Research article

Putnam-Fuglede type theorem for class $\mathcal{A}_k$ operators

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Abstract: We will call $U \in B(X)$ as an operator of class $\mathcal{A}_k$ if for some integer $k$, the following inequality is satisfied:

$$|U^{k+1}|^{\frac{1}{2k+1}} \geq |U|^2.$$  

In the present article, some basic spectral properties of this class are given, also the asymmetric Putnam-Fuglede theorem and the range kernel orthogonality for class $\mathcal{A}_k$ operators are proved.

Keywords: Putnam-Fuglede theorem; hyponormal operator; class $\mathcal{A}_k$ operator

Mathematics Subject Classification: 47B47, 47A30, 47B20

1. Introduction

Spectral theory has a key important role in the modern functional analysis and its applications in various fields [4, 15]. Basically, it is incorporated with specific inverse operators, their common properties and their dealings with the original operators. Such inverse operators play a major role in solving systems of linear algebraic equations, differential and Sylvester equations.

Everywhere in this paper, a complex Hilbert space of infinite dimension with the inner product $\langle \cdot, \cdot \rangle$ will be denoted by $X$ and $B(X)$ indicates the algebra of all linear bounded operators which act on $X$. Spectrum, approximate spectrum, residual spectrum, and point spectrum of an operator $U$ will be denoted by $\sigma(U)$, $\sigma_a(U)$, $\sigma_r(U)$, and $\sigma_p(U)$, respectively. The kernel of an operator $U$ will be denoted by $\ker(U)$ and the range by $\text{ran}(U)$.

For each operator $U \in B(X)$, we set, as usual $|U| = (U^*U)^{1/2}$, and review the following standard (familiar) definitions:

$U$ is normal if $U^*U = UU^*$, and
$U$ is hyponormal if $|U^*|^2 \leq |U|^2$,

(i.e. equivalently, if $\|U^* x\| \leq \|U x\|$ for every $x \in X$).

An operator $U \in B(X)$ is said to be of class $\mathcal{A}$ if and only if $|U|^2 \geq |U|^2$.

The class of hyponormal operator has been studied by many authors. In recent years this class has been generalized, in some sense, to the larger sets of so called class $p-$hyponormal, log $-$hyponormal [21], $w-$hyponormal [2] and class $\mathcal{A}$ operators [19].

**Definition 1.** An operator $U \in B(X)$ is said to be class $\mathcal{A}_k$ operator if

$$|U^{k+1}|^{\frac{1}{k+1}} \geq |U|^2,$$

holds for some integer $k$.

The class $\mathcal{A}$ coincides with class $\mathcal{A}_1$ when $k = 1$.

**Example 2.** If $U \in B(X)$ is a bilateral shift operator with weights $\{\alpha_n\}$, $\alpha_n \neq 0$, then $U$ is class $\mathcal{A}_k$ if and only if

$$|\alpha_{n+1}| \cdots |\alpha_{n+k}| \geq |\alpha_n|^k.$$

Our first goal is to prove that the class $\mathcal{A}$ shares many properties with that of hyponormal operators. The following inclusions give the relationships between these operators

hyponormal $\subset$ $p$-hyponormal $\subset$ log $-$hyponormal $\subset$ $w$-hyponormal $\subset$ class $\mathcal{A}$ $\subset$ class $\mathcal{A}_k$.

The generalized derivation $\delta_{U,T} : B(X) \rightarrow B(X)$ for $U, T \in B(X)$ is defined by $\delta_{U,T}(H) = UH - HT$ for $H \in B(X)$, and we note $\delta_{U,U} = \delta_U$. If the following inequality

$$\|T - (UH - HU)\| \geq \|T\|,$$

holds for all $T \in \ker\delta_U$ and for all $H \in B(X)$, then we remark that the range of $\delta_U$ is orthogonal to the kernel of $\delta_U$.

The familiar Putnam-Fuglede’s theorem affirms that if both $U \in B(X)$ and $T \in B(X)$ are normal operators and $UH = HT$ for some $H \in B(X)$, then $U^*H = HT^*$ (see [17]). This theorem attracted attention of many researchers and they extended it for several nonnormal classes of operators (see [2–4, 10, 12–15, 18, 19, 21–23]).

In this article, our second goal is extend this theorem to class $\mathcal{A}_k$ operators and prove the range kernel orthogonality for class $\mathcal{A}_k$ operators.

Let $U \in B(X)$ and let $\{e_n\}$ be an orthonormal basis of a Hilbert space $X$. The Hilbert-Schmidt norm is given by

$$\|U\|_2 = \left(\sum_{n=1}^{\infty} \|Ue_n\|^2\right)^{\frac{1}{2}}.$$

An operator $U$ is called to be a Hilbert-Schmidt operator if $\|U\|_2 < \infty$ (see [8] for details). $C_2(X)$ denotes a set of all Hilbert-Schmidt operators. For $T, U \in B(X)$, the operator $\Gamma_{T,U}$ defined as $\Gamma_{T,U} : C_2(X) \ni H \mapsto THU \in C_2(X)$ has been studied in [6]. It is known that $\|\Gamma\| \leq \|T\|\|U\|$ and $(\Gamma_{T,U})^*H = T^*HU^* = \Gamma_{T^*,U^*}H$. If $U \geq 0$ and $T \geq 0$, then $\Gamma_{U,T} \geq 0$. For more information see [6].

We organise our paper as follows: Section 2 deals with some properties for class $A_k$ operators which will be needed to prove our main results. We present our main theorems, like the asymmetric Putnam-Fuglede’s theorem for some $A_k$ class operators and also some orthogonality results in section 3.

2. Materials and method

Properties of class $A_k$ operators

**Theorem 3.** [11] If $U \in B(X)$ is a p-hyponormal or a log-hyponormal operator, then $U$ is class $A_k$ operator, for each positive integer $k$.

**Corollary 4.** Every hyponormal operator is a class $A_k$ operator.

**Theorem 5.** [11] If $U \in B(X)$ is an invertible class $A_k$, then $U$ is class $A_k$ operator for every $k$.

A number $\lambda \in \mathbb{C}$ is said to be in the joint spectrum of operator $U$ if there exist a joint eigenvector $v$ corresponding to $U$ and $U^*v$ such that $Uv = \lambda v$ and $U^*v = \bar{\lambda}v$, where $\bar{\lambda}$ is the complex conjugate of $\lambda$. We will denote the joint point spectrum and the point spectrum of operator $U$ by $\sigma_{jp}(U)$ and $\sigma_p(U)$, respectively.

**Theorem 6.** Let $U \in B(X)$ be a class $A_k$ operator. Then the following hold

(i) If $Uv = \lambda v$, $\lambda \neq 0$, then $U^*v = \bar{\lambda}v$,

(ii) $\sigma_{jp}(U) - \{0\} = \sigma_p(U) - \{0\}$,

(iii) Let $Uv = \lambda v$ and $Uw = \mu w$ with $\lambda \neq \mu$. Then $v \perp w$.

**Proof.**

(i) We have that the following

$$|\lambda|^2\|v\|^2 = \|Uv\|^2$$

$$= \langle |U|^2v, v \rangle$$

$$\leq \langle |U^{k+1}v\|^{\frac{2}{k+1}}v, v \rangle$$

$$\leq \langle |U^{k+1}v, v\|^{\frac{2}{k+1}}\|v\|^{\frac{2}{k+1}}$$

$$\leq \|U^{k+1}v\|^{\frac{2}{k+1}}\|v\|^{\frac{2}{k+1}}$$

$$= \left(|\lambda|^2\|v\|^2\right)^{\frac{1}{k+1}}\|v\|^{\frac{2}{k+1}}$$

$$= |\lambda|^2\|v\|^2$$

follow from using Holder-McCarthy and Schwarz’s inequalities.

Hence

$$|\lambda|^2\langle v, v \rangle = \langle U^*Uv, v \rangle = \langle |U^{k+1}v|^{\frac{2}{k+1}}v, v \rangle.$$ 

Since $|U^{k+1}v|^{\frac{2}{k+1}}v$ and $v$ are linearly independent [16], we get

$$|U^{k+1}v|^{\frac{2}{k+1}}v = |\lambda|^2v.$$
Also, \[ ||(|U^{k+1}|^{2^*} - U^*U)^{1/2}v||^2 = \langle(|U^{k+1}|^{2^*} - U^*U)v, v \rangle = 0. \]

Therefore \[ U^*Uv = |U^{k+1}|^{2^*}v = |\lambda|^2v, \]
and so \[ (U - \lambda)^*v = 0. \]

(ii) We can easily see that (ii) follows from the definition of the joint point spectrum and (i).

(iii) Let \( Uv = \lambda v \) and \( Uw = \mu w \), then

\[
\langle Uv, w \rangle = \langle \lambda v, w \rangle = \lambda \langle v, w \rangle = \langle v, U^*w \rangle = \langle v, \bar{\mu}w \rangle = \mu \langle v, w \rangle.
\]

Since \( \lambda \neq \mu \), then \( \langle v, w \rangle = 0 \), i.e., \( v \perp w. \)

\[ \square \]

**Definition 7.** We say that \( U \in B(\mathcal{H}) \) is finite if the distance \( \text{dist}(I, \text{ran}(\delta_U)) \geq 1 \) from the identity to the range of \( \delta_U \).

**Definition 8.** If \( U \in B(\mathcal{H}) \), we denote by \( \sigma_{ra}(U) \) the reducissant approximate spectrum, the set of scalars \( \lambda \) for which there is a normalized sequence \( \{x_n\} \subset \mathcal{H} \) verifying

\[
(U - \lambda)x_n \to 0 \quad \text{and} \quad (U - \lambda)^*x_n \to 0
\]

**Proposition 9.** [1] Let \( U \in B(\mathcal{H}) \), if \( \sigma_{ra} \) is not empty, then \( U \) is finite.

**Proposition 10.** (Berberian Technique) [5]
Let \( \mathcal{H} \) be a complex Hilbert space, then there is a Hilbert space \( \mathcal{K} \supset \mathcal{H} \) and \( \varphi : B(\mathcal{H}) \to B(\mathcal{K}) \) \((U \mapsto \tilde{U})\) satisfying: \( \varphi \) is an *-isomorphism preserving the order such that:

(i) \( \varphi(U^*) = \varphi(U)^* \), \( \varphi(I) = I \);
(ii) \( \varphi(\alpha U + \beta V) = \alpha \varphi(U) + \beta \varphi(V) \), \( \varphi(UV) = \varphi(U)\varphi(V) \);
(iii) \( ||\varphi(U)|| = ||U|| \);
(iv) \( \varphi(U) \leq \varphi(V) \) if \( U \leq V \), for all \( U, V \in B(\mathcal{H}) \), \( \alpha, \beta \in \mathbb{C} \);
(v) \( \sigma(U) = \sigma(\tilde{U}) \), \( \sigma_{rd}(U) = \sigma_{rd}(\tilde{U}) = \sigma_p(\tilde{U}) \).

**Proposition 11.** If \( U \in B(\mathcal{H}) \) is a class \( \mathcal{A}_k \), then \( \varphi(U) \) is a class \( \mathcal{A}_k \).

**Proof.** By using Berberian technique, we prove easily that

\[
|\varphi(U)^{k+1}|^{2^*} = |\varphi(U^{k+1})|^{2^*} = \varphi(|U^{k+1}|^{2^*}) \geq \varphi(|U|^2)
\]
this means that \( \varphi(U) \) is a class \( A_k \).

**Proposition 12.** If \( U \in B(H) \) is a class \( A_k \), then \( U \) is finite.

**Proof.** From Proposition 11 \( \varphi(U) \) is a class \( A_k \), with \( \sigma_a(U) = \sigma_a(\tilde{U}) = \sigma_p(U) \) using Berberian technique, since \( \sigma_a(U) \) is never empty and \( \sigma_{jp}(U) - \{0\} = \sigma_p(U) - \{0\} \), so by Theorem 6, it follows that \( \sigma_{ra}(U) \neq \emptyset \) implying \( U \) is finite.

**Proposition 13.** If \( U \in A_k \), then \( U^* \notin \operatorname{ran}(\delta_U) \).

**Proof.** Let \( \lambda \in \sigma_{ra} - \{0\} \neq \emptyset \), then there is a normalized sequence \( \{x_n\} \) such that
\[
(U - \lambda)x_n \rightarrow 0 \quad \text{and} \quad (U - \lambda)^*x_n \rightarrow 0
\]
and let \( X \in B(H) \), then
\[
\|UX - UX - U^*\| = \|(U - \lambda)X - X(U - \lambda) - (U^* - \overline{\lambda}) - \overline{\lambda}\|
\geq \|((U - \lambda)X_n, x_n) - (X(U - \lambda)x_n, x_n) - ((U^* - \overline{\lambda}) - \overline{\lambda})\|
\]
letting \( n \rightarrow \infty \), we get \( \|UX - UX - U^*\| \geq |\lambda| \) implying \( U^* \notin \operatorname{ran}(\delta_U) \).

**Proposition 14.** If \( U \) is a class \( A_k \) and \( N \) is a normal operator such that \( UN = NU \), then for every \( \lambda \in \sigma_p(N) \)
\[
|\lambda| \leq \operatorname{dist}(N, \operatorname{ran}(\delta_U))
\]

**Proof.** Let \( \lambda \in \sigma_p(N) \) and \( M_\lambda \) be the eigenspace associated to \( \lambda \). Since \( NU = UN \), then \( U^*N = NU^* \) by Putnam-Fuglede Theorem. Hence \( M_\lambda \) reduces orthogonally \( U \) and \( N \). Let \( T \in B(H) \), we can write \( U, N \) and \( T \) according to the decomposition \( H = M_\lambda \oplus M_\lambda^\perp \) as follows:
\[
U = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}, \quad N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}, \quad \text{and} \quad U = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}.
\]
We have
\[
\|N + UT - TU\| = \left\| \begin{bmatrix} \lambda + U_1T_1 - T_1U_1 & \ast \\ \ast & \ast \end{bmatrix} \right\|
\geq \|\lambda + U_1T_1 - T_1U_1\|
\geq |\lambda| \left\| \begin{bmatrix} I + U_1\left(\frac{T_1}{\lambda}\right) - \left(\frac{T_1}{\lambda}\right) \end{bmatrix} \right\|
\leq |\lambda|.
\]

**Proposition 15.** If \( U \) is a class \( A_k \), then for every normal operator \( N \) such that \( UN = NU \), we have \( \|N\| \leq \operatorname{dist}(N, \operatorname{ran}(\delta_U)) \).
Proof. Let $\lambda \in \sigma(N) = \sigma_d(N)$ [1], from proposition 10, $\tilde{N}$ is normal and $\tilde{U}$ is a class $\mathcal{A}_k$, $\tilde{N}U = \tilde{N}\tilde{U} = \tilde{U}\tilde{N}$, also $\lambda \in \sigma_p(\tilde{N})$. Applying proposition (14), we get for every $T \in B(H)$

$$|\lambda| \leq ||\tilde{N} + \tilde{U}\tilde{T} - \tilde{T}\tilde{U}|_{\text{vert}} = ||N + UT - TU||$$

Therefore

$$\sup_{\lambda \in \sigma(N)} |\lambda| = ||\tilde{N}|| = ||N|| \leq ||N + UT - TU||.$$

□

We will denote by $U \otimes T$, the tensor product of some non-zero operators $U, T \in B(X)$, on the product space $X \oplus X$. We can see the importance the tensor product operation $U \oplus T$ as it preserves many properties of $U, T \in B(X)$. It can be checked that the tensor product of operators $U$ and $T$ i.e. $U \otimes T$ is hyponormal if and only if $U$ and $T$ are hyponormal [9].

We will obtain an analogous result for class $\mathcal{A}_k$ operators in this section. Before stating our main theorems, we need some preliminary results.

Lemma 16. [20] Let $U_1, U_2 \in B(X), T_1, T_2 \in B(X)$ be non-negative operators. If $U_1$ and $T_1$ are non-zero, then the following assertions are equivalent

1. $U_1 \oplus T_1 \leq U_2 \oplus T_2$
2. There exists $c > 0$ for which $U_1 \leq U_2$ and $T_1 \leq c^{-1}T_2$.

Lemma 17. If $U, T \in B(X)$ are class $\mathcal{A}_k$ operators, then $U \oplus T$ is class $\mathcal{A}_k$ operator.

Proof. Since $U$ and $T$ are class $\mathcal{A}_k$ operators, then

$$|(U \oplus T)^{k+1}|_{\mathcal{T}}^{\frac{2}{k+1}} \leq |U^{k+1}|_{\mathcal{T}}^{\frac{2}{k+1}} \oplus |T^{k+1}|_{\mathcal{T}}^{\frac{2}{k+1}}$$

$$\geq |U|^2 \oplus |T|^2$$

$$= |U \oplus T|.$$

Hence $U \oplus T$ is a class $\mathcal{A}_k$ operator. □

Theorem 18. [11] If $U$ is a class $\mathcal{A}_k$ operator and $M$ is an invariant subspace of $U$, the restriction $U|_M$ is also a class $\mathcal{A}_k$.

3. Main results

In the following, we prove that if $H$ is a Hilbert-Schmidt operator, $U$ is a class $\mathcal{A}_k$ operator and $T^*$ is an invertible class $\mathcal{A}$ following the relation $UH = HT$, then $U^*H = HT^*$.

Theorem 19. Let $U$ and $T \in B(X)$. Then $\Gamma_{U,T}$ is a class $\mathcal{A}_k$ operator on $C_2(X)$ if and only if $U$ and $T^*$ belong to $\mathcal{A}_k$ operators.

Proof. The unitary operator

$$U : C_2(X) \to X \oplus X$$
defined by

\[ (v \oplus w)^* = v \oplus w \]

induces the \(*\)-isomorphism

\[ \psi : B(C_2(X)) \to B(X \oplus X) \]

by a map

\[ H \mapsto UHU^*. \]

Then we can obtain

\[ \psi(\Gamma_{U,T}) = U \oplus T^*, \]

see [7] for details. This completes the proof by Lemma 17. \( \square \)

**Theorem 20.** Let \( U \) be a class \( \mathcal{A}_k \) operator and \( T^* \) an invertible class \( \mathcal{A} \) operator. If \( UH = HT \) for some \( H \in C_2(X) \), then \( U^*H = HT^* \).

**Proof.** Let \( \Gamma \) be defined on \( C_2(X) \) by

\[ \Gamma(V) = UVT^{-1}. \]

The operator \( T \) is an invertible class \( \mathcal{A} \), then \( T \) is a class \( \mathcal{A}_k \) by Theorem 5.

Since \( U \) and \((T^{-1})^* = (T^*)^{-1}\) are \( \mathcal{A}_k \) operators, we have by Theorem 19, we can say that \( \Gamma \) is also an \( \mathcal{A}_k \) operator. Moreover,

\[ \Gamma(H) = UHT^{-1} = H \]

because of \( UH = HT \). Hence, \( H \) is an eigenvector of \( \Gamma \). By Theorem 6, we have

\[ \Gamma^*(H) = U^*H(T^{-1})^* = H, \]

that is,

\[ U^*H = HT^* \]

as desired. \( \square \)

**Corollary 21.** Let \( U \in B(X) \) be a class \( \mathcal{A} \) and \( T^* \) be an invertible class \( \mathcal{A} \) such that \( UH = HT \) for some \( H \in C_2(X) \). Then, \( U^*H = HT^* \).

**Corollary 22.** Let \( U \in B(X) \) be hyponormal and \( T^* \) be an invertible class \( \mathcal{A} \) such that \( UH = HT \) for some \( H \in C_2(X) \). Then, \( U^*H = HT^* \).

**Corollary 23.** Let \( U \in B(X) \) be a class \( \mathcal{A}_k \) and \( T^* \) be an invertible hyponormal such that \( UH = HT \) for some \( H \in C_2(X) \). Then, \( U^*H = HT^* \).

**Corollary 24.** Let \( U \in B(X) \) be a class \( \mathcal{A} \) and \( T^* \) be an invertible hyponormal such that \( UH = HT \) for some \( H \in C_2(X) \). Then, \( U^*H = HT^* \).

Now, we are ready to extend the orthogonality results to some class \( \mathcal{A}_k \) operators.
Theorem 25. Let $U, T \in B(X)$ and $V \in C_2(X)$. Then

$$\|\delta_{U,T}(H) + V\|_2^2 = \|\delta_{U,T}(H)\|_2^2 + \|V\|_2^2,$$  
(3.1)

and

$$\|\delta^*_{U,T}(H) + V\|_2^2 = \|\delta^*_{U,T}(H)\|_2^2 + \|V\|_2^2,$$  
(3.2)

if and only if $\delta_{U,T}(V) = 0 = \delta^*_{U^*,T^*}(V)$ for all $V \in C_2(X)$.

Proof. It is known that the Hilbert-Schmidt class $C_2(X)$ is a Hilbert space. Note that

$$\|\delta_{U,T}(H) + V\|_2^2 = \|\delta_{U,T}\|_2^2 + \|V\|_2^2 + \text{Re}\langle\delta_{U,T}(H), V\rangle$$

and

$$\|\delta^*_{U,T}(H) + V\|_2^2 = \|\delta^*_{U,T}\|_2^2 + \|V\|_2^2 + \text{Re}\langle H, \delta^*_{U,T}(V)\rangle.$$  
(3.3)

Hence by the equality $\delta_{U,T}(V) = 0 = \delta^*_{U^*,T^*}(V)$, we obtain (3.1) and (3.2). So, this completes the proof as our claim is verified. □

Corollary 26. Let $U, T$ be operators in $B(X)$ and $V \in C_2(X)$. Then

$$\|\delta_{U,T}(H) + V\|_2^2 = \|\delta_{U,T}(H)\|_2^2 + \|V\|_2^2$$

and

$$\|\delta^*_{U,T}(H) + V\|_2^2 = \|\delta^*_{U,T}(H)\|_2^2 + \|V\|_2^2$$

if either of the following hold

(i) $U$ is a class $A_k$ and $(T^*)^{-1}$ is a class $A$;
(ii) $U$ is a class $A$ and $(T^*)^{-1}$ is a class $A$;
(iii) $U$ is hyponormal and $(T^*)^{-1}$ is a class $A$;
(iv) $U$ is a class $A_k$ and $(T^*)^{-1}$ is hyponormal.

4. Discussions

The basic properties of class $A_k$ are studied and discussed. The Putnam-Fuglede Theorem plays an important role in operator theory. We proved that the Putnam-Fuglede Theorem for class $A_k$ operators holds in the Hilbert-Schmidt case. Also, range-kernel results for the generalized derivations induced by certain $A_k$ classes are obtained.

5. Conclusions

The questions which logically arise after this study are as follows:

1. Is the Putnam-Fuglede Theorem remains true for $A_k$ class in any Hilbert space $H$?
2. Is the Putnam-Fuglede Theorem remains true for $A_k$ class in any bilateral ideal in $B(H)$?
Acknowledgments

At the end of this paper we would like to thank the referee for his useful remarks. The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through General Research Project grant number (G.R.P-119-38).

Conflict of interest

The author declares no conflict of interest.

Authors’ contributions

All authors drafted the manuscript, and they read and approved the final manuscript.

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