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Research article

The modified Kies-Fréchet distribution: Properties, inference and application

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Abstract: The present article introduces a new distribution called the modified Kies-Fréchet (MKF) distribution that extends the Fréchet distribution and provides two new sub-models called modified Kies inverse-exponential and modified Kies inverse-Rayleigh distributions. The MKF model can provide left-skewed, symmetrical, right-skewed, J-shaped, and reversed-J shaped densities. The MKF density was expressed as a linear combination of Fréchet densities. We derive some basic mathematical properties of the MKF model. The MKF parameters are estimated using some classical estimation methods called, the maximum likelihood, Anderson-Darling, least-squares, Cramér-von Mises, and weighted least squares. The performances of these estimators were explored by a detailed simulation study. Finally, the flexibility of the MKF model is checked using a real data set, showing that it can provide close fit as compared with other competing models.

Keywords: Fréchet distribution; modified Kies family; Cramér-von Mises estimation; generating function
Mathematics Subject Classification: 60E05, 62F10

1. Introduction

The Fréchet distribution is considered one of the important models in extreme value theory and its applications are common in several areas such as accelerated life testing floods, earthquakes, rainfall, wind speeds, queues in supermarkets, and sea waves. Other important applications can explored in Kotz and Nadarajah [1], Harlow [2], Nadarajah and Kotz [3], and Zaharim et al. [4].

Several generalized forms of the Fréchet distribution have developed to model data in many applied areas. For example, the exponentiated-Fréchet, beta-Fréchet, transmuted-Fréchet, Marshall-Olkin-Fréchet, gamma Fréchet, transmuted exponentiated Fréchet, Kumaraswamy-Fréchet, transmuted Marshall-Olkin Fréchet, Weibull Fréchet, beta exponential Fréchet, Kummaraswamy exponentiated Fréchet, odd Lindely-Fréchet, Burr X Fréchet, and odd Lomax Fréchet by Nadarajah and Kotz [5], Barreto-Souza et al. [6], Mahmoud and Mandouh [7], Krishna et al. [8], da Silva et al. [9], Elbatal et al. [10], Mead and Abd-Eltawab [11], Afify et al. [12], Afify et al. [13], Mead et al. [14], Mansour et al. [15, 16], Abouelmagd et al. [17], and Hamed et al. [18], respectively.

In this article, we introduce a new model called the modified Kies-Fréchet (MKF) distribution to extend the Fréchet model and increase its flexibility in modeling real data. The three-parameter MKF distribution is constructed based on the modified Kies generator (MK-G) introduced by Al-Babtain et al. [19] by taking the Fréchet distribution as a baseline in the KMK-G family.

The MKF distribution can give a better fit than some other rival models, each one having the same or more number of parameters. The MKF distribution can provide left-skewed, symmetrical, right-skewed, J-shaped, and reversed-J shaped densities, and unimodal, decreasing, increasing and reversed-J shaped hazard functions. Another important aim of this paper is to study the estimation of the MKF parameters using classical estimators such as maximum likelihood estimators (MLEs), Anderson-Darling estimators (ADEs), least-squares estimators (LSEs), Cramér-von Mises estimators (CVMEs), and weighted least squares estimators (WLSEs). These estimators were compared using an extensive simulation results to assess their performances using small and large samples.

The cumulative distribution function (CDF) of the Fréchet distribution, with shape parameter λ and scale parameter α, takes the form

$$G(x;\lambda,\alpha) = \exp\left[-\left(\alpha^{\lambda}x^{-\lambda}\right)\right],\ x > 0\ \lambda,\alpha > 0. \tag{1.1}$$

Its probability density function (PDF) reduces to

$$g(x;\lambda,\alpha) = \lambda\alpha^{\lambda}x^{-\lambda-1}\exp\left[-\left(\alpha^{\lambda}x^{-\lambda}\right)\right],\ x > 0\ \lambda,\alpha > 0. \tag{1.2}$$

Al-Babtain et al. [19] defined a new generator of distributions to develop a more flexible extensions from any baseline model with CDF, $G(x;\boldsymbol{\varphi})$. The CDF of the MK-G family with a positive shape parameter σ takes the form

$$F(x;\sigma,\boldsymbol{\varphi}) = 1 - \exp\left\{-\left[\frac{G(x;\boldsymbol{\varphi})}{1-G(x;\boldsymbol{\varphi})}\right]^{\sigma}\right\},\ x > 0\ \sigma > 0. \tag{1.3}$$

The PDF and hazard rate function (HRF) of the MK-G family are

$$f(x;\sigma,\boldsymbol{\varphi}) = \frac{\sigma g(x;\boldsymbol{\varphi})G(x;\boldsymbol{\varphi})^{\sigma-1}}{\left[1-G(x;\boldsymbol{\varphi})\right]^{\sigma+1}}\exp\left\{-\left[\frac{G(x;\boldsymbol{\varphi})}{1-G(x;\boldsymbol{\varphi})}\right]^{\sigma}\right\},\ x > 0\ \sigma > 0 \tag{1.4}$$

and

$$h(x;\sigma,\boldsymbol{\varphi}) = \frac{\sigma g(x;\boldsymbol{\varphi})G(x;\boldsymbol{\varphi})^{\sigma-1}}{\left[1-G(x;\boldsymbol{\varphi})\right]^{\sigma+1}},\ x > 0\ \sigma > 0.$$

The rest of this article is outlined as follows. The MKF distribution was defined in Section 2. Its basic mathematical properties were derived in Section 3. Five methods of estimation were presented in Section 4. The performance of these estimation methods was explored using simulation results in Section 5. A real data set from the medicine field was analyzed to show the importance of the MKF distribution in Section 6. We presented some conclusions in Section 7.

2. The MKF distribution

The CDF of MKF distribution follows, by inserting the CDF of the the Fréchet distribution (1.1) in Eq (1.3), as

$$F(x;\sigma,\lambda,\alpha) = 1 - \exp\left\{-\left[\exp\left(\alpha^{\lambda}x^{-\lambda}\right) - 1\right]^{-\sigma}\right\}, \tag{2.1}$$

The corresponding PDF of the MKF model follows, by inserting the CDF and PDF of the Fréchet distribution (1.1) and (1.2) in Eq (1.4), as

$$f(x;\sigma,\lambda,\alpha) = \frac{\sigma\lambda\alpha^{\lambda}x^{-\lambda-1}\exp\left[-\sigma\left(\alpha^{\lambda}x^{-\lambda}\right)\right]}{\{1-\exp\left[-\left(\alpha^{\lambda}x^{-\lambda}\right)\right]\}^{\sigma+1}}\exp\left\{-\left[\exp\left(\alpha^{\lambda}x^{-\lambda}\right) - 1\right]^{-\sigma}\right\}. \tag{2.2}$$

A random variable X with PDF (2.2) can be denoted by $X \sim \text{MKF}(\sigma,\lambda,\alpha)$. By setting $\lambda = 1$ in (2.2), we obtain modified Kies inverse-exponential distribution. By setting $\lambda = 2$ in (2.2), the MKF model reduces to the modified Kies inverse-Rayleigh distribution. The survival function (SF) and HRF of the MKF distribution are

$$S(x;\sigma,\lambda,\alpha) = \exp\left\{-\left[\exp\left(\alpha^{\lambda}x^{-\lambda}\right) - 1\right]^{-\sigma}\right\}, \tag{2.3}$$

$$h(x;\sigma,\lambda,\alpha) = \frac{\sigma\lambda\alpha^{\lambda}x^{-\lambda-1}\exp\left[-\sigma\left(\alpha^{\lambda}x^{-\lambda}\right)\right]}{\{1-\exp\left[-\left(\alpha^{\lambda}x^{-\lambda}\right)\right]\}^{\sigma+1}}. \tag{2.4}$$

Plots of the PDF and HRF of MKF distribution are displayed in Figures 1 and 2, respectively.

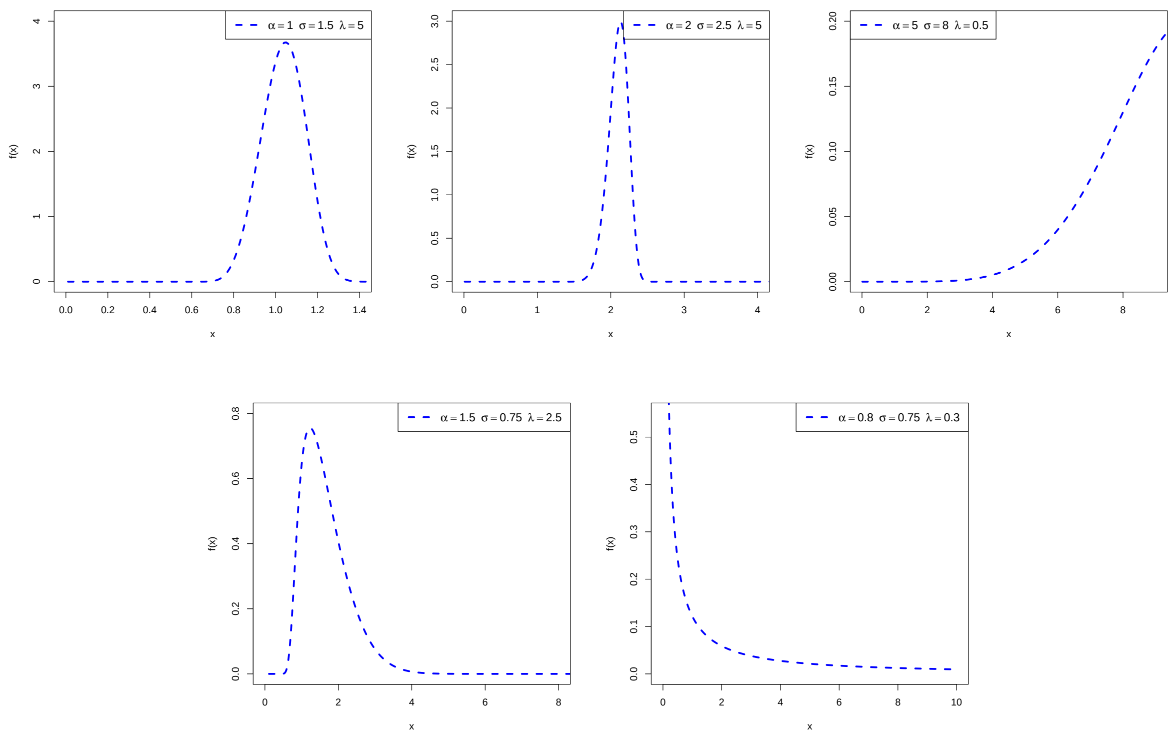

Figure 1. Some possible shapes for the KMF density for different selected values of its parameters.

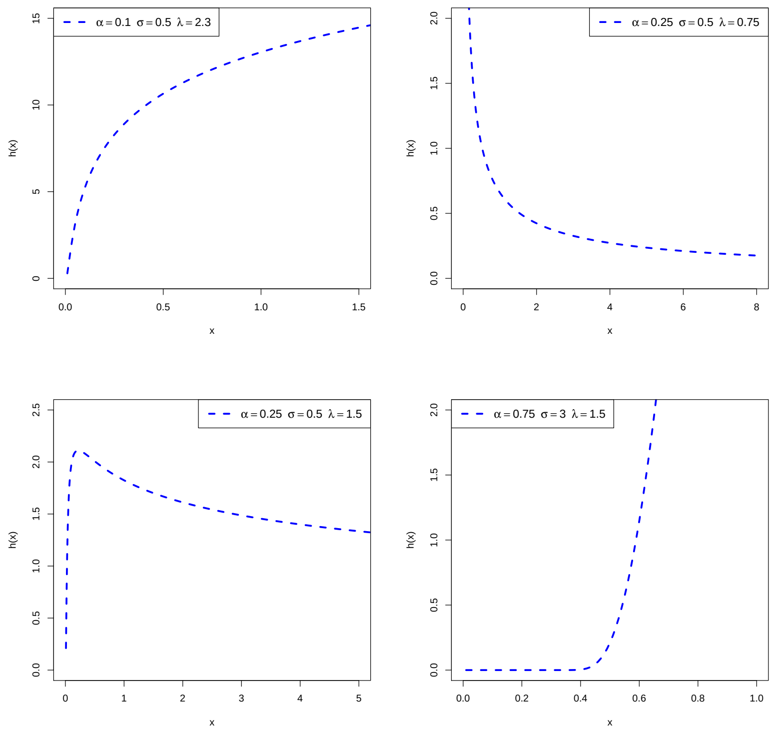

Figure 2. Some possible shapes for the KMF hazard function for different selected values of its parameters.

3. Mathematical properties

3.1. *Quantile function*

The quantile function (QF) of the MKF distribution is derived by determining the inverse function of the CDF (2.1) as

$$Q(p) = \alpha \log^{-\frac{1}{\lambda}} \left(- \left\{ -[-\log(1-p)]^{1/\sigma} - 1 \right\} [-\log(1-p)]^{-1/\sigma} \right), \quad 0 < p < 1. \tag{3.1}$$

The first, second, and third quartiles of the MKF distribution are obtained by setting $p = 0.25, 0.5,$ and 0.75, respectively, in (3.1).

Let p follow uniform distribution $(0, 1)$, hence the QF is used for generating random data sets of size n from the MKF distribution as follows:

$$x_i = \alpha \log^{-\frac{1}{\lambda}} \left(- \left\{ -[-\log(1-p_i)]^{1/\sigma} - 1 \right\} [-\log(1-p_i)]^{-1/\sigma} \right), \quad i = 1, 2, \ldots, n.$$

3.2. *Mixture representation*

The PDF (2.2) of the MKF model can be expressed as

$$f(x) = \sigma \lambda \alpha^{\lambda} \left(\frac{1}{x} \right)^{\lambda+1} \exp\left[\left(\alpha^{\lambda} x^{-\lambda} \right) \right] \left[\frac{\exp\left[- \left(\alpha^{\lambda} x^{-\lambda} \right) \right]}{1 - \exp\left[- (\alpha^{\lambda} x^{-\lambda}) \right]} \right]^{\sigma+1} e^{-\left[\frac{\exp\left[- \left(\alpha^{\lambda} x^{-\lambda} \right) \right]}{1 - \exp\left[- \left(\alpha^{\lambda} x^{-\lambda} \right) \right]} \right]^{\sigma}}. \tag{3.2}$$

By expanding the last term in Eq (3.2) which represents an exponential function, we have

$$e^{-\left[\frac{\exp\left[-\left(\alpha^{\lambda}x^{-\lambda}\right)\right]}{1-\exp\left[-\left(\alpha^{\lambda}x^{-\lambda}\right)\right]}\right]^{\sigma}} = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{e^{-k\sigma\left(\frac{\alpha}{x}\right)^{\lambda}}}{\left[1-\exp\left[-\left(\alpha^{\lambda}x^{-\lambda}\right)\right]\right]^{k\sigma}}$$

Inserting the last expansion in Eq (3.2) and, after some algebra, we can write

$$f(x) = \sigma\lambda\alpha^{\lambda}\left(\frac{1}{x}\right)^{\lambda+1}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}e^{-(k\sigma+\sigma)\left(\frac{\alpha}{x}\right)^{\lambda}}\left[1-\exp\left[-\left(\alpha^{\lambda}x^{-\lambda}\right)\right]\right]^{-(k\sigma+\sigma+1)}.$$

By expanding the binomial terms, we obtain

$$f(x) = \sigma\lambda\alpha^{\lambda}\left(\frac{1}{x}\right)^{\lambda+1}\sum_{j,k=0}^{\infty}\frac{(-1)^k[k\sigma+\sigma+1]^{(j)}}{k!\,j!}e^{-(k\sigma+\sigma+j)\left(\frac{\alpha}{x}\right)^{\lambda}}.$$

where $a^{(j)} = \Gamma(a+j)/\Gamma(a)$ is the rising factorial defined for any real a.

The last equation can be expressed as

$$f(x) = \sum_{j,k=0}^{\infty}\Phi_{i,j}\lambda\alpha^{\lambda}(k\sigma+\sigma+j)x^{-\lambda-1}e^{-(k\sigma+\sigma+j)\left(\frac{\alpha}{x}\right)^{\lambda}}, \tag{3.3}$$

where

$$\Phi_{i,j} = \frac{\sigma(-1)^k[k\sigma+\sigma+1]^{(j)}}{k!\,j!(k\sigma+\sigma+j)}.$$

Equation (3.3) can be reduced in the following form

$$f(x) = \sum_{j,k=0}^{\infty}\Phi_{i,j}h_{k\sigma+\sigma+j}(x), \tag{3.4}$$

where $h_{k\sigma+\sigma+j}(x)$ is the Fréchet density with scale parameter $\alpha[k\sigma+\sigma+j]^{1/\lambda}$ and shape parameter λ.

3.3. Moments

The rth moments of the MKF distribution follows as

$$\begin{aligned}\mu'_r = E(X^r) &= \int_0^{\infty}x^r f(x)dx = \sum_{j,k=0}^{\infty}\Phi_{i,j}\int_0^{\infty}x^r h_{k\sigma+\sigma+j}(x)\\ &= \sum_{j,k=0}^{\infty}\Phi_{i,j}\Gamma\left(1-\frac{r}{\lambda}\right)\left[\alpha(\sigma+j+\sigma k)^{1/\lambda}\right]^r,\quad r<\lambda.\end{aligned}$$

The first four original moments of the MKF distribution can be obtained by replacing $r = 1, 2, 3,$ and 4.

The nth central moment of X, say μ_n, follows as

$$\mu_n = E(x-\mu)^n = \sum_{k=0}^{\infty}(-1)^k\binom{n}{k}\mu_1'^k\mu'_{n-k}$$

The cumulants (k_n) of X can be obtained as the following

$$k_n = \mu'_n - \sum_{k=0}^{n-1}\binom{n-1}{k-1}k_r\mu'_{n-r}$$

3.4. Generating function

The moment generating function (MGF) of the MKF distribution follows as

$$M(t)=\sum_{j,k,r=0}^{\infty}\Phi_{i,j}\frac{t^r}{r!}\int_0^{\infty}x^r h_{k\sigma+\sigma+j}(x)=\sum_{j,k,r=0}^{\infty}\Phi_{i,j}\frac{t^r}{r!}\Gamma\left(1-\frac{r}{\lambda}\right)\left[\alpha(\sigma+j+\sigma k)^{1/\lambda}\right]^r,\ \ r<\lambda.$$

The characteristic function for the MKF distribution is obtained from the above equation by setting $t = it$ Another formula for the MKF MGF is derived in terms of the Wright generalized hyper geometric function as follows. The MGF of Fréchet distribution is given by

$$M(t)=\int_0^{\infty}e^{tx}f(x)dx=\lambda\alpha^{\lambda}\int_0^{\infty}e^{tx}x^{-\lambda-1}e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}dx,$$

by setting $y = x^{-1}$ and by expanding the first exponential, we have

$$M(t)=\sum_{m=0}^{\infty}\frac{\alpha^m t^m}{m!}\Gamma\left(\frac{\lambda-m}{\lambda}\right).$$

Consider the Wright generalized hyper geometric function defined by

$${}_p\Omega_q\left[\begin{array}{c}(\alpha_1,A_1),\ldots,(\alpha_p,A_p)\\(\lambda_1,B_1),\ldots,(\lambda_q,B_q)\end{array};x\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(\alpha_j+A_jn)x^n}{\prod_{j=1}^{q}\Gamma(\lambda_j+B_jn)n!}.$$

Hence, the MGF of the Fréchet distribution follows as (Afify et al. [13])

$$M(t)={}_1\Omega_0\left[\begin{array}{c}(1,-\lambda^{-1})\\-\end{array};\alpha t\right].$$

Then, the MGF of the MKF distribution can be expressed as

$$M(t)=\sum_{j,k=0}^{\infty}\Phi_{i,j}\,{}_1\Omega_0\left[\begin{array}{c}(1,-\lambda^{-1})\\-\end{array};\left[\alpha(\sigma+j+\sigma k)^{1/\lambda}\right]t\right].$$

3.5. Order statistics

The PDF and CDF of the ith order statistic for the MKF distribution are

$$\begin{aligned}f_{i:n}(x)&=\frac{n!}{(i-1)!\,(n-i)!}\,[1-F(x)]^{n-i}\,[F(x)]^{i-1}\,f(x)\\&=\frac{\sigma\lambda n!\alpha^{\lambda}\left(\frac{1}{x}\right)^{\lambda+1}e^{\left(\frac{\alpha}{x}\right)^{\lambda}}\left[\frac{1}{e^{\left(\frac{\alpha}{x}\right)^{\lambda}}-1}\right]^{\sigma+1}\left\{1-e^{-\left[\frac{1}{e^{\left(\frac{\alpha}{x}\right)^{\lambda}}-1}\right]^{\sigma}}\right\}^{i}\left\{e^{-\left[\frac{1}{e^{\left(\frac{\alpha}{x}\right)^{\lambda}}-1}\right]^{\sigma}}\right\}^{n-i}}{\Gamma(i)\Gamma(-i+n+1)\left\{e^{\left[\frac{1}{e^{\left(\frac{\alpha}{x}\right)^{\lambda}}-1}\right]^{\sigma}}-1\right\}},\end{aligned}$$

$$\begin{aligned} F_{i:n}(x) &= \sum_{r=i}^{n} \binom{n}{r} (F(x))^r (1-F(x))^{n-r} \\ &= \binom{n}{i} \left\{ 1 - e^{-\left[\frac{1}{e^{\left(\frac{\alpha}{x}\right)^\lambda}-1}\right]^\sigma} \right\}^i \left\{ e^{-\left[\frac{1}{e^{\left(\frac{\alpha}{x}\right)^\lambda}-1}\right]^\sigma} \right\}^{n-i} {}_2F_1 \left\{ 1, i-n; i+1; 1 - e^{\left[\frac{1}{-1+e^{\left(\frac{\alpha}{x}\right)^\lambda}}\right]^\sigma} \right\}, \end{aligned}$$

where ${}_2F_1 \left\{ 1, i-n; i+1; 1 - e^{\left[\frac{1}{-1+e^{\left(\frac{\alpha}{x}\right)^\lambda}}\right]^\sigma} \right\}$ is a hyper geometric function.

3.6. Incomplete moments

The sth incomplete moment of the MKF distribution is

$$\begin{aligned} \Psi_s(t) &= \int_0^t x^s f(x) dx = \sum_{j,k=0}^{t} \Phi_{i,j} \int_0^\infty x^s h_{k\sigma+\sigma+j}(x) \\ &= \alpha^s \sum_{j,k=0}^{t} \Phi_{i,j} (k\sigma + \sigma + j)^{\frac{s}{\lambda}} \gamma \left[1 - \frac{s}{\lambda} (k\sigma + \sigma + j) \left(\frac{\alpha}{t} \right)^\lambda \right]. \end{aligned}$$

The first incomplete moment, $\Psi_1(t)$, which follows from the above equation with $s = 1$, can be used to calculate the Bonferroni and Lorenz curves which are defined by $L(p) = \Psi_1(t)/\mu_1'$ and $B(p) = \Psi_1(x_p)/(p\mu_1')$, respectively, where x_p can be evaluated numerically using Eq (3.2) for a certain probability p. Further, $\Psi_1(t)$ can be used to obtain the mean waiting time and mean residual life which are given by $M_1(t) = t - \Psi_1(t)/F(t)$ and $m_1(t) = [1 - \Psi_1(t)]/S(t) - t$.

4. Methods of estimation

This section is devoted to estimating the MKF parameters using several estimators including the MLEs, ADEs, CVMEs, LSEs and WLSEs. Many authors have studied the estimation of the model parameters using different estimation methods. For example, Al-Mofleh et al. [20], Afify and Mohamed [21], Aldahlan and Afify [22], and Afify et al. [23] for the generalized Ramos-Louzada, extended odd Weibull exponential, odd exponentiated half-logistic exponential distribution, and alpha power exponentiated exponential distributions, respectively.

4.1. Maximum likelihood

Let $x_1, x_2, \ldots, x_n$ be a random sample from the PDF (2.2) with size n, then the log-likelihood function reduces to

$$L = \sum_{i=1}^{n} \left[\alpha x_i^{-\lambda} - \left(\frac{1}{e^{\alpha x_i^{-\lambda}} - 1} \right)^\sigma \right] - (\sigma + 1) \sum_{i=1}^{n} \log \left(e^{\alpha x_i^{-\lambda}} - 1 \right) - (\lambda + 1) \sum_{i=1}^{n} \log(x_i) + n \log(\alpha\sigma\lambda). \quad (4.1)$$

Differentiating Eq (4.1) with respect to α, σ and λ, and equating to zero, one can obtain

$$\frac{\partial L}{\partial \alpha} = -(\sigma+1)\sum_{i=1}^{n}\frac{x_i^{-\lambda}e^{\alpha x_i^{-\lambda}}}{e^{\alpha x_i^{-\lambda}}-1} + \sum_{i=1}^{n}\left[\sigma x_i^{-\lambda}e^{\alpha x_i^{-\lambda}}\left(\frac{1}{e^{\alpha x_i^{-\lambda}}-1}\right)^{\sigma+1} + x_i^{-\lambda}\right] + \frac{n}{\alpha} = 0,$$

$$\frac{\partial L}{\partial \sigma} = \sum_{i=1}^{n}-\left(\frac{1}{e^{\alpha x_i^{-\lambda}}-1}\right)^{\sigma}\log\left(\frac{1}{e^{\alpha x_i^{-\lambda}}-1}\right) - \sum_{i=1}^{n}\log\left(e^{\alpha x_i^{-\lambda}}-1\right) + \frac{n}{\sigma} = 0,$$

$$\begin{aligned}\frac{\partial L}{\partial \lambda} &= -(\sigma+1)\sum_{i=1}^{n}-\frac{\alpha x_i^{-\lambda}\log(x_i)\,e^{\alpha x_i^{-\lambda}}}{e^{\alpha x_i^{-\lambda}}-1} - \sum_{i=1}^{n}\log(x_i) + \frac{n}{\lambda} \\ &+ \sum_{i=1}^{n}\left[\alpha\sigma x_i^{-\lambda}\log(x_i)\left(-e^{\alpha x_i^{-\lambda}}\right)\left(\frac{1}{e^{\alpha x_i^{-\lambda}}-1}\right)^{\sigma+1} - \alpha x_i^{-\lambda}\log(x_i)\right] = 0.\end{aligned}$$

The ML estimators of the MKF parameters were obtained by solving the above equations which can be done numerically using different statistical programs such as R or Mathematica.

4.2. Least-squares and weighted least-squares

Let $x_{1:n}, x_{2:n}, \ldots, x_{2:n}$ be the order statistics of a random sample of size n from the MKF distribution. Hence, the LSEs of the MKF parameters follows by minimizing:

$$O = \sum_{i=1}^{n}\left[F(x_{i:n}) - \frac{i}{n+1}\right]^2 = \sum_{i=1}^{n}\left\{1 - e^{-\left[\frac{1}{e^{\left(\frac{\alpha}{x_{i:n}}\right)^{\lambda}}-1}\right]^{\sigma}} - \frac{i}{n+1}\right\}^2,$$

The LSEs of the parameters of the MKF model can also be derived by solving the following nonlinear equations:

$$\sum_{i=1}^{n}\left\{1 - e^{-\left[\frac{1}{e^{\left(\frac{\alpha}{x_{i:n}}\right)^{\lambda}}-1}\right]^{\sigma}} - \frac{i}{n+1}\right\}\varDelta_s(x_{i:n}) = 0, \quad s = 1,2,3,$$

where

$$\varDelta_1(x_{i:n}) = \frac{\partial}{\partial \alpha}F(x_{i:n}) = -\frac{\sigma\lambda\left(\frac{\alpha}{x_{i:n}}\right)^{\lambda} e^{\left(\frac{\alpha}{x_{i:n}}\right)^{\lambda}-\left(\frac{1}{e^{\left(\frac{\alpha}{x_{i:n}}\right)^{\lambda}}-1}\right)^{\sigma}}\left(\frac{1}{e^{\left(\frac{\alpha}{x_{i:n}}\right)^{\lambda}}-1}\right)^{\sigma+1}}{\alpha}, \tag{4.2}$$

$$\varDelta_2(x_{i:n}) = \frac{\partial}{\partial \sigma}F(x_{i:n}) = e^{-\left(\frac{1}{e^{\left(\frac{\alpha}{x_{i:n}}\right)^{\lambda}}-1}\right)^{\sigma}}\left(\frac{1}{e^{\left(\frac{\alpha}{x_{i:n}}\right)^{\lambda}}-1}\right)^{\sigma}\log\left(\frac{1}{e^{\left(\frac{\alpha}{x_{i:n}}\right)^{\lambda}}-1}\right), \tag{4.3}$$

$$\varDelta_3(x_{i:n}) = \frac{\partial}{\partial \lambda}F(x_{i:n}) = \sigma\left(\frac{\alpha}{x_{i:n}}\right)^{\lambda}\log\left(\frac{\alpha}{x_{i:n}}\right)\left(-e^{\left(\frac{\alpha}{x_{i:n}}\right)^{\lambda}-\left(\frac{1}{e^{\left(\frac{\alpha}{x_{i:n}}\right)^{\lambda}}-1}\right)^{\sigma}}\right)\left(\frac{1}{e^{\left(\frac{\alpha}{x_{i:n}}\right)^{\lambda}}-1}\right)^{\sigma+1}. \tag{4.4}$$

The WLSEs of the MKF parameters can be obtained by minimizing:

$$W=\sum_{i=1}^{n}\frac{(n+2)/,/,(n+1)^{2}}{i\ (n-i+1)}\left[F(x_{i:n})-\frac{i}{n+1}\right]^{2}=\sum_{i=1}^{n}\frac{(n+2)\ (n+1)^{2}}{i\ (n-i+1)}\left\{1-e^{-\left[\frac{1}{e^{\left(\frac{\alpha}{x_{i:n}}\right)^{\lambda}}-1}\right]^{\sigma}}-\frac{i}{n+1}\right\}^{2}.$$

Further, the WLSEs of the parameters of the MKF model are obtained by solving the following nonlinear equations:

$$\sum_{i=1}^{n}\frac{(n+2)\ (n+1)^{2}}{i\ (n-i+1)}\left[F(x_{i:n})-\frac{i}{n+1}\right]\varDelta_{s}(x_{i:n})=0,$$

where $\varDelta_{s}(x_{i:n})$, $s=1,2,3$ are given by (4.2)–(4.4), respectively.

4.3. Anderson-Darling estimation

The ADEs of the MKF parameters can be calculated by minimizing:

$$A=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)[\log F(x_{i:n})+\log S\,(x_{i:n})].$$

The ADEs can also be obtained by solving the following nonlinear equations:

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\varDelta_{s}(x_{i:n})}{F(x_{i:n})}-\frac{\varDelta_{s}(x_{n+1-i:n})}{S\,(x_{n+1-i:n})}\right]=0,$$

where $\varDelta_{s}(x_{i:n})$, $s=1,2,3$ are given by (4.2)–(4.4), respectively.

4.4. Cramér-von Mises estimation

The CVMEs of the MKF parameters were obtained by minimizing:

$$CV=\frac{1}{12n}+\sum_{i=1}^{n}\left[F(x_{i:n})-\frac{2i-1}{2n}\right]^{2}=\frac{1}{12n}+\sum_{i=1}^{n}\left\{1-e^{-\left[\frac{1}{e^{\left(\frac{\alpha}{x_{i:n}}\right)^{\lambda}}-1}\right]^{\sigma}}-\frac{2i-1}{2n}\right\}^{2},$$

or by solving the following nonlinear equations

$$\sum_{i=1}^{n}\left\{1-e^{-\left[\frac{1}{e^{\left(\frac{\alpha}{x_{i:n}}\right)^{\lambda}}-1}\right]^{\sigma}}-\frac{2i-1}{2n}\right\}\varDelta_{s}(x_{i:n})=0,$$

where $\varDelta_{s}(x_{i:n})$, $s=1,2,3$ are given by (4.2)–(4.4), respectively.

5. Simulation results

This section is devoted to exploring the performance of the proposed estimators in estimating the MKF parameters based on simulation results which are conducted using the R software at https://cran.r-project.org/. We considered some different sample sizes, $n = \{20, 30, 50, 100, 200, 400\}$, and some parametric values of the MKF parameters, where $\alpha = \{0.25, 0.5, 0.75, 1.5, 2, 2.5, 3\}$, $\sigma = \{0.25, 0.5, 0.75, 1.5, 2, 3\}$ and $\lambda = \{0.25, 0.5, 0.75, 1.5, 2, 2.5, 3\}$. We generated $n = 5000$ random samples from the MKF distribution which were used to calculate the average values of the estimates (AVEs) and their associated average values of the absolute biases (AVBs), average values of the mean square error (MSEs), and average values of the mean relative estimates (MREs) for all parameter combinations and different sample sizes.

The AVBs, MSEs, and MREs can be determined by the following equations:

$$AVBs = \frac{1}{N}\sum_{i=1}^{N}|\widehat{\boldsymbol{\psi}} - \boldsymbol{\psi}|, \quad MSEs = \frac{1}{N}\sum_{i=1}^{N}(\widehat{\boldsymbol{\psi}} - \boldsymbol{\psi})^2, \quad MREs = \frac{1}{N}\sum_{i=1}^{N}|\widehat{\boldsymbol{\psi}} - \boldsymbol{\psi}|/\boldsymbol{\psi},$$

where $\boldsymbol{\psi} = (\alpha, \sigma, \lambda)'$.

The simulation results about the AVEs, AVBs, MSEs, and MREs of the MKF parameters using the five estimation approaches were listed in Tables A1–A8 (see Appendix A). It is noted that the estimated parameters of the of the MKF distribution from all studied estimation methods are entirely good, where, the estimated parameters are very close to the true parameter values, showing small AVBs, MSEs and MREs for all considered cases. The five estimators illustrate the consistency property, that is, the AVBs, MSEs and MREs decrease as n increases, for all considered cases. In summary, we can say that the MLEs, ADEs, CVMEs, LSEs and WLSEs methods perform very well in estimating the parameters of the MKF distribution.

Now we provide some plots for the results in Table A8, as an example, to help the reader reading the results, in this table and other simulation tables, easily. Figures 3–5 display the AVBs, MSEs and MREs for the three parameters $\alpha = 3$, $\sigma = 1.5$ and $\lambda = 0.5$ for different estimators and sample sizes. The plots in Figures 3–5 show graphically that the AVBs, MSEs and MREs decay to zero with the increasing in the sample size for all parameter combinations. It is shown that the maximum likelihood method is the best estimation method for estimating the MKF parameters, so it is adopted in the application section.

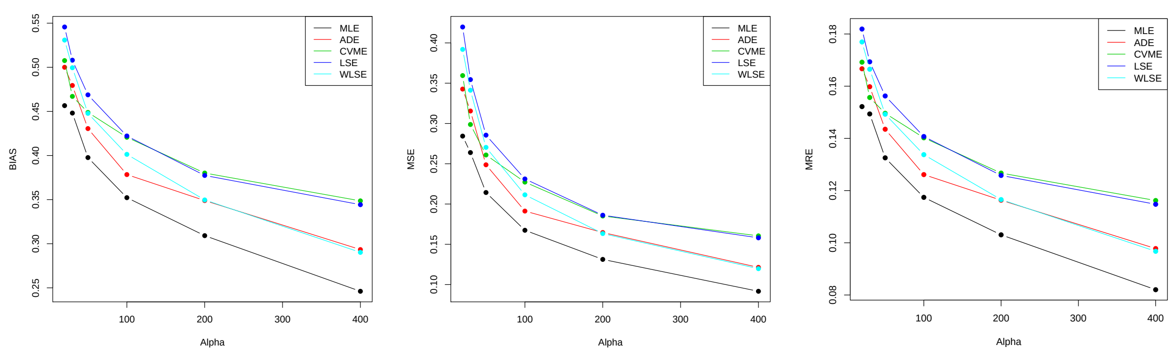

Figure 3. The AVBs, MSEs and MREs for various estimators of the parameter $\alpha = 3$.

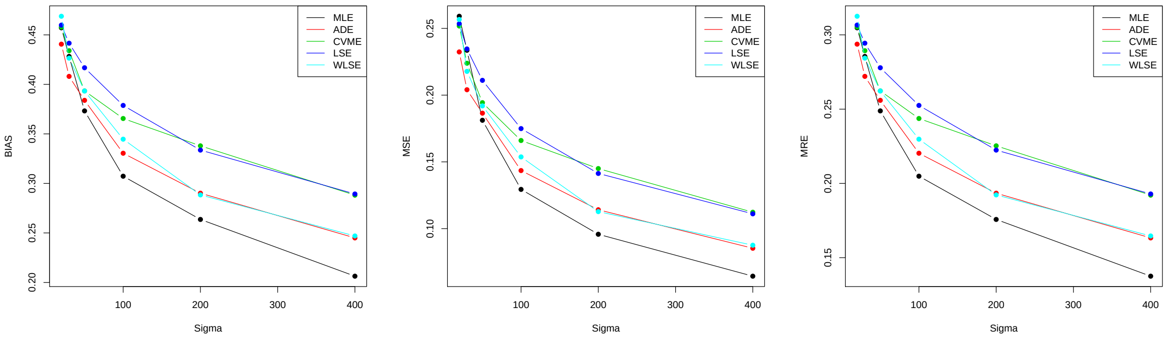

Figure 4. The AVBs, MSEs and MREs for various estimators of the parameter $\sigma = 1.5$.

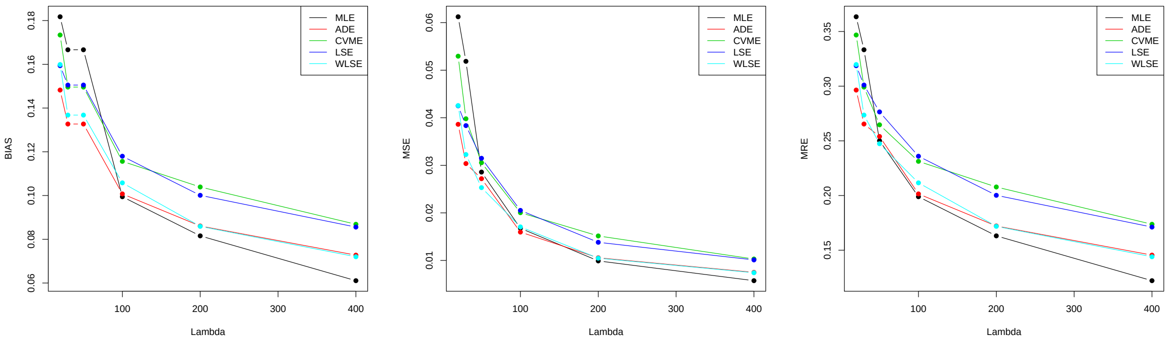

Figure 5. The AVBs, MSEs and MREs for various estimators of the parameter $\lambda = 0.5$.

6. An application

In this section, we consider a real data set about survival times (in weeks) of 33 patients who suffering from acute Myelogeneous Leukaemia. The data were reported in Feigl and Zelen [24]: 2, 3, 8, 4, 3, 30, 4, 1, 5, 16, 108, 121, 4, 39, 26, 22, 56, 1, 65, 56, 65, 17, 7, 16, 22, 3, 4, 43, 65, 156, 143, 100, 134. These data were analyzed by Nassar et al. [25] and Sen et al. [26].

We compared the MKF distribution with some other well-known rival distributions such as, Fréchet (F), exponentiated-Fréchet (ExF) [5], Marshall-Olkin Fréchet (MOF) [8], beta exponential-Fréchet (BEF) [14], and Kumaraswamy Marshall-Olkin Fréchet (KMOF) [27] distributions.

The MKF model and its competing models were compared based on some measures such as, Akaike information (A-IC), consistent Akaike information (CA-IC), Bayesian information (B-IC), and Hannan-Quinn information (HQ-IC) criteria. Further measures are the Anderson Darling (AN-DA), Cramér-von Mises (CRvMI), and Kolmogorov-Smirnov (KO-SM) with its associated p-values.

Table 1 reports the estimated parameters by the maximum likelihood and the standard errors of estimates (SEs) for the MKF distribution and other studied models. The values of the A-IC, CA-IC, B-IC, HQ-IC, AN-DA, CRvMI, KO-SM and p-value for the MKF model and other rival models were reported in Table 2. The values in Table 2 illustrate that the MKF model provides the best fit to the analyzed data. The MKF model has the lowest values for the A-IC, CA-IC, B-IC, HQ-IC, AN-DA, CRvMI, KO-SM measures and the largest value for the KO-SM p-value among all fitted distributions, hence it can be chosen as the best model to fit the Leukaemia data.

Table 1. Parameters estimates and SEs for the MKF model and other studied models.

Model	Estimates (SEs)				
MKF	$\hat{\alpha}$ = 14.0097478 (4.1105727)	$\hat{\sigma}$ = 0.9576292 (0.5697766)	$\hat{\lambda}$ = 0.5116603 (0.2479026)		
ExF	$\hat{\alpha}$ = 1426.60000 (3605.6610)	$\hat{\lambda}$ = 0.2490925 (0.07074903)	$\hat{\theta}$ = 13.7459997 (13.505500)		
MOF	$\hat{\alpha}$ = 1.9073027 (1.8015605)	$\hat{\lambda}$ = 0.9874992 (0.1859656)	$\hat{\theta}$ = 8.0679981 (11.1476834)		
BEF	$\hat{\alpha}$ = 0.11620340 (0.02650809)	$\hat{\beta}$ = 4.36408103 (0.02531044)	$\hat{\delta}$ = 0.04369861 (0.04889537)	$\hat{a}$ = 9.38488433 (2.70925294)	$\hat{b}$ = 6.40558974 (9.69673079)
KMOF	$\hat{\alpha}$ = 31946.70 (670.3244)	$\hat{\beta}$ = 0.6074154 (0.1067216)	$\hat{\delta}$ = 4.8060140 (6.1024980)	$\hat{a}$ = 1.014608 (1.390991)	$\hat{b}$ = 13724.50 (10687.2300)
F	$\hat{\alpha}$ = 7.8651853 (2.09130917)	$\hat{\lambda}$ = 0.6944123 (0.09149117)			

Table 2. The A-IC, CA-IC, B-IC, HQ-IC, AN-DA, CRvMI, KO-SM and p-value for the MKF model and other rival models.

Model	A-IC	CA-IC	B-IC	HQ-IC	AN-DA	CRvMI	KO-SM	p-Value
MKF	311.4870	312.3146	315.9765	312.9976	0.09286145	0.5922105	0.1303864	0.6288292
ExF	313.7879	314.6155	318.2774	315.2985	0.11150910	0.7050999	0.1353264	0.5813256
MOF	315.3784	316.2060	319.8680	316.8890	0.12887670	0.7977654	0.1359999	0.5749112
BEF	319.9048	322.1270	327.3873	322.4224	0.13931250	0.8549722	0.1400324	0.5369602
KMOF	314.8036	317.0259	322.2862	317.3213	0.09465571	0.6269345	0.1404210	0.5333498
F	315.9971	316.3971	318.9901	317.0041	0.16011280	0.9759212	0.1490254	0.4561218

The plots of fitted density, distribution and survival functions, and probability-probability (P-P) plot for the MKF distribution were displayed in Figure 6. The P-P plots and estimated distribution functions plots for the MKF model and other rival models are displayed in Figures 7 and 8, respectively.

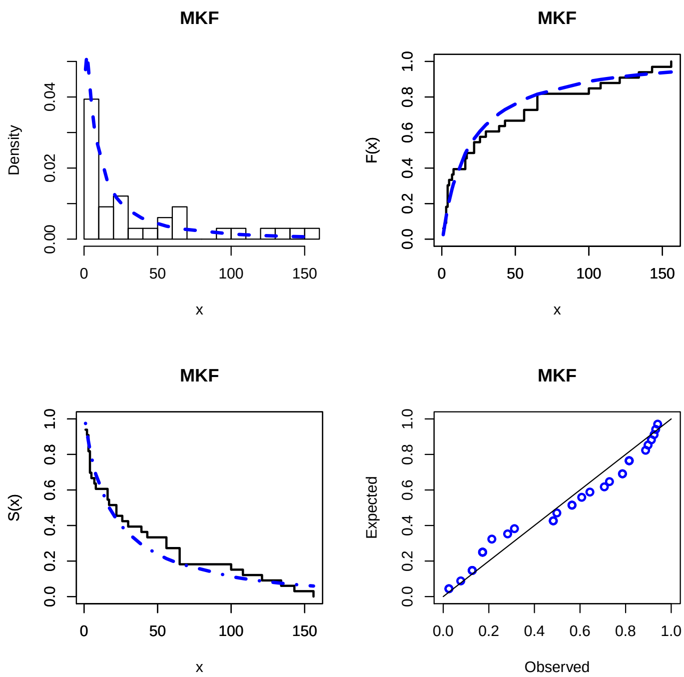

Figure 6. The fitted density, distribution and survival functions and P-P plots of the MKF distribution.

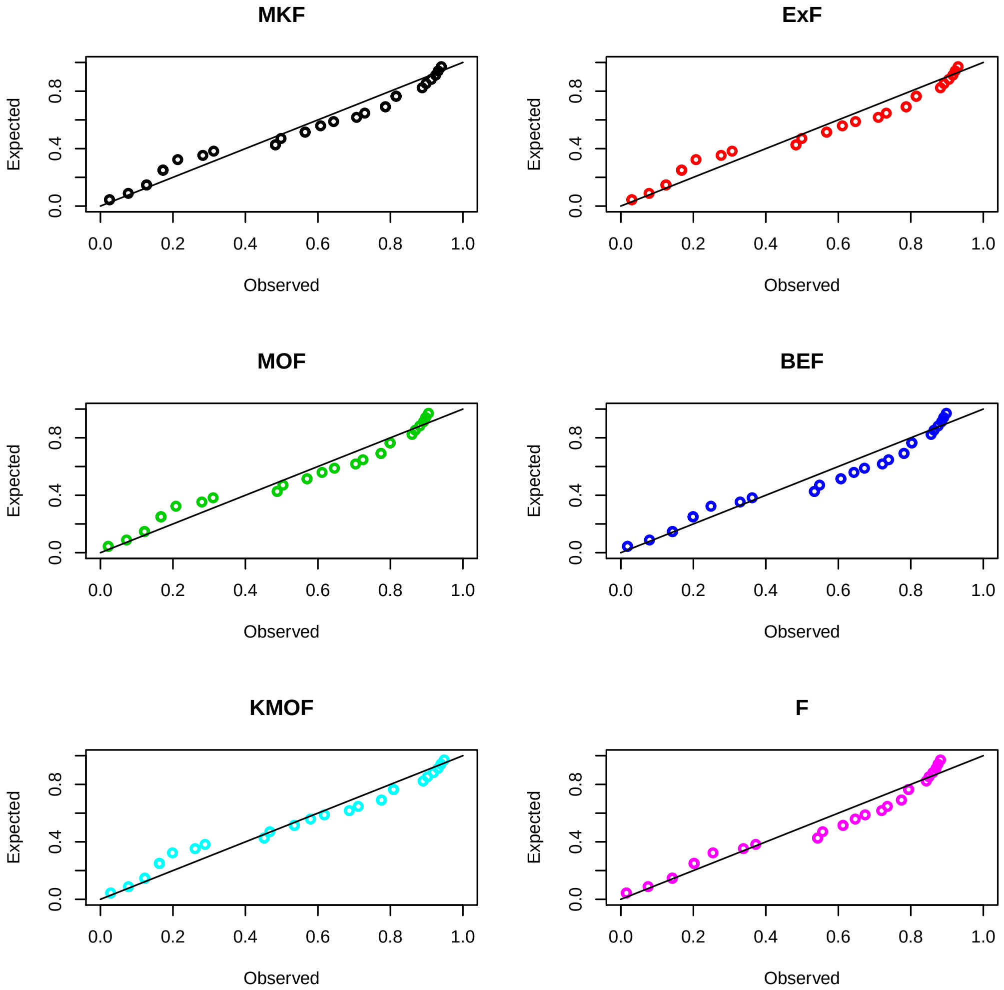

Figure 7. The P-P plots for the MKF distribution and other fitted distributions.

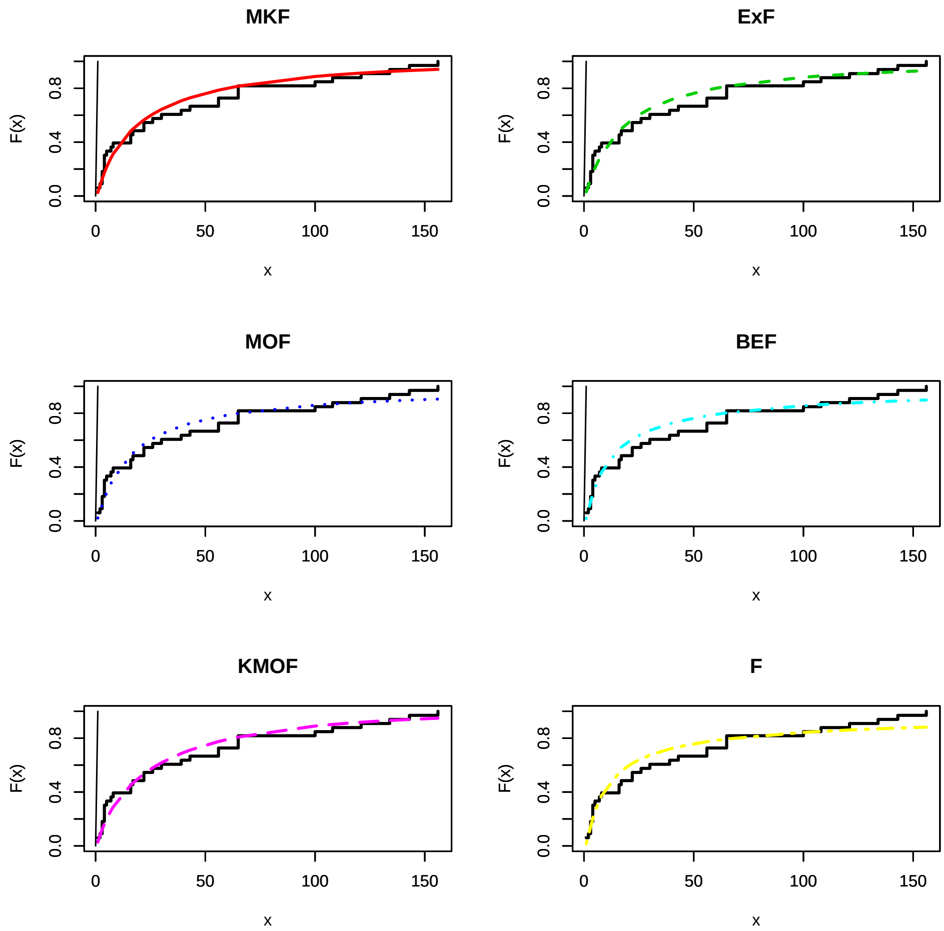

Figure 8. The estimated distribution functions for the MKF distribution and other fitted distributions.

7. Conclusions

In this paper, we proposed a more flexible model called the modified Kies-Fréchet (MKF) distribution to improve the fit of the Fréchet distribution. The MKF distribution includes two new sub-models called modified Kies inverse-exponential and modified Kies inverse-Rayleigh distributions. Some of its key mathematical properties were provided in explicit expressions. The MKF density function was expressed as a linear mixture of Fréchet densities. The parameters of the MKF distribution were estimated using five classical estimators, called the maximum likelihood estimators, Anderson-Darling estimators, least-squares estimators, Cramér-von Mises estimators, and weighted least squares estimators. The conducted simulation results showed that all estimation methods perform very well in estimating the MKF parameters. The importance of the MKF distribution was illustrated practically using a real data set, showing it can provide adequate fit as compared with other rival models.

Conflict of interest

There is no conflict of interest declared by the authors.

Appendix A

Table A1. Simulation results of the AVEs, AVBs, MSEs and MREs of MKF distribution for ($\alpha = 0.25,\ \sigma = 0.5,\ \lambda = 0.75$).

n	Est.	Est. Par.	MLEs	ADEs	CVMEs	LSEs	WLSEs
20	AVEs	$\hat{\alpha}$	0.24246	0.25894	0.25728	0.25701	0.25988
		$\hat{\sigma}$	0.51004	0.56637	0.57471	0.57714	0.55683
		$\hat{\lambda}$	0.93832	0.79679	0.87993	0.82093	0.81587
	AVBs	$\hat{\alpha}$	0.07206	0.07236	0.07797	0.07705	0.07645
		$\hat{\sigma}$	0.19211	0.20693	0.24247	0.24554	0.22550
		$\hat{\lambda}$	0.30892	0.23899	0.32657	0.30899	0.27722
	MSEs	$\hat{\alpha}$	0.00867	0.00975	0.01144	0.01039	0.01069
		$\hat{\sigma}$	0.05740	0.06773	0.08523	0.08840	0.07642
		$\hat{\lambda}$	0.24881	0.09584	0.23491	0.20615	0.15269
	MREs	$\hat{\alpha}$	0.28824	0.28943	0.31190	0.30818	0.30582
		$\hat{\sigma}$	0.38421	0.41386	0.48495	0.49109	0.45100
		$\hat{\lambda}$	0.41189	0.31866	0.43542	0.41199	0.36963
30	AVEs	$\hat{\alpha}$	0.24734	0.25008	0.25357	0.25144	0.25233
		$\hat{\sigma}$	0.51947	0.53620	0.56136	0.55238	0.56265
		$\hat{\lambda}$	0.83747	0.80030	0.81095	0.80233	0.78396
	AVBs	$\hat{\alpha}$	0.05947	0.05941	0.06092	0.06014	0.06043
		$\hat{\sigma}$	0.15000	0.16340	0.20473	0.21066	0.19841
		$\hat{\lambda}$	0.19793	0.19181	0.24101	0.25016	0.23092
	MSEs	$\hat{\alpha}$	0.00581	0.00571	0.00613	0.00593	0.00639
		$\hat{\sigma}$	0.03865	0.04512	0.06703	0.06917	0.06295
		$\hat{\lambda}$	0.07667	0.06650	0.10107	0.11165	0.09897
	MREs	$\hat{\alpha}$	0.23786	0.23765	0.24367	0.24056	0.24172
		$\hat{\sigma}$	0.30000	0.32681	0.40946	0.42133	0.39681
		$\hat{\lambda}$	0.26390	0.25575	0.32134	0.33355	0.30789
50	AVEs	$\hat{\alpha}$	0.24806	0.25152	0.25117	0.24709	0.25036
		$\hat{\sigma}$	0.50838	0.54002	0.53613	0.55059	0.53758
		$\hat{\lambda}$	0.80350	0.76304	0.79616	0.76156	0.76427
	AVBs	$\hat{\alpha}$	0.04542	0.04481	0.04900	0.04586	0.04696
		$\hat{\sigma}$	0.11330	0.13920	0.16565	0.16772	0.14406
		$\hat{\lambda}$	0.14093	0.15458	0.19114	0.18491	0.16300
	MSEs	$\hat{\alpha}$	0.00334	0.00307	0.00369	0.00334	0.00353
		$\hat{\sigma}$	0.02105	0.03397	0.04563	0.04803	0.03602
		$\hat{\lambda}$	0.03669	0.03777	0.06249	0.05498	0.04397
	MREs	$\hat{\alpha}$	0.18168	0.17924	0.19601	0.18345	0.18785
		$\hat{\sigma}$	0.22660	0.27839	0.33129	0.33543	0.28812
		$\hat{\lambda}$	0.18790	0.20611	0.25486	0.24654	0.21734
100	AVEs	$\hat{\alpha}$	0.24864	0.25239	0.25094	0.24733	0.25035
		$\hat{\sigma}$	0.51097	0.52428	0.53218	0.53172	0.51933
		$\hat{\lambda}$	0.76833	0.75023	0.75523	0.75425	0.75268
	AVBs	$\hat{\alpha}$	0.03103	0.03349	0.03539	0.03434	0.03274
		$\hat{\sigma}$	0.08017	0.09615	0.11406	0.11946	0.09409
		$\hat{\lambda}$	0.09351	0.10609	0.12824	0.13161	0.10976
	MSEs	$\hat{\alpha}$	0.00154	0.00178	0.00208	0.00186	0.00164
		$\hat{\sigma}$	0.01140	0.01631	0.02421	0.02623	0.01568
		$\hat{\lambda}$	0.01510	0.01771	0.02634	0.02877	0.01966
	MREs	$\hat{\alpha}$	0.12411	0.13394	0.14155	0.13734	0.13097
		$\hat{\sigma}$	0.16035	0.19229	0.22812	0.23892	0.18817
		$\hat{\lambda}$	0.12468	0.14145	0.17099	0.17549	0.14634
200	AVEs	$\hat{\alpha}$	0.24922	0.24930	0.24924	0.24787	0.24912
		$\hat{\sigma}$	0.49940	0.50695	0.51425	0.51243	0.50714
		$\hat{\lambda}$	0.76494	0.75660	0.75537	0.75366	0.75611
	AVBs	$\hat{\alpha}$	0.02273	0.02263	0.02331	0.02330	0.02292
		$\hat{\sigma}$	0.05526	0.06312	0.07842	0.07818	0.06409
		$\hat{\lambda}$	0.06623	0.07581	0.09208	0.09141	0.07343
	MSEs	$\hat{\alpha}$	0.00084	0.00080	0.00086	0.00086	0.00085
		$\hat{\sigma}$	0.00503	0.00640	0.01025	0.01070	0.00694
		$\hat{\lambda}$	0.00712	0.00880	0.01337	0.01310	0.00874
	MREs	$\hat{\alpha}$	0.09093	0.09053	0.09324	0.09319	0.09168
		$\hat{\sigma}$	0.11052	0.12624	0.15684	0.15636	0.12819
		$\hat{\lambda}$	0.08831	0.10108	0.12277	0.12188	0.09791
400	AVEs	$\hat{\alpha}$	0.25050	0.25094	0.25002	0.24913	0.25117
		$\hat{\sigma}$	0.49932	0.50013	0.50515	0.50777	0.50322
		$\hat{\lambda}$	0.75725	0.75681	0.75302	0.74865	0.75225
	AVBs	$\hat{\alpha}$	0.01574	0.01634	0.01710	0.01699	0.01649
		$\hat{\sigma}$	0.03837	0.04410	0.05372	0.05523	0.04496
		$\hat{\lambda}$	0.04548	0.05128	0.06297	0.06339	0.05246
	MSEs	$\hat{\alpha}$	0.00039	0.00041	0.00046	0.00044	0.00042
		$\hat{\sigma}$	0.00233	0.00314	0.00458	0.00508	0.00325
		$\hat{\lambda}$	0.00328	0.00416	0.00639	0.00624	0.00435
	MREs	$\hat{\alpha}$	0.06295	0.06536	0.06841	0.06797	0.06598
		$\hat{\sigma}$	0.07674	0.08819	0.10743	0.11047	0.08991
		$\hat{\lambda}$	0.06065	0.06837	0.08397	0.08452	0.06994

Table A2. Simulation results of the AVEs, AVBs, MSEs and MREs of MKF distribution for ($\alpha = 0.5,\ \sigma = 0.25,\ \lambda = 2.5$).

n	Est.	Est. Par.	MLEs	ADEs	CVMEs	LSEs	WLSEs
20	AVEs	$\hat{\alpha}$	0.48755	0.50660	0.49872	0.49801	0.50525
		$\hat{\sigma}$	0.27226	0.31992	0.40484	0.52070	0.33891
		$\hat{\lambda}$	2.99098	2.68317	2.78788	2.58407	2.62747
	AVBs	$\hat{\alpha}$	0.07912	0.08242	0.09167	0.08992	0.08022
		$\hat{\sigma}$	0.09823	0.13992	0.22869	0.34565	0.16463
		$\hat{\lambda}$	0.89010	0.79395	1.01501	0.92334	0.84445
	MSEs	$\hat{\alpha}$	0.01033	0.01186	0.01398	0.01486	0.01085
		$\hat{\sigma}$	0.01980	0.36559	1.30552	2.95185	0.51636
		$\hat{\lambda}$	1.70458	1.19926	2.00777	1.58503	1.34030
	MREs	$\hat{\alpha}$	0.15825	0.16483	0.18333	0.17984	0.16043
		$\hat{\sigma}$	0.39293	0.55968	0.91478	1.38260	0.65851
		$\hat{\lambda}$	0.35604	0.31758	0.40600	0.36934	0.33778
30	AVEs	$\hat{\alpha}$	0.49346	0.50643	0.50000	0.50531	0.50519
		$\hat{\sigma}$	0.26144	0.27966	0.31185	0.35343	0.28308
		$\hat{\lambda}$	2.77674	2.58479	2.63187	2.54645	2.58147
	AVBs	$\hat{\alpha}$	0.06293	0.06773	0.07070	0.07350	0.06843
		$\hat{\sigma}$	0.07450	0.08533	0.12058	0.16475	0.09182
		$\hat{\lambda}$	0.63079	0.60040	0.72183	0.74506	0.65581
	MSEs	$\hat{\alpha}$	0.00661	0.00756	0.00823	0.00953	0.00748
		$\hat{\sigma}$	0.01012	0.01837	0.08361	0.90573	0.01836
		$\hat{\lambda}$	0.83983	0.61669	0.95918	0.97279	0.74071
	MREs	$\hat{\alpha}$	0.12585	0.13546	0.14140	0.14700	0.13687
		$\hat{\sigma}$	0.29800	0.34130	0.48233	0.65901	0.36728
		$\hat{\lambda}$	0.25231	0.24016	0.28873	0.29802	0.26232
50	AVEs	$\hat{\alpha}$	0.49742	0.50492	0.50368	0.50455	0.50530
		$\hat{\sigma}$	0.26071	0.26067	0.27846	0.27406	0.26557
		$\hat{\lambda}$	2.62098	2.56693	2.58504	2.55773	2.53701
	AVBs	$\hat{\alpha}$	0.05039	0.05180	0.05384	0.05728	0.05441
		$\hat{\sigma}$	0.05347	0.05831	0.07806	0.07581	0.06203
		$\hat{\lambda}$	0.44108	0.44766	0.55668	0.56433	0.48167
	MSEs	$\hat{\alpha}$	0.00399	0.00440	0.00455	0.00522	0.00487
		$\hat{\sigma}$	0.00507	0.00612	0.01696	0.01729	0.00727
		$\hat{\lambda}$	0.33303	0.33693	0.53577	0.55137	0.40669
	MREs	$\hat{\alpha}$	0.10079	0.10361	0.10767	0.11456	0.10883
		$\hat{\sigma}$	0.21389	0.23323	0.31223	0.30323	0.24810
		$\hat{\lambda}$	0.17643	0.17906	0.22267	0.22573	0.19267
100	AVEs	$\hat{\alpha}$	0.49634	0.50033	0.49970	0.50347	0.50301
		$\hat{\sigma}$	0.25258	0.25524	0.25809	0.26109	0.25703
		$\hat{\lambda}$	2.58172	2.53480	2.56649	2.51416	2.51127
	AVBs	$\hat{\alpha}$	0.03337	0.03443	0.03757	0.03933	0.03705
		$\hat{\sigma}$	0.03764	0.03893	0.04795	0.04838	0.04030
		$\hat{\lambda}$	0.30760	0.31272	0.38226	0.37775	0.31577
	MSEs	$\hat{\alpha}$	0.00179	0.00189	0.00223	0.00243	0.00215
		$\hat{\sigma}$	0.00230	0.00258	0.00403	0.00416	0.00276
		$\hat{\lambda}$	0.15848	0.16256	0.23653	0.23104	0.16189
	MREs	$\hat{\alpha}$	0.06675	0.06886	0.07514	0.07865	0.07409
		$\hat{\sigma}$	0.15056	0.15570	0.19180	0.19351	0.16118
		$\hat{\lambda}$	0.12304	0.12509	0.15290	0.15110	0.12631
200	AVEs	$\hat{\alpha}$	0.49828	0.50208	0.50146	0.50191	0.50091
		$\hat{\sigma}$	0.25112	0.25380	0.25465	0.25422	0.25516
		$\hat{\lambda}$	2.53723	2.50007	2.51771	2.50813	2.50596
	AVBs	$\hat{\alpha}$	0.02322	0.02579	0.02784	0.02839	0.02582
		$\hat{\sigma}$	0.02557	0.02756	0.03186	0.03254	0.02927
		$\hat{\lambda}$	0.19686	0.22129	0.25708	0.26715	0.23193
	MSEs	$\hat{\alpha}$	0.00085	0.00108	0.00121	0.00124	0.00104
		$\hat{\sigma}$	0.00102	0.00123	0.00169	0.00176	0.00143
		$\hat{\lambda}$	0.06516	0.07606	0.10316	0.11084	0.08471
	MREs	$\hat{\alpha}$	0.04645	0.05158	0.05567	0.05677	0.05164
		$\hat{\sigma}$	0.10228	0.11026	0.12743	0.13018	0.11706
		$\hat{\lambda}$	0.07874	0.08852	0.10283	0.10686	0.09277
400	AVEs	$\hat{\alpha}$	0.49977	0.50113	0.50120	0.50029	0.50016
		$\hat{\sigma}$	0.25063	0.25134	0.25276	0.25139	0.25116
		$\hat{\lambda}$	2.52019	2.50535	2.50555	2.51024	2.51101
	AVBs	$\hat{\alpha}$	0.01694	0.01826	0.02023	0.01901	0.01921
		$\hat{\sigma}$	0.01864	0.01978	0.02302	0.02129	0.01971
		$\hat{\lambda}$	0.14436	0.16023	0.19138	0.17183	0.15823
	MSEs	$\hat{\alpha}$	0.00128	0.00051	0.00064	0.00058	0.00057
		$\hat{\sigma}$	0.00059	0.00061	0.00084	0.00072	0.00062
		$\hat{\lambda}$	0.03313	0.03975	0.05759	0.04761	0.03945
	MREs	$\hat{\alpha}$	0.03389	0.03652	0.04047	0.03803	0.03843
		$\hat{\sigma}$	0.07457	0.07911	0.09208	0.08514	0.07883
		$\hat{\lambda}$	0.05774	0.06409	0.07655	0.06873	0.06329

Table A3. Simulation results of the AVEs, AVBs, MSEs and MREs of MKF distribution for ($\alpha = 0.75,\ \sigma = 1.50,\ \lambda = 0.25$).

n	Est.	Est. Par.	MLEs	ADEs	CVMEs	LSEs	WLSEs
20	AVEs	$\hat{\alpha}$	1.00593	0.96752	1.04549	0.97531	0.94757
		$\hat{\sigma}$	1.27065	1.41598	1.36488	1.37985	1.41866
		$\hat{\lambda}$	0.38557	0.33769	0.40509	0.36890	0.34655
	AVBs	$\hat{\alpha}$	0.44510	0.45876	0.52610	0.49877	0.45186
		$\hat{\sigma}$	0.55269	0.54289	0.60998	0.61042	0.57774
		$\hat{\lambda}$	0.16451	0.13685	0.20305	0.18194	0.15838
	MSEs	$\hat{\alpha}$	0.44253	0.44075	0.60219	0.51996	0.42784
		$\hat{\sigma}$	0.38535	0.35994	0.45902	0.45273	0.40900
		$\hat{\lambda}$	0.06382	0.04812	0.12164	0.09369	0.07347
	MREs	$\hat{\alpha}$	0.59346	0.61168	0.70147	0.66503	0.60248
		$\hat{\sigma}$	0.36846	0.36193	0.40665	0.40695	0.38516
		$\hat{\lambda}$	0.65804	0.54742	0.81219	0.72774	0.63352
30	AVEs	$\hat{\alpha}$	0.92849	0.87637	0.96379	0.93913	0.90927
		$\hat{\sigma}$	1.34376	1.45943	1.39994	1.42535	1.42896
		$\hat{\lambda}$	0.33679	0.30526	0.35593	0.33944	0.32267
	AVBs	$\hat{\alpha}$	0.36695	0.35801	0.42558	0.45044	0.39032
		$\hat{\sigma}$	0.49176	0.49645	0.55811	0.56575	0.53879
		$\hat{\lambda}$	0.11983	0.10542	0.15176	0.14887	0.12787
	MSEs	$\hat{\alpha}$	0.26550	0.23928	0.37513	0.40975	0.28298
		$\hat{\sigma}$	0.30270	0.30223	0.38155	0.39118	0.35086
		$\hat{\lambda}$	0.03122	0.02529	0.06146	0.05734	0.03977
	MREs	$\hat{\alpha}$	0.48927	0.47735	0.56744	0.60058	0.52043
		$\hat{\sigma}$	0.32784	0.33096	0.37207	0.37717	0.35919
		$\hat{\lambda}$	0.47932	0.42168	0.60706	0.59547	0.51148
50	AVEs	$\hat{\alpha}$	0.86175	0.84628	0.91566	0.85984	0.86045
		$\hat{\sigma}$	1.43114	1.46858	1.42908	1.44881	1.46311
		$\hat{\lambda}$	0.29650	0.28948	0.32194	0.30421	0.29311
	AVBs	$\hat{\alpha}$	0.28803	0.30661	0.37298	0.35190	0.31742
		$\hat{\sigma}$	0.41588	0.44386	0.50856	0.50255	0.46490
		$\hat{\lambda}$	0.08200	0.08651	0.12008	0.11189	0.09241
	MSEs	$\hat{\alpha}$	0.14616	0.16493	0.25326	0.22202	0.18585
		$\hat{\sigma}$	0.21884	0.24436	0.31609	0.30638	0.26619
		$\hat{\lambda}$	0.01251	0.01479	0.03375	0.02974	0.01797
	MREs	$\hat{\alpha}$	0.38404	0.40882	0.49730	0.46920	0.42323
		$\hat{\sigma}$	0.27725	0.29591	0.33904	0.33504	0.30993
		$\hat{\lambda}$	0.32798	0.34603	0.48033	0.44754	0.36964
100	AVEs	$\hat{\alpha}$	0.80417	0.79117	0.82653	0.80361	0.78594
		$\hat{\sigma}$	1.47689	1.49188	1.48656	1.50036	1.51030
		$\hat{\lambda}$	0.27304	0.27140	0.28550	0.27770	0.27127
	AVBs	$\hat{\alpha}$	0.22284	0.23303	0.28520	0.27651	0.25217
		$\hat{\sigma}$	0.34087	0.37559	0.43840	0.43749	0.40404
		$\hat{\lambda}$	0.05916	0.06382	0.08392	0.07871	0.07063
	MSEs	$\hat{\alpha}$	0.07996	0.08368	0.13055	0.12181	0.10020
		$\hat{\sigma}$	0.15441	0.17846	0.23629	0.23018	0.20031
		$\hat{\lambda}$	0.00595	0.00666	0.01349	0.01095	0.00847
	MREs	$\hat{\alpha}$	0.29712	0.31070	0.38027	0.36868	0.33623
		$\hat{\sigma}$	0.22724	0.25039	0.29226	0.29166	0.26936
		$\hat{\lambda}$	0.23665	0.25528	0.33568	0.31485	0.28250
200	AVEs	$\hat{\alpha}$	0.78378	0.76016	0.80589	0.78278	0.76163
		$\hat{\sigma}$	1.48304	1.53270	1.48645	1.51956	1.52528
		$\hat{\lambda}$	0.26436	0.25750	0.27271	0.26545	0.25844
	AVBs	$\hat{\alpha}$	0.17216	0.18207	0.22797	0.23326	0.19469
		$\hat{\sigma}$	0.26674	0.30340	0.36214	0.37773	0.31941
		$\hat{\lambda}$	0.04271	0.04692	0.06281	0.06309	0.05096
	MSEs	$\hat{\alpha}$	0.04454	0.04983	0.07800	0.08147	0.05861
		$\hat{\sigma}$	0.10056	0.12364	0.16738	0.17962	0.13292
		$\hat{\lambda}$	0.00293	0.00333	0.00639	0.00642	0.00403
	MREs	$\hat{\alpha}$	0.22955	0.24276	0.30396	0.31101	0.25959
		$\hat{\sigma}$	0.17783	0.20227	0.24143	0.25182	0.21294
		$\hat{\lambda}$	0.17086	0.18766	0.25124	0.25237	0.20384
400	AVEs	$\hat{\alpha}$	0.76483	0.75732	0.76327	0.75074	0.74062
		$\hat{\sigma}$	1.50900	1.52198	1.53067	1.53141	1.54889
		$\hat{\lambda}$	0.25594	0.25564	0.25881	0.25578	0.25083
	AVBs	$\hat{\alpha}$	0.12371	0.15143	0.19470	0.18336	0.15221
		$\hat{\sigma}$	0.20557	0.24853	0.31287	0.30379	0.25136
		$\hat{\lambda}$	0.03129	0.03820	0.04994	0.04729	0.03827
	MSEs	$\hat{\alpha}$	0.02370	0.03300	0.05226	0.04796	0.03344
		$\hat{\sigma}$	0.06263	0.08817	0.12837	0.12310	0.09008
		$\hat{\lambda}$	0.00155	0.00219	0.00374	0.00324	0.00214
	MREs	$\hat{\alpha}$	0.16495	0.20190	0.25961	0.24447	0.20295
		$\hat{\sigma}$	0.13705	0.16569	0.20858	0.20252	0.16758
		$\hat{\lambda}$	0.12517	0.15279	0.19975	0.18918	0.15306

Table A4. Simulation results of the AVEs, AVBs, MSEs and MREs of MKF distribution for ($\alpha = 2,\ \sigma = 0.75,\ \lambda = 0.5$).

n	Est.	Est. Par.	MLEs	ADEs	CVMEs	LSEs	WLSEs
20	AVEs	$\hat{\alpha}$	2.05680	2.05817	2.07909	2.12735	2.03386
		$\hat{\sigma}$	0.72787	0.79140	0.79019	0.76922	0.79167
		$\hat{\lambda}$	0.62488	0.56383	0.62279	0.59999	0.57124
	AVBs	$\hat{\alpha}$	0.64378	0.62739	0.68779	0.73820	0.63640
		$\hat{\sigma}$	0.26299	0.28598	0.32082	0.32323	0.31186
		$\hat{\lambda}$	0.19502	0.18005	0.24185	0.24068	0.20726
	MSEs	$\hat{\alpha}$	0.74875	0.71862	0.88284	1.01030	0.72051
		$\hat{\sigma}$	0.09166	0.10521	0.12710	0.12931	0.12180
		$\hat{\lambda}$	0.07419	0.06415	0.14535	0.13579	0.09617
	MREs	$\hat{\alpha}$	0.32189	0.31369	0.34389	0.36910	0.31820
		$\hat{\sigma}$	0.35065	0.38130	0.42776	0.43098	0.41581
		$\hat{\lambda}$	0.39005	0.36009	0.48370	0.48135	0.41452
30	AVEs	$\hat{\alpha}$	2.01458	2.08344	2.01726	2.00958	1.98973
		$\hat{\sigma}$	0.76110	0.80077	0.79464	0.79818	0.78199
		$\hat{\lambda}$	0.57303	0.53891	0.58068	0.55794	0.55187
	AVBs	$\hat{\alpha}$	0.50821	0.54367	0.55618	0.58091	0.52199
		$\hat{\sigma}$	0.22809	0.25468	0.29572	0.30010	0.26627
		$\hat{\lambda}$	0.14727	0.15174	0.19446	0.19536	0.17167
	MSEs	$\hat{\alpha}$	0.42299	0.52928	0.52888	0.57044	0.43274
		$\hat{\sigma}$	0.07372	0.08854	0.11152	0.11363	0.09478
		$\hat{\lambda}$	0.04284	0.04056	0.07983	0.08126	0.06021
	MREs	$\hat{\alpha}$	0.25410	0.27183	0.27809	0.29045	0.26100
		$\hat{\sigma}$	0.30412	0.33957	0.39430	0.40013	0.35503
		$\hat{\lambda}$	0.29453	0.30348	0.38891	0.39072	0.34334
50	AVEs	$\hat{\alpha}$	2.02238	2.01934	2.01917	1.98828	1.98916
		$\hat{\sigma}$	0.75782	0.78855	0.78460	0.79825	0.79266
		$\hat{\lambda}$	0.53943	0.51976	0.54623	0.52918	0.51801
	AVBs	$\hat{\alpha}$	0.39328	0.42677	0.44769	0.44475	0.42216
		$\hat{\sigma}$	0.18211	0.20829	0.24153	0.25548	0.22707
		$\hat{\lambda}$	0.10647	0.11405	0.14769	0.14904	0.12532
	MSEs	$\hat{\alpha}$	0.25234	0.28922	0.34119	0.31888	0.29845
		$\hat{\sigma}$	0.04954	0.06274	0.08103	0.08930	0.07187
		$\hat{\lambda}$	0.01988	0.01991	0.04107	0.03924	0.02609
	MREs	$\hat{\alpha}$	0.19664	0.21338	0.22384	0.22237	0.21108
		$\hat{\sigma}$	0.24282	0.27771	0.32204	0.34064	0.30275
		$\hat{\lambda}$	0.21293	0.22811	0.29539	0.29809	0.25064
100	AVEs	$\hat{\alpha}$	1.99366	1.98468	1.96247	1.98641	1.97702
		$\hat{\sigma}$	0.76280	0.77984	0.79948	0.77928	0.77869
		$\hat{\lambda}$	0.51780	0.50640	0.50901	0.51690	0.51046
	AVBs	$\hat{\alpha}$	0.28485	0.29588	0.32529	0.31817	0.30985
		$\hat{\sigma}$	0.13511	0.15768	0.19446	0.19666	0.17114
		$\hat{\lambda}$	0.07497	0.08129	0.10663	0.10897	0.09499
	MSEs	$\hat{\alpha}$	0.12825	0.13666	0.16512	0.15786	0.15015
		$\hat{\sigma}$	0.03005	0.03974	0.05676	0.05747	0.04548
		$\hat{\lambda}$	0.00933	0.01078	0.01761	0.01945	0.01416
	MREs	$\hat{\alpha}$	0.14243	0.14794	0.16265	0.15908	0.15493
		$\hat{\sigma}$	0.18014	0.21024	0.25928	0.26221	0.22818
		$\hat{\lambda}$	0.14993	0.16259	0.21325	0.21793	0.18998
200	AVEs	$\hat{\alpha}$	2.00851	1.98088	1.99122	1.98491	1.97904
		$\hat{\sigma}$	0.75710	0.77554	0.77779	0.77676	0.76633
		$\hat{\lambda}$	0.50812	0.49961	0.50509	0.50205	0.50547
	AVBs	$\hat{\alpha}$	0.20252	0.21170	0.23718	0.23310	0.20535
		$\hat{\sigma}$	0.09181	0.11559	0.14334	0.14287	0.12120
		$\hat{\lambda}$	0.05021	0.06064	0.07646	0.07643	0.06390
	MSEs	$\hat{\alpha}$	0.06400	0.07004	0.08702	0.08699	0.06908
		$\hat{\sigma}$	0.01405	0.02300	0.03442	0.03388	0.02434
		$\hat{\lambda}$	0.00412	0.00580	0.00924	0.00935	0.00655
	MREs	$\hat{\alpha}$	0.10126	0.10585	0.11859	0.11655	0.10267
		$\hat{\sigma}$	0.12241	0.15413	0.19112	0.19049	0.16160
		$\hat{\lambda}$	0.10042	0.12128	0.15292	0.15285	0.12780
400	AVEs	$\hat{\alpha}$	2.00818	2.00051	1.97779	1.98719	1.98355
		$\hat{\sigma}$	0.74795	0.76093	0.77403	0.76498	0.76207
		$\hat{\lambda}$	0.50644	0.50085	0.49831	0.50153	0.50072
	AVBs	$\hat{\alpha}$	0.14300	0.14692	0.17031	0.16744	0.14532
		$\hat{\sigma}$	0.06618	0.07961	0.10713	0.10490	0.08463
		$\hat{\lambda}$	0.03657	0.04206	0.05731	0.05728	0.04487
	MSEs	$\hat{\alpha}$	0.03229	0.03392	0.04656	0.04475	0.03496
		$\hat{\sigma}$	0.00711	0.01044	0.01927	0.01777	0.01191
		$\hat{\lambda}$	0.00218	0.00283	0.00520	0.00511	0.00318
	MREs	$\hat{\alpha}$	0.07150	0.07346	0.08515	0.08372	0.07266
		$\hat{\sigma}$	0.08824	0.10615	0.14284	0.13987	0.11285
		$\hat{\lambda}$	0.07313	0.08411	0.11462	0.11457	0.08973

Table A5. Simulation results of the AVEs, AVBs, MSEs and MREs of MKF distribution for ($\alpha = 1.5,\ \sigma = 2,\ \lambda = 3$).

n	Est.	Est. Par.	MLEs	ADEs	CVMEs	LSEs	WLSEs
20	AVEs	$\hat{\alpha}$	1.49708	1.49155	1.49329	1.48774	1.48550
		$\hat{\sigma}$	2.06722	2.08264	2.13286	2.07500	2.14191
		$\hat{\lambda}$	3.12955	3.00256	3.04832	2.94523	2.88974
	AVBs	$\hat{\alpha}$	0.04290	0.04451	0.04502	0.04718	0.04943
		$\hat{\sigma}$	0.36820	0.40793	0.41151	0.44407	0.42715
		$\hat{\lambda}$	0.47235	0.51026	0.49321	0.56512	0.56576
	MSEs	$\hat{\alpha}$	0.00290	0.00317	0.00325	0.00373	0.00388
		$\hat{\sigma}$	0.16667	0.19906	0.20226	0.22979	0.21163
		$\hat{\lambda}$	0.26033	0.31999	0.30851	0.39710	0.39434
	MREs	$\hat{\alpha}$	0.02860	0.02967	0.03001	0.03145	0.03295
		$\hat{\sigma}$	0.18410	0.20397	0.20576	0.22203	0.21358
		$\hat{\lambda}$	0.15745	0.17009	0.16440	0.18837	0.18859
30	AVEs	$\hat{\alpha}$	1.49928	1.49072	1.49614	1.48778	1.49030
		$\hat{\sigma}$	2.07551	2.12670	2.12839	2.12723	2.11408
		$\hat{\lambda}$	3.10195	2.94544	3.02958	2.90533	2.92803
	AVBs	$\hat{\alpha}$	0.03602	0.03932	0.03738	0.04204	0.04193
		$\hat{\sigma}$	0.35182	0.39843	0.38825	0.42552	0.41490
		$\hat{\lambda}$	0.45956	0.50115	0.49378	0.54518	0.52158
	MSEs	$\hat{\alpha}$	0.00204	0.00248	0.00225	0.00281	0.00269
		$\hat{\sigma}$	0.15414	0.18535	0.18326	0.20771	0.19988
		$\hat{\lambda}$	0.24598	0.30192	0.29629	0.36821	0.32887
	MREs	$\hat{\alpha}$	0.02401	0.02621	0.02492	0.02803	0.02795
		$\hat{\sigma}$	0.17591	0.19921	0.19412	0.21276	0.20745
		$\hat{\lambda}$	0.15319	0.16705	0.16459	0.18173	0.17386
50	AVEs	$\hat{\alpha}$	1.50041	1.49187	1.49203	1.49124	1.48762
		$\hat{\sigma}$	2.02565	2.09450	2.12020	2.10928	2.11673
		$\hat{\lambda}$	3.10147	2.97661	2.99565	2.95308	2.92450
	AVBs	$\hat{\alpha}$	0.03153	0.03414	0.03359	0.03530	0.03533
		$\hat{\sigma}$	0.32871	0.37250	0.37720	0.38537	0.38283
		$\hat{\lambda}$	0.45561	0.49195	0.47769	0.49982	0.49316
	MSEs	$\hat{\alpha}$	0.00153	0.00181	0.00180	0.00196	0.00201
		$\hat{\sigma}$	0.13664	0.16644	0.17125	0.17606	0.17419
		$\hat{\lambda}$	0.23707	0.28526	0.27306	0.29594	0.29169
	MREs	$\hat{\alpha}$	0.02102	0.02276	0.02239	0.02353	0.02356
		$\hat{\sigma}$	0.16436	0.18625	0.18860	0.19268	0.19142
		$\hat{\lambda}$	0.15187	0.16398	0.15923	0.16661	0.16439
100	AVEs	$\hat{\alpha}$	1.49874	1.49299	1.49320	1.49217	1.49367
		$\hat{\sigma}$	2.04080	2.08388	2.08958	2.08875	2.09273
		$\hat{\lambda}$	3.05362	2.97275	2.98585	2.95244	2.96477
	AVBs	$\hat{\alpha}$	0.02803	0.02843	0.02809	0.03081	0.02901
		$\hat{\sigma}$	0.32397	0.35359	0.36964	0.37444	0.35533
		$\hat{\lambda}$	0.43840	0.46329	0.47728	0.49737	0.46313
	MSEs	$\hat{\alpha}$	0.00115	0.00121	0.00119	0.00141	0.00126
		$\hat{\sigma}$	0.13175	0.15051	0.16096	0.16506	0.15101
		$\hat{\lambda}$	0.21996	0.24786	0.25891	0.28108	0.24767
	MREs	$\hat{\alpha}$	0.01868	0.01895	0.01873	0.02054	0.01934
		$\hat{\sigma}$	0.16198	0.17680	0.18482	0.18722	0.17767
		$\hat{\lambda}$	0.14613	0.15443	0.15909	0.16579	0.15438
200	AVEs	$\hat{\alpha}$	1.49968	1.49379	1.49271	1.49407	1.49427
		$\hat{\sigma}$	2.02705	2.07669	2.09043	2.06946	2.07336
		$\hat{\lambda}$	3.04767	2.96494	2.97241	2.98740	2.99099
	AVBs	$\hat{\alpha}$	0.02557	0.02661	0.02702	0.02728	0.02662
		$\hat{\sigma}$	0.29787	0.32865	0.34930	0.35249	0.33164
		$\hat{\lambda}$	0.40785	0.43552	0.46284	0.46697	0.44427
	MSEs	$\hat{\alpha}$	0.00089	0.00100	0.00102	0.00104	0.00099
		$\hat{\sigma}$	0.11244	0.13291	0.14609	0.14703	0.13435
		$\hat{\lambda}$	0.19569	0.22148	0.24010	0.24545	0.22384
	MREs	$\hat{\alpha}$	0.01705	0.01774	0.01801	0.01819	0.01775
		$\hat{\sigma}$	0.14893	0.16432	0.17465	0.17624	0.16582
		$\hat{\lambda}$	0.13595	0.14517	0.15428	0.15566	0.14809
400	AVEs	$\hat{\alpha}$	1.49666	1.49538	1.49550	1.49322	1.49404
		$\hat{\sigma}$	2.04446	2.05495	2.05819	2.08693	2.07480
		$\hat{\lambda}$	3.00505	2.98990	3.00554	2.95791	2.96924
	AVBs	$\hat{\alpha}$	0.02155	0.02414	0.02549	0.02642	0.02436
		$\hat{\sigma}$	0.27037	0.30129	0.32410	0.33315	0.30622
		$\hat{\lambda}$	0.36003	0.40477	0.43399	0.44313	0.40656
	MSEs	$\hat{\alpha}$	0.00066	0.00079	0.00085	0.00093	0.00081
		$\hat{\sigma}$	0.09788	0.11497	0.12741	0.13526	0.11779
		$\hat{\lambda}$	0.16304	0.19685	0.21618	0.22612	0.19807
	MREs	$\hat{\alpha}$	0.01437	0.01610	0.01700	0.01761	0.01624
		$\hat{\sigma}$	0.13518	0.15064	0.16205	0.16657	0.15311
		$\hat{\lambda}$	0.12001	0.13492	0.14466	0.14771	0.13552

Table A6. Simulation results of the AVEs, AVBs, MSEs and MREs of MKF distribution for ($\alpha = 0.5,\ \sigma = 3,\ \lambda = 1.5$).

n	Est.	Est. Par.	MLEs	ADEs	CVMEs	LSEs	WLSEs
20	AVEs	$\hat{\alpha}$	0.51110	0.50452	0.50687	0.49975	0.50129
		$\hat{\sigma}$	2.84484	2.90324	2.95703	2.94414	2.93359
		$\hat{\lambda}$	1.73804	1.63995	1.67207	1.58381	1.59424
	AVBs	$\hat{\alpha}$	0.02808	0.02839	0.02815	0.02996	0.03061
		$\hat{\sigma}$	0.57755	0.60083	0.55854	0.58481	0.61985
		$\hat{\lambda}$	0.36864	0.36457	0.35749	0.36313	0.36476
	MSEs	$\hat{\alpha}$	0.00113	0.00119	0.00118	0.00135	0.00137
		$\hat{\sigma}$	0.41407	0.44199	0.41385	0.44965	0.47240
		$\hat{\lambda}$	0.16430	0.16186	0.15681	0.16242	0.16199
	MREs	$\hat{\alpha}$	0.05617	0.05677	0.05630	0.05993	0.06123
		$\hat{\sigma}$	0.19252	0.20028	0.18618	0.19494	0.20662
		$\hat{\lambda}$	0.24576	0.24305	0.23833	0.24209	0.24317
30	AVEs	$\hat{\alpha}$	0.50992	0.50372	0.50603	0.50097	0.50336
		$\hat{\sigma}$	2.82286	2.90442	2.94179	2.88214	2.91018
		$\hat{\lambda}$	1.71322	1.62649	1.65937	1.60009	1.62279
	AVBs	$\hat{\alpha}$	0.02686	0.02625	0.02568	0.02755	0.02573
		$\hat{\sigma}$	0.57998	0.59003	0.55519	0.60695	0.61163
		$\hat{\lambda}$	0.36016	0.33802	0.34553	0.36264	0.34393
	MSEs	$\hat{\alpha}$	0.00099	0.00098	0.00094	0.00109	0.00097
		$\hat{\sigma}$	0.40367	0.41892	0.38642	0.46733	0.44634
		$\hat{\lambda}$	0.15831	0.14403	0.14974	0.15861	0.14788
	MREs	$\hat{\alpha}$	0.05373	0.05251	0.05135	0.05511	0.05146
		$\hat{\sigma}$	0.19333	0.19668	0.18506	0.20232	0.20388
		$\hat{\lambda}$	0.24011	0.22535	0.23035	0.24176	0.22928
50	AVEs	$\hat{\alpha}$	0.50843	0.50349	0.50672	0.50469	0.50271
		$\hat{\sigma}$	2.82857	2.91508	2.90799	2.87814	2.92957
		$\hat{\lambda}$	1.68650	1.61484	1.65545	1.62727	1.60479
	AVBs	$\hat{\alpha}$	0.02391	0.02406	0.02376	0.02511	0.02480
		$\hat{\sigma}$	0.57968	0.58001	0.57571	0.59350	0.58404
		$\hat{\lambda}$	0.33133	0.32534	0.33716	0.34216	0.32989
	MSEs	$\hat{\alpha}$	0.00078	0.00079	0.00077	0.00086	0.00085
		$\hat{\sigma}$	0.39265	0.39416	0.39877	0.42361	0.39512
		$\hat{\lambda}$	0.14093	0.13455	0.14342	0.14505	0.13699
	MREs	$\hat{\alpha}$	0.04782	0.04812	0.04752	0.05021	0.04959
		$\hat{\sigma}$	0.19323	0.19334	0.19190	0.19783	0.19468
		$\hat{\lambda}$	0.22089	0.21689	0.22477	0.22810	0.21993
100	AVEs	$\hat{\alpha}$	0.50706	0.50493	0.50465	0.50427	0.50477
		$\hat{\sigma}$	2.83864	2.89847	2.92031	2.88817	2.90402
		$\hat{\lambda}$	1.65954	1.62224	1.62376	1.61747	1.61925
	AVBs	$\hat{\alpha}$	0.02205	0.02154	0.02202	0.02181	0.02216
		$\hat{\sigma}$	0.55194	0.56351	0.57979	0.58647	0.57504
		$\hat{\lambda}$	0.31017	0.30589	0.31775	0.31640	0.31243
	MSEs	$\hat{\alpha}$	0.00062	0.00061	0.00064	0.00063	0.00064
		$\hat{\sigma}$	0.35463	0.36757	0.38127	0.39771	0.37468
		$\hat{\lambda}$	0.12553	0.12227	0.13051	0.12969	0.12579
	MREs	$\hat{\alpha}$	0.04410	0.04307	0.04404	0.04362	0.04432
		$\hat{\sigma}$	0.18398	0.18784	0.19326	0.19549	0.19168
		$\hat{\lambda}$	0.20678	0.20393	0.21183	0.21093	0.20829
200	AVEs	$\hat{\alpha}$	0.50558	0.50350	0.50568	0.50259	0.50325
		$\hat{\sigma}$	2.86600	2.91993	2.87709	2.96257	2.91744
		$\hat{\lambda}$	1.62979	1.59738	1.63633	1.58400	1.59933
	AVBs	$\hat{\alpha}$	0.02023	0.02009	0.02122	0.02089	0.02066
		$\hat{\sigma}$	0.52468	0.52371	0.57396	0.54908	0.54290
		$\hat{\lambda}$	0.28682	0.28285	0.31003	0.29407	0.29008
	MSEs	$\hat{\alpha}$	0.00052	0.00052	0.00057	0.00056	0.00054
		$\hat{\sigma}$	0.32747	0.32321	0.37146	0.34380	0.33617
		$\hat{\lambda}$	0.11042	0.10603	0.12427	0.11285	0.11013
	MREs	$\hat{\alpha}$	0.04045	0.04019	0.04244	0.04177	0.04132
		$\hat{\sigma}$	0.17489	0.17457	0.19132	0.18303	0.18097
		$\hat{\lambda}$	0.19121	0.18857	0.20668	0.19604	0.19339
400	AVEs	$\hat{\alpha}$	0.50453	0.50292	0.50326	0.50335	0.50311
		$\hat{\sigma}$	2.89575	2.93326	2.92487	2.91962	2.93486
		$\hat{\lambda}$	1.60481	1.58644	1.59648	1.59592	1.58537
	AVBs	$\hat{\alpha}$	0.01852	0.01907	0.02032	0.02008	0.01942
		$\hat{\sigma}$	0.48248	0.50266	0.53756	0.53456	0.51456
		$\hat{\lambda}$	0.25855	0.26614	0.28713	0.28785	0.27380
	MSEs	$\hat{\alpha}$	0.00044	0.00045	0.00050	0.00049	0.00047
		$\hat{\sigma}$	0.28279	0.29824	0.33219	0.33208	0.30824
		$\hat{\lambda}$	0.09143	0.09513	0.10773	0.10769	0.09876
	MREs	$\hat{\alpha}$	0.03704	0.03814	0.04065	0.04016	0.03885
		$\hat{\sigma}$	0.16083	0.16755	0.17919	0.17819	0.17152
		$\hat{\lambda}$	0.17237	0.17742	0.19142	0.19190	0.18254

Table A7. Simulation results of the AVEs, AVBs, MSEs and MREs of MKF distribution for ($\alpha = 2.5,\ \sigma = 0.75,\ \lambda = 2$).

n	Est.	Est. Par.	MLEs	ADEs	CVMEs	LSEs	WLSEs
20	AVEs	$\hat{\alpha}$	2.50035	2.49012	2.49907	2.49517	2.49348
		$\hat{\sigma}$	0.76208	0.79057	0.81367	0.78928	0.79952
		$\hat{\lambda}$	2.15379	2.02499	2.06550	1.98444	1.97428
	AVBs	$\hat{\alpha}$	0.18799	0.20556	0.19647	0.20547	0.19257
		$\hat{\sigma}$	0.16149	0.17687	0.19102	0.19561	0.19259
		$\hat{\lambda}$	0.36272	0.37729	0.39415	0.41352	0.40943
	MSEs	$\hat{\alpha}$	0.05396	0.06323	0.06029	0.06406	0.05770
		$\hat{\sigma}$	0.03339	0.03871	0.04367	0.04598	0.04399
		$\hat{\lambda}$	0.16068	0.17545	0.18967	0.20971	0.20435
	MREs	$\hat{\alpha}$	0.07520	0.08222	0.07859	0.08219	0.07703
		$\hat{\sigma}$	0.21531	0.23583	0.25469	0.26082	0.25678
		$\hat{\lambda}$	0.18136	0.18865	0.19708	0.20676	0.20471
30	AVEs	$\hat{\alpha}$	2.51173	2.50088	2.50737	2.48902	2.48947
		$\hat{\sigma}$	0.75539	0.78378	0.78796	0.77583	0.79215
		$\hat{\lambda}$	2.13756	2.01953	2.06134	1.99950	1.98464
	AVBs	$\hat{\alpha}$	0.16375	0.16914	0.16991	0.16641	0.16814
		$\hat{\sigma}$	0.14749	0.16559	0.17398	0.18243	0.17648
		$\hat{\lambda}$	0.34352	0.35095	0.37537	0.39446	0.37451
	MSEs	$\hat{\alpha}$	0.04001	0.04449	0.04474	0.04311	0.04291
		$\hat{\sigma}$	0.02847	0.03466	0.03733	0.04094	0.03866
		$\hat{\lambda}$	0.14811	0.15471	0.17418	0.19211	0.17334
	MREs	$\hat{\alpha}$	0.06550	0.06766	0.06796	0.06656	0.06726
		$\hat{\sigma}$	0.19666	0.22078	0.23198	0.24324	0.23531
		$\hat{\lambda}$	0.17176	0.17548	0.18769	0.19723	0.18725
50	AVEs	$\hat{\alpha}$	2.48999	2.49856	2.49670	2.49306	2.48788
		$\hat{\sigma}$	0.75967	0.77867	0.78894	0.77548	0.77986
		$\hat{\lambda}$	2.09046	2.01668	2.02640	2.01653	1.99815
	AVBs	$\hat{\alpha}$	0.12332	0.12628	0.13285	0.13486	0.13220
		$\hat{\sigma}$	0.13927	0.15354	0.16315	0.17236	0.16192
		$\hat{\lambda}$	0.30943	0.32992	0.34693	0.37156	0.34497
	MSEs	$\hat{\alpha}$	0.02365	0.02529	0.02769	0.02804	0.02803
		$\hat{\sigma}$	0.02593	0.03044	0.03374	0.03656	0.03341
		$\hat{\lambda}$	0.12484	0.13732	0.14987	0.16787	0.15129
	MREs	$\hat{\alpha}$	0.04933	0.05051	0.05314	0.05394	0.05288
		$\hat{\sigma}$	0.18569	0.20473	0.21754	0.22981	0.21589
		$\hat{\lambda}$	0.15471	0.16496	0.17347	0.18578	0.17248
100	AVEs	$\hat{\alpha}$	2.49510	2.49129	2.49067	2.49252	2.48670
		$\hat{\sigma}$	0.75716	0.77784	0.77929	0.76744	0.77038
		$\hat{\lambda}$	2.05810	2.00571	2.01693	2.01468	2.01257
	AVBs	$\hat{\alpha}$	0.08699	0.08770	0.09612	0.09807	0.08737
		$\hat{\sigma}$	0.11904	0.13060	0.14324	0.14673	0.13419
		$\hat{\lambda}$	0.26097	0.27667	0.30096	0.31871	0.28231
	MSEs	$\hat{\alpha}$	0.01167	0.01232	0.01472	0.01504	0.01227
		$\hat{\sigma}$	0.01996	0.02358	0.02714	0.02826	0.02436
		$\hat{\lambda}$	0.09327	0.10318	0.11907	0.13118	0.10582
	MREs	$\hat{\alpha}$	0.03480	0.03508	0.03845	0.03923	0.03495
		$\hat{\sigma}$	0.15873	0.17413	0.19098	0.19563	0.17892
		$\hat{\lambda}$	0.13048	0.13834	0.15048	0.15936	0.14115
200	AVEs	$\hat{\alpha}$	2.49695	2.49489	2.49065	2.48651	2.49251
		$\hat{\sigma}$	0.75697	0.76049	0.77669	0.77600	0.76897
		$\hat{\lambda}$	2.02875	2.00969	2.00140	1.99045	1.99801
	AVBs	$\hat{\alpha}$	0.06195	0.06581	0.07055	0.06974	0.06502
		$\hat{\sigma}$	0.09255	0.10053	0.12558	0.12544	0.10684
		$\hat{\lambda}$	0.19896	0.22185	0.27234	0.26371	0.23129
	MSEs	$\hat{\alpha}$	0.00608	0.00670	0.00772	0.00783	0.00662
		$\hat{\sigma}$	0.01307	0.01523	0.02182	0.02206	0.01717
		$\hat{\lambda}$	0.05992	0.07160	0.10034	0.09516	0.07790
	MREs	$\hat{\alpha}$	0.02478	0.02633	0.02822	0.02790	0.02601
		$\hat{\sigma}$	0.12340	0.13404	0.16744	0.16725	0.14245
		$\hat{\lambda}$	0.09948	0.11092	0.13617	0.13186	0.11564
400	AVEs	$\hat{\alpha}$	2.49675	2.49679	2.49690	2.49612	2.49841
		$\hat{\sigma}$	0.75370	0.75972	0.76114	0.76201	0.75913
		$\hat{\lambda}$	2.01851	2.00531	2.01740	2.00823	2.00237
	AVBs	$\hat{\alpha}$	0.04191	0.04763	0.05313	0.05026	0.04564
		$\hat{\sigma}$	0.06699	0.07943	0.10179	0.10013	0.08019
		$\hat{\lambda}$	0.14719	0.17179	0.21919	0.21899	0.17212
	MSEs	$\hat{\alpha}$	0.00279	0.00362	0.00445	0.00391	0.00331
		$\hat{\sigma}$	0.00717	0.00981	0.01540	0.01467	0.01003
		$\hat{\lambda}$	0.03378	0.04426	0.06964	0.06748	0.04496
	MREs	$\hat{\alpha}$	0.01677	0.01905	0.02125	0.02010	0.01825
		$\hat{\sigma}$	0.08932	0.10591	0.13572	0.13351	0.10692
		$\hat{\lambda}$	0.07360	0.08589	0.10959	0.10950	0.08606

Table A8. Simulation results of the AVEs, AVBs, MSEs and MREs of MKF distribution for ($\alpha = 3,\ \sigma = 1.5,\ \lambda = 0.5$).

n	Est.	Est. Par.	MLEs	ADEs	CVMEs	LSEs	WLSEs
20	AVEs	$\hat{\alpha}$	3.04042	2.94437	2.98177	2.90235	2.90349
		$\hat{\sigma}$	1.47265	1.59793	1.54169	1.58595	1.59216
		$\hat{\lambda}$	0.60927	0.52788	0.57235	0.51581	0.52278
	AVBs	$\hat{\alpha}$	0.45655	0.50003	0.50761	0.54569	0.53086
		$\hat{\sigma}$	0.45714	0.44061	0.45792	0.45985	0.46879
		$\hat{\lambda}$	0.18174	0.14823	0.17340	0.15936	0.15989
	MSEs	$\hat{\alpha}$	0.28432	0.34271	0.35945	0.41976	0.39195
		$\hat{\sigma}$	0.25908	0.23236	0.25177	0.25344	0.25667
		$\hat{\lambda}$	0.06124	0.03862	0.05295	0.04251	0.04256
	MREs	$\hat{\alpha}$	0.15218	0.16668	0.16920	0.18190	0.17695
		$\hat{\sigma}$	0.30476	0.29374	0.30528	0.30657	0.31252
		$\hat{\lambda}$	0.36348	0.29645	0.34680	0.31872	0.31978
30	AVEs	$\hat{\alpha}$	3.03775	2.91184	2.95362	2.90742	2.88305
		$\hat{\sigma}$	1.44721	1.56769	1.57072	1.56068	1.59682
		$\hat{\lambda}$	0.59687	0.52505	0.54609	0.52447	0.51689
	AVBs	$\hat{\alpha}$	0.44812	0.47940	0.46702	0.50806	0.49950
		$\hat{\sigma}$	0.42821	0.40809	0.43400	0.44159	0.42650
		$\hat{\lambda}$	0.16665	0.13270	0.14959	0.15051	0.13678
	MSEs	$\hat{\alpha}$	0.26387	0.31547	0.29874	0.35446	0.34126
		$\hat{\sigma}$	0.23361	0.20405	0.22389	0.23458	0.21779
		$\hat{\lambda}$	0.05187	0.03037	0.03979	0.03835	0.03227
	MREs	$\hat{\alpha}$	0.14937	0.15980	0.15567	0.16935	0.16650
		$\hat{\sigma}$	0.28547	0.27206	0.28933	0.29439	0.28433
		$\hat{\lambda}$	0.33330	0.26539	0.29918	0.30103	0.27357
50	AVEs	$\hat{\alpha}$	3.01660	2.96517	2.99173	2.94328	2.92531
		$\hat{\sigma}$	1.51045	1.52920	1.54201	1.56954	1.57164
		$\hat{\lambda}$	0.54923	0.53261	0.53856	0.52426	0.51577
	AVBs	$\hat{\alpha}$	0.39766	0.43053	0.44884	0.46878	0.44781
		$\hat{\sigma}$	0.37321	0.38389	0.39339	0.41680	0.39331
		$\hat{\lambda}$	0.12506	0.12700	0.13233	0.13824	0.12369
	MSEs	$\hat{\alpha}$	0.21431	0.24870	0.26089	0.28551	0.27032
		$\hat{\sigma}$	0.18117	0.18648	0.19435	0.21108	0.19194
		$\hat{\lambda}$	0.02858	0.02719	0.03055	0.03148	0.02531
	MREs	$\hat{\alpha}$	0.13255	0.14351	0.14961	0.15626	0.14927
		$\hat{\sigma}$	0.24881	0.25593	0.26226	0.27786	0.26221
		$\hat{\lambda}$	0.25012	0.25400	0.26466	0.27647	0.24738
100	AVEs	$\hat{\alpha}$	3.00918	2.95263	2.98508	2.95277	2.93609
		$\hat{\sigma}$	1.49344	1.55455	1.53705	1.55689	1.56763
		$\hat{\lambda}$	0.53391	0.51251	0.52510	0.51319	0.50606
	AVBs	$\hat{\alpha}$	0.35222	0.37840	0.42068	0.42216	0.40132
		$\hat{\sigma}$	0.30728	0.33048	0.36555	0.37874	0.34467
		$\hat{\lambda}$	0.09945	0.10074	0.11563	0.11797	0.10580
	MSEs	$\hat{\alpha}$	0.16743	0.19126	0.22722	0.23115	0.21143
		$\hat{\sigma}$	0.12939	0.14349	0.16601	0.17493	0.15372
		$\hat{\lambda}$	0.01676	0.01596	0.02004	0.02052	0.01705
	MREs	$\hat{\alpha}$	0.11741	0.12613	0.14023	0.14072	0.13377
		$\hat{\sigma}$	0.20486	0.22032	0.24370	0.25249	0.22978
		$\hat{\lambda}$	0.19891	0.20148	0.23126	0.23594	0.21161
200	AVEs	$\hat{\alpha}$	3.00672	2.95258	2.96766	2.94901	2.95522
		$\hat{\sigma}$	1.50774	1.55155	1.55024	1.56299	1.55488
		$\hat{\lambda}$	0.51851	0.50514	0.51431	0.50495	0.50286
	AVBs	$\hat{\alpha}$	0.30916	0.34895	0.38014	0.37741	0.34965
		$\hat{\sigma}$	0.26362	0.29012	0.33793	0.33365	0.28840
		$\hat{\lambda}$	0.08156	0.08607	0.10391	0.10010	0.08591
	MSEs	$\hat{\alpha}$	0.13128	0.16462	0.18515	0.18612	0.16336
		$\hat{\sigma}$	0.09578	0.11414	0.14499	0.14134	0.11275
		$\hat{\lambda}$	0.00990	0.01055	0.01515	0.01381	0.01047
	MREs	$\hat{\alpha}$	0.10305	0.11632	0.12671	0.12580	0.11655
		$\hat{\sigma}$	0.17575	0.19342	0.22529	0.22243	0.19227
		$\hat{\lambda}$	0.16312	0.17214	0.20782	0.20019	0.17183
400	AVEs	$\hat{\alpha}$	2.98851	2.96122	2.96180	2.95013	2.95449
		$\hat{\sigma}$	1.51918	1.54874	1.55352	1.55327	1.54420
		$\hat{\lambda}$	0.50705	0.50092	0.50461	0.50335	0.50194
	AVBs	$\hat{\alpha}$	0.24610	0.29338	0.34860	0.34428	0.29022
		$\hat{\sigma}$	0.20633	0.24497	0.28814	0.28934	0.24695
		$\hat{\lambda}$	0.06107	0.07275	0.08686	0.08558	0.07201
	MSEs	$\hat{\alpha}$	0.09170	0.12137	0.16050	0.15807	0.11979
		$\hat{\sigma}$	0.06438	0.08530	0.11228	0.11108	0.08756
		$\hat{\lambda}$	0.00574	0.00749	0.01029	0.01011	0.00741
	MREs	$\hat{\alpha}$	0.08203	0.09779	0.11620	0.11476	0.09674
		$\hat{\sigma}$	0.13755	0.16331	0.19210	0.19289	0.16463
		$\hat{\lambda}$	0.12213	0.14551	0.17372	0.17116	0.14402

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