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Research article

Inclusion relations of q-Bessel functions associated with generalized conic domain

Shahid Khan[1,*], **Saqib Hussain**[2] **and Maslina Darus**[3,*]

[1] Department of Mathematics, Riphah International University Islamabad 44000, Pakistan
[2] Department of Mathematics, COMSATS University Islamabad, Abbottabad Campus 22060, Pakistan
[3] Department of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Bangi 43600, Malaysia

* **Correspondence:** Email: shahidmath761@gmail.com, maslina@ukm.edu.my.

Abstract: In this paper, we investigate the geometric properties of Jackson and Hahn-Exton q-Bessel functions and perform their normalization for the analyticity in open unit disk E. By applying normalized Jackson and Hahn-Exton q-Bessel functions and idea of convolution we introduce a new operator and define new family of subclasses of analytic functions related with generalized conic domain. For these subclasses of analytic functions, we investigate inclusion relations and integral preserving properties. Also we will use q-Bernardi integral operator to discuss some applications of our main results.

Keywords: analytic functions; q-calculus; q-derivative; q-starlike functions; q-convex functions; q-Bessel functions; generalized conic domain $\Omega_{k,q}$
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1. Introduction and definition

Let us consider the following second-order linear homogeneous differential equation

$$z^2 w''(z) + zw'(z) + \left(z^2 - u^2\right) w(z) = 0 \qquad (u \in \mathbb{C}). \tag{1.1}$$

The differential equation in (1.1) is famous Bessel's differential equation. Its solution is denoted by $J_u(z)$ and known as Bessel function. The familiar representation of $J_u(z)$ is given by (1.2) and is defined by particular solution of (1.1) as follows:

$$J_u(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(u+n+1)} \left(\frac{z}{2}\right)^{2n+u} \qquad (z \in \mathbb{C}), \tag{1.2}$$

where Γ is the familiar Euler Gamma function. For a comprehensive study of Bessel function of first kind, see [9, 30].

Let $\mathcal{A}$ represents the class of all those functions which are analytic in the open unit disk

$$E = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and having the series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in E). \tag{1.3}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting the functions that is univalent in E and satisfy the normalized conditions

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

Let two functions f and g are analytic in E, then f is subordinate to g , (written as $f \prec g$), if there exists a Schwarz function $h(z)$, which is analytic in E with

$$h(0) = 0 \qquad \text{and} \qquad |h(z)| < 1,$$

such that

$$f(z) = g(h(z)).$$

If g is univalent in E, then

$$f(z) \prec g(z) \Leftrightarrow f(0) = 0 = g(0) \text{ and } f(E) \subset g(E).$$

For $f \in \mathcal{A}$, given by (1.3) and another function g, given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

then the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z).$$

In [34], Robertson introduced the class of starlike ($\mathcal{S}^*$) and class of convex (C) functions and be defined as:

$$\mathcal{S}^* = f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \text{ and } C = f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0.$$

It can easily seen that

$$f \in C \iff zf^{'} \in \mathcal{S}^*.$$

After that Srivastava and Owa investigated these subclasses in [43].

Let $f \in \mathcal{A}$ and $g \in \mathcal{S}^*$, is said to be close to convex ($\mathcal{K}$) functions if and only if

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > 0.$$

Furthermore, Kanas and Wisniowska in [12] introduced subclasses of k-uniformly convex $(k - \mathcal{UCV})$ and $(k - \mathcal{ST})$ and be defined as:

$$k - \mathcal{UCV} = \left\{ f \in \mathcal{A} : \ \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right|, \ z \in E, \ k \geq 0 \right\}$$

and

$$k - \mathcal{ST} = \left\{ f \in \mathcal{A} : \ \Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right|, \ z \in E, \ k \geq 0 \right\}.$$

Note that

$$f \in k - \mathcal{UCV} \quad \Longleftrightarrow \quad zf' \in k - \mathcal{ST}.$$

For the further developments Kanas and Srivastava studied these subclasses $(k - \mathcal{UCV})$ and $(k - \mathcal{ST})$ of analytic functions in [11]. For particular value of $k = 1$, then $k - \mathcal{UCV} = \mathcal{UCV}$ and $k - \mathcal{ST} = \mathcal{S}^*$

Kanas and Wisniowska [13, 14] (see also [11] and [15]) defined these subclasses of analytic functions subject to the conic domain Ω_k, where

$$\Omega_k = a + ib : a^2 > k^2 \left\{(a-1)^2 + b^2\right\}, \ a > 0, \ \ k \geq 0.$$

For $k = 0$, the domain Ω_k presents the right half plane, for $0 < k < 1$, the domain Ω_k presents hyperbola, for $k = 1$ its presents parabola and an ellipse for $k > 1$.

For this conic domain, the following functions play the role of extremal functions.

$$p_k(z) = \begin{cases} \phi_1(z) & \text{for} \quad k = 0, \\ \phi_2(z) & \text{for} \quad k = 1, \\ \phi_3(z) & \text{for} \quad 0 < k < 1, \\ \phi_4(z) & \text{for} \quad k > 1, \end{cases} \tag{1.4}$$

where

$$\phi_1(z)) = \frac{1+z}{1-z},$$

$$\phi_2(z) = 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2,$$

$$\phi_3(z) = 1 + \frac{2}{1-k^2}\sinh^2\left\{\left(\frac{2}{\pi}\arccos k\right)\arctan h\sqrt{z}\right\},$$

$$\phi_4(z) = 1 + \frac{1}{k^2-1}\sin\left(\frac{\pi}{2R(t)}\int_0^{\frac{y(z)}{\sqrt{t}}}\frac{dx}{\sqrt{1-x^2}\sqrt{1-t^2x^2}}\right) + \frac{1}{k^2-1}$$

and

$$y(z) = \frac{z - \sqrt{t}}{1 - \sqrt{t}z} \qquad \{t \in (0,1)\}$$

is chosen such that

$$k = \cosh\left(\pi R'(t)/(4R(t))\right).$$

Here $R(t)$ is Legender's complete elliptic integral of first kind (see [13, 14]).

Since the q-calculus is being vastly used in different areas of mathematics and physics it is of great interest of researchers. In the study of Geometric Function Theory, the versatile applications of q-derivative operator make it remarkably significant. Initially, in the year 1990, Ismail et al. [5] gave the idea of q-starlike functions. Nevertheless, a firm foothold of the usage of the q-calculus in the context of Geometric Function Theory was effectively established, and the use of the generalized basic (or q-) hypergeometric functions in Geometric Function Theory was made by Srivastava (see for detail [37]). For the study of various families of analytic and univalent function, the quantum (or q-) calculus has been used as a important tools. Jackson [7, 8] first defined the q-derivative and integral operator as well as provided some of their applications. The q-Ruscheweyh differential operator was defined by Kanas and Raducanu in [10]. Recently, by using the concept of convolution Srivastava [40] introduced q-Noor integral operator and studied some of its applications. Many q-differential and q-integral operators can be written in term of convolution, for detail we refer [4,23,36,39,41] see also [16, 18]. Moreover, Srivastava et al. (see, for example, [35, 44, 45]) published a set of articles in which they concentrated upon the classes of q-starlike functions related with the Janowski functions from several different aspects. Additionally specking, a recently-published survey-cum-expository review article by Srivastava [38] is potentially useful for researchers and scholars working on these topics. In this survey-cum-expository review article [38], the mathematical explanation and applications of the fractional q-calculus and the fractional q-derivative operators in Geometric Function Theory was systematically investigated. For other recent investigations involving the q-calculus, one may refer to [1, 19, 22, 24, 25, 31–33] and [17]. We remark in passing that, in the above-cited recently-published survey-cum-expository review article [38], the so-called (p, q)-calculus was exposed to be a rather trivial and inconsequential variation of the classical q-calculus, the additional parameter p being redundant or superfluous (see, for details, [38, p. 340]). In order to have a better understanding of the present article we provide some notation and concepts of quantum (or q-) calculus used in this article.

Definition 1. ([10]). Let $q \in (0, 1)$ and define the q-number $[\eta]_q$ as:

$$\begin{aligned} [\eta]_q &= \frac{1-q^\eta}{1-q}, & \eta \in \mathbb{C}, \\ &= 1 + q + ... + q^{n-1}, & \eta = n \in \mathbb{N}, \\ [0] &= 0, & \eta = 0. \end{aligned}$$

Definition 2. Let $q \in (0, 1)$, $n \in \mathbb{N}$ and define the q-factorial $[n]_q!$

$$[n]_q! = [1]_q [2]_q ... [n]_q \quad \text{and} \quad [0]_q! = 1.$$

Definition 3. The q-generalized Pochhammer symbol $[a]_{n,q}$ be defined as:

$$[a]_{n,q} = \prod_{k=1}^{n} \left(1 - aq^{k-1}\right), \quad n \in \mathbb{N}$$

and

$$[a]_{\infty,q} = \prod_{k=1}^{\infty} \left(1 - aq^{k-1}\right).$$

Definition 4. The q-Gamma function $\Gamma_q(n)$ is defined by

$$\Gamma_q(n) = \frac{[q,q]_\infty}{[q^n,q]_\infty}(1-q)^{n-1}.$$

The q-Gamma function $\Gamma_q(n)$ satisfies the following functional equation

$$\Gamma_q(n+1) = \left(\frac{1-q^n}{1-q}\right)\Gamma_q(n).$$

Definition 5. ([7]). For $f \in \mathcal{A}$, and the q-derivative operator or q-difference operator be defined as:

$$\begin{aligned} D_q f(z) &= \frac{f(z) - f(qz)}{(1-q)z} \qquad (z \in E), \\ D_q f(z) &= 1 + \sum_{n=2}^{\infty} [n]_q \, a_n z^{n-1} \end{aligned} \tag{1.5}$$

and

$$D_q z^n = [n]_q \, z^{n-1}.$$

Definition 6. ([5]). An analytic function $f \in \mathcal{S}_q^*$ if

$$f(0) = f^{'}(0) = 1, \tag{1.6}$$

and

$$\left|\frac{zD_q f(z)}{f(z)} - \frac{1}{1-q}\right| \leq \frac{1}{1-q}, \tag{1.7}$$

we can rewrite the conditions (1.7) as follows, (see [46]).

$$\frac{zD_q f(z)}{f(z)} \prec \frac{1+z}{1-qz}.$$

Here Serivastava et al. [39] (see also [42]) defined the following definition by making use of quantum (or q-) calculus, principle of subordination and general conic domain $\Omega_{k,q}$ as:

Definition 7. ([39]) . Let $k \geq 0$ and $q \in (0,1)$. A function $p(z)$ is said to be in the class $k - \mathcal{P}_q$ if and only if

$$p(z) \prec p_{k,q}(z) = \frac{2p_k(z)}{(1+q) + (1-q)\,p_k(z)} \tag{1.8}$$

and $p_k(z)$ is given by (1.4).

Geometrically, the function $p(z) \in k - \mathcal{P}_q$ takes all values from the domain $\Omega_{k,q}$ which is defined as follows:

$$\Omega_{k,q} = \left\{w : \Re\left(\frac{(1+q)\,w}{2+(q-1)\,w}\right) > k\left|\frac{(1+q)\,w}{2+(q-1)\,w} - 1\right|\right\}.$$

Remark 1. We see that

$$k-\mathcal{P}_q \subseteq \mathcal{P}\left(\frac{2k}{2k+1+q}\right)$$

and

$$\Re(p(z)) > \Re(p_{k,q}(z)) > \frac{2k}{2k+1+q}.$$

For $q \to 1-$, then we have

$$k-\mathcal{P}_q = \mathcal{P}\left(\frac{k}{k+1}\right),$$

where the class $\mathcal{P}\left(\frac{k}{k+1}\right)$ introduced by Kanas and Wisniowska [13] and therefore,

$$\Re(p(z)) > \Re(p_k(z)) > \frac{k}{k+1}.$$

Also for $k = 0$ and $q \to 1-$, we have

$$k-\mathcal{P}_q = \mathcal{P}$$

and

$$\Re(p(z)) > 0.$$

Remark 2. For $q \to 1-$, then $\Omega_{k,q} = \Omega_k$, where domain Ω_k introduced by Kanas and Wisniowska in [13].

By Applying q-derivative operator we introduce new subclasses of q-starlike functions, q-convex functions, q-close to convex functions and q-quasi-convex functions as follows:

Definition 8. [42] For $f \in \mathcal{A}$, $k \geq 0$, then $f \in k-\mathcal{ST}_q$ if and only if

$$\frac{zD_qf(z)}{f(z)} \prec p_{k,q}(z). \tag{1.9}$$

Definition 9. [42] For $f \in \mathcal{A}$, $k \geq 0$, then $f \in k-\mathcal{UCV}_q$ if and only if

$$\frac{D_q\left(zD_qf(z)\right)}{D_qf(z)} \prec p_{k,q}(z).$$

It can easily seen that

$$f \in k-\mathcal{UCV}_q \quad \text{iff} \quad zD_qf \in k-\mathcal{ST}_q. \tag{1.10}$$

Definition 10. [42] For $f \in \mathcal{A}$, $k \geq 0$, then $f \in k-\mathcal{UCC}_q$ if and only if

$$\frac{zD_qf(z)}{g(z)} \prec p_{k,q}(z), \quad \text{for some } g(z) \in k-\mathcal{ST}_q.$$

Definition 11. [42] For $f \in \mathcal{A}$, $k \geq 0$, then $f \in k-\mathcal{UQV}_q$ if and only if

$$\frac{D_q\left(zD_qf(z)\right)}{D_qg(z)} \prec p_{k,q}(z), \quad \text{for some } g(z) \in k-\mathcal{UCC}_q.$$

Remark 3. For $q \to 1-$, then all theses newly defined subclasses reduces to the well-known subclasses of analytic functions introduced in [29].

The Jackson q-Bessel functions and the Hahn-Exton q-Bessel functions are, respectively, defined by

$$J_u^1(z,q) = \frac{\left[q^{u+1},q\right]_\infty}{[q,q]_\infty} \sum_{n=1}^{\infty} \frac{(-1)^n\, q^{n(n+u)}}{[q,q]_n \left[q^{u+1},q\right]_n} \left(\frac{z}{2}\right)^{2n+u}$$

and

$$J_u^2(z,q) = \frac{\left[q^{u+1},q\right]_\infty}{[q,q]_\infty} \sum_{n=1}^{\infty} \frac{(-1)^n\, q^{\frac{1}{2}n(n+u)}}{[q,q]_n \left[q^{u+1},q\right]_n} z^{2n+u},$$

where $z \in \mathbb{C},\ u > -1,\ q \in (0,1)$. The functions $J_u^1(z,q)$ and $J_u^2(z,q)$ are the q-extensions of the classical Bessel functions of the first kind. For more study about q-extensions of Bessel functions (see [6, 20, 21]). Since neither $J_u^1(z,q)$ nor $J_u^2(z,q)$ belongs to $\mathcal{A}$, first we perform normalizations of $J_u^1(z,q)$ and $J_u^2(z,q)$ as:

$$\begin{aligned} f_u^1(z,q) &= 2^u C_u(q) z^{1-n-u} J_u^1(z,q) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n\, q^{n(n+u)}}{4^n\, [q,q]_n \left[q^{u+1},q\right]_n} z^{n+1} \\ &= z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\, q^{(n-1)(n-1+u)}}{4^{n-1}\, [q,q]_{n-1} \left[q^{u+1},q\right]_{n-1}} z^n. \end{aligned}$$

Similarly

$$\begin{aligned} f_u^2(z,q) &= C_u(q) z^{1-n-u} J_u^2(z,q) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n\, q^{\frac{1}{2}n(n+u)}}{[q,q]_n \left[q^{u+1},q\right]_n} z^{n+1}, \\ &= z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\, q^{\frac{1}{2}(n-1)(n-1+u)}}{[q,q]_{n-1} \left[q^{u+1},q\right]_{n-1}} z^n, \end{aligned}$$

where

$$C_u(q) = \frac{[q,q]_\infty}{\left[q^{u+1},q\right]_\infty},\ \ z \in \mathbb{C},\ u > -1,\ q \in (0,1).$$

Now clearly, the functions $f_u^1(z,q)$ and $f_u^2(z,q) \in \mathcal{A}$.

Now, by using the above idea of convolution and normalized Jackson and Hahn-Exton q-Bessel functions, we introduce a new operators B_u^q and $B_{u,1}^q$ as follows:

$$B_u^q f(z) = f_u^1(z,q) * f(z) = z + \sum_{n=2}^{\infty} \varphi_1 a_n z^n \tag{1.11}$$

and

$$B_{u,1}^q f(z) = f_u^2(z,q) * f(z) = z + \sum_{n=2}^{\infty} \varphi_2 a_n z^n, \tag{1.12}$$

where

$$\varphi_1 = \frac{(-1)^{n-1}\, q^{(n-1)(n-1+u)}}{4^{n-1}\, [q,q]_{n-1}\, [q^{u+1},q]_{n-1}}$$

and

$$\varphi_2 = \frac{(-1)^{n-1}\, q^{\frac{1}{2}(n-1)(n-1+u)}}{[q,q]_{n-1}\, [q^{u+1},q]_{n-1}}.$$

From the definition (1.11) and (1.12), it can easy to verify that

$$zD_q\left(B_{u+1}^q f(z)\right) = \left(\frac{[u]_q}{q^u} + 1\right) B_u^q f(z) - \frac{[u]_q}{q^u} B_{u+1}^q f(z) \tag{1.13}$$

and

$$zD_q\left(B_{u+1,1}^q f(z)\right) = \left(\frac{[u]_q}{q^u} + 1\right) B_{u,1}^q f(z) - \frac{[u]_q}{q^u} B_{u+1,1}^q f(z).$$

Finally Noor et al. introduced q-Bernardi integral operator [28], which is defined by

$$L_\lambda^q = L_\lambda^q f(z) = \frac{[\lambda+1]_q}{z^\lambda} \int_0^z t^{\lambda-1} f(t) d_q t, \quad \lambda > -1.$$

Remark 4. For $q \to 1-$, then $L_\lambda^q = L$, introduced by Bernardi in [2].

2. Preliminary results

Here we gave the generalization of two lemmas which was introduced in [3, 27].

Lemma 1. *Let $h(z)$ be an analytic and convex univalent in E with*

$$\Re\left(vh(z) + \alpha\right) > 0 \quad (v, \alpha \in \mathbb{C}) \quad \textit{and} \quad h(0) = 1.$$

If $p(z)$ is analytic in E and $p(0) = 1$, then

$$p(z) + \frac{zD_q p(z)}{vp(z) + \alpha} \prec h(z), \quad z \in E, \tag{2.1}$$

then

$$p(z) \prec h(z).$$

Proof. Suppose that $h(z)$ is analytic and convex univalent in E and $p(z)$ is analytic in E. Letting $q \to 1-$, in (2.1), we have

$$p(z) + \frac{zp'(z)}{vp(z) + \alpha} \prec h(z), \quad z \in E,$$

then by Lemma in [26], we have

$$p(z) \prec h(z).$$

□

Lemma 2. *Let an analytic functions $p(z)$ and $g(z)$ in open unit disk E with*

$$\Re p(z) > 0 \quad \textit{and} \quad h(0) = g(0).$$

suppose that $h(z)$ be convex functions in E and let $U \geq 0$, then

$$Uz^2 D_q^2 g(z) + p(z)g(z) \prec h(z) \tag{2.2}$$

then

$$g(z) \prec h(z), \quad z \in E.$$

Proof. Suppose that $h(z)$ is convex in the open unit disk E. Let $p(z)$ and $g(z)$ is analytic in E with $\Re p(z) > 0$ and $h(0) = g(0)$. Letting $q \to 1-$, in (2.2), we have

$$Uz^2 g''(z) + p(z)g(z) \prec h(z), \quad z \in E,$$

then by Lemma in [27], we have

$$g(z) \prec h(z).$$

□

3. Main results

Theorem 1. *Let $h(z)$ be convex univalent in E with $\Re(h(z)) > 0$ and $h(0) = 1$. If a function $f \in \mathcal{A}$ satisfies the condition*

$$\frac{zD_q\left(B_u^q f(z)\right)}{B_u^q f(z)} \prec h(z), \quad z \in E,$$

then

$$\frac{zD_q\left(B_{u+1}^q f(z)\right)}{B_{u+1}^q f(z)} \prec h(z), \quad z \in E.$$

Proof. Let

$$p(z) = \frac{zD_q\left(B_{u+1}^q f(z)\right)}{B_{u+1}^q f(z)}. \tag{3.1}$$

where p is an analytic function in E with $p(0) = 1$. By using (1.13) into (3.1), we have

$$p(z) = \left(\frac{[u]_q}{q^u} + 1\right)\frac{zB_u^q f(z)}{B_{u+1}^q f(z)} - \frac{[u]_q}{q^u}.$$

Differentiating logarithmically with respect to z, we have

$$p(z) + \frac{zD_q p(z)}{p(z) + \frac{[u]_q}{q^u}} = \frac{zB_u^q f(z)}{B_{u+1}^q f(z)}.$$

By using Lemma 1, we get required result. □

By taking $q \to 1-$, in Theorem 1, then we have the following result.

Corollary 1. *Let $h(z)$ be convex univalent in E with $\Re(h(z) > 0$ and $h(0) = 1$. If a function $f \in \mathcal{A}$ satisfies the condition*

$$\frac{z\left(B_u f(z)\right)'}{B_u f(z)} \prec h(z), \quad z \in E,$$

then

$$\frac{z\left(B_{u+1} f(z)\right)'}{B_{u+1} f(z)} \prec h(z), \quad z \in E.$$

Theorem 2. *Let $f \in \mathcal{A}$. If $B_u^q f(z) \in k - \mathcal{ST}_q$, then $B_{u+1}^q f(z) \in k - \mathcal{ST}_q$.*

Proof. Let

$$p(z) = \frac{zD_q\left(B_{u+1}^q f(z)\right)}{B_{u+1}^q f(z)}.$$

From (1.13), we have

$$\left(\frac{[u]_q}{q^u} + 1\right)\frac{zB_u^q f(z)}{B_{u+1}^q f(z)} = p(z) + \frac{[u]_q}{q^u}.$$

Differentiating logarithmically with respect to z, we have

$$\frac{zD_q B_u^q f(z)}{B_u^q f(z)} = p(z) + \frac{zD_q p(z)}{p(z) + \frac{[u]_q}{q^u}} \prec p_{k,q}(z).$$

Since $p_{k,q}(z)$ is convex univalent in E given by (1.8) and

$$\Re\left(p_{k,q}(z)\right) > \frac{2k}{2k+1+q}.$$

The proof of the theorem 2 follows by Theorem 1 and condition (1.9). □

For $q \to 1-$, in Theorem 2, then we have the following result.

Corollary 2. *Let $f \in \mathcal{A}$. If $B_u f(z) \in k - \mathcal{ST}$, then $B_{u+1} f(z) \in k - \mathcal{ST}$.*

Theorem 3. *Let $f \in \mathcal{A}$. If $B_u^q f(z) \in k - \mathcal{UCV}_q$, then $B_{u+1}^q f(z) \in k - \mathcal{UCV}_q$.*

Proof. By virtue of (1.10), and Theorem 2, we get

$$\begin{aligned} B_u^q f(z) \in \; & k - \mathcal{UCV}_q \Leftrightarrow zD_q\left(B_u^q f(z)\right) \in k - \mathcal{ST}_q \\ \Leftrightarrow \; & B_u^q zD_q f(z) \in k - \mathcal{ST}_q \\ \Rightarrow \; & B_{u+1}^q zD_q f(z) \in k - \mathcal{ST}_q \\ \Leftrightarrow \; & B_{u+1}^q f(z) \in k - \mathcal{UCV}_q. \end{aligned}$$

Hence Theorem 3 is complete. □

For $q \to 1-$, in Theorem 3, then we have the following result.

Corollary 3. *Let $f \in \mathcal{A}$. If $B_u f(z) \in k - \mathcal{UCV}$, then $B_{u+1} f(z) \in k - \mathcal{UCV}$.*

Theorem 4. *Let $f \in \mathcal{A}$. If $B_u^q f(z) \in k - \mathcal{UCC}_q$, then $B_{u+1}^q f(z) \in k - \mathcal{UCC}_q$.*

Proof. Since

$$B_u^q f(z) \in k - \mathcal{UCC}_q,$$

then

$$\frac{zD_q B_u^q f(z)}{B_u^q g(z)} \prec p_{k,q}(z), \quad \text{for some} \quad B_u^q g(z) \in k - \mathcal{ST}_q. \tag{3.2}$$

Letting

$$h(z) = \frac{zD_q B_{u+1}^q f(z)}{B_{u+1}^q g(z)}$$

and

$$H(z) = \frac{zD_q B_{u+1}^q g(z)}{B_{u+1}^q g(z)}.$$

We see that $h(z), H(z) \in \mathcal{A}$, in E with $h(0) = H(0) = 1$. By using Theorem 2, we have

$$B_{u+1}^q g(z) \in k - \mathcal{ST}_q$$

and

$$\Re\left(H(z)\right) > \frac{2k}{2k+1+q}.$$

Also note that

$$zD_q B_{u+1}^q f(z) = h(z)\left(B_{u+1}^q g(z)\right). \tag{3.3}$$

Differentiating both sides of (3.3), we obtain

$$\begin{aligned} \frac{zD_q\left(zD_q B_{u+1}^q f(z)\right)}{B_{u+1}^q g(z)} &= \frac{zD_q B_{u+1}^q g(z)}{B_{u+1}^q g(z)} h(z) + zD_q h(z) \\ &= H(z)h(z) + zD_q h(z). \end{aligned} \tag{3.4}$$

By using the identity (1.13), we get

$$\begin{aligned} \frac{zD_q B_u^q f(z)}{B_u^q g(z)} &= \frac{B_u^q zD_q f(z)}{B_u^q g(z)} \\ &= \frac{zD_q\left(B_{u+1}^q zD_q f(z)\right) + \frac{[u]_q}{q^u}\left(B_{u+1}^q zD_q f(z)\right)}{zD_q\left(B_{u+1}^q g(z)\right) + \frac{[u]_q}{q^u} B_{u+1}^q g(z)} \\ &= \frac{\frac{zD_q\left(B_{u+1}^q zD_q f(z)\right)}{B_{u+1}^q g(z)} + \frac{[u]_q}{q^u}\frac{\left(B_{u+1}^q zD_q f(z)\right)}{B_{u+1}^q g(z)}}{\frac{zD_q\left(B_{u+1}^q g(z)\right)}{B_{u+1}^q g(z)} + \frac{[u]_q}{q^u}} \end{aligned}$$

$$= \quad h(z) + \frac{zD_q h(z)}{H(z) + \frac{[u]_q}{q^u}}. \tag{3.5}$$

From (3.2), (3.4), and (3.5), we conclude that

$$h(z) + \frac{zD_q h(z)}{H(z) + \frac{[u]_q}{q^u}} \prec p_{k,q}(z).$$

On letting $U = 0$ and $B(z) = \frac{1}{H(z)+\frac{[u]_q}{q^u}}$, we have

$$\Re(B(z)) = \frac{\Re\left(H(z) + \frac{[u]_q}{q^u}\right)}{\left|H(z) + \frac{[u]_q}{q^u}\right|^2} > 0.$$

Apply Lemma 2, we have

$$h(z) \prec p_{k,q}(z),$$

where $p_{k,q}(z)$ given by (1.8). Hence Theorem 4 is complete. □

We can prove Theorem 5 by using a similar argument of Theorem 4

Theorem 5. *Let $f \in \mathcal{A}$. If $B_u^q f(z) \in k - \mathcal{UQC}_q$, then $B_{u+1}^q f(z) \in k - \mathcal{UQC}_q$.*

Now in Theorem 6, we study the closure properties of the q-Bernardi integral operator L_λ^q.

Theorem 6. *Let $f \in \mathcal{A}$ and $\lambda > -\left(\frac{2k}{2k+1+q}\right)$. If $B_u^q f(z) \in k - \mathcal{ST}_q$, then $L_\lambda^q\left(B_u^q f(z)\right) \in k - \mathcal{ST}_q$.*

Proof. From the definition of $L_\lambda^q f(z)$ and the linearity of the operator B_u^q, we have

$$zD_q\left(B_u^q L_\lambda^q f(z)\right) = (1+\lambda) B_u^q f(z) - \lambda B_u^q L_\lambda^q f(z). \tag{3.6}$$

Substituting $p(z) = \frac{zD_q\left(B_u^q L_\lambda^q f(z)\right)}{B_u^q L_\lambda^q f(z)}$ in (3.6), we have

$$p(z) = (1+\lambda)\frac{B_u^q f(z)}{B_u^q L_\lambda^q f(z)} - \lambda. \tag{3.7}$$

Differentiating (3.7) with respect to z, we have

$$\begin{aligned} \frac{zD_q\left(B_u^q f(z)\right)}{B_u^q f(z)} &= \frac{zD_q\left(B_u^q L_\lambda^q f(z)\right)}{B_u^q L_\lambda^q f(z)} + \frac{zD_q p(z)}{p(z)+\lambda} \\ &= p(z) + \frac{zD_q p(z)}{p(z)+\lambda}. \end{aligned}$$

By Lemma 1, $p(z) \prec p_{k,q}(z)$, since $\Re\left(p_{k,q}(z) + \lambda\right) > 0$. This completes the proof of Theorem 6. □

By a similar argument we can prove Theorem 7 as below.

Theorem 7. *Let $f \in \mathcal{A}$ and $\lambda > -\left(\frac{2k}{2k+1+q}\right)$. If $B_u^q f(z) \in k - \mathcal{UCV}_q$, then $L_\lambda^q\left(B_u^q f(z)\right) \in k - \mathcal{UCV}_q$.*

Theorem 8. *Let $f \in \mathcal{A}$ and $\lambda > -\left(\frac{2k}{2k+1+q}\right)$. If $B_u^q f(z) \in k-\mathcal{UCC}_q$, then $L_\lambda^q\left(B_u^q f(z)\right) \in k-\mathcal{UCC}_q$.*

Proof. By definition, there exists a function

$$B_u^q g(z) \in k-\mathcal{ST}_q,$$

so that

$$\frac{zD_q B_u^q f(z)}{B_u^q g(z)} \prec p_{k,q}(z). \tag{3.8}$$

Now from (3.6), we have

$$\begin{aligned}
\frac{zD_q\left(B_u^q f(z)\right)}{B_u^q g(z)} &= \frac{zD_q\left(B_u^q L_\lambda^q\left(zD_q f(z)\right)\right)+\lambda\left(B_u^q L_\lambda^q\left(zD_q f(z)\right)\right)}{zD_q\left(B_u^q L_\lambda^q g(z)\right)+\lambda B_u^q L_\lambda^q\left(g(z)\right)} \\
&= \frac{\frac{zD_q\left(B_u^q L_\lambda^q\left(zD_q f(z)\right)\right)}{B_u^q L_\lambda^q(g(z))}+\lambda\frac{\left(B_u^q L_\lambda^q\left(zD_q f(z)\right)\right)}{B_u^q L_\lambda^q(g(z))}}{\frac{zD_q\left(B_u^q L_\lambda^q g(z)\right)}{B_u^q L_\lambda^q(g(z))}+\lambda}.
\end{aligned} \tag{3.9}$$

Since $B_u^q g(z) \in k-\mathcal{ST}_q$, by Theorem 6, we have $L_\lambda^q\left(B_u^q g(z)\right) \in k-\mathcal{ST}_q$. Taking

$$H(z) = \frac{zD_q\left(B_u^q L_\lambda^q g(z)\right)}{B_u^q\left(L_\lambda^q g(z)\right)}.$$

We see that $H(z) \in \mathcal{A}$ in E with $H(0) = 1$, and

$$\Re\left(H(z)\right) > \frac{2k}{2k+1+q}.$$

Now for

$$h(z) = \frac{zD_q\left(B_u^q L_\lambda^q f(z)\right)}{B_u^q\left(L_\lambda^q g(z)\right)}.$$

Thus we obtain

$$zD_q\left(B_u^q L_\lambda^q f(z)\right) = h(z) B_u^q\left(L_\lambda^q g(z)\right). \tag{3.10}$$

Differentiating both sides of (3.10), we obtain

$$\begin{aligned}
\frac{zD_q\left(B_u^q D_q\left(zL_\lambda^q f(z)\right)\right)}{B_u^q\left(L_\lambda^q g(z)\right)} &= \frac{zD_q\left(B_u^q\left(L_\lambda^q g(z)\right)\right)}{B_u^q\left(L_\lambda^q g(z)\right)} h(z) + zD_q h(z) \\
&= H(z)h(z) + zD_q h(z).
\end{aligned} \tag{3.11}$$

Therefore from (3.9) and (3.11), we obtain

$$\frac{zD_q\left(B_u^q f(z)\right)}{B_u^q g(z)} = \frac{zD_q h(z) + H(z)h(z) + \lambda h(z)}{H(z)+\lambda}.$$

This in conjunction with (3.8) leads to

$$h(z)+\frac{zD_qh(z)}{H(z)+\lambda}\prec p_{k,q}(z). \tag{3.12}$$

On letting $U=0$ and $B(z)=\frac{1}{H(z)+\lambda}$, we have

$$\Re\left(B(z)\right)=\frac{\Re\left(H(z)+\lambda\right)}{|H(z)+\lambda|^2}>0.$$

Apply Lemma 2, we have

$$h(z)\prec p_{k,q}(z).$$

where $p_{k,q}(z)$ given by (1.8). Hence Theorem 8 is complete. □

We can prove Theorem 9 by using a similar argument of Theorem 8.

Theorem 9. *Let $f\in\mathcal{A}$ and $\lambda>-\left(\frac{2k}{2k+1+q}\right)$. If $B_u^qf(z)\in k-\mathcal{UQC}_q$, then $L_\lambda^q\left(B_u^qf(z)\right)\in k-\mathcal{UQC}_q$.*

4. Conclusions

Our present investigation is motivated by the well-established potential for the usages of the basic (or q-) calculus and the fractional basic (or q-) calculus in Geometric Function Theory as described in a recently-published survey-cum-expository review article by Srivastava [38]. We have studied new family of analytic functions involving the Jackson and Hahn-Exton q-Bessel functions and investigate their inclusion relationships and certain integral preserving properties bounded by generalized conic domain $\Omega_{k,q}$. Also we discussed some applications of our main results by using the q-Bernardi integral operator. The convolution operator $B_{u,1}^q$, which are defined by (1.12) will indeed apply to any attempt to produce the rather straightforward results which we have presented in this paper.

Basic (or q-) series and basic (or q-) polynomials, especially the basic (or q-) hypergeometric functions and basic (or q-) hypergeometric polynomials, are applicable particularly in several diverse areas (see, for example, [38, p. 328]).

Moreover, in this recently-published survey-cum-expository review article by Srivastava [38], the so-called (p,q)-calculus was exposed to be a rather trivial and inconsequential variation of the classical q-calculus, the additional parameter p being redundant (see, for details, [38, p. 340]). This observation by Srivastava [38] will indeed apply also to any attempt to produce the rather straightforward (p,q)-variations of the results which we have presented in this paper.

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Conflicts of interest

The authors declare that they have no competing interests.

References

1. Q. Z. Ahmad, N. Khan, M. Raza, M. Tahir, B. Khan, Certain q-difference operators and their applications to the subclass of meromorphic q-starlike functions, *Filomat*, **33** (2019), 3385–3397.
2. S. D. Bernardi, Convex and starlike univalent functions, *Trans. Am. Math. Soc.*, **135** (1969), 429–446.
3. P. Eeinigenburg, S. S. Miller, P. T. Mocanu, M. D. Reade, General inequalities, *Birkhäuser, Basel*, **64** (1983), 339–348.
4. S. Hussain, S. Khan, M. A. Zaighum, M. Darus, Applications of a q-Salagean type operator on multivalent function, *J. Inequalities Appl.*, **1** (2018), 301–312.
5. M. E. H. Ismail, E. Merkes, D. Styer, A generalization of starlike functions, *Complex Variables, Theory Appl.*, **14** (1990), 77–84.
6. M. E. H. Ismail, M. E. Muldoon, On the variation with respect to a parameter of zeros of Bessel and q-Bessel functions, *J. Math. Anal. Appl.*, **135** (1988), 187–207.
7. F. H. Jackson, On q-functions and a certain difference operator, *Trans. R. Soc. Edinburgh*, **46** (1908), 253–281.
8. F. H. Jackson, On q-definite integrals, *Q. J. Pure Appl. Math.*, **41** (1910), 193–203.
9. S. Kanas, S. R. Mondal, A. D. Mohammed, Relations between the generalized Bessel functions and the Janowski class, *Math. Inequalities Appl.*, **21** (2018), 165–178.
10. S. Kanas, D. Raducanu, Some class of analytic functions related to conic domains, *Math. Slovaca*, **64** (2014), 1183–1196.
11. S. Kanas, H. M. Srivastava, Linear operators associated with k-uniform convex functions, *Integr. Transforms Spec. Funct.*, **9** (2000), 121–132.
12. S. Kanas, A. Wisniowska, Conic regions and k-uniform convexity II, *Zeszyty Nauk. Politech. Rzeszowskiej Mat.*, **22** (1998), 65–78.
13. S. Kanas, A. Wisniowska, Conic regions and k-uniform convexity, *J. Comput. Appl. Math.*, **105** (1999), 327–336.
14. S. Kanas, A. Wisniowska, Conic domains and k-starlike functions, *Rev. Roum. Math. Pure Appl.*, **45** (2000), 647–657.
15. N. Khan, B. Khan, Q. Z. Ahmad, S. Ahmad, Some convolution properties of multivalent analytic functions, *AIMS Math.,* **2** (2017), 260–268.
16. B. Khan, Z. G. Liu, H. M. Srivastava, N. Khan, M. Darus, M. Tahir, A study of some families of multivalent q-starlike functions involving higher-order q-Derivatives, *Mathematics*, **8** (2020), 1–12.
17. N. Khan, M. Shafiq, M. Darus, B. Khan, Q. Z. Ahmad, Upper bound of the third Hankel determinant for a subclass of q-starlike functions associated with Lemniscate of Bernoulli, *J. Math. Inequal.*, **14** (2020), 51–63.

18. B. Khan, H. M. Srivastava, N. Khan, M. Darus, M. Tahir, Q. Z. Ahmad, Coefficient estimates for a subclass of analytic functions associated with a certain leaf-like domain, *Mathematics*, **8** (2020), 1–15.

19. B. Khan, H. M Srivastava, M. Tahir, M. Darus, Q. Z. Ahmad, N. Khan, Applications of a certain q-integral operator to the subclasses of analytic and bi-univalent functions, *AIMS Math.*, **6** (2020), 1024–1039.

20. H. T. Koelink, R. F. Swarttouw, On the zeros of the Hahn-Exton q-Bessel function and associated q-Lommel polynomials, *J. Math. Anal. Appl.*, **186** (1994), 690–710.

21. T. H. Koornwinde, R. F. Swarttouw, On q-analogues of the hankel and fourier transforms, *Trans. Amer. Math. Soc.*, **333** (1992), 445–461.

22. S. Mahmood, Q. Z. Ahmad, H. M. Srivastava, N. Khan, B. Khan, M. Tahir, A certain subclass of meromorphically q-starlike functions associated with the Janowski functions, *J. Inequalities Appl.*, **2019** (2019), 1–11.

23. A. Mahammed, M. Darus, A generalized operator involving the q-hypergeometric function, *Matematichki Vesnik*, **65** (2013), 454–465.

24. S. Mahmood, N. Raza, E. S. A. Abujarad, G. Srivastava, H. M. Srivastava, S. N. Malik, Geometric properties of certain classes of analytic functions associated with a q-integral operator, *Symmetry*, **11** (2019), 1–14.

25. S. Mahmood, H. M. Srivastava, N. Khan, Q. Z. Ahmad, B. Khan, I. Ali, Upper bound of the third Hankel determinant for a subclass of q-Starlike functions, *Symmetry*, **11** (2019), 1–13.

26. S. S. Miller, P. T. Mocanu, Differential subordinations and univalent functions, *Mich. Math. J.*, **28** (1981), 157–172.

27. S. S. Miller, P. T. Mocanu, Differential subordinations and inequalities in complex plane, *J. Differ. Equations*, **67** (1987), 199–211.

28. K. I. Noor, S. Riaz, M. A. Noor, On q-Bernardi integral operator, *TWMS J. Pure Appl. Math.*, **8** (2017), 3–11.

29. C. Ramachandran, S. Annamalai, S. Sivasubramanian, Inclusion relations for Bessel functions for domains bounded by conical domains, *Adv. Differ. Equations*, **2014** (2014), 288.

30. M. Raza, M. D. Din, S. N. Malik, Certain geometric properties of normalized wright functions, *J. Funct. Spaces*, **2016** (2016), 1896154.

31. M. S. Rehman, Q. Z. Ahmad, B. Khan, M. Tahir, N. Khan, Generalisation of certain subclasses of analytic and univalent functions, *Maejo Internat. J. Sci. Technol.*, **13** (2019), 1–9.

32. M. S. Rehman, Q. Z. Ahmad, H. M. Srivastava, B. Khan, N. Khan, Partial sums of generalized q-Mittag-Leffler functions, *AIMS Math.*, **5** (2019), 408–420.

33. M. S. Rehman, Q. Z. Ahmad, H. M. Srivastava, N. Khan, M. Darus, B. Khan, Applications of higher-order q-derivatives to the subclass of q-starlike functions associated with the Janowski functions, *AIMS Math.*, **6** (2020), 1110–1125.

34. M. S. Robertson, On the theory of univalent functions, *Ann. Math.*, **37** (1936), 374–408.

35. M. Shafiq, N. Khan, H. M. Srivastava, B. Khan, Q. Z. Ahmad, M. Tahir, Generalisation of close-to-convex functions associated with Janowski functions, *Maejo Internat. J. Sci. Technol.*, **14** (2020), 141–155.

36. L. Shi, Q. Khan, G. Srivastava, J. L. Liu, M. Arif, A study of multivalent q-starlike functions connected with circular domain, *Mathematics*, **7** (2019), 670.

37. H. M. Srivastava, Univalent functions, fractional calculus, and associated generalized hypergeometric functions, In: H. M. Srivastava, S. Owa, *Univalent functions, Fractional Calculus, and Their Applications: Srivastava*, New York: John Wiley & Sons, (1989), 329–354.

38. H. M. Srivastava, Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis, *Iran. J. Sci. Technol. Trans. A: Sci.*, **44** (2020), 327–344.

39. H. M. Srivastava, Q. Z. Ahmad, N. Khan, N. Khan, B. Khan, Hankel and toeplitz determinants for a subclass of q-starlike functions associated with a general conic domain, *Mathematics,* **7** (2019), 1–15.

40. H. M. Srivastava, S. Khan, Q. Z. Ahmad, N. Khan, S. Hussain, The Faber polynomial expansion method and its application to the general coefficient problem for some subclasses of bi-univalent functions associated with a certain q-integral operator, *Stud. Univ. Babes-Bolyai Math.*, **63** (2018), 419–436.

41. H. M. Srivastava, B. Khan, N. Khan, Q. Z. Ahmad, Coefficient inequalities for q-starlike functions associated with the Janowski functions, *Hokkaido Math. J.,* **48** (2019), 407–425.

42. H. M. Srivastava, B. Khan, N. Khan, Q. Z. Ahmad, M. Tahir, A generalized conic domain and its applications to certain subclasses of analytic functions, *Rocky Mountain J. Math.*, **49** (2019), 2325–2346.

43. H. M. Srivastava, S. Owa, *Current Topics in Analytic Function Theory*, River Edge: World Scientific, 1992.

44. H. M. Srivastava, M. Tahir, B. Khan, Q. Z. Ahmad, N. Khan, Some general classes of q-starlike functions associated with the Janowski functions, *Symmetry*, **11** (2019), 1–14.

45. H. M. Srivastava, M. Tahir, B. Khan, Q. Z. Ahmad, N. Khan, Some general families of q-starlike functions associated with the Janowski functions, *Filomat*, **33** (2019), 2613–2626.

46. H. E. O. Uçar, Coefficient inequality for q-starlike functions, *Appl. Math. Comput.,* **76** (2016), 122–126.