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Research article

Lie symmetry analysis of fractional ordinary differential equation with neutral delay

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Abstract: In this paper, Lie symmetry analysis method is employed to solve the fractional ordinary differential equation with neutral delay. The Lie symmetries for the fractional ordinary differential equation with neutral delay are obtained, and the group classification of the equation is established. The obtained Lie symmetries are used to construct the exact solutions of the fractional ordinary differential equation with neutral delay.

Keywords: fractional ordinary differential equation; neutral delay; lie symmetry analysis method; invariant solutions
Mathematics Subject Classification: 34A08, 35B06, 47E99

1. Introduction

Fractional calculus has developed rapidly in recent decades. It has been successfully applied in many aspects of science and technology. At the same time, many related masterpieces emerged, such as S. Samko et al. [1], I. Podlubny [2], R. Hilfer [3], A. Kilbas et al. [4], etc.

Time delay is a common phenomenon in real world [5]. In order to describe the models more accurately in many practical systems, we need to take fractional calculus and delay into consideration together. Therefore, it has a significance to study the solutions and their characteristics of fractional differential equation with delay.

Many analytic techniques have been developed to deal with fractional differential equations (FDEs). Among them, the Lie symmetry analysis method is an effective technique to derive exact solutions of FDEs. This method was initially advocated by Norwegian mathematician Sophus Lie in the beginning of nineteenth century and was further developed by Ovsianikov [6] and others [7–12]. The Lie symmetry analysis method of differential equations has been extended to FDEs by Gazizov et al. [13] (see also [14,15]). The effectiveness of this method has widely been demonstrated in variety of nonlinear fractional partial differential equations occurring in different areas of applied science (see [16–18]).

Recently, there are some literatures studying delay fractional differential equations by Lie symmetry technique (see [19–21]). Complete Lie group classifications of first- and second-order delay ordinary differential equations are obtained by Dorodnitsyn et al. [22] and Pue-on and Meleshko [23], respectively.

In [19], Aminu M. Nass studied the Lie symmetry analysis of fractional ordinary differential equations with neutral delay as follows

$$D_x^{\alpha} y(x) = a(x)y(x-\tau) + b(x)y'(x-\tau) + d(x)y(x) + g(x).$$

The author use symmetry analysis method, establish infinite dimension symmetry algebras and obtain the exact solutions to the equation. Aminu M. Nass [20], Kassimu Mpungu and Aminu M. Nass [21] presented the complete Lie group classifications of delay fractional differential equations.

In this paper, we continue the study and extend the Lie symmetry analysis to the following equation

$$D_x^{\alpha} y(x) = a(x)y(x-\tau) + b(x)y'(x-\tau) + d(x)y(x) + e(x)y'(x) + g(x), \qquad (FODE)$$

where the coefficients $a(x), b(x), d(x), e(x)$ and $g(x)$ are arbitrary functions with respect to independent variable x, the delay $\tau > 0$, and $0 < \alpha < 1$. This equation appears in many fields, such as population dynamics, prey-predator systems, viscoelasticity, heat flow and so on. we set out to obtain the Lie symmetries of (FODE) by the Lie point symmetry approach.

Moreover, we carry out the complete group classification of the equation, and get some concrete periodic invariant solutions to (FODE).

This paper is organized as follows. In Section 2, we recall the definition of the Riemann-Liouville fractional derivative and some relevant properties. In Section 3, we compute Lie symmetries for the fractional ordinary differential equation (FODE) and complete the group classification of the equation. In Section 4, some group invariant solutions for the (FODE) are constructed. The conclusion is given in the last section.

2. Preliminaries

In this section, we recall some standard definitions and notations in fractional calculus. For convenience, we suggest that one refers to [1–4] for details.

Definition 2.1. *The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of the function $f(t) \in L^1([a,b], \mathbb{R}_+)$, denoted by ${}_aI_t^{\alpha}$, is defined by*

$$ {}_aI_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(x)}{(t-x)^{1-\alpha}} dx, \quad t > a, $$

$$ {}_aI_t^{0} f(t) = f(t), $$

where $\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx$ is the Gamma function.

Definition 2.2. *The Riemann-Liouville fractional differential operator of order $\alpha > 0$ of the function f(t)$\in L^1([a,b], \mathbb{R}_+)$, denoted by ${}_aD_t^\alpha$, is defined by*

$$ {}_aD_t^\alpha f(t) = D_a^n I_t^{n-\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx, & n-1 < \alpha < n, n \in \mathbb{N} \\ f^{(n)}(t), & \alpha = n \in \mathbb{N} \end{cases} $$

for $t > a$.

If $\alpha = 0$, then ${}_aD_t^\alpha f(t) = f(t)$.

Some properties for the Riemann-Liouville fractional derivative and integral are as follows:

$$ {}_0I_t^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} t^{\beta+\alpha}, \quad \alpha > 0, \beta > -1, t > 0, $$

$$ {}_0D_t^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, \quad \alpha > 0, \beta > -1, t > 0, $$

$$ {}_0I_t^\alpha {}_0D_t^\alpha f(t) = f(t) - \sum_{k=1}^{n} \frac{[{}_0D_t^{\alpha-k} f(t)]_{t=0}}{\Gamma(\alpha-k+1)} t^{\alpha-k}, \quad t > 0, n-1 \le \alpha < n, n \in \mathbb{N}, $$

$$ {}_0D_t^\alpha {}_0I_t^\alpha f(t) = f(t), \quad \alpha > 0, t > 0. $$

The generalized Leibnitz rule for the Riemann-Liouville fractional derivative has the following form

$$ {}_aD_t^\alpha (f(t)g(t)) = \sum_{k=0}^{+\infty} \binom{\alpha}{k} {}_aD_t^{\alpha-k} f(t) D_t^k g(t), \quad \alpha > 0, t > a, $$

where $\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\Gamma(k+1)}$.

If $a = 0$, we denote ${}_aD_t^\alpha f(t) = D_t^\alpha f(t)$ for simplicity.

3. Lie symmetries of the fractional ordinary differential equation

Consider the fractional ordinary differential equation with neutral delay as follows,

$$ D_x^\alpha y(x) = a(x)y(x-\tau) + b(x)y'(x-\tau) + d(x)y(x) + e(x)y'(x) + g(x), \tag{3.1} $$

where the coefficients $a(x), b(x), d(x), e(x)$ and $g(x)$ are arbitrary functions with respect to independent variable x, the delay $\tau > 0$,and $0 < \alpha < 1$.

We assume that the FODE (3.1) is invariant under the one-parameter(ϵ) Lie group of continuous point transformations in (x, y) plane, i.e.,

$$ \begin{aligned} \bar{x} &= x + \epsilon\xi(x,y) + o(\epsilon) \\ \bar{y} &= y + \epsilon\eta(x,y) + o(\epsilon) \\ \bar{y_\tau} &= y_\tau + \epsilon\eta_\tau + o(\epsilon) \\ \bar{y'} &= y' + \epsilon\eta^1 + o(\epsilon) \\ \bar{y'_\tau} &= y'_\tau + \epsilon\eta_\tau^1 + o(\epsilon) \end{aligned} \tag{3.2} $$

$$D^{\alpha}_{\bar{x}}\bar{y} = D^{\alpha}_{x}y + \epsilon\eta^{\alpha} + o(\epsilon)$$

for some smooth functions $\xi(x, y)$ and $\eta(x, y)$ known as infinitesimals, and $y_\tau = y(x - \tau)$, $\xi_\tau = \xi(x - \tau, y(x - \tau))$, $\eta_\tau = \eta(x - \tau, y(x - \tau))$.

According to the Lie group theory, the group generator X of the point transformations (3.2) is expressed as

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}. \tag{3.3}$$

So the prolongation of the above group generator X has the form

$$prX = X + \eta^{1}\frac{\partial}{\partial y'} + \eta^{1}_{\tau}\frac{\partial}{\partial y'_{\tau}} + \eta^{\alpha}\frac{\partial}{\partial y^{\alpha}}, \tag{3.4}$$

where

$$\eta^{1} = D_x(\eta - y'\xi) + \xi D_x(y'),\ \ \eta^{1}_{\tau} = D_x(\eta_\tau - y'_\tau\xi_\tau) + \xi_\tau D_x(y'_\tau),\ \ \eta^{\alpha} = D^{\alpha}_{x}(\eta - y'\xi) + \xi D_x(D^{\alpha}_{x}(y))$$

and D_x is the total derivative with respect to x.

The prolongation of Lie-Bäcklund generator [10] equivalent to infinitesimal prolongation (3.4) is

$$\bar{X} = \zeta\frac{\partial}{\partial y} + \zeta_\tau\frac{\partial}{\partial y_\tau} + \zeta^{1}\frac{\partial}{\partial y'} + \zeta^{1}_{\tau}\frac{\partial}{\partial y'_{\tau}} + \zeta^{\alpha}\frac{\partial}{\partial y^{\alpha}} \tag{3.5}$$

where

$$\zeta = \eta - y'\xi,\ \ \zeta_\tau = \eta_\tau - y'_\tau\xi_\tau,\ \ \zeta^{1} = D_x(\eta - y'\xi),\ \ \zeta^{1}_{\tau} = D_x(\eta_\tau - y'_\tau\xi_\tau),\ \ \zeta^{\alpha} = D^{\alpha}_{x}(\eta - y'\xi).$$

From the definition of total derivative, we have

$$\zeta^{1} = D_x(\eta - y'\xi) = \eta_x + (\eta_y - \xi_x)y' - \xi_y y'^{2} - \xi y'', \tag{3.6}$$

$$\zeta^{1}_{\tau} = D_x(\eta_\tau - y'_\tau\xi_\tau) = \eta^{\tau}_{x} + (\eta^{\tau}_{y} - \xi^{\tau}_{x})y'_\tau - \xi^{\tau}_{y}y'^{2}_{\tau} - \xi_\tau y_\tau''. \tag{3.7}$$

Since

$$\eta^{\alpha} = D^{\alpha}_{x}(\eta - y'\xi) + \xi D_x(D^{\alpha}_{x}(y)) = D^{\alpha}_{x}(\eta) - \sum_{n=0}^{+\infty}\binom{\alpha}{n+1}D^{\alpha-n}_{x}(y)D^{n+1}_{x}(\xi),$$

we obtain

$$\zeta^{\alpha} = D^{\alpha}_{x}(\eta - y'\xi) = D^{\alpha}_{x}(\eta) - \sum_{n=0}^{+\infty}\binom{\alpha}{n+1}D^{\alpha-n}_{x}(y)D^{n+1}_{x}(\xi) - \xi D^{\alpha+1}_{x}(y). \tag{3.8}$$

Remark: The infinitesimal transformations (3.2) should conserve the structure of the Riemann-Liouville fractional derivative operator, of which, the lower limit in the integral is fixed. Therefore, the manifold $x = 0$ should be invariant with respect to transformations (3.2). The invariance condition arrives at

$$\xi(x, y)|_{x=0} = 0.$$

The one-parameter Lie symmetry transformations (3.2) are admitted by FODE (3.1), if the following invariance criterion holds,

$$\bar{X}(D^{\alpha}_{x}y(x) - a(x)y(x - \tau) - b(x)y'(x - \tau) - d(x)y(x) - e(x)y'(x) - g(x))|_{(3.1)} = 0, \tag{3.9}$$

where the infinitesimal generator $\bar{X}$ is defined in (3.5). Eq (3.9) can be abbreviated as

$$\zeta^{\alpha} - a\zeta_{\tau} - b\zeta_{\tau}^{1} - d\zeta - e\zeta^{1}|_{(3.1)} = 0, \tag{3.10}$$

which is known as the determining equation.

Put

$$D_x^{(\alpha+1)}y = a'y_{\tau} + ay'_{\tau} + b'y'_{\tau} + by''_{\tau} + d'y + dy' + e'y' + ey'' + g'$$

into (3.10) and let coefficients of y, y_{τ} and their derivatives in the determining Eq (3.10) to be zero, we can obtain the over-determined system of differential equations as follows,

$$\xi^{\tau} = \xi \tag{3.11}$$

$$\xi_y^{\tau} = \xi_y = 0 \tag{3.12}$$

$$\binom{\alpha}{n}\frac{\partial^n \eta_y}{\partial x^n} = \binom{\alpha}{n+1}D_x^{n+1}(\xi), n \in \mathbb{N} \tag{3.13}$$

from the coefficients of y''_{τ}, y'^2, y'^2_{τ} and $D_x^{\alpha-n}(y)$. From (3.11), (3.12) and (3.13), we get

$$\xi(x, y) = \xi(x), \quad \eta(x, y) = \psi_1(x)y + \psi_2(x), \tag{3.14}$$

where $\xi(x)$, $\psi_1(x)$ are periodic functions with period τ, i.e.,

$$\xi(x - \tau) = \xi(x), \quad \psi_1(x - \tau) = \psi_1(x),$$

and $\psi_2(x)$ is an arbitrary function. Put (3.14) into the over-determined system, the simplified forms are

$$-\alpha\xi_x e - \xi e' + \xi_x e = 0 \tag{3.15}$$

$$-\alpha\xi_x b - \xi b' + \xi_x^{\tau} b = 0 \tag{3.16}$$

$$-\alpha\xi_x d - \xi d' - e\frac{\mathrm{d}\psi_1}{\mathrm{d}x} = 0 \tag{3.17}$$

$$-\alpha\xi_x a - \xi a' - b\frac{\mathrm{d}\psi_1^{\tau}}{\mathrm{d}x} = 0 \tag{3.18}$$

$$-\alpha\xi_x g - \xi g' + g\psi_1 - a\psi_2^{\tau} - b\frac{\mathrm{d}\psi_2^{\tau}}{\mathrm{d}x} - d\psi_2 - e\frac{\mathrm{d}\psi_2}{\mathrm{d}x} - e\psi_1 + D_x^{\alpha}(\psi_2) = 0. \tag{3.19}$$

From (3.13), (3.14), (3.17) and $\xi(x, y)|_{x=0} = 0$, we obtain

(i) : $\xi(x) = 0, \ \psi_1(x) = c_1$

(ii) : $\xi(x) = c_1 \sin\frac{2\pi x}{\tau}, \ \psi_1(x) = c_1\frac{2\pi(\alpha-n)}{\tau(n+1)}\cos\frac{2\pi x}{\tau}, n \in \mathbb{N}$

(iii) : $\xi(x) = c_1 \sin\frac{2(n+1)\pi x}{\tau}, \ \psi_1(x) = c_1\frac{2\pi(\alpha-n)}{\tau}\cos\frac{2(n+1)\pi x}{\tau}, n \in \mathbb{N}$.

3.1. Case (i): $\xi(x) = 0, \psi_1(x) = c_1$

From $\xi(x, y) = 0$, $\eta(x, y) = \psi_1(x)y + \psi_2(x) = c_1 y + \psi_2(x)$, we can get the following group generator of the FODE (3.1):

$$X_1 = (c_1 y + \psi_2(x))\frac{\partial}{\partial y},$$

where, c_1 is an arbitrary constant, $\psi_2(x)$ satisfies Eq (3.19). For arbitrary functions $a(x), b(x), d(x), e(x)$ and $g(x)$, it is also difficult to obtain $\psi_2(x)$ from Eq (3.19). But for some special functions $a(x), b(x), d(x), e(x)$ and $g(x)$, we can get some concrete Lie symmetries.

For example, if $\psi_2(x)$ is an arbitrary function, we get

$$X_\infty = \psi_2(x)\frac{\partial}{\partial y}$$

with functions $a(x), b(x), d(x), e(x)$ and $g(x)$ satisfying Eq (3.19).

If $\psi_2(x) = 0$, we get

$$X_1 = y\frac{\partial}{\partial y}$$

with functions $e(x), g(x)$ satisfying $e(x) = g(x)$.

If $\psi_2(x) = c_4 \neq 0$, we get

$$X_1 = y\frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial y}$$

with function $g(x)$ satisfying

$$g(x) = \frac{c_4}{c_1}a(x) + \frac{c_4}{c_1}d(x) + e(x) - \frac{c_4}{c_1}\frac{x^{-\alpha}}{\Gamma(1-\alpha)}.$$

In what follows, we present Lie symmetries for some special functions $a(x), b(x), d(x), e(x)$ and $g(x)$.

3.2. Case (ii): $\xi(x) = c_1 \sin\frac{2\pi x}{\tau}, \psi_1(x) = c_1\frac{2\pi(\alpha-n)}{\tau(n+1)}\cos\frac{2\pi x}{\tau}, n \in \mathbb{N}$

By (3.15) and (3.16), we obtain

$$b(x) = e(x) = c_2(\sin\frac{2\pi x}{\tau})^{1-\alpha}. \tag{3.20}$$

Put (3.14) into (3.17) and (3.18), we can get

$$a(x) = d(x) = (\sin\frac{2\pi x}{\tau})^{-\alpha}[-c_2\frac{2\pi(\alpha-n)}{\tau(n+1)}\cos\frac{2\pi x}{\tau} + c_3] \tag{3.21}$$

Therefore, the FODE (3.1) admits infinite dimensional Lie algebra, which is spanned by the following group generators

$$X_1 = \sin\frac{2\pi x}{\tau}\frac{\partial}{\partial x} + y\frac{2\pi(\alpha-n)}{\tau(n+1)}\cos\frac{2\pi x}{\tau}\frac{\partial}{\partial y}, \quad X_\infty = \psi_2(x)\frac{\partial}{\partial y}$$

for any functions $g(x)$ and $\psi_2(x)$ satisfying Eq (3.19), and the coefficient functions $a(x), b(x), d(x)$ and $e(x)$ defined in (3.20) and (3.21) with period τ .

Particularly, for (3.19), we assume ψ_2 = constant = c_4, and obtain the following two forms of $g(x)$, i.e.,

$$g(x) = \begin{cases} (\sin\frac{2\pi x}{\tau})^{-\frac{n(\alpha+1)}{n+1}}[-\frac{c_2(\alpha-n)}{2n+1-\alpha}(\sin\frac{2\pi x}{\tau})^{\frac{2n+1-\alpha}{n+1}}+c_5], \quad \psi_2=0 & (3.22)\\ (\sin\frac{2\pi x}{\tau})^{-\frac{n(\alpha+1)}{n+1}}[\frac{c_4}{c_1\Gamma(1-\alpha)}\int(\sin\frac{2\pi x}{\tau})^{\frac{n\alpha-1}{n+1}}\mathrm{d}x-\frac{2c_4c_3}{c_1}\int(\sin\frac{2\pi x}{\tau})^{-\frac{\alpha+1}{n+1}}\mathrm{d}x & \\ -\frac{2c_4c_2}{c_1}(\sin\frac{2\pi x}{\tau})^{\frac{n-\alpha}{n+1}}-\frac{c_2(\alpha-n)}{2n+1-\alpha}(\sin\frac{2\pi x}{\tau})^{\frac{2n+1-\alpha}{n+1}}+c_6], \quad \psi_2=c_4\neq 0 & (3.23)\end{cases}$$

Therefore, if $\psi_2 = c_4 = 0$, the FODE (3.1) admits one-dimension Lie algebra, which is spanned by

$$X_1 = \sin\frac{2\pi x}{\tau}\frac{\partial}{\partial x} + y\frac{2\pi(\alpha-n)}{\tau(n+1)}\cos\frac{2\pi x}{\tau}\frac{\partial}{\partial y}$$

for given functions $a(x), b(x), d(x), e(x)$ and $g(x)$ defined in (3.20), (3.21) and (3.22).

In the same way, if $\psi_2 = c_4 \neq 0$, the FODE (3.1) admits one extra dimensional Lie algebra spanned by the infinitesimal operators

$$X_1 = \sin\frac{2\pi x}{\tau}\frac{\partial}{\partial x} + y\frac{2\pi(\alpha-n)}{\tau(n+1)}\cos\frac{2\pi x}{\tau}\frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial y}$$

for given functions $a(x), b(x), d(x), e(x)$ and $g(x)$ defined in (3.20), (3.21) and (3.23). Note, the $c_i (i = 1, \cdots, 6)$ are arbitrary constants.

3.3. *Case (iii):* $\xi(x) = c_1\sin\frac{2(n+1)\pi x}{\tau}, \psi_1(x) = c_1\frac{2\pi(\alpha-n)}{\tau}\cos\frac{2(n+1)\pi x}{\tau}, n\in\mathbb{N}$

In this case, we can obtain

$$b(x) = e(x) = c_2(\sin\frac{2(n+1)\pi x}{\tau})^{1-\alpha} \tag{3.24}$$

$$a(x) = d(x) = (\sin\frac{2(n+1)\pi x}{\tau})^{-\alpha}[-c_2\frac{2\pi(\alpha-n)}{\tau}\cos\frac{2(n+1)\pi x}{\tau}+c_3] \tag{3.25}$$

from Eqs (3.15)–(3.18). Therefore, the FODE (3.1) admits infinite dimensional Lie algebra, which is spanned by the following infinitesimal operators

$$X_1 = \sin\frac{2(n+1)\pi x}{\tau}\frac{\partial}{\partial x} + y\frac{2\pi(\alpha-n)}{\tau}\cos\frac{2(n+1)\pi x}{\tau}\frac{\partial}{\partial y}, \quad X_\infty = \psi_2(x)\frac{\partial}{\partial y}$$

for any function $g(x)$ and $\psi_2(x)$ satisfying Eq (3.19), and the coefficient functions $a(x), b(x), d(x)$and $e(x)$ defined in (3.20) and (3.21) with period τ .

As same as Case (ii), we assume ψ_2 = constant = c_4, and obtain the following two forms of $g(x)$, i.e.,

$$g(x) = \begin{cases} (\sin\frac{2(n+1)\pi x}{\tau})^{-\frac{n(\alpha+1)}{n+1}}[-\frac{c_2(\alpha-n)}{2n+1-\alpha}(\sin\frac{2(n+1)\pi x}{\tau})^{\frac{2n+1-\alpha}{n+1}}+c_5], \quad \psi_2=0 & (3.26)\\ (\sin\frac{2(n+1)\pi x}{\tau})^{-\frac{n(\alpha+1)}{n+1}}[\frac{c_4}{c_1\Gamma(1-\alpha)}\int(\sin\frac{2(n+1)\pi x}{\tau})^{\frac{n\alpha-1}{n+1}}\mathrm{d}x- & \\ \frac{2c_4c_3}{c_1}\int(\sin\frac{2(n+1)\pi x}{\tau})^{-\frac{\alpha+1}{n+1}}\mathrm{d}x-\frac{2c_4c_2}{c_1}(\sin\frac{2(n+1)\pi x}{\tau})^{\frac{n-\alpha}{n+1}}- & \\ \frac{c_2(\alpha-n)}{2n+1-\alpha}(\sin\frac{2(n+1)\pi x}{\tau})^{\frac{2n+1-\alpha}{n+1}}+c_6], \quad \psi_2=c_4\neq 0 & (3.27)\end{cases}$$

Therefore, if $\psi_2 = c_4 = 0$, FODE (3.1) admits one-dimension Lie algebra, which is spanned by

$$X_1 = \sin\frac{2(n+1)\pi x}{\tau}\frac{\partial}{\partial x} + y\frac{2\pi(\alpha-n)}{\tau}\cos\frac{2(n+1)\pi x}{\tau}\frac{\partial}{\partial y}$$

for given functions $a(x), b(x), d(x), e(x)$ and $g(x)$ defined in (3.24), (3.25) and (3.26).

If $\psi_2 = c_4 \neq 0$, FODE (3.1) admits one extra dimensional Lie algebra spanned by the differential operators

$$X_1 = \sin\frac{2(n+1)\pi x}{\tau}\frac{\partial}{\partial x} + y\frac{2\pi(\alpha-n)}{\tau}\cos\frac{2(n+1)\pi x}{\tau}\frac{\partial}{\partial y},\ \ X_2 = \frac{\partial}{\partial y}$$

for given functions $a(x), b(x), d(x), e(x)$ and $g(x)$ defined in (3.24), (3.25) and (3.27), where $c_i(i = 1, \cdots, 6)$ are arbitrary constants.

4. Invariant solutions

In this section, we use the admitted group generators to construct some analytical solutions for fractional ordinary differential equation with neutral delay (FODE).

4.1. Case (i): $\xi(x) = 0, \psi_1(x) = c_1$

According to the Lie group theory, we can not obtain a new solution, when $\psi_2(x) = 0$ with

$$X_1 = y\frac{\partial}{\partial y},$$

$\psi_2(x) = c_4 \neq 0$ with

$$X_1 = y\frac{\partial}{\partial y},\ \ X_2 = \frac{\partial}{\partial y},$$

and $\psi_2(x)$ is an arbitrary function with

$$X_\infty = \psi_2(x)\frac{\partial}{\partial y}.$$

4.2. Case (ii): $\xi(x) = c_1 \sin\frac{2\pi x}{\tau}, \psi_1(x) = c_1\frac{2\pi(\alpha-n)}{\tau(n+1)}\cos\frac{2\pi x}{\tau}, n \in \mathbb{N}$

4.2.1. $\psi_2 = 0$

The characteristic equation associated with the group generator X_1 is

$$\frac{\mathrm{d}x}{\sin\frac{2\pi x}{\tau}} = \frac{\mathrm{d}y}{y\frac{2\pi(\alpha-n)}{\tau(n+1)}\cos\frac{2\pi x}{\tau}}, \tag{4.1}$$

where

$$X_1 = \sin\frac{2\pi x}{\tau}\frac{\partial}{\partial x} + y\frac{2\pi(\alpha-n)}{\tau(n+1)}\cos\frac{2\pi x}{\tau}\frac{\partial}{\partial y}.$$

Solving (4.1), we have

$$y(x) = c(\sin\frac{2\pi x}{\tau})^{\frac{\alpha-n}{n+1}}, n \in \mathbb{N}, \tag{4.2}$$

where c is a constant.

Graphical representations of the periodic solutions (4.2) are given in Figures 1–3. The behaviors of the solutions depend on the delay (τ), the coefficient (c), the fractional order (α) and the natural number (n). For example, in Figure 1, we set $\tau = 1$, $c = 0.8$, $\alpha = 0.5$ and $n = 1$.

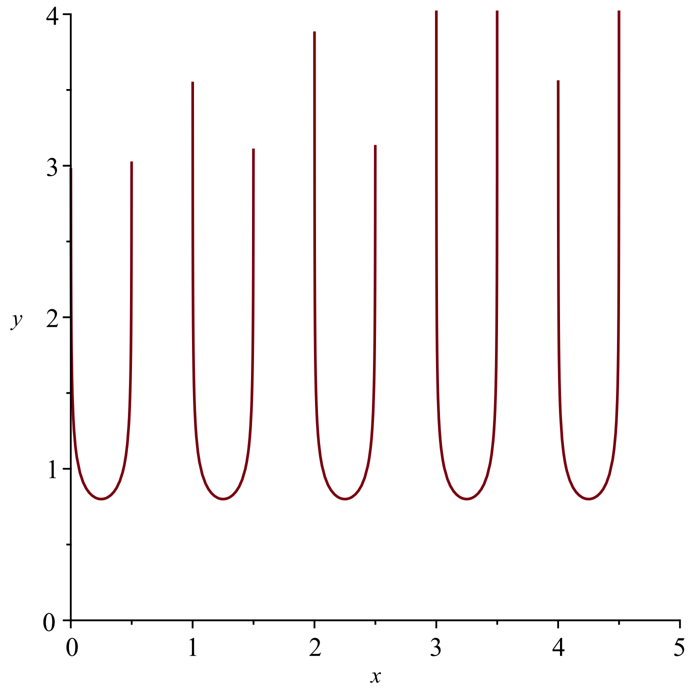

Figure 1. Graph of solutions (4.2) with $\tau = 1$, $c = 0.8$, $\alpha = 0.5$ and $n = 1$.

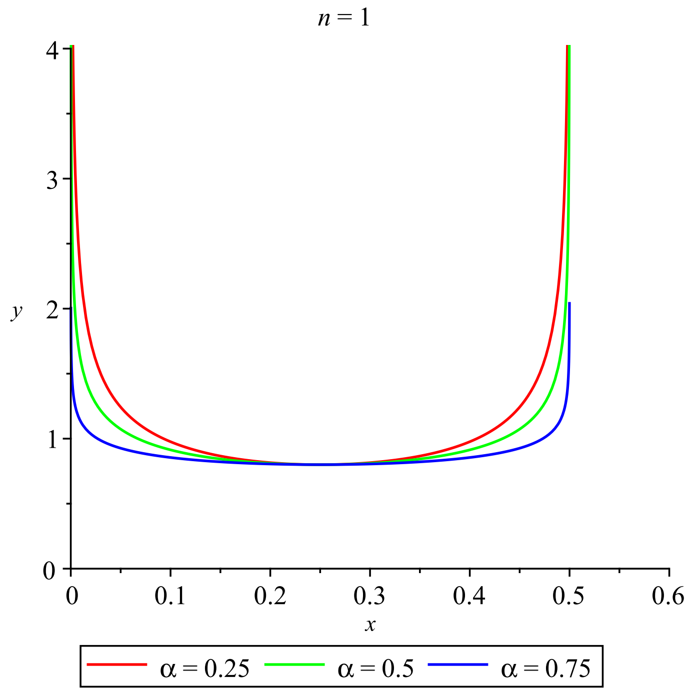

Figure 2. Graph of solutions (4.2) in a period with different fractional-order and $\tau = 1$, $c = 0.8$, $n = 1$.

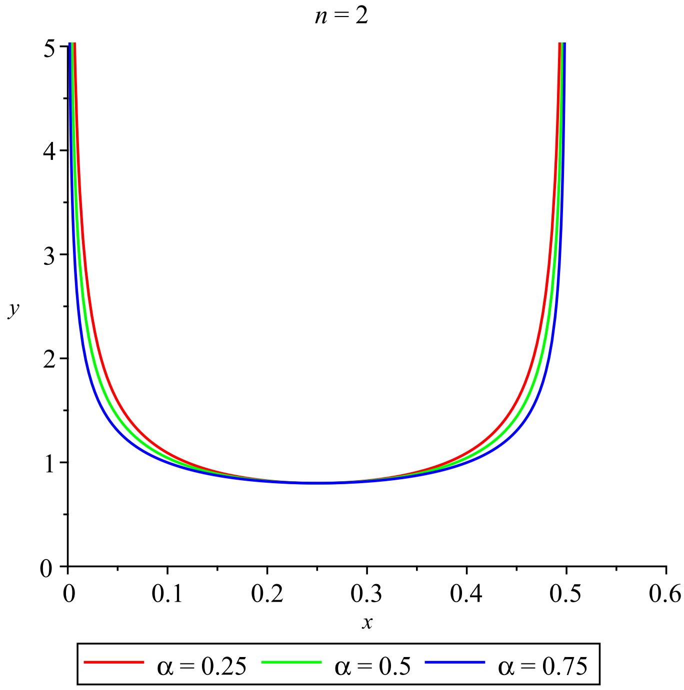

Figure 3. Graph of solutions (4.2) in a period with different fractional-order and $\tau = 1$, $c = 0.8$, $n = 2$.

4.2.2. $\psi_2 = c_4 \neq 0$

In this case, considering the linear combinations of the group generators X_1 and X_2, i.e.,

$$X_1 + X_2 = \sin\frac{2\pi x}{\tau}\frac{\partial}{\partial x} + (y\frac{2\pi(\alpha - n)}{\tau(n+1)}\cos\frac{2\pi x}{\tau} + 1)\frac{\partial}{\partial y},$$

we can get the characteristic equation associated with the admitted operator $X_1 + X_2$ as follows,

$$\frac{\mathrm{d}x}{\sin\frac{2\pi x}{\tau}} = \frac{\mathrm{d}y}{y\frac{2\pi(\alpha-n)}{\tau(n+1)}\cos\frac{2\pi x}{\tau} + 1}, \tag{4.3}$$

Its solutions are

$$y(x) = (\sin\frac{2\pi x}{\tau})^{\frac{\alpha-n}{n+1}}[\int(\sin\frac{2\pi x}{\tau})^{-\frac{\alpha+1}{n+1}}\mathrm{d}x + c], n \in \mathbb{N}, \tag{4.4}$$

where c is a constant.

4.3. Case (iii): $\xi(x) = c_1\sin\frac{2(n+1)\pi x}{\tau}, \psi_1(x) = c_1\frac{2\pi(\alpha-n)}{\tau}\cos\frac{2(n+1)\pi x}{\tau}, n \in \mathbb{N}$

4.3.1. $\psi_2 = 0$

The characteristic equation associated with the group generator X_1 is

$$\frac{\mathrm{d}x}{\sin\frac{2(n+1)\pi x}{\tau}} = \frac{\mathrm{d}y}{y\frac{2\pi(\alpha-n)}{\tau}\cos\frac{2(n+1)\pi x}{\tau}}, \tag{4.5}$$

where

$$X_1 = \sin\frac{2(n+1)\pi x}{\tau}\frac{\partial}{\partial x} + y\frac{2\pi(\alpha-n)}{\tau}\cos\frac{2(n+1)\pi x}{\tau}\frac{\partial}{\partial y}.$$

Solving (4.5), we can get invariant solutions

$$y(x) = c(\sin\frac{2(n+1)\pi x}{\tau})^{\frac{\alpha-n}{n+1}}, n \in \mathbb{N}, \tag{4.6}$$

where c is a constant.

Graphical representations of the periodic solutions (4.6) are displayed in Figures 4–6. In Figure 4, we set $\tau = 1$, $c = 0.8$, $\alpha = 0.5$ and $n = 1$.

4.3.2. $\psi_2 = c_4 \neq 0$

In this case, considering the linear combinations of the group generators X_1 and X_2, i.e.,

$$X_1 + X_2 = \sin\frac{2(n+1)\pi x}{\tau}\frac{\partial}{\partial x} + (y\frac{2\pi(\alpha-n)}{\tau}\cos\frac{2(n+1)\pi x}{\tau} + 1)\frac{\partial}{\partial y},$$

we can get the characteristic equation associated with the admitted operator $X_1 + X_2$ as follows,

$$\frac{\mathrm{d}x}{\sin\frac{2(n+1)\pi x}{\tau}} = \frac{\mathrm{d}y}{y\frac{2\pi(\alpha-n)}{\tau}\cos\frac{2(n+1)\pi x}{\tau} + 1}. \tag{4.7}$$

Solving (4.7), we can obtain the following invariant solutions,

$$y(x) = (\sin\frac{2(n+1)\pi x}{\tau})^{\frac{\alpha-n}{n+1}}[\int(\sin\frac{2(n+1)\pi x}{\tau})^{-\frac{\alpha+1}{n+1}}\mathrm{d}x + c], n \in \mathbb{N}, \tag{4.8}$$

where c is a constant.

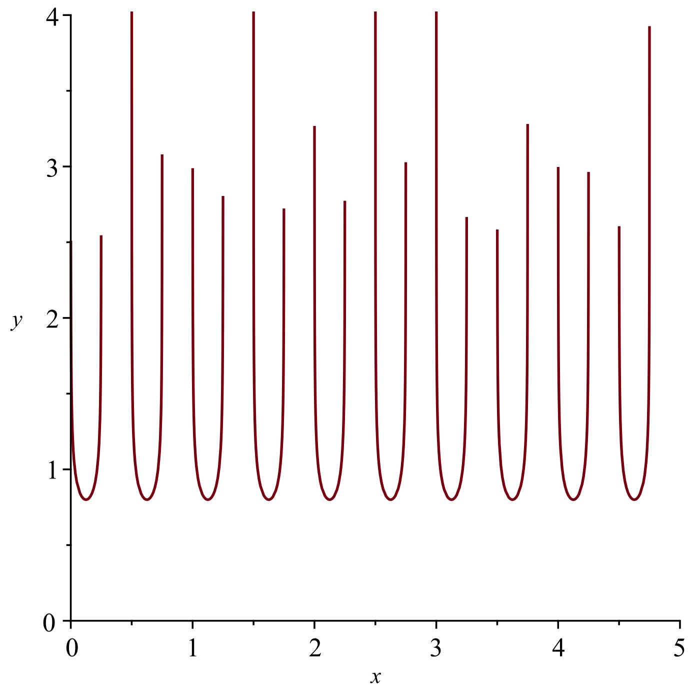

Figure 4. Graph of solutions (4.6) with $\tau = 1$, $c = 0.8$, $\alpha = 0.5$ and $n = 1$.

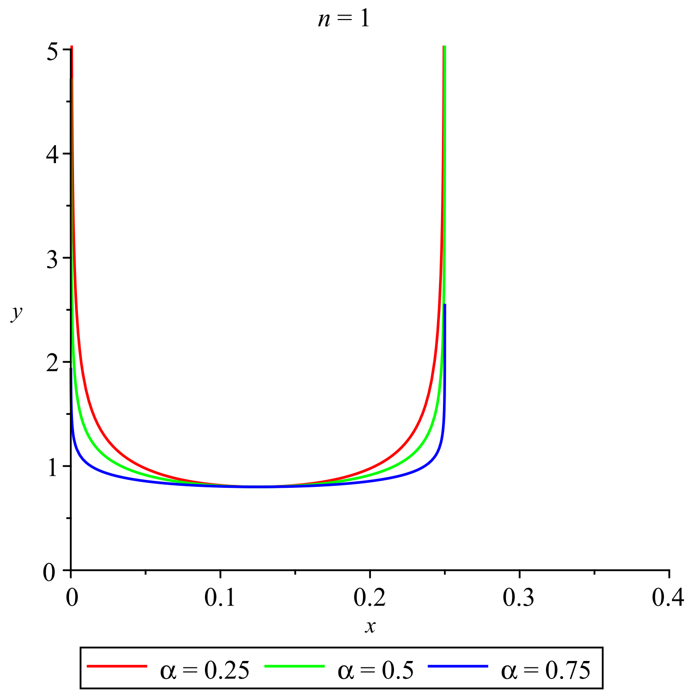

Figure 5. Graph of solutions (4.6) in a period with different fractional-order and $\tau = 1$, $c = 0.8$, $n = 1$.

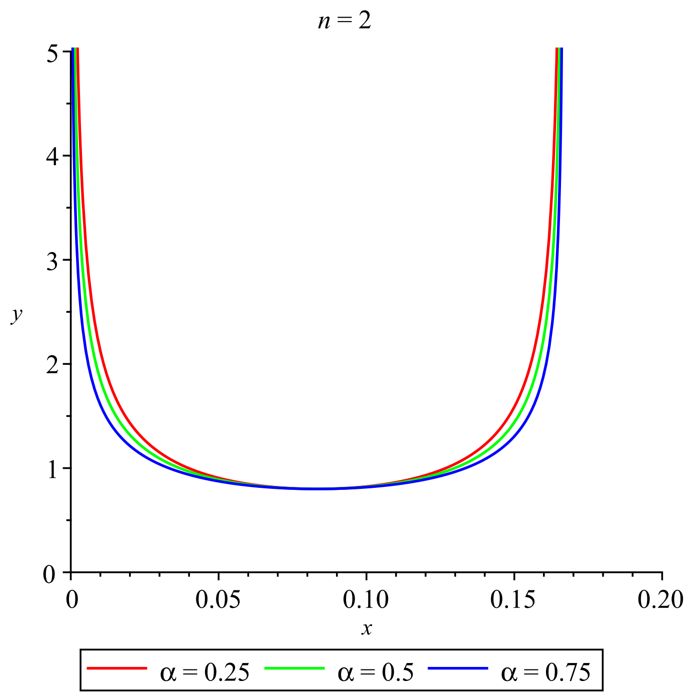

Figure 6. Graph of solutions (4.6) in a period with different fractional-order and $\tau = 1$, $c = 0.8$, $n = 2$.

5. Conclusions

In this paper, it is proved that Lie symmetry technique is a powerful method to analysis the fractional ordinary differential equation with neutral delay, i.e.,

$$D_x^\alpha y(x) = a(x)y(x-\tau) + b(x)y'(x-\tau) + d(x)y(x) + e(x)y'(x) + g(x).$$

Firstly, we obtain the infinitesimal symmetries for the considered equation, and carry out a group classification. Secondly, we use the obtained Lie symmetries to construct the invariant solutions. Finally, we get some periodic solutions of the fractional ordinary differential equation with neutral delay.

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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