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Research article

Extremal solutions of $\varphi-$Caputo fractional evolution equations involving integral kernels

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Abstract: This paper deals with the existence and uniqueness of solution for the Cauchy problem of $\varphi-$Caputo fractional evolution equations involving Volterra and Fredholm integral kernels. We derive a mild solution in terms of semigroup and construct a monotone iterative sequence for extremal solutions under a noncompactness measure condition of the nonlinearity. These results can be reduced to previous works with the classical Caputo fractional derivative. Furthermore, we give an example of initial-boundary value problem for the time-fractional parabolic equation to illustrate the application of the results.

Keywords: fractional evolution equation; monotone iterative technique; upper and lower solutions; Volterra operator; Fredholm operator
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1. Introduction

Fractional derivatives and integrals are generalization of derivatives and integrals to arbitrary non-integer orders. The theory of fractional calculus has been studied and applied to be valuable tools in the investigation and explanation of many phenomena in various fields such as chemistry, physics [1, 2], engineering [3], economics [4], control theory [5], epidemiology [6–8], etc (see [9–11]). Differential equations involving time-fractional derivatives are more realistic to explain some phenomena than those of integer order in time because it can describe the rate of change that depends on the past state. Consequently, fractional derivatives have been investigated on qualitative and numerical aspects [12, 13] for describing physical phenomena.

There are various definitions of fractional derivatives and integrals, which include Riemann-Liouville, Caputo, Hilfer, Riesz, Erdelyi-Kober, Hadamard, etc [1, 9–11]. Among these definitions, Caputo derivative and Riemann-Liouville are widely used by many researchers. In addition, they are derived from the corresponding fractional integral operators. The variation of fractional calculus in many different forms of fractional derivative and integral operators arises from various special functions. In one direction, the integral operators have been extended to include the weight function φ as a definition of generalized Caputo fractional derivative by Almeida [14]. This definition has the advantage in terms of accuracy in mathematical modeling if a function φ is appropriately selected. Later, Jarad and Abdeljawad [15] constructed the Laplace transform and its inverse operator which depends on another function φ for solving some fractional differential systems in the notion of φ-Caputo fractional derivative.

Over the past years, there has been an essential development in fractional evolution equations since many problems occurring in science, engineering and economy can be formulated by fractional evolution equations. Evolution equations are generally used to interpret the changing and evolving over time of the system. For instance, reaction-diffusion equations in chemical physics and biology [16, 17], Schrödinger equations in quantum mechanics [18, 19], Navier-Stokes equation in fluid mechanics [20], and Black-Scholes equation in finance [21] are common examples of fractional evolution equations. The development in the theory of fractional evolution equations is an essential branch of fractional calculus ranging from the study of existence, uniqueness, stability [22, 23], numerical techniques [24] and mathematical modeling [25, 26]. In particular, the existence and uniqueness theorems for fractional evolution equations have been extensively studied by means of semigroup theory and fixed point theorems [24, 27–36].

Various types of fixed point theorems are extensively used as fundamental tools for proving the existence and uniqueness of solutions for fractional evolution equations. However, some fixed point theorems are non-constructive results. As we all know, the monotone iterative method [37] is a flexible and efficient technique that provides both existence and constructive results for nonlinear differential equations [38–43] in terms of the lower and upper solutions. Furthermore, it can contribute to several comparison results which are applicable tools for the study. In this work, we emphasize on using the monotone iterative method involving the construction of upper and lower solutions.

In 2020, Gou and Li [44] investigated the existence and uniqueness of mild solutions for impulsive fractional evolution equations of Volterra and Fredholm types in an ordered Banach space E subject to the periodic boundary condition by means of monotone iterative method for the problem

$$\begin{cases} {}^{C}D_{0}^{\alpha}u(t)+Au(t)=f(t,u(t),Gu(t),Hu(t)), \qquad t\in J, t\neq t_k \\ \Delta u|_{t=t_k}=I_k(u(t_k)), k=1,2,\ldots,m, \\ u(0)=u(\omega) \end{cases}$$

where ${}^{C}D_{0}^{\alpha}$ is the classical Caputo fractional derivative of order $0<\alpha<1$ with the lower limit zero, $A:D(A)\subset E\to E$ is a closed linear operator and $-A$ generates a C_0-semigroup $\{\mathscr{T}(t)\}_{t\geq 0}$ in E; $f\in C(J\times E\times E\times E,E)$ is a function, $I_k\in C(E,E)$ is an impulsive function, $k=1,2,\ldots,m$; $J=[0,\omega]$, $J'=J\setminus\{t_1,t_2,\ldots,t_m\}$, $J_0=[0,t_1]$, $J_k=(t_k,t_{k+1}]$, the $\{t_k\}$ satisfy $0=t_0<t_1<t_2<\cdots<t_m<t_{m+1}=\omega$, $m\in\mathbb{N}$; $\Delta u(t_k)=u(t_k^+)-u(t_k^-)$, $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t=t_k$, respectively. The operators G and H are Volterra integral operator and Fredholm integral operator, respectively, which are defined by

$$Gu(t)=\int_0^t g(t,s)u(s)ds \tag{1.1}$$

where the integral kernel $g \in C(\Omega, \mathbb{R}^+)$ and $\Omega = \{(t,s) \in \mathbb{R}^2 \,|\, 0 \le s \le t \le T\}$, and

$$Hu(t) = \int_0^T h(t,s)u(s)ds \tag{1.2}$$

with the integral kernel $h \in C(\overline{\Omega}, \mathbb{R}^+)$ and $\overline{\Omega} = \{(t,s) \in \mathbb{R}^2 \,|\, 0 \le t,s \le T\}$.

Recently, Derbazi et al. [45] studied the existence and uniqueness of extremal solutions for fractional differential equations involving the $\varphi-$Caputo derivative subject to an initial condition:

$$\begin{cases} {}^C D_a^{\alpha;\varphi} u(t) = f(t,u(t)), & t \in [a,b] \\ u(a) = a^* \end{cases}$$

where ${}^C D_a^{\alpha;\varphi}$ is the $\varphi-$Caputo fractional derivative of order $0 < \alpha < 1$, $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function and $a^* \in \mathbb{R}$.

Inspired by [44, 45], some monotone conditions and noncompactness measure conditions of nonlinearity f, we use the monotone iterative technique to establish the existence of solutions of fractional evolution equations in an order Banach space E given by

$$\begin{cases} {}^C D_{t_0}^{\alpha;\varphi} u(t) = Au(t) + f(t,u(t),Gu(t),Hu(t)), & t > t_0 \\ u(t_0) = u_0 \end{cases} \tag{1.3}$$

where $0 < \alpha < 1$, $0 \le t_0 \le t \le T < \infty$ and $u_0 \in E$. Here $A : D(A) \subset E \to E$ is the infinitesimal generator of an analytic semigroup of uniformly bounded linear operators $\{\mathscr{T}(t)\}_{t\ge 0}$ on E. The nonlinearity $f : [t_0,T] \times E \times E \times E \to E$ is a function involving the Volterra integral operator G and the Fredholm integral operator H defined by

$$Gu(t) = \int_{t_0}^t g(t,s)u(s)ds \tag{1.4}$$

where the integral kernel $g \in C(\Omega, \mathbb{R}^+)$ and $\Omega = \{(t,s) \in \mathbb{R}^2 \,|\, t_0 \le s \le t \le T\}$, and

$$Hu(t) = \int_{t_0}^T h(t,s)u(s)ds \tag{1.5}$$

with the integral kernel $h \in C(\overline{\Omega}, \mathbb{R}^+)$ and $\overline{\Omega} = \{(t,s) \in \mathbb{R}^2 \,|\, t_0 \le t,s \le T\}$.

The motivation for this work is taken by Derbazi et al. [45] and we apply the same techniques used in [45]. However, the generalization of this problem to our work involves evolution operator A. Hence, in order to establish the existence of solutions, it is required to derive the form of fundamental solution in terms of a semigroup induced by resolvent with respect to the weight function φ. Moreover, we notice that our problem (1.3) can be reduced to the work of Derbazi et al. [45] when the evolution operator A, and the operators G and H are taken to be zero operators on Banach space $E = \mathbb{R}$.

In this paper, we aim to derive a mild solution for the problem (1.3) in terms of semigroup depending on a function φ from Caputo fractional derivative. In addition, we construct lower and upper solutions to prove the existence and uniqueness results of mild solution for the problem (1.3) under the condition that $\{\mathscr{T}(t)\}_{t\ge 0}$ do not require compactness by using the monotone iterative technique. Moreover, the results obtained in this work are in the abstract form based on a more general definition of $\varphi-$Caputo fractional derivative so that it can be extended and generalized some results in the literature such as the impulsive evolution equations [42, 44], the evolution equations with delay and nonlocal conditions.

This manuscript is organized as follows. In Section 2, we recall basic concepts for fractional calculus and some known results used in the later. In section 3, we construct a mild solution of the Cauchy problem (1.3) in the form of operator semigroup involving a function φ which is obtained from the generalized Caputo derivative and then give the definitions of lower and upper solutions. Next, we investigate the existence and uniqueness results of mild solutions for the Cauchy problem (1.3) under the assumption that $\{\mathscr{T}(t)\}_{t\geq 0}$ does not require compactness by using the monotone iterative method in Section 4. Moreover, we provide an example to illustrate the results obtained in Section 5 and conclusion in Section 6.

2. Preliminaries

In this section, we recall some notations and definitions of fractional calculus and give auxiliary results which will be used in the sequel.

We begin by introducing some properties of cones on real Banach spaces E.

In cone $\mathscr{P}$, a partially ordered $\leq$ is defined which means if $x \leq y$ if and only if $y - x \in \mathscr{P}$. If $x \leq y$ and $x \neq y$, then we denote $x < y$ or $y > x$.

Definition 2.1. *[46] The cone $\mathscr{P}$ is called*

(N) *normal if there exists a constant $N > 0$ such that $\|x_1\| \leq N\|x_2\|$ if $\theta \leq x_1 \leq x_2$, for all $x_1, x_2 \in E$. The least positive number satisfying above is called the normal constant of $\mathscr{P}$.*

(R) *regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}_{n\geq 1}$ is a sequence such that*

$$x_1 \leq x_2 \leq \cdots \leq y$$

for some $y \in E$, then there is $x \in E$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$.

Lemma 2.2. *[46, 47]*

(i). Every regular cone is normal.

(ii). The cone $\mathscr{P}$ is regular if and only if every decreasing sequence which is bounded from below is convergent.

Theorem 2.3. *[48] Let E be a weakly complete Banach space and $\mathscr{P}$ a cone in E. Then, $\mathscr{P}$ is normal if and only if $\mathscr{P}$ is regular.*

Definition 2.4. *An operator family $\mathbb{S}(t)(t \geq 0)$ is said to be a positive operator in E if for any $u \in \mathscr{P}$ and $t \geq 0$ such that $\mathbb{S}(t)u \geq \theta$.*

Here, we assume that E is an ordered Banach space with the norm $\|\cdot\|$ and the partial order $\leq$, whose positive cone $\mathscr{P} = \{x \in E : x \geq \theta\}$ (θ is the zero element of E) is normal with normal constant $N > 0$.

Let $C([t_0, T], E)$ be the Banach space of all continuous maps from $[t_0, T]$ to E with the norm $\|u\|_C = \sup_{t\in[t_0,T]} \|u(t)\|$. For $x_1, x_2 \in C([t_0, T], E)$, $x_1 \leq x_2$ if and only if $x_1(t) \leq x_2(t)$ for all

$t \in [t_0, T]$. For $v, w \in C([t_0, T], E)$, denote the ordered interval $[v, w] = \{u \in C([t_0, T], E) : v \leq u \leq w\}$ and $[v(t), w(t)] = \{u \in E : v(t) \leq u \leq w(t)\}$ for all $t \in [t_0, T]$.

Next, we briefly highlight the definition and some basic properties of the $\varphi-$Caputo fractional derivative which are used throughout this paper.

Definition 2.5. *(φ-Riemann-Liouville fractional integral, [14]) Let $\alpha > 0$, $u \in L^1([a,b])$ and $\varphi \in C^1([a,b])$ be a function such that $\varphi'(t) > 0$ for all $t \in [a,b]$. The φ-Riemann-Liouville fractional integral of order α of a function u with respect to another function φ is defined by*

$$(I_a^{\alpha;\varphi} u)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (\varphi(t) - \varphi(s))^{\alpha-1} u(s) \varphi'(s) ds. \tag{2.1}$$

The above definition can be reduced to the classical Riemann-Liouville fractional integral when $\varphi(t) = t$.

Definition 2.6. *(φ-Riemann-Liouville fractional derivative, [14]) Let $\alpha > 0$, $n \in \mathbb{N}$ and $u, \varphi \in C^n([a,b])$ be two functions such that $\varphi'(t) > 0$, for all $t \in [a,b]$. The φ-Riemann-Liouville fractional derivative of a function u of order α is defined by*

$$\begin{aligned}
(D_a^{\alpha;\varphi} u)(t) &= \left(\frac{1}{\varphi'(t)} \frac{d}{dt}\right)^n \left(I_a^{n-\alpha;\varphi} u\right)(t) \\
&= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\varphi'(t)} \frac{d}{dt}\right)^n \int_a^t (\varphi(t) - \varphi(s)^{n-\alpha-1} u(s) \varphi'(s) ds
\end{aligned}$$

where $n = [\alpha] + 1$.

Definition 2.7. *(φ-Caputo fractional derivative, [14, 15]) Let $\alpha > 0$, $n \in \mathbb{N}$ and $u, \varphi \in C^n([a,b])$ be two functions such that $\varphi'(t) > 0$ for all $t \in [a,b]$. The φ-Caputo fractional derivative of a function u of order α is defined by*

$$\begin{aligned}
\left({}^C D_a^{\alpha;\varphi} u\right)(t) &= \left(I_a^{n-\alpha;\varphi} u^{[n]}\right)(t) \\
&= \frac{1}{\Gamma(n-\alpha)} \int_a^t (\varphi(t) - \varphi(s))^{n-\alpha-1} u^{[n]}(s) \varphi'(s) ds
\end{aligned}$$

where $n = [\alpha] + 1$ and $u^{[n]}(t) := \left(\frac{1}{\varphi'(t)} \frac{d}{dt}\right)^n u(t)$ on $[a,b]$.

Lemma 2.8. *[14] Let $\alpha > 0$. If $u \in C^n([a,b])$ then*

$$I_a^{\alpha;\varphi} \left({}^C D_a^{\alpha;\varphi} u(t)\right) = u(t) - \sum_{k=0}^{n-1} \frac{u^{[k]}(a^+)}{k!} (\varphi(t) - \varphi(a))^k.$$

In particular, given $\alpha \in (0,1)$, we have

$$I_a^{\alpha;\varphi} \left({}^C D_a^{\alpha;\varphi} u(t)\right) = u(t) - u(a).$$

Definition 2.9. *[15] Let u and φ be real valued functions on $[a,\infty)$ such that $\varphi(t)$ is continuous and $\varphi'(t)>0$ on $[a,\infty)$. The generalized Laplace transform of u is defined by*

$$\mathscr{L}_{\varphi}\left\{u(t)\right\}(s)=\int_{a}^{\infty}e^{-s(\varphi(t)-\varphi(a))}u(t)\varphi'(t)dt$$

for all s.

Definition 2.10. *[15] Let u and v be piecewise continuous functions on an interval $[a,b]$ and of exponential order. The generalized convolution of u and v is defined as*

$$\left(u*_{\varphi}v\right)(t)=\int_{a}^{t}u(\tau)v\left(\varphi^{-1}\left(\varphi(t)+\varphi(a)-\varphi(\tau)\right)\right)\varphi'(\tau)d\tau.$$

Theorem 2.11. *(Gronwall's inequality, [49, 50]) Let $\varphi\in C^{1}([a,b])$ be a function such that $\varphi'(t)>0$ for all $t\in[a,b]$. Suppose that*

(i) u and v are nonnegative and integrable functions;

(ii) w is nonnegative continuous and nondecreasing function on $[a,b]$

with

$$u(t)\leq v(t)+w(t)\int_{a}^{t}\left(\varphi(t)-\varphi(s)\right)^{\alpha-1}u(s)\varphi'(s)ds.$$

Then

$$u(t)\leq v(t)+\int_{a}^{t}\sum_{k=1}^{\infty}\frac{\left[w(t)\Gamma(\alpha)\right]^{k}}{\Gamma(n\alpha)}\left(\varphi(t)-\varphi(s)\right)^{k\alpha-1}v(s)\varphi'(s)ds$$

for all $t\in[a,b]$.

Definition 2.12. *[51, 52] Let $0<\alpha<1$ and $z\in\mathbb{C}$. The function ϕ_{α} defined by*

$$\phi_{\alpha}(z)=\sum_{k=0}^{\infty}\frac{(-z)^{k}}{k!\Gamma(-\alpha k+1-\alpha)}=\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-z)^{k}\Gamma\left(\alpha(k+1)\right)\sin\left(\pi(k+1)\alpha\right)}{k!}$$

is called Wright type function.

Proposition 2.13. *[51, 52] The Wright type function ϕ_{α} is an entire function and has the following properties:*

(i) $\phi_{\alpha}(\theta)\geq 0$ for $\theta\geq 0$ and $\int_{0}^{\infty}\phi_{\alpha}(\theta)d\theta=1$;

(ii) $\int_{0}^{\infty}\phi_{\alpha}(\theta)\theta^{r}d\theta=\frac{\Gamma(1+r)}{\Gamma(1+\alpha r)}$ for $r>-1$;

(iii) $\int_{0}^{\infty}\phi_{\alpha}(\theta)e^{-z\theta}d\theta=E_{\alpha}(-z)$, $z\in\mathbb{C}$;

(iv) $\alpha\int_{0}^{\infty}\theta\phi_{\alpha}(\theta)e^{-z\theta}d\theta=E_{\alpha,\alpha}(-z)$, $z\in\mathbb{C}$

where $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}$ *is the Mittag-Leffler function with* $z \in \mathbb{C}$ *and* $\alpha > 0$.

Now, we recall the definition and some properties of Kuratowski measure of noncompactness.

Definition 2.14. *[53] Let B be a bounded set of Banach space E. The Kuratowski measure of noncompactness* $\mu(\cdot)$ *is defined by*

$$\mu(B) := \inf\{\varepsilon > 0 : B \subset \cup_{i=1}^{n} B_j, diam(B_i) < \varepsilon\}$$

where $diam(B_i) = \sup\{|y-x| : x, y \in B_i\}$ *for* $i = 1, 2, \dots n \in \mathbb{N}$.

Lemma 2.15. *[54] Let C and D be bounded subsets of a Banach space E. The noncompactness measure which satisfies the following properties:*

(i) D is precompact if and only if $\mu(D) = 0$*;*

(ii) $\mu(C \cup D) = \max\{\mu(C), \mu(D)\}$ *;*

(iii) $\mu(C + D) \leq \mu(C) + \mu(D)$*;*

(iv) $\mu(\lambda C) = |\lambda|\mu(C)$ *where* $\lambda \in \mathbb{R}$*;*

(v) Let X be another Banach space. If $\mathbb{S} : D(\mathbb{S}) \subset E \to X$ *satisfies Lipschitz continuity with constant L, then*

$$\mu(\mathbb{S}(B)) \leq L\mu(B)$$

for any bounded subset $B \subset D(\mathbb{S})$.

Lemma 2.16. *[54] If* $B \subset C([t_0, T], E)$ *is bounded and equicontinuity, then* $\mu(B(t))$ *is continuous on* $[t_0, T]$*, and*

$$\mu(B) = \sup_{t \in [t_0, T]} \mu(B(t))$$

where $B(t) = \{u(t) : u \in B\}$ *for all* $t \in [t_0, T]$.

Lemma 2.17. *[55] If* $B \subset C([t_0, T], E)$ *is bounded and equicontinuous, then* $\mu(B(t))$ *is continuous on* $[t_0, T]$*, and*

$$\mu\left(\left\{\int_{t_0}^{T} u(t)dt \,|\, u \in B\right\}\right) \leq \int_{t_0}^{T} \mu(B(t))\, dt.$$

Lemma 2.18. *[56] If* $B = \{u_n\}_{n=1}^{\infty} \subset C([t_0, T], E)$ *be a bounded and countable set, then* $\mu(B(t))$ *is Lebesgue integral on* $[t_0, T]$*, and*

$$\mu\left(\left\{\int_{t_0}^{T} u_n(t)dt \mid n \in \mathbb{N}\right\}\right) \leq 2\int_{t_0}^{T} \mu(B(t))\, dt.$$

Throughout this work, A is assumed to be the infinitesimal generator of a strongly continuous semigroup (i.e., C_0-semigroup) of uniformly bounded linear operators $\{\mathscr{T}(t)\}_{t\geq 0}$ on E with

$$M = \sup_{t \in [0,\infty)} \|\mathscr{T}(t)\| \qquad \text{for some } M \geq 1.$$

3. A mild solution for $\varphi-$Caputo fractional evolution equations

In this section, we derive the mild solution of the Cauchy problem (1.3) based on the semigroup theory and generalized Laplace transform.

Lemma 3.1. *Assume $v \in C([t_0,T],E)$ and $0<\alpha<1$. The mild solution of the linear Cauchy problem*

$$\begin{cases} {}^{C}D_{t_0}^{\alpha;\varphi}u(t) = Au(t)+v(t), & t>t_0 \\ u(t_0)=u_0 \in E \end{cases} \tag{3.1}$$

is given by

$$u(t) = S_{\alpha;\varphi}(t,t_0)u_0 + \int_{t_0}^{t} (\varphi(t)-\varphi(s))^{\alpha-1} T_{\alpha;\varphi}(t,s)v(s)\varphi'(s)ds, \quad t \in [t_0,T] \tag{3.2}$$

where the operators $S_{\alpha;\varphi}(t,s)$ and $T_{\alpha;\varphi}(t,s)$ are defined by

$$S_{\alpha;\varphi}(t,s)u = \int_0^\infty \phi_\alpha(\theta)\mathscr{T}\left((\varphi(t)-\varphi(s))^\alpha\theta\right)ud\theta \tag{3.3}$$

and

$$T_{\alpha;\varphi}(t,s)u = \alpha\int_0^\infty \theta\phi_\alpha(\theta)\mathscr{T}\left((\varphi(t)-\varphi(s))^\alpha\theta\right)ud\theta \tag{3.4}$$

for $0 \le s \le t \le T$ and $u \in E$.

Proof. The proof follows similar ideas as in [57]. Firstly, we apply the Definition 2.7 and Lemma 2.8 into the Cauchy problem (3.1). It can be rewritten the Cauchy problem (3.1) in form of the integral representation as

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} (\varphi(t)-\varphi(\tau))^{\alpha-1}(Au(\tau)+v(\tau))\varphi'(\tau)d\tau. \tag{3.5}$$

Taking the generalized Laplace transforms to both sides of (3.5), we get that for $\lambda>0$,

$$\begin{aligned} U(\lambda) &= \lambda^{\alpha-1}(\lambda^\alpha I-A)^{-1}u_0 + (\lambda^\alpha I-A)^{-1}V(\lambda) \\ &= \lambda^{\alpha-1}\int_0^\infty e^{-\lambda^\alpha s}\mathscr{T}(s)u_0ds + \int_0^\infty e^{-\lambda^\alpha s}\mathscr{T}(s)V(\lambda)ds \\ &= \alpha\int_0^\infty (\lambda\eta)^{\alpha-1}e^{-(\lambda\eta)^\alpha}\mathscr{T}(\eta^\alpha)u_0d\eta + \alpha\int_0^\infty \eta^{\alpha-1}e^{-(\lambda\eta)^\alpha}\mathscr{T}(\eta^\alpha)V(\lambda)d\eta \\ &=: J_1+J_2 \end{aligned}$$

where

$$U(\lambda) = \int_{t_0}^\infty e^{-\lambda(\varphi(\tau)-\varphi(t_0))}u(\tau)\varphi'(\tau)d\tau$$

and

$$V(\lambda) = \int_{t_0}^\infty e^{-\lambda(\varphi(\tau)-\varphi(t_0))}v(\tau)\varphi'(\tau)d\tau.$$

Substituting $\eta = \varphi(t) - \varphi(t_0)$ into J_1 and J_2 gives

$$\begin{aligned}
J_1 &= \alpha \int_{t_0}^{\infty} \lambda^{\alpha-1} (\varphi(t) - \varphi(t_0))^{\alpha-1} e^{-(\lambda(\varphi(t)-\varphi(t_0)))^{\alpha}} \mathscr{T}\left((\varphi(t) - \varphi(t_0))^{\alpha}\right) u_0 \varphi'(t) dt \\
&= \int_{t_0}^{\infty} -\frac{1}{\lambda} \frac{d}{dt} \left(e^{-(\lambda(\varphi(t)-\varphi(t_0)))^{\alpha}}\right) \mathscr{T}\left((\varphi(t) - \varphi(t_0))^{\alpha}\right) u_0 dt
\end{aligned}$$

and

$$\begin{aligned}
J_2 &= \int_{t_0}^{\infty} \alpha (\varphi(t) - \varphi(t_0))^{\alpha-1} e^{-(\lambda(\varphi(t)-\varphi(t_0)))^{\alpha}} \\
&\qquad \times \mathscr{T}\left((\varphi(t) - \varphi(t_0))^{\alpha}\right) V(\lambda) \varphi'(t) dt \\
&= \int_{t_0}^{\infty} \int_{t_0}^{\infty} \alpha (\varphi(t) - \varphi(t_0))^{\alpha-1} e^{-(\lambda(\varphi(t)-\varphi(t_0)))^{\alpha}} \\
&\qquad \times \mathscr{T}\left((\varphi(t) - \varphi(t_0))^{\alpha}\right) e^{-(\lambda(\varphi(s)-\varphi(t_0)))} v(s) \varphi'(s) \varphi'(t) ds dt.
\end{aligned}$$

The following one-sided stable probability density in [2] is considered by

$$\rho_{\alpha}(\theta) = \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^{k-1} \theta^{-\alpha k - 1} \frac{\Gamma(\alpha k + 1)}{k!} \sin(k\pi\alpha), \qquad \theta \in (0, \infty)$$

whose integration is given by

$$\int_0^{\infty} e^{-\lambda\theta} \rho_{\alpha}(\theta) d\theta = e^{-\lambda^{\alpha}} \qquad \text{for} \quad 0 < \alpha < 1. \tag{3.6}$$

Applying (3.6) to J_1 and J_2, it follows that

$$\begin{aligned}
J_1 &= \int_{t_0}^{\infty} \int_0^{\infty} \theta \rho_{\alpha}(\theta) e^{-\lambda(\varphi(t)-\varphi(t_0))\theta} \mathscr{T}\left((\varphi(t) - \varphi(t_0))^{\alpha}\right) u_0 \varphi'(t) d\theta dt \\
&= \int_{t_0}^{\infty} e^{-\lambda(\varphi(t)-\varphi(t_0))} \left(\int_0^{\infty} \rho_{\alpha}(\theta) \mathscr{T}\left(\frac{(\varphi(t) - \varphi(t_0))^{\alpha}}{\theta^{\alpha}}\right) u_0 d\theta\right) \varphi'(t) dt
\end{aligned}$$

and

$$\begin{aligned}
J_2 &= \int_{t_0}^{\infty} \int_{t_0}^{\infty} \int_0^{\infty} \alpha (\varphi(t) - \varphi(t_0))^{\alpha-1} \rho_{\alpha}(\theta) e^{-\lambda(\varphi(t)-\varphi(t_0))\theta} \\
&\qquad \times \mathscr{T}\left((\varphi(t) - \varphi(t_0))^{\alpha}\right) e^{-\lambda(\varphi(s)-\varphi(t_0))} v(s) \varphi'(s) \varphi'(t) d\theta ds dt \\
&= \int_{t_0}^{\infty} \int_{t_0}^{\infty} \int_0^{\infty} \alpha e^{-\lambda(\varphi(t)+\varphi(s)-2\varphi(t_0))} \frac{(\varphi(t) - \varphi(t_0))^{\alpha-1}}{\theta^{\alpha}} \rho_{\alpha}(\theta) \\
&\qquad \times \mathscr{T}\left(\frac{(\varphi(t) - \varphi(t_0))^{\alpha}}{\theta^{\alpha}}\right) v(s) \varphi'(s) \varphi'(t) d\theta ds dt \\
&= \int_{t_0}^{\infty} \int_{t}^{\infty} \int_0^{\infty} \alpha e^{-\lambda(\varphi(\tau)-\varphi(t_0))} \rho_{\alpha}(\theta) \frac{(\varphi(t) - \varphi(t_0))^{\alpha-1}}{\theta^{\alpha}} \\
&\qquad \times \mathscr{T}\left(\frac{(\varphi(t) - \varphi(t_0))^{\alpha}}{\theta^{\alpha}}\right) v(\varphi^{-1}(\varphi(\tau) - \varphi(t) + \varphi(t_0))) \varphi'(\tau) \varphi'(t) d\theta d\tau dt
\end{aligned}$$

$$= \int_{t_0}^{\infty} \int_{t_0}^{\tau} \int_0^{\infty} \alpha e^{-\lambda(\varphi(\tau)-\varphi(t_0))} \rho_\alpha(\theta) \frac{(\varphi(t)-\varphi(t_0))^{\alpha-1}}{\theta^\alpha}$$
$$\times \mathscr{T}\left(\frac{(\varphi(t)-\varphi(t_0))^\alpha}{\theta^\alpha}\right) v(\varphi^{-1}(\varphi(\tau)-\varphi(t)+\varphi(t_0)))\varphi'(\tau)\varphi'(t)d\theta dt d\tau$$
$$= \int_{t_0}^{\infty} e^{-\lambda(\varphi(\tau)-\varphi(t_0))}$$
$$\times \left(\int_{t_0}^{\tau}\int_0^{\infty} \alpha\rho_\alpha(\theta)\frac{(\varphi(\tau)-\varphi(s))^{\alpha-1}}{\theta^\alpha}\mathscr{T}\left(\frac{(\varphi(\tau)-\varphi(s))^\alpha}{\theta^\alpha}\right) v(s)\varphi'(s)d\theta ds\right)\varphi'(\tau)d\tau.$$

It follows that

$$U(\lambda) = \int_{t_0}^{\infty} e^{-\lambda(\varphi(t)-\varphi(t_0))}\left(\int_0^{\infty}\rho_\alpha(\theta)\mathscr{T}\left(\frac{(\varphi(t)-\varphi(t_0))^\alpha}{\theta^\alpha}\right)u_0 d\theta\right)\varphi'(t)dt$$
$$+ \int_{t_0}^{\infty} e^{-\lambda(\varphi(\tau)-\varphi(t_0))}$$
$$\times \left(\int_{t_0}^{\tau}\int_0^{\infty} \alpha\rho_\alpha(\theta)\frac{(\varphi(\tau)-\varphi(s))^{\alpha-1}}{\theta^\alpha}\mathscr{T}\left(\frac{(\varphi(\tau)-\varphi(s))^\alpha}{\theta^\alpha}\right) v(s)\varphi'(s)d\theta ds\right)\varphi'(\tau)d\tau.$$

Hence, we apply the inverse Laplace transform to get

$$u(t) = \int_0^{\infty}\rho_\alpha(\theta)\mathscr{T}\left(\frac{(\varphi(t)-\varphi(t_0))^\alpha}{\theta^\alpha}\right)u_0 d\theta$$
$$+ \int_{t_0}^{t}\int_0^{\infty}\alpha\rho_\alpha(\theta)\frac{(\varphi(t)-\varphi(s))^{\alpha-1}}{\theta^\alpha}\mathscr{T}\left(\frac{(\varphi(t)-\varphi(s))^\alpha}{\theta^\alpha}\right)v(s)\varphi'(s)d\theta ds$$
$$= \int_0^{\infty}\phi_\alpha(\theta)\mathscr{T}\left((\varphi(t)-\varphi(t_0))^\alpha\theta\right)u_0 d\theta$$
$$+ \int_{t_0}^{t}(\varphi(t)-\varphi(s))^{\alpha-1}\left(\int_0^{\infty}\alpha\theta\phi_\alpha(\theta)\mathscr{T}\left((\varphi(t)-\varphi(t_0))^\alpha\theta\right)d\theta\right)v(s)\varphi'(s)ds$$
$$:= S_{\alpha;\varphi}(t,t_0)u_0 + \int_{t_0}^{t}(\varphi(t)-\varphi(s))^{\alpha-1}T_{\alpha;\varphi}(t,s)v(s)\varphi'(s)ds$$

where $\phi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\rho_\alpha(\theta^{-\frac{1}{\alpha}})$ is the probability density function defined on $(0,\infty)$. □

Lemma 3.2. *[57] The operators $S_{\alpha;\varphi}$ and $T_{\alpha;\varphi}$ have the following properties:*

(i) For any fixed $0 \le s \le t$, $S_{\alpha;\varphi}(t,s)$ and $T_{\alpha;\varphi}(t,s)$ are bounded linear operators with

$$\|S_{\alpha;\varphi}(t,s)(u)\| \le M\|u\| \qquad \text{and} \qquad \|T_{\alpha;\varphi}(t,s)(u)\| \le \frac{\alpha M}{\Gamma(1+\alpha)}\|u\| = \frac{M}{\Gamma(\alpha)}\|u\|$$

for all $u \in E$.

(ii) The operators $S_{\alpha;\varphi}(t,s)$ and $T_{\alpha;\varphi}(t,s)$ are strongly continuous for all $0 \le s \le t$, that is, for every $u \in E$ and $0 \le s \le t_1 < t_2 \le T$ we have

$$\|S_{\alpha;\varphi}(t_2,s)u - S_{\alpha;\varphi}(t_1,s)u\| \to 0 \qquad \text{and} \qquad \|T_{\alpha;\varphi}(t_2,s)u - T_{\alpha;\varphi}(t_1,s)u\| \to 0$$

as $t_1 \to t_2$.

Definition 3.3. *A function $u \in C([t_0,T],E)$ is called a mild solution of* (1.3) *if it satisfies*

$$u(t) = S_{\alpha;\varphi}(t,t_0)u_0 + \int_{t_0}^{t} (\varphi(t)-\varphi(s))^{\alpha-1} T_{\alpha;\varphi}(t,s) f(s,u(s),Gu(s),Hu(s))\varphi'(s)ds$$

where the operators $S_{\alpha;\varphi}$ and $T_{\alpha;\varphi}$ are defined by (3.3) *and* (3.4)*, respectively.*

From Definition 2.4, if $\mathscr{T}(t)(t \geq 0)$ is a positive semigroup generated by $-A$, f and u_0 are nonnegaive, then the mild solution $u \in C([t_0,T],E)$ of Cauchy problem (1.3) satisfies $u \geq \theta$.

Definition 3.4. *A function $\underline{u} \in C([t_0,T],E)$ is called a lower solution of problem* (1.3) *and satisfies*

$$\begin{cases} {}^{C}D_{t_0}^{\alpha;\varphi}\underline{u}(t) \leq A\underline{u}(t) + f(t,\underline{u}(t),G\underline{u}(t),H\underline{u}(t)), & t \in (t_0,T] \\ \underline{u}(t_0) \leq u_0. \end{cases} \tag{3.7}$$

Analogously, a function $\overline{u} \in C([t_0,T],E)$ is called a upper solution of problem (1.3) *and satisfies*

$$\begin{cases} {}^{C}D_{t_0}^{\alpha;\varphi}\overline{u}(t) \geq A\overline{u}(t) + f(t,\overline{u}(t),G\overline{u}(t),H\overline{u}(t)), & t \in (t_0,T] \\ \overline{u}(t_0) \geq u_0. \end{cases} \tag{3.8}$$

4. Main results

Before stating and proving the main results, we introduce following assumptions:

(H_1) There exists lower and upper solutions $\underline{u}_0, \overline{u}_0 \in C([t_0,T],E)$ of Cauchy problem (1.3) respectively, such that $\underline{u}_0 \leq \overline{u}_0$.

(H_2) The nonlinear term f is a function in $C([t_0,T] \times E \times E \times E, E)$ and there exists a nonnegative constant C such that

$$f(t,u_2,v_2,w_2) - f(t,u_1,v_1,w_2) \geq -C(u_2-u_1),$$

for any $t \in [t_0,T]$, $\underline{u}_0(t) \leq u_1 \leq u_2 \leq \overline{u}_0(t)$, $G\underline{u}_0(t) \leq v_1 \leq v_2 \leq G\overline{u}_0(t)$ and $H\underline{u}_0(t) \leq w_1 \leq w_2 \leq H\overline{u}_0(t)$.

(H_3) There exist nonnegative constants L_1, L_2, L_3 such that for any bounded and countable sets $B_1, B_2, B_3 \subset E$

$$\mu(\{f(t,B_1,B_2,B_3)\}) \leq L_1\mu(B_1) + L_2\mu(B_2) + L_3\mu(B_3),$$

for $t \in [t_0,T]$.

(H_4) There are nonnegative constants S_1, S_2, S_3 such that

$$f(t,u_2,v_2,w_2) - f(t,u_1,v_1,w_1) \leq S_1(u_2-u_1) + S_2(v_2-v_1) + S_3(w_2-w_1),$$

for any $t \in [t_0,T]$, $\underline{u}_0(t) \leq u_1 \leq u_2 \leq \overline{u}_0(t)$, $G\underline{u}_0(t) \leq v_1 \leq v_2 \leq G\overline{u}_0(t)$ and $H\underline{u}_0(t) \leq w_1 \leq w_2 \leq H\overline{u}_0(t)$.

For convenience, we write $G^* = \max_{(t,s)\in\Omega}|g(t,s)|$, and $H^* = \max_{(t,s)\in\overline{\Omega}}|h(t,s)|$.

Theorem 4.1. *Let E be an ordered Banach space, whose positive cone $\mathscr{P}$ is normal with normal constant N. Assume that (H_1)-(H_3) holds with $\mathscr{T}(t)(t \geq 0)$ is positive and*

$$R := \frac{2M(\varphi(T)-\varphi(t_0))^{\alpha}}{\Gamma(\alpha+1)}\Big(L_1 + 2G^*L_2T + 2H^*L_3T + C\Big) < 1.$$

Then, the Cauchy problem (1.3) *has the minimal and maximal mild solutions between $\underline{u}_0$ and $\overline{u}_0$ which can be iteratively constructed by monotone sequence starting from $\underline{u}_0$ and $\overline{u}_0$, respectively.*

Proof. Let $D = [\underline{u}_0, \overline{u}_0] = \{v \in C([t_0,T],E)\,|\,\underline{u}_0 \leq v \leq \overline{u}_0\}$ and we define an operator $\mathscr{Q}: D \to C([t_0,T],E)$ by

$$\begin{aligned}\mathscr{Q}u(t) &= S_{\alpha;\varphi}(t,t_0)u_0\\ &\quad + \int_{t_0}^{t}(\varphi(t)-\varphi(s))^{\alpha-1}T_{\alpha;\varphi}(t,s)\left[f(s,u(s),Gu(s),Hu(s)) + Cu(s)\right]\varphi'(s)ds.\end{aligned}$$

First, we will verify that $\mathscr{Q}: D \to D$ is monotone increasing. For $u_1, u_2 \in D$ and $u_1 \leq u_2$, by the positivity of operators $S_{\alpha;\varphi}(t,s)$ and $T_{\alpha;\varphi}(t,s)$ for $t_0 \leq s \leq t \leq T$, and (H_2), we have

$$\begin{aligned}\mathscr{Q}u_1(t) &= S_{\alpha;\varphi}(t,t_0)u_0 + \int_{t_0}^{t}(\varphi(t)-\varphi(s))^{\alpha-1}\\ &\qquad\qquad \times T_{\alpha;\varphi}(t,s)\left[f(s,u_1(s),Gu_1(s),Hu_1(s)) + Cu_1(s)\right]\varphi'(s)ds\\ &\leq S_{\alpha;\varphi}(t,t_0)u_0 + \int_{t_0}^{t}(\varphi(t)-\varphi(s))^{\alpha-1}\\ &\qquad\qquad \times T_{\alpha;\varphi}(t,s)\left[f(s,u_2(s),Gu_2(s),Hu_2(s)) + Cu_2(s)\right]\varphi'(s)ds\\ &= \mathscr{Q}u_2(t)\end{aligned}$$

which implies $\mathscr{Q}u_1 \leq \mathscr{Q}u_2$. Let $\rho(t) = {}^{C}D_{t_0}^{\alpha;\varphi}\underline{u}_0(t) - A\underline{u}_0(t) + C\underline{u}_0(t)$. By Definition 3.4, we obtain $\rho(t) \leq f(t,\underline{u}_0(t),G\underline{u}_0(t),H\underline{u}_0(t)) + C\underline{u}_0(t)$, for $t \in [t_0,T]$. From Lemma 3.1, and the positivity of operators $S_{\alpha;\varphi}(t,s)$ and $T_{\alpha;\varphi}(t,s)$ for $t_0 \leq s \leq t \leq T$, we have

$$\begin{aligned}\underline{u}_0(t) &= S_{\alpha;\varphi}(t,t_0)u_0 + \int_{t_0}^{t}(\varphi(t)-\varphi(s))^{\alpha-1}T_{\alpha;\varphi}(t,s)\rho(s)\varphi'(s)ds\\ &\leq S_{\alpha;\varphi}(t,t_0)u_0 + \int_{t_0}^{t}(\varphi(t)-\varphi(s))^{\alpha-1}\\ &\qquad\qquad \times T_{\alpha;\varphi}(t,s)\left[f(s,\underline{u}_0(s),G\underline{u}_0(s),H\underline{u}_0(s)) + C\underline{u}_0(s)\right]\varphi'(s)ds\\ &= \mathscr{Q}\underline{u}_0(t) \qquad \text{for } t \in [t_0,T],\end{aligned}$$

and hence $\underline{u}_0 \leq \mathscr{Q}\underline{u}_0$. Similarly, we can show that $\mathscr{Q}\overline{u}_0 \leq \overline{u}_0$. This implies that for $u \in D$

$$\underline{u}_0 \leq \mathscr{Q}\underline{u}_0 \leq \mathscr{Q}u \leq \mathscr{Q}\overline{u}_0 \leq \overline{u}_0.$$

Hence, $\mathscr{Q}$ is an increasing monotonic operator.

Now, we define two sequences $\{\underline{u}_n\}$ and $\{\overline{u}_n\}$ in D by the iterative scheme

$$\underline{u}_n = \mathscr{Q}\underline{u}_{n-1} \qquad \text{and} \qquad \overline{u}_n = \mathscr{Q}\overline{u}_{n-1}, \quad n \in \mathbb{N}. \tag{4.1}$$

Then, by the monotonicity of $\mathscr{Q}$, it follows that

$$\underline{u}_0 \le \underline{u}_1 \le \cdots \underline{u}_n \le \cdots \le \overline{u}_n \le \cdots \le \overline{u}_1 \le \overline{u}_0. \tag{4.2}$$

Next, we claim that $\{\underline{u}_n\}$ and $\{\overline{u}_n\}$ are uniformly convergent in $[t_0,T]$. Let $B = \{\underline{u}_n \mid n \in \mathbb{N}\}$ and $B_0 = \{\underline{u}_{n-1} \mid n \in \mathbb{N}\}$. Then $B_0 = B \cup \{\underline{u}_0\}$ and hence $\mu(B(t)) = \mu(B_0(t))$ for $t \in [t_0,T]$.

In view of (4.2), since the positive cone $\mathscr{P}$ is normal, then B_0 and B are bounded in $C([t_0,T],E)$.

Now, we prove that $\mathscr{Q}(B)$ is equicontinuous. For any $u \in D$, by (H_2), we have

$$f(t,\underline{u}_0,G\underline{u}_0,H\underline{u}_0)+C\underline{u}_0 \le f(t,u,Gu,Hu)+Cu \le f(t,\overline{u}_0,G\overline{u}_0,H\overline{u}_0)+C\overline{u}_0.$$

By the normality of the positive cone $\mathscr{P}$, there exists a constant $K>0$ such that

$$\|f(t,u,Gu,Hu)+Cu\| \le K \qquad \text{for } u \in E.$$

For any $\underline{u}_n \in B$ and $t_0 \le t_1 < t_2 \le T$, we have

$$\begin{aligned}
&\|(\mathscr{Q}\underline{u}_n)(t_2)-(\mathscr{Q}\underline{u}_n)(t_1)\| \\
&\le \|S_{\alpha;\varphi}(t_2,t_0)u_0 - S_{\alpha;\varphi}(t_1,t_0)u_0\| \\
&\quad + \Bigg\| \int_{t_0}^{t_2} (\varphi(t_2)-\varphi(s))^{\alpha-1} T_{\alpha;\varphi}(t_2,s)\,[f(s,\underline{u}_n(s),G\underline{u}_n(s),H\underline{u}_n(s))+C\underline{u}_n(s)]\,\varphi'(s)ds \\
&\qquad - \int_{t_0}^{t_1} (\varphi(t_1)-\varphi(s))^{\alpha-1} T_{\alpha;\varphi}(t_1,s)\,[f(s,\underline{u}_n(s),G\underline{u}_n(s),H\underline{u}_n(s))+C\underline{u}_n(s)]\,\varphi'(s)ds \Bigg\| \\
&= \|S_{\alpha;\varphi}(t_2,t_0)u_0 - S_{\alpha;\varphi}(t_1,t_0)u_0\| \\
&\quad + \Bigg\| \int_{t_0}^{t_1} (\varphi(t_2)-\varphi(s))^{\alpha-1} T_{\alpha;\varphi}(t_2,s)\,[f(s,\underline{u}_n(s),G\underline{u}_n(s),H\underline{u}_n(s))+C\underline{u}_n(s)]\,\varphi'(s)ds \\
&\qquad + \int_{t_1}^{t_2} (\varphi(t_2)-\varphi(s))^{\alpha-1} T_{\alpha;\varphi}(t_2,s)\,[f(s,\underline{u}_n(s),G\underline{u}_n(s),H\underline{u}_n(s))+C\underline{u}_n(s)]\,\varphi'(s)ds \\
&\qquad + \int_{t_0}^{t_1} (\varphi(t_1)-\varphi(s))^{\alpha-1} T_{\alpha;\varphi}(t_2,s)\,[f(s,\underline{u}_n(s),G\underline{u}_n(s),H\underline{u}_n(s))+C\underline{u}_n(s)]\,\varphi'(s)ds \\
&\qquad - \int_{t_0}^{t_1} (\varphi(t_1)-\varphi(s))^{\alpha-1} T_{\alpha;\varphi}(t_2,s)\,[f(s,\underline{u}_n(s),G\underline{u}_n(s),H\underline{u}_n(s))+C\underline{u}_n(s)]\,\varphi'(s)ds \\
&\qquad - \int_{t_0}^{t_1} (\varphi(t_1)-\varphi(s))^{\alpha-1} T_{\alpha;\varphi}(t_1,s)\,[f(s,\underline{u}_n(s),G\underline{u}_n(s),H\underline{u}_n(s))+C\underline{u}_n(s)]\,\varphi'(s)ds \Bigg\| \\
&\le \|S_{\alpha;\varphi}(t_2,t_0)u_0 - S_{\alpha;\varphi}(t_1,t_0)u_0\| \\
&\quad + \Bigg\| \int_{t_1}^{t_2} (\varphi(t_2)-\varphi(s))^{\alpha-1} T_{\alpha;\varphi}(t_2,s)\,(f(s,\underline{u}_n(s),G\underline{u}_n(s),H\underline{u}_n(s))+C\underline{u}_n(s))\,\varphi'(s)ds \Bigg\|
\end{aligned}$$

$$
\begin{aligned}
&+\left\|\int_{t_0}^{t_1}\left[(\varphi(t_2)-\varphi(s))^{\alpha-1}-(\varphi(t_1)-\varphi(s))^{\alpha-1}\right]\right.\\
&\qquad\qquad \left.\times T_{\alpha;\varphi}(t_2,s)\left[f(s,\underline{u}_n(s),G\underline{u}_n(s),H\underline{u}_n(s))+C\underline{u}_n(s)\right]\varphi'(s)ds\right\|\\
&+\left\|\int_{t_0}^{t_1}(\varphi(t_1)-\varphi(s))^{\alpha-1}\right.\\
&\qquad\qquad \left.\times\left[T_{\alpha;\varphi}(t_2,s)-T_{\alpha;\varphi}(t_1,s)\right]\left(f(s,\underline{u}_n(s),G\underline{u}_n(s),H\underline{u}_n(s))+C\underline{u}_n(s)\right)\varphi'(s)ds\right\|\\
&=: J_1+J_2+J_3+J_4.
\end{aligned}
$$

By Lemma 3.2, it is clear that $J_1 \to 0$ as $t_1 \to t_2$ and we obtain

$$J_2 \le \frac{MK}{\Gamma(\alpha+1)}\left(\varphi(t_2)-\varphi(t_1)\right)^{\alpha}$$

and

$$J_3 \le \frac{MK}{\Gamma(\alpha+1)}\left[(\varphi(t_2)-\varphi(t_0))^{\alpha}-(\varphi(t_1)-\varphi(t_0))^{\alpha}-(\varphi(t_2)-\varphi(t_1))^{\alpha}\right]$$

and hence $J_2 \to 0$ and $J_3 \to 0$ as $t_2 \to t_1$. For $t_1 = 0$ and $0 < t_2 \le T$, it is easy to see that $J_4 = 0$. Then, for any $\varepsilon \in (0,t_1)$, we have

$$
\begin{aligned}
J_4 \le &\left\|\int_{t_0}^{t_1-\varepsilon}(\varphi(t_1)-\varphi(s))^{\alpha-1}\left[T_{\alpha;\varphi}(t_2,s)-T_{\alpha;\varphi}(t_1,s)\right]\right.\\
&\qquad\qquad \left.\left(f(s,\underline{u}_n(s),G\underline{u}_n(s),H\underline{u}_n(s))+C\underline{u}_n(s)\right)\varphi'(s)ds\right\|\\
&+\left\|\int_{t_1-\varepsilon}^{t_1}(\varphi(t_1)-\varphi(s))^{\alpha-1}\left[T_{\alpha;\varphi}(t_2,s)-T_{\alpha;\varphi}(t_1,s)\right]\right.\\
&\qquad\qquad \left.\left(f(s,\underline{u}_n(s),G\underline{u}_n(s),H\underline{u}_n(s))+C\underline{u}_n(s)\right)\varphi'(s)ds\right\|\\
\le &\frac{K}{\alpha}\left[(\varphi(t_1)-\varphi(t_0))^{\alpha}-(\varphi(t_1)-\varphi(t_1-\varepsilon))^{\alpha}\right]\sup_{t_0\le s<t_1-\varepsilon}\|T_{\alpha;\varphi}(t_2,s)-T_{\alpha;\varphi}(t_1,s)\|\\
&+\frac{2MK}{\Gamma(\alpha+1)}\left[(\varphi(t_1)-\varphi(t_1-\varepsilon))^{\alpha}\right]
\end{aligned}
$$

By Lemma 3.2, it follows that $J_4 \to 0$ as $t_2 \to t_1$ and $\varepsilon \to 0$. Thus, we obtain

$$\|(\mathscr{Q}u)(t_2)-(\mathscr{Q}u)(t_1)\| \to 0 \qquad \text{independently of } u \in D \text{ as } t_2 \to t_1.$$

which means that $\mathscr{Q}(B)$ is equicontinuous.

For $t \in [t_0, T]$, by Lemma 2.18 we have

$$\mu(GB_0(t)) = \mu\left(\left\{\int_{t_0}^{t} g(t,s)u_{n-1}(s)ds\right\}_{n=1}^{\infty}\right) \leq 2G^*T \sup_{t\in[t_0,T]} \mu\{B_0(t)\}$$

and

$$\mu(HB_0(t)) = \mu\left(\left\{\int_{t_0}^{T} h(t,s)u_{n-1}(s)ds\right\}_{n=1}^{\infty}\right) \leq 2H^*T \sup_{t\in[t_0,T]} \mu\{B_0(t)\}.$$

Since the sequence $\{\underline{u}_{n-1}(t_0)\}_{n=1}^{\infty}$ is convergent, we obtain $\mu\left(\{\underline{u}_{n-1}(t_0)\}_{n=1}^{\infty}\right) = 0$. For any $t \in [t_0, T]$, by (H_3) and Lemma 2.17 and Lemma 2.18 we have

$$\begin{aligned}
&\mu(B(t)) \\
&= \mu(B_0(t)) \\
&= \mu\Bigg(\Bigg\{S_{\alpha;\varphi}(t,t_0)u_0 + \int_{t_0}^{t} (\varphi(t)-\varphi(s))^{\alpha-1} T_{\alpha;\varphi}(t,s) \\
&\qquad\qquad [f(s,u_{n-1}(s),Gu_{n-1}(s),Hu_{n-1}(s)) + Cu_{n-1}(s)]\,\varphi'(s)ds\Bigg\}_{n=1}^{\infty}\Bigg) \\
&\leq \mu\left(\left\{S_{\alpha;\varphi}(t,t_0)u_0\right\}\right) \\
&\qquad + \mu\Bigg(\Bigg\{\int_{t_0}^{t} (\varphi(t)-\varphi(s))^{\alpha-1} T_{\alpha;\varphi}(t,s) \\
&\qquad\qquad [f(s,u_{n-1}(s),Gu_{n-1}(s),Hu_{n-1}(s)) + Cu_{n-1}(s)]\,\varphi'(s)ds\Bigg\}_{n=1}^{\infty}\Bigg) \\
&\leq 2\int_{t_0}^{t} \mu\Big(\Big\{(\varphi(t)-\varphi(s))^{\alpha-1} \\
&\qquad \times T_{\alpha;\varphi}(t,s)\,[f(s,u_{n-1}(s),Gu_{n-1}(s),Hu_{n-1}(s)) + Cu_{n-1}(s)]\,\varphi'(s)\Big\}_{n=1}^{\infty}\Big)ds \\
&\leq \frac{2M}{\Gamma(\alpha)}\int_{t_0}^{t} (\varphi(t)-\varphi(s))^{\alpha-1} \\
&\qquad \times \mu\left(\left\{[f(s,u_{n-1}(s),Gu_{n-1}(s),Hu_{n-1}(s)) + Cu_{n-1}(s)]\right\}_{n=1}^{\infty}\right)\varphi'(s)ds \\
&\leq \frac{2M}{\Gamma(\alpha)}\int_{t_0}^{t} (\varphi(t)-\varphi(s))^{\alpha-1} \Bigg(\left(L_1\mu\left(\left\{u_{n-1}(s)\right\}_{n=1}^{\infty}\right) + L_2\mu\left(\left\{Gu_{n-1}(s)\right\}_{n=1}^{\infty}\right)\right) \\
&\qquad\qquad + L_3\mu\left(\left\{Hu_{n-1}(s)\right\}_{n=1}^{\infty}\right)\Big) + \mu\left(\left\{Cu_{n-1}(s)\right\}_{n=1}^{\infty}\right)\Bigg)\varphi'(s)ds \\
&\leq \frac{2M}{\Gamma(\alpha)}(L_1 + 2G^*L_2T + 2H^*L_3T + C)\sup_{t\in[t_0,T]}\mu(B_0(t))\int_{t_0}^{t} (\varphi(t)-\varphi(s))^{\alpha-1}\varphi'(s)ds \\
&\leq \frac{2M(\varphi(T)-\varphi(t_0))^{\alpha}}{\Gamma(\alpha+1)}\Big(L_1 + 2G^*L_2T + 2H^*L_3T + C\Big)\sup_{t\in[t_0,T]}\mu(B_0(t))
\end{aligned}$$

$$=: R \sup_{t\in[t_0,T]} \mu\left(B_0(t)\right).$$

Since $\{\mathscr{Q}\underline{u}_n\}_{n=0}^{\infty}$ is equicontinuous on $[t_0,T]$ and by Lemma 2.16, we get

$$\mu\left(B\right) \leq R\mu\left(B\right).$$

Since $R<1$, we obtain $\mu\left(B\right)=0$. Hence the set B is relatively compact in E and so there is a convergent subsequence of $\{\underline{u}_n\}$ in E. Combining this with the monotonicity (4.2), we can prove that $\{\underline{u}_n\}$ itself is convergent, i.e., $\lim_{n\to\infty}\underline{u}_n(t)=\underline{u}(t), t\in[t_0,T]$. Similarly, we can prove that $\lim_{n\to\infty}\overline{u}_n(t)=\overline{u}(t), t\in[t_0,T]$.

For any $t\in[t_0,T]$, we see that

$$\begin{aligned}\underline{u}_n(t) &= \mathscr{Q}\underline{u}_{n-1}(t)\\ &= S_{\alpha;\varphi}(t,t_0)u_0+\int_{t_0}^{t}\left(\varphi(t)-\varphi(s)\right)^{\alpha-1}\\ &\qquad \times T_{\alpha;\varphi}(t,s)\left[f(s,\underline{u}_{n-1}(s),G\underline{u}_{n-1}(s),H\underline{u}_{n-1}(s))+C\underline{u}_{n-1}(s)\right]\varphi'(s)ds.\end{aligned}$$

Taking $n\to\infty$ in the above equality, by the Lebesgue dominated convergence theorem, we obtain

$$\underline{u}=\mathscr{Q}\underline{u} \qquad \text{and} \qquad \underline{u}\in C([t_0,T],E).$$

Similarly, we can prove that there exists $\overline{u}\in C([t_0,T],E)$ such that $\overline{u}=\mathscr{Q}\overline{u}$.

Combining this fact with monotonicity (4.2) we notice that

$$\underline{u}_0\leq\underline{u}\leq u\leq\overline{u}\leq\overline{u}_0.$$

Now, we will claim that $\underline{u}$ and $\overline{u}$ are the minimal and maximal fixed points of $\mathscr{Q}$ on $[\underline{u}_0,\overline{u}_0]$, respectively. For any $u\in D$ and u is a fixed point of $\mathscr{Q}$, we have

$$\underline{u}_1=\mathscr{Q}\underline{u}_0\leq\mathscr{Q}u=u\leq\mathscr{Q}\overline{u}_0=\overline{u}_1.$$

By induction, we obtain $\underline{u}_n\leq u\leq\overline{u}_n$ for all $n\in\mathbb{N}$. From (4.2) and taking the limit as $n\to\infty$, we conclude that

$$\underline{u}\leq u\leq\overline{u}.$$

Thus, $\underline{u}$ and $\overline{u}$ are minimal and maximal mild solutions of the Cauchy problem (1.3) on $[\underline{u}_0,\overline{u}_0]$, respectively, and $\underline{u}$, $\overline{u}$ can be obtained by the iterative scheme (4.1) starting from $\underline{u}_0$ and $\overline{u}_0$, respectively. □

Corollary 4.2. *Let E be an ordered Banach space, whose positive cone $\mathscr{P}$ is regular and (H_1)-(H_3) hold with $\mathscr{T}(t)(t\geq 0)$ is positive. Then, the Cauchy problem* (1.3) *has the minimal and maximal mild solutions between $\underline{u}_0$ and $\overline{u}_0$ which can be iteratively constructed by monotone sequence starting from $\underline{u}_0$ and $\overline{u}_0$, respectively.*

Proof. As (H_2) and (H_3) are satisfied, we have that the sequences $\{\underline{u}_n\}$ and $\{\overline{u}_n\}$ defined by (4.1) satisfies (4.2). Since the positive cone $\mathscr{P}$ is regular, we obtain that any monotonic and ordered-bounded sequence is convergent, and hence there are $\underline{u}^*, \overline{u}^* \in C([t_0,T],E)$ such that

$$\lim_{n\to\infty} \underline{u}_n = \underline{u}^* \qquad \text{and} \qquad \lim_{n\to\infty} \overline{u}_n = \overline{u}^*.$$

It follows from the proof of Theorem 4.1 that the statement of this theorem is satisfied. □

Corollary 4.3. *Suppose E is a partially ordered and weakly sequentially complete Banach space with normal positive cone $\mathscr{P}$. Assume that (H_1)-(H_2) hold with $\mathscr{T}(t)(t \geq 0)$ is positive. Then, the Cauchy problem* (1.3) *has extremal mild solutions in* $[\underline{u}_0, \overline{u}_0]$.

Proof. Since E is an ordered and weakly sequentially complete Banach space, the cone $\mathscr{P}$ is regular by Theorem 2.3. By the proof of Theorem 4.1, we have that the sequences $\{\underline{u}_n\}$ and $\{\overline{u}_n\}$ defined by (4.1) satisfies (4.2).

Let $\{u_n\}$ be an increasing or a decreasing sequence with $\{u_n\} \subset [\underline{u}_0(t), \overline{u}_0(t)]$. Then by the condition (H_2), the sequence $\{f(t,u_n,Gu_n,Hu_n)+Cu_n\}$ is a monotonic and order-bounded sequence, so $\mu\{f(t,u_n,Gu_n,Hu_n)+Cu_n\} = 0$. Thus, by the properties of the measure of noncompactness, we obtain

$$\mu\{f(t,u_n,Gu_n,Hu_n)\} \leq \mu\{f(t,u_n,Gu_n,Hu_n)+Cu_n\} + \mu\{Cu_n\} = 0.$$

Hence, the condition (H_3) holds.

By the proof of Theorem 4.1, we obtain that the sequences are uniformly convergent. Let $\underline{u}(t) = \lim_{n\to\infty} \underline{u}_n(t)$ and $\overline{u}(t) = \lim_{n\to\infty} \overline{u}_n(t)$, for $t \in [t_0,T]$. By Lebesgue dominated convergence theorem, we obtain

$$\underline{u} = \mathscr{Q}\underline{u} \qquad \text{and} \qquad \overline{u} = \mathscr{Q}\overline{u}$$

with $\underline{u}, \overline{u} \in C([t_0,T],E)$. Hence $\underline{u}$ and $\overline{u}$ are a mild solutions for (1.3). If $u \in D$ and $u = \mathscr{Q}u$, then

$$\underline{u}_1 = \mathscr{Q}\underline{u}_0 \leq u = \mathscr{Q}u \leq \mathscr{Q}\overline{u}_0 = \overline{u}_1.$$

From the process of induction, $\underline{u}_n \leq u \leq \overline{u}_n$ and $\underline{u}_0 \leq \underline{u} \leq u \leq \overline{u} \leq \overline{u}_0$ as $n \to \infty$. This means $\underline{u}$ is the minimal and $\overline{u}$ is the maximal mild solution for (1.3). □

Next, we will prove the uniqueness of solution of the Cauchy problem (1.3) by using monotone technique. To this end, we replace (H_3) by (H_4) .

Theorem 4.4. *Let E be an ordered Banach space, whose positive cone $\mathscr{P}$ is normal with normal constant N. Assume that $\mathscr{T}(t)(t \geq 0)$ is positive and the assumption (H_1)-(H_2) and (H_4) hold. If*

$$\tilde{R} := \frac{2M(\varphi(T)-\varphi(t_0))^{\alpha}}{\Gamma(\alpha+1)} \Big(NS_1 + NC + 2G^*NS_2T + 2H^*NS_3T + 2C \Big) < 1,$$

then the Cauchy problem (1.3) *has the unique mild solution between between $\underline{u}_0$ and $\overline{u}_0$ which can be iteratively constructed by monotone sequence starting from $\underline{u}_0$ and $\overline{u}_0$, respectively.*

Proof. For $t \in [t_0, T]$, let $\{u_n\} \subset [\underline{u}_0, \overline{u}_0]$, $\{v_n\} \subset [G\underline{u}_0, G\overline{u}_0]$ and $\{w_n\} \subset [H\underline{u}_0, H\overline{u}_0]$ be an increasing sequence. For $m, n = 1, 2, \dots$ with $m > n$, by (H_2) and (H_4), we have

$$\begin{aligned}\theta &\le f(t, u_m, v_m, w_m) - f(t, u_n, v_n, Hw_n) + C(u_m - u_n)\\ &\le (S_1 + C)(u_m - u_n) + S_2(v_m - v_n) + S_3(w_m - w_n).\end{aligned}$$

By the normality of cone $\mathscr{P}$ it follows that

$$\begin{aligned}&\|f(t, u_m, v_m, w_m) - f(t, u_n, v_n, w_n)\|\\ &\quad \le (NS_1 + NC + C)\|u_m - u_n\| + NS_2\|v_m - v_n\| + NS_3\|w_m - w_n\|.\end{aligned}$$

From the definition of the measure of noncompactness, we have

$$\begin{aligned}\mu(f(t, u_n, Gu_n, Hu_n) &\le (NS_1 + NC + C)\mu(u_n) + NS_2\mu(v_n) + NS_3\mu(w_n)\\ &:= L_1\mu(u_n) + L_2\mu(v_n) + L_3\mu(w_n)\end{aligned}$$

where $L_1 = NS_1 + NC + C$, $L_2 = NS_2$ and $L_3 = NS_3$. Hence, (H_3) holds. Therefore, by Theorem 4.1, the Cauchy problem (1.3) has the minimal mild solution $\underline{u}$ and the maximal mild solution $\overline{u}$ on $D = [\underline{u}_0, \overline{u}_0]$. In view of the proof of Theorem 4.1, we show that $\underline{u} = \overline{u}$. For $t \in [t_0, T]$, by the positivity of operator $T_{\alpha;\varphi}$, we have

$$\begin{aligned}\theta &\le \overline{u} - \underline{u}\\ &= \mathscr{Q}\overline{u} - \mathscr{Q}\underline{u}\\ &= \int_{t_0}^{t} (\varphi(t) - \varphi(s))^{\alpha-1} T_{\alpha;\varphi}(t,s)\Big[f(s, \overline{u}(s), G\overline{u}(s), H\overline{u}(s))\\ &\qquad\qquad - f(s, \underline{u}(s), G\underline{u}(s), H\underline{u}(s)) + C(\overline{u}(s) - \underline{u}(s))\Big]\varphi'(s)ds\\ &\le \int_{t_0}^{t} (\varphi(t) - \varphi(s))^{\alpha-1} T_{\alpha;\varphi}(t,s)\Big[S_1(\overline{u}(s) - \underline{u}(s)) + S_2(G\overline{u}(s) - G\underline{u}(s))\\ &\qquad\qquad + S_3(H\overline{u}(s) - H\underline{u}(s)) + C(\overline{u}(s) - \underline{u}(s))\Big]\varphi'(s)ds.\end{aligned}$$

Since the positive cone $\mathscr{P}$ is normal, we obtain

$$\begin{aligned}&\|\overline{u} - \underline{u}\|\\ &\le N\Bigg\|\int_{t_0}^{t} (\varphi(t) - \varphi(s))^{\alpha-1} T_{\alpha;\varphi}(t,s)\Big[S_1(\overline{u}(s) - \underline{u}(s)) + S_2(G\overline{u}(s) - G\underline{u}(s))\\ &\qquad\qquad + S_3(H\overline{u}(s) - H\underline{u}(s)) + C(\overline{u}(s) - \underline{u}(s))\Big]\varphi'(s)ds\Bigg\|\\ &\le NM(S_1 + S_2G^* + S_3H^*)\int_{t_0}^{t} (\varphi(t) - \varphi(s))^{\alpha-1}\|\overline{u}(s) - \underline{u}(s)\|\varphi'(s)ds.\end{aligned}$$

By Theorem 2.11, we get $\underline{u} = \overline{u}$ on $[t_0, T]$. Hence, $\underline{u} = \overline{u}$ is the the unique mild solution of the Cauchy problem (1.3) on D. By the proof of Theorem 4.1, the solution can be obtained by a monotone iterative procedure starting from $\underline{u}_0$ or $\overline{u}_0$. □

Similar to Corollary 4.2 and Corollary 4.3, we obtain the following result.

Corollary 4.5. *Assume that $\mathscr{T}(t)(t \geq 0)$ is positive and the assumption (H_1)-(H_2) and (H_4) hold. One of the following conditions is satisfied:*

(i) E is an ordered Banach space, whose positive cone $\mathscr{P}$ is regular;

(ii) E is an ordered and weakly sequentially complete Banach space, whose positive cone $\mathscr{P}$ is normal with normal constant N,

then the Cauchy problem (1.3) *has the unique mild solution between $\underline{u}_0$ or $\overline{u}_0$, which can be obtained which can be iteratively constructed by monotone sequence starting from $\underline{u}_0$ and $\overline{u}_0$, respectively.*

5. Example

We consider the following initial-boundary value problem of time-fractional parabolic partial differential equation with nonlinear source term:

$$\begin{cases} {}^{C}D_0^{\alpha;\varphi}u(x,t) - \Delta u(x,t) = \dfrac{\sin(\pi t)}{2(1+e^t)}u(x,t) + \dfrac{\cos^2(t)(\varphi(t)-\varphi(0))^\alpha}{3\Gamma(1-\alpha)} \\ \qquad\qquad + \frac{1}{50}\int_0^t (t-s)u(x,s)ds + \frac{e^{-4t}}{34}\int_0^1 e^{-|t-s|}u(x,s)ds, \quad x \in [0,\pi], t \in (0,1], \\ u(0,t) = u(\pi,t) = 0 \quad t \in [0,1], \\ u(x,0) = u_0(x), \quad x \in (0,\pi), \end{cases} \tag{5.1}$$

where $\alpha \in (0,1)$ and $u_0 \geq 0$.

Let $E = L^2([0,\pi])$ and $\mathscr{P} = \{y \in E \mid y \geq \theta\}$. Then $\mathscr{P}$ is normal cone in Banach space E with normal constant $N = 1$. Define the operator $A : D(A) \subset E \to E$ as follows:

$$Au = \Delta u$$

with the domain

$$D(A) = \Big\{v \in E \mid v, \frac{\partial v}{\partial x} \text{ are absolutely continuous}, \frac{\partial^2 v}{\partial x^2} \in E, v(0) = v(\pi) = 0\Big\}.$$

It is well known that A generates an analytic semigroup of uniformly bounded analytic semigroup $\{\mathscr{T}(t)\}_{t\geq 0}$ in E with $\mathscr{T}(t)$ is positive and $\|\mathscr{T}(t)\| \leq 1$ for $t \geq 0$.

Further, for any $t \in [0,1]$, we define

$$u(t) = u(\cdot,t), \qquad g(t,s) = t-s \quad \text{for } 0 \leq s,t \leq 1,$$

$$h(t,s) = e^{-|t-s|} \quad \text{for } 0 \leq s \leq t \leq 1,$$

$$Gu(t) = \int_0^t g(t,s)u(\cdot,s)ds, \qquad Hu(t) = \int_0^1 h(t,s)u(\cdot,s)ds,$$

and

$$f(t,u(t),Gu(t),Hu(t))=\frac{\sin(\pi t)}{2(1+e^t)}u(t)+\frac{\cos^2(t)(\varphi(t)-\varphi(0))^\alpha}{3\Gamma(1-\alpha)}+\frac{1}{50}Gu(t)+\frac{e^{-4t}}{34}Hu(t)$$

Then the problem (5.1) can be reformulated as the Cauchy problem (1.3) in E.

Let $\underline{u}_0(x,t)=0$ for $(x,t)\in[0,\pi]\times[0,1]$. Then

$$f(t,\underline{u}(x,t),G\underline{u}(x,t),H\underline{u}(x,t))\geq 0, \qquad \text{for } (x,t)\in[0,\pi]\times[0,1].$$

Let $\overline{u}_0=v$ be the positive solution of the following problem:

$$\begin{cases} {}^C D_0^{\alpha;\varphi}v(x,t)-\Delta v(x,t)=\dfrac{1}{2(1+e^t)}v(x,t)+\dfrac{(\varphi(t)-\varphi(0))^\alpha}{3\Gamma(1-\alpha)} \\ \qquad\qquad +\frac{1}{50}\int_0^t(t-s)v(x,s)ds+\frac{1}{34}\int_0^1 e^{-|t-s|}v(x,s)ds, \quad x\in[0,\pi],t\in(0,1], \\ v(0,t)=v(\pi,t)=0 \quad t\in[0,1], \\ v(x,0)=u_0(x) \quad x\in(0,\pi) \end{cases}$$

which can be obtained by modifying the proof of Theorem 5.1 in [57]. It is clearly seen that $\underline{u}$ and $\overline{u}$ are lower and upper solutions, respectively and $\underline{u}_0\leq\overline{u}_0$.

Suppose that $\{u_n\}\subset[\underline{u}_0,\overline{u}_0]$ is a monotone increasing sequence. Then, we have that for each $n\leq m$

$$\begin{aligned} 0 &\leq f(t,u_m,Gu_m,Hu_m)-f(t,u_n,Gu_n,Hu_n) \\ &\leq \frac{1}{4}(u_m-u_n)+\frac{1}{50}(Gu_m-Gu_n)+\frac{1}{34}(Hu_m-Hu_n). \end{aligned}$$

By normality of $\mathscr{P}$, we have

$$\begin{aligned} &\|f(t,u_m,Gu_m,Hu_m)-f(t,u_n,Gu_n,Hu_n)\| \\ &\leq \frac{1}{4}\|u_m-u_n\|+\frac{1}{50}\|Gu_m-Gu_n\|+\frac{1}{34}\|Hu_m-Hu_n\| \end{aligned}$$

and hence by Lemma 2.15

$$\mu(f(t,u_n,Gu_n,Hu_n))\leq\frac{1}{4}\mu(u_n)+\frac{1}{50}\mu(Gu_n)+\frac{1}{34}\mu(Hu_n).$$

This implies that the conditions (H_2) - (H_3) are satisfies with $L_1=\dfrac{1}{4}, L_2=\dfrac{1}{50}$ and $L_3=\dfrac{1}{34}$.

For example $\alpha=\frac{3}{7}$ and $\varphi(t)=2^t$. Then, upon computation, we get

$$R:=\frac{2M}{\Gamma(\alpha+1)}\left(L_1+2G^*L_2T+2H^*L_3T+C\right)\left(\varphi(T)-\varphi(0)\right)^\alpha\approx 0.7873<1.$$

where $G^*=H^*=T=1$ and $C=0$. Therefore, by Theorem 4.1, we obtain that the minimal and maximal mild solutions for the Cauchy problem (5.1) are between the lower solution $\underline{u}_0$ and upper solution $\overline{u}_0$.

Moreover, the condition (H_4) is satisfied with $S_1=\dfrac{1}{4}$, $S_2=\dfrac{1}{50}$ and $S_3=\dfrac{1}{34}$. Then, the Cauchy problem (5.1) has a unique mild solution between the lower and upper solutions by Theorem 4.4.

6. Conclusions

This paper investigates the existence and uniqueness results of mild solutions for $\varphi-$Caputo fractional integro-differential evolution equations. The method is inspired by using the monotone iterative technique involving lower and upper solutions, some existence and uniqueness result of mild solutions for $\varphi-$Caputo fractional integro-differential evolution equations has been proved. Here, the compactness condition of C_0-semigroup $\{\mathscr{T}(t)\}_{t\geq 0}$ does not require.

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Conflict of interest

The authors declare no conflict of interest in this paper.

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