde{H} := \int_a^b (L + \lambda G)dt$, that is, $\widetilde{H}(\overline{x}+\eta) \geq \widetilde{H}(\overline{x})$. So, if $x \in C^1([a,b],\mathbb{R})$ is any function such that $x(a) = \overline{x}(a)$ and $x(b) = \overline{x}(b)$, then

$$\int_a^b L[x](t)dt + \lambda \int_a^b G[x](t)dt \geq \int_a^b L[\overline{x}](t) + \lambda \int_a^b G[\overline{x}](t)dt.$$

If we restrict to the integral constraint, we have

$$\int_a^b L[x](t)dt + \lambda k \geq \int_a^b L[\overline{x}](t)dt + \lambda k.$$

Therefore,

$$\int_a^b L[x](t)dt \geq \int_a^b L[\overline{x}](t)dt,$$

this is, $\mathcal{J}(x) \geq \mathcal{J}(\overline{x})$. The remaining cases are proven in a similar way. □

Theorem 3.9. *(Sufficient optimality conditions for variational problems with an holonomic constraint) Consider the functional $\mathcal{J}$ defined in* (3.14)*, where the Lagrangian function L is convex (resp. concave) in $[a,b] \times \mathbb{R}^6$, and function $\lambda : [a,b] \to \mathbb{R}$ given by formula* (3.20)*. Then, each solution $\overline{x} = (\overline{x}_1, \overline{x}_2)$ of equations* (3.15) *minimizes (resp. maximizes) the functional $\mathcal{J}$, subject to the constraints $x(a) = \overline{x}(a)$ and $x(b) = \overline{x}(b)$, and the holonomic restriction* (3.12)*. Also, if $x(a)$ is free, then each solution $\overline{x}$ of* *the fractional equations* (3.15) *and* (3.16) *minimizes (resp. maximizes)* $\mathcal{J}$ *subject to* (3.12). *If* $x(b)$ *is free, then each solution* $\overline{x}$ *of the fractional equations* (3.15) *and* (3.17) *minimizes (resp. maximizes)* $\mathcal{J}$ *subject to* (3.12).

Proof. We shall give the proof only for the case where L is convex; the concave case is analogous. Let $\eta_1, \eta_2 \in C^1([a,b],\mathbb{R})$ be arbitrary functions, where $\eta = (\eta_1, \eta_2)$ is a differentiable function with $\eta(a) = (0,0) = \eta(b)$. Since L is convex, we have

$$\begin{aligned}
&\mathcal{J}(\overline{x}+\eta) - \mathcal{J}(\overline{x}) \\
&\geq \int_a^b \Big(\partial_2 L[\overline{x}](t)\eta_1(t) + \partial_3 L[\overline{x}](t)\eta_2(t) + \partial_4 L[\overline{x}](t)\,{}^C\mathrm{D}_{a+}^{\phi(\alpha),\psi}\eta_1(t) \\
&\qquad +\partial_5 L[\overline{x}](t)\,{}^C\mathrm{D}_{a+}^{\phi(\alpha),\psi}\eta_2(t) + \partial_6 L[\overline{x}](t)\,{}^C\mathrm{D}_{b-}^{\varphi(\alpha),\psi}\eta_1(t) + \partial_7 L[\overline{x}](t)\,{}^C\mathrm{D}_{b-}^{\varphi(\alpha),\psi}\eta_2(t)\Big)dt.
\end{aligned}$$

Using the integration by parts formulae, we obtain

$$\begin{aligned}
&\mathcal{J}(\overline{x}+\eta) - \mathcal{J}(\overline{x}) \\
&\geq \int_a^b \left(\partial_2 L[\overline{x}](t) + \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi}\frac{\partial_4 L[\overline{x}](t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi}\frac{\partial_6 L[\overline{x}](t)}{\psi'(t)}\right)\psi'(t)\right)\eta_1(t)dt \\
&+ \int_a^b \left(\partial_3 L[\overline{x}](t) + \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi}\frac{\partial_5 L[\overline{x}](t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi}\frac{\partial_7 L[\overline{x}](t)}{\psi'(t)}\right)\psi'(t)\right)\eta_2(t)dt \\
&+\left[\left(\mathrm{I}_{b-}^{1-\phi(\alpha),\psi}\frac{\partial_4 L[\overline{x}](t)}{\psi'(t)}\right)\eta_1(t)\right]_{t=a}^{t=b} - \left[\left(\mathrm{I}_{a+}^{1-\varphi(\alpha),\psi}\frac{\partial_6 L[\overline{x}](t)}{\psi'(t)}\right)\eta_1(t)\right]_{t=a}^{t=b} \\
&+\left[\left(\mathrm{I}_{b-}^{1-\phi(\alpha),\psi}\frac{\partial_5 L[\overline{x}](t)}{\psi'(t)}\right)\eta_2(t)\right]_{t=a}^{t=b} - \left[\left(\mathrm{I}_{a+}^{1-\varphi(\alpha),\psi}\frac{\partial_7 L[\overline{x}](t)}{\psi'(t)}\right)\eta_2(t)\right]_{t=a}^{t=b}.
\end{aligned}$$

Using Eqs (3.19) and (3.20), we get

$$\begin{aligned}
&\mathcal{J}(\overline{x}+\eta) - \mathcal{J}(\overline{x}) \\
&\geq \int_a^b \left(\partial_2 L[\overline{x}](t) + \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi}\frac{\partial_4 L[\overline{x}](t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi}\frac{\partial_6 L[\overline{x}](t)}{\psi'(t)}\right)\psi'(t)\right. \\
&\left.+\lambda(t)\partial_2 g(t,\overline{x}(t))\right)\eta_1(t)dt + \left[\left(\mathrm{I}_{b-}^{1-\phi(\alpha),\psi}\frac{\partial_4 L[\overline{x}](t)}{\psi'(t)}\right)\eta_1(t)\right]_{t=a}^{t=b} \\
&-\left[\left(\mathrm{I}_{a+}^{1-\varphi(\alpha),\psi}\frac{\partial_6 L[\overline{x}](t)}{\psi'(t)}\right)\eta_1(t)\right]_{t=a}^{t=b} + \left[\left(\mathrm{I}_{b-}^{1-\phi(\alpha),\psi}\frac{\partial_5 L[\overline{x}](t)}{\psi'(t)}\right)\eta_2(t)\right]_{t=a}^{t=b} \\
&-\left[\left(\mathrm{I}_{a+}^{1-\varphi(\alpha),\psi}\frac{\partial_7 L[\overline{x}](t)}{\psi'(t)}\right)\eta_2(t)\right]_{t=a}^{t=b}.
\end{aligned} \tag{3.24}$$

Since $\eta(a) = (0,0) = \eta(b)$, we obtain

$$\begin{aligned}
&\mathcal{J}(\overline{x}+\eta) - \mathcal{J}(\overline{x}) \\
&\geq \int_a^b \left(\partial_2 L[\overline{x}](t) + \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi}\frac{\partial_4 L[\overline{x}](t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi}\frac{\partial_6 L[\overline{x}](t)}{\psi'(t)}\right)\psi'(t)\right. \\
&\qquad\qquad \left.+\lambda(t)\partial_2 g(t,\overline{x}(t))\right)\eta_1(t)dt = 0,
\end{aligned}$$

since $\overline{x}$ is a solution of the fractional Euler-Lagrange equation (3.15) for $i = 1$, for all $t \in [a, b]$. If $x(a)$ is free, then by considering $\eta(a) \neq (0, 0)$ and $\eta(b) = (0, 0)$ in (3.24), we get

$$\begin{aligned}
&\mathcal{J}(\overline{x} + \eta) - \mathcal{J}(\overline{x}) \\
&\geq \int_a^b \left(\partial_2 L[\overline{x}](t) + \left(\mathrm{D}_{b-}^{\phi(\alpha),\psi} \frac{\partial_4 L[\overline{x}](t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a+}^{\varphi(\alpha),\psi} \frac{\partial_6 L[\overline{x}](t)}{\psi'(t)}\right)\psi'(t) \right. \\
&\left. +\lambda(t)\partial_2 g(t, \overline{x}(t))\right)\eta_1(t)dt + \left(\mathrm{I}_{a+}^{1-\varphi(\alpha),\psi} \frac{\partial_6 L[\overline{x}]}{\psi'}(a) - \mathrm{I}_{b-}^{1-\phi(\alpha),\psi} \frac{\partial_4 L[\overline{x}]}{\psi'}(a)\right)\eta_1(a) \\
&+\left(\mathrm{I}_{a+}^{1-\varphi(\alpha),\psi} \frac{\partial_7 L[\overline{x}]}{\psi'}(a) - \mathrm{I}_{b-}^{1-\phi(\alpha),\psi} \frac{\partial_5 L[\overline{x}]}{\psi'}(a)\right)\eta_2(a) = 0,
\end{aligned}$$

since $\overline{x}$ is a solution of the fractional equations (3.15) and (3.16). Similarly, if $x(b)$ is free, then by considering $\eta(a) = (0, 0)$ and $\eta(b) \neq (0, 0)$ in (3.24), and since $\overline{x}$ is a solution of the fractional equations (3.15) and (3.17), we conclude that $\mathcal{J}(\overline{x} + \eta) - \mathcal{J}(\overline{x}) \geq 0$, proving the desired result. □

4. Illustrative examples

In this section we provide three examples in order to illustrate our results.

Example 1. *Suppose we want to minimize the following functional*

$$\begin{aligned}
\mathcal{J}(x) \quad = \quad & \int_0^1 \left((x(t) - (\psi(t) - \psi(0))^4)^2 \right. \\
& \left. + \left({}^C\mathrm{D}_{0+}^{\phi(\alpha),\psi} x(t) + \frac{(\psi(t) - \psi(0))^3 - (\psi(t) - \psi(0))^4}{\ln(\psi(t) - \psi(0))} \right)^2 \right) dt,
\end{aligned}$$

in the class of functions $C^1([0, 1], \mathbb{R})$ *subject to the restriction* $x(0) = 0$*, where* $\phi : [0, 1] \to [0, 1]$ *is defined by* $\phi(\alpha) = \dfrac{\Gamma(5 - \alpha)}{4!}$*. From Theorem 3.2, every local extremizer* x *of functional* $\mathcal{J}$ *such that*

$$t \mapsto \mathrm{D}_{1-}^{\phi(\alpha),\psi} \frac{\partial_3 L[x](t)}{\psi'(t)} \tag{4.1}$$

is continuous on $[0, 1]$*, satisfies the following necessary conditions*

$$x(t) - (\psi(t) - \psi(0))^4 + \left(\mathrm{D}_{1-}^{\phi(\alpha),\psi} \frac{{}^C\mathrm{D}_{0+}^{\phi(\alpha),\psi} x(t) + \dfrac{(\psi(t) - \psi(0))^3 - (\psi(t) - \psi(0))^4}{\ln(\psi(t) - \psi(0))}}{\psi'(t)} \right) \psi'(t) = 0, \tag{4.2}$$

for all $t \in [0, 1]$ *and, at* $t = 1$*,*

$$\mathrm{I}_{1-}^{1-\phi(\alpha),\psi} \frac{{}^C\mathrm{D}_{0+}^{\phi(\alpha),\psi} x(t) + \dfrac{(\psi(t) - \psi(0))^3 - (\psi(t) - \psi(0))^4}{\ln(\psi(t) - \psi(0))}}{\psi'(t)} = 0. \tag{4.3}$$

Note that the function $\overline{x} : [0, 1] \to \mathbb{R}$ *defined by* $\overline{x}(t) = (\psi(t) - \psi(0))^4$ *is such that*

$${}^C\mathrm{D}_{0+}^{\alpha,\psi} \overline{x}(t) = \frac{4!}{\Gamma(5 - \alpha)} (\psi(t) - \psi(0))^{4-\alpha} \quad (\textit{by Lemma 2.4}).$$

Thus,

$$^{C}\mathrm{D}_{0+}^{\phi(\alpha),\psi}\overline{x}(t) = \frac{-(\psi(t)-\psi(0))^3 + (\psi(t)-\psi(0))^4}{\ln(\psi(t)-\psi(0))},$$

and therefore $\overline{x}$ *satisfies condition* (4.1)*, the Euler-Lagrange equation* (4.2)*, and the natural boundary condition* (4.3)*. Since the Lagrangian function is convex, by Theorem 3.7, we conclude that* $\overline{x}$ *is a minimizer of* $\mathcal{J}$*.*

Example 2. *Suppose we want to minimize the following functional*

$$\mathcal{J}(x) \;=\; \int_0^1 \left(t^2 + \left((\psi(1)-\psi(t))^{\alpha} \cdot {}^{C}\mathrm{D}_{1-}^{\varphi(\alpha),\psi}x(t) - (\psi(1)-\psi(t))^{\alpha+1}\right)^2\right)dt,$$

in the class of functions $C^1([0,1],\mathbb{R})$ *subject to the restriction* $x(1)=0$*, where* $\varphi:[0,1]\to[0,1]$ *is defined by* $\varphi(\alpha)=\dfrac{2}{\Gamma(\alpha+2)}$*. From Theorem 3.2, every local extremizer* x *of functional* $\mathcal{J}$ *such that*

$$t \mapsto \mathrm{D}_{0+}^{\varphi(\alpha),\psi}\frac{\partial_4 L[x](t)}{\psi'(t)} \tag{4.4}$$

is continuous on $[0,1]$*, satisfies the following necessary conditions*

$$\left(\mathrm{D}_{0+}^{\varphi(\alpha),\psi}\frac{\left((\psi(1)-\psi(t))^{\alpha}\cdot{}^{C}\mathrm{D}_{1-}^{\varphi(\alpha),\psi}x(t)-(\psi(1)-\psi(t))^{\alpha+1}\right)(\psi(1)-\psi(t))^{\alpha}}{\psi'(t)}\right)\psi'(t)=0, \tag{4.5}$$

for all $t\in[0,1]$ *and*

$$\mathrm{I}_{0+}^{1-\varphi(\alpha),\psi}\frac{\left((\psi(1)-\psi(t))^{\alpha}\cdot{}^{C}\mathrm{D}_{1-}^{\varphi(\alpha),\psi}x(t)-(\psi(1)-\psi(t))^{\alpha+1}\right)(\psi(1)-\psi(t))^{\alpha}}{\psi'(t)} = 0,\ \text{at } t=0. \tag{4.6}$$

Note that, by Lemma 2.4, if $\overline{x}:[0,1]\to\mathbb{R}$ *is defined by* $\overline{x}(t)=\dfrac{(\psi(1)-\psi(t))^{\alpha+1}}{2}$*, then*

$$^{C}\mathrm{D}_{1-}^{\alpha,\psi}\overline{x}(t)=\frac{\Gamma(\alpha+2)}{2}(\psi(1)-\psi(t)).$$

Thus,

$$^{C}\mathrm{D}_{1-}^{\varphi(\alpha),\psi}\overline{x}(t)=\psi(1)-\psi(t),$$

and therefore $\overline{x}$ *satisfies condition* (4.4)*, the Euler-Lagrange equation* (4.5)*, and the natural boundary condition* (4.6)*. Since the Lagrangian function is convex, by Theorem 3.7,* $\overline{x}$ *is indeed a minimizer of* $\mathcal{J}$*.*

Example 3. *Consider now the following problem*

$$\mathcal{J}(x)=\int_0^1\left(x^2(t)+(\psi(1)-\psi(t))^{6\alpha+2}+\left({}^{C}\mathrm{D}_{1-}^{\varphi(\alpha),\psi}x(t)\right)^2 + \frac{1}{4}\left(\frac{(\psi(1)-\psi(t))^3-\psi(1)+\psi(t)}{\ln(\psi(1)-\psi(t))}\right)^2\right)dt\to\min,$$

in the class of functions $C^1([0,1],\mathbb{R})$, subject to the restriction $x(1)=0$ and to the integral constraint

$$\mathcal{I}(x) = \int_0^1 \left(x(t)(\psi(1)-\psi(t))^{3\alpha+1} + \frac{1}{2}\left(\frac{(\psi(1)-\psi(t))^3-\psi(1)+\psi(t)}{\ln(\psi(1)-\psi(t))}\right) {}^C\mathrm{D}_{1-}^{\varphi(\alpha),\psi} x(t)\right) dt = k,$$

where

$$k = \int_0^1 \left(\frac{1}{4}\left(\frac{(\psi(1)-\psi(t))^3-\psi(1)+\psi(t)}{\ln(\psi(1)-\psi(t))}\right)^2 + (\psi(1)-\psi(t))^{6\alpha+2}\right) dt,$$

and $\varphi:[0,1]\to[0,1]$ is defined by $\varphi(\alpha)=\dfrac{\Gamma(2\alpha+2)}{\Gamma(3\alpha+2)}$. Consider the function $\overline{x}:[0,1]\to\mathbb{R}$ defined by $\overline{x}(t)=(\psi(1)-\psi(t))^{3\alpha+1}$. Then, by Lemma 2.4,

$${}^C\mathrm{D}_{1-}^{\alpha,\psi}\overline{x}(t) = \frac{\Gamma(3\alpha+2)}{\Gamma(2\alpha+2)}(\psi(1)-\psi(t))^{2\alpha+1}$$

and so

$${}^C\mathrm{D}_{1-}^{\varphi(\alpha),\psi}\overline{x}(t) = \frac{1}{2}\left(\frac{(\psi(1)-\psi(t))^3-\psi(1)+\psi(t)}{\ln(\psi(1)-\psi(t))}\right).$$

Let

$$H := \left(x(t)-(\psi(1)-\psi(t))^{3\alpha+1}\right)^2 + \left({}^C\mathrm{D}_{1-}^{\varphi(\alpha),\psi} x(t) - \frac{1}{2}\left(\frac{(\psi(1)-\psi(t))^3-\psi(1)+\psi(t)}{\ln(\psi(1)-\psi(t))}\right)\right)^2.$$

Therefore, $\overline{x}$ satisfies the Euler-Lagrange equation with respect to the Hamiltonian H:

$$x(t)-(\psi(1)-\psi(t))^{3\alpha+1} + \left(\mathrm{D}_{0+}^{\varphi(\alpha),\psi} \frac{{}^C\mathrm{D}_{1-}^{\varphi(\alpha),\psi} x(t) - \frac{1}{2}\left(\frac{(\psi(1)-\psi(t))^3-\psi(1)+\psi(t)}{\ln(\psi(1)-\psi(t))}\right)}{\psi'(t)}\right)\psi'(t) = 0, \tag{4.7}$$

for all $t\in[0,1]$ and the transversality condition

$$\mathrm{I}_{0+}^{1-\varphi(\alpha),\psi} \frac{{}^C\mathrm{D}_{1-}^{\varphi(\alpha),\psi} x(t) - \frac{1}{2}\left(\frac{(\psi(1)-\psi(t))^3-\psi(1)+\psi(t)}{\ln(\psi(1)-\psi(t))}\right)}{\psi'(t)} = 0, \quad t=0. \tag{4.8}$$

Thus, $\overline{x}$ satisfies the necessary conditions of Theorem 3.3 with $\lambda=-2$. Since the Hamiltonian function H is convex, by Theorem 3.8, a solution of equations (4.7) and (4.8) is actually a minimizer of $\mathcal{J}$ subject to the previous integral constraint. Hence,

$$\overline{x}(t) = (\psi(1)-\psi(t))^{3\alpha+1}$$

is a solution of the proposed problem.

5. Conclusions and future work

In this work we generalized some of the results presented in [4] and [6], by considering in the Lagrangian functional a new fractional derivative that combines the two ones given in those papers. Namely, we deduced necessary and sufficient optimality conditions for variational problems with or without isoperimetric and holonomic restrictions.

For future, we intend to generalize the results presented in this paper, by considering variational problems with higher-order derivatives and delayed arguments. Also, we intend to study variational problems of Herglotz type involving the new distributed-order fractional derivatives with arbitrary kernels introduced in this paper.

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