Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(4): 3583–3591.
DOI:10.3934/math.2021213
Received: 22 September 2020
Accepted: 07 January 2021
Published: 21 January 2021

Research article

On spinor construction of Bertrand curves

Tülay Erişir*

Department of Mathematics, Erzincan Binali Yıldırım University, Erzincan, 24100, Turkey

* **Correspondence:** Email: tulay.erisir@erzincan.edu.tr.

Abstract: Spinors permeate all of modern physics and have also an important place in mathematics. Spinors are used intensively in modern theoretical physics and differential geometry. In this study, spinors are used for a different representation of differential geometry in $\mathbb{E}^3$. The goal of this study is also the spinor structure lying at the basis of differential geometry. In this paper, firstly, spinors are introduced algebraically. Then, the spinor construction of Bertrand curves is defined. Moreover, the angle notion for these spinors is given. In this way, a different geometric construction of spinors is established in this paper.

Keywords: spinors; Bertrand curves
Mathematics Subject Classification: 15A66, 53A04

1. Introduction

Spinors are structures used in many fields of applied sciences. In physics, the theory of spinors is especially used in applications electron spin and theory of relativity in quantum mechanics. The wave function of a particle whose spin is $1/2$ is called spinor. In addition to that, the application of spinors in electromagnetic theory is very important. The construction to obtain a spin structure on a 4-dimensional space-time are extended to obtain spinors in physics, such as the Dirac spinor. The Dirac representation of the column vectors in Heisenberg representation corresponds to $Ket \equiv |\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$. Here, ψ_1 and ψ_2 are complex numbers [7, 16, 24]. The Pauli spin matrices mentioned here are with dimension $2x2$, and the two-component complex column matrices acting on these matrices are spinors. Spinors can be evaluated as concrete objects when choosing Cartesian coordinates. For example, in three dimensional Euclidean space, spinors can be generated by choosing Pauli spin matrices corresponding to the three coordinate axes.

In quantum theory, spin is represented by Pauli matrices. Also, Pauli matrices are a representation of Clifford spin algebra and therefore all properties of Pauli matrices derive from basic algebra. One of the most important studies giving the relationship between Clifford algebra and spinors is [23]. Clifford algebras constitute a highly intuitive formalism, having an intimate relationship to quantum field theory. Among the existing approaches to Clifford algebras and spinors this book is unique in that it provides a didactical presentation of the topic. Also, the first person to study spinors in geometric terms is Cartan, one of the founders of Clifford algebra. Cartan studied spinors in a geometric sense [4]. Cartan's study gives the spinor representation of the basic geometric definitions. So, it is a very impressive reference in terms of the geometry of the spinors. In Cartan's study, it was emphasized that the set of isotropic vectors in the vector space $\mathbb{C}^3$ forms a two-dimensional surface in the space $\mathbb{C}^2$. It is seen that each isotropic vector in the complex vector space $\mathbb{C}^3$ corresponds to two vectors in the space $\mathbb{C}^2$. Conversely; these vectors in space $\mathbb{C}^2$ correspond to the same isotropic vector. Cartan expressed that these complex vectors with two-dimensional [4].

In 1927, when Pauli introduced his spin matrices, he first applied the spinors to mathematical physics [20]. After that, the relationship between the spinors and the Lorentz group was demonstrated by Dirac, and the fully relativistic theory of electron spin was discovered [8]. In 1930, Juvet and Sauter represented the spinor space as the left ideal of a matrix algebra [13, 18, 21]. In further detail, instead of taking the spinors as complex valued two-dimensional column vectors, as Pauli did, if the spinors are taken as complex-valued 2x2-dimensional matrices where the left column of the elements are not zero, then the spinor space becomes a minimal left ideal in Mat (2, C) [17]. Later, in differential geometry the spinor representation of curve theory was expressed by Torres del Castillo and Barrales [6]. That study is one of the basic references of our study. Then, the spinor formulation of the Darboux frame on a directed surface and Bishop frame of curves in $\mathbb{E}^3$ were given in [15, 22]. After that, the hyperbolic spinor corresponding to the Frenet frame of curves in Minkowski space $\mathbb{E}^3_1$ was expressed by Ketenci et al. [14]. Moreover, Erişir et. al obtained the spinor formulation of the alternative frame of a curve in Minkowski space, and the spinor formulation of the relationship between Frenet frame and Bishop frame [10]. Thus, Balcı et al. gave the hyperbolic spinor formulation of the Darboux frame in $\mathbb{E}^3_1$ [1]. Finally, Erişir and Kardağ have shown a new representation of the Involute Evolute curves in $\mathbb{E}^3$ with the help of spinors [11].

In this study, answer to the following question has been sought: "What is the geometric construction of spinors for Bertrand curves?". For this, firstly, the spinor characterization of unit speed and not unit speed curve is calculated. Then, Bertrand curves have been considered and these curves have been represented by spinors in the space $\mathbb{C}^2$. Later, considering the relations between Bertrand curves, the spinor relations corresponding to these curves have been obtained. After that, the answer of question "If the angle between the tangent vectors of the Bertrand curves is θ, what is the angle between the spinors corresponding to these curves?" has been found. In this way, a different geometric construction of the spinors has been shown.

2. Preliminaries

A curve is considered as differentiable at the each point of an open interval. Thus, one can construct a set of mutually orthogonal unit vectors on this curve. These vectors are called tangent, normal and binormal unit vectors or the Serret-Frenet frame, collectively. So, let us consider that the regular curve (α) which is differentiable function so that $\alpha : I \to \mathbb{E}^3$, $(I \subseteq \mathbb{R})$ has the arbitrary parameter t. Moreover, for $\forall t \in I$ the Frenet vectors on the point $\alpha(t)$ of the curve (α) are given by $\{\boldsymbol{T}(t), \boldsymbol{N}(t), \boldsymbol{B}(t)\}$.

So, these Frenet vectors are obtained by the equations $\boldsymbol{T}(t) = \frac{1}{\|\alpha'(t)\|}\alpha'(t)$, $\boldsymbol{N}(t) = \boldsymbol{B}(t) \times \boldsymbol{T}(t)$ and $\boldsymbol{B}(t) = \frac{1}{\|\alpha'(t)\times\alpha''(t)\|}(\alpha'(t) \times \alpha''(t))$ where "$'$" is the derivative with respect to arbitrary parameter t, κ and τ are the curvature and torsion of this curve (α) [12]. Moreover, the Frenet derivative formulas of this curve (α) are given by $\boldsymbol{T}' = \|\alpha'\| \kappa\boldsymbol{N}$, $\boldsymbol{N}' = \|\alpha'\| (-\kappa\boldsymbol{T} + \tau\boldsymbol{B})$, $\boldsymbol{B}' = -\|\alpha'\| \tau\boldsymbol{N}$ [12]. Consider that the parameter of the curve (α) is the arc-length parameter s. So, if there is the equation $\|\dot{\alpha}(s)\| = 1$ on the point $\forall s \in I$ of the curve $\alpha : I \to \mathbb{E}^3$, $(I \subseteq \mathbb{R})$, the curve (α) is called a curve parameterized by the arc-length or a unit speed curve. Thus, the Frenet vectors of the curve (α) parameterized by arc-length can be obtained by $\boldsymbol{T}(s) = \dot{\alpha}(s)$, $\boldsymbol{N}(s) = \frac{1}{\|\ddot{\alpha}(s)\|}\ddot{\alpha}(s)$ and $\boldsymbol{B}(s) = \boldsymbol{T}(s) \times \boldsymbol{N}(s)$ where "$.$" is the derivative with respect to arc-length parameter s. Moreover, the Frenet formulas of this curve are as $\dot{\boldsymbol{T}} = \kappa\boldsymbol{N}, \dot{\boldsymbol{N}} = -\kappa\boldsymbol{T} + \tau\boldsymbol{B}, \dot{\boldsymbol{B}} = -\tau\boldsymbol{N}$ [12].

French mathematician Saint-Venant raised the question in 1845 on whether the principal normal vectors of a curve α on a ruled surface produced by the principal normal vectors of a curve α in three dimensional Euclidean space can be the same as the principal normal vectors of a second curve. This question was answered by an article published by Bertrand in 1850. For this reason, the curves with this feature started to be called Bertrand curves after that study. Bertrand expressed that to have a second curve with this property, there is a linear relationship between the curvature and torsion of the original curve. These curve pairs, which are a remarkable subject for scientists, have been treated in many studies [2, 3, 5, 9, 12, 19]. So, for the Bertrand curves, the following definition and theorems can be given.

Definition 2.1. Let $\alpha,\beta : I \to \mathbb{E}^3$ be two curves and the Frenet vector fields of these curves be $\{\boldsymbol{T},\boldsymbol{N},\boldsymbol{B}\}$ and $\{\boldsymbol{T}^*,\boldsymbol{N}^*,\boldsymbol{B}^*\}$, respectively. If the vector pair $\{\boldsymbol{N},\boldsymbol{N}^*\}$ is linearly dependent, this pair (α,β) is called Bertrand curves [12].

Theorem 2.2. *Let the curves $\alpha,\beta : I \to \mathbb{E}^3$ be Bertrand curves. Then, the distance between mutual points of these curves is constant [12].*

Theorem 2.3. *If the curves $\alpha,\beta : I \to \mathbb{E}^3$ are considered Bertrand curves then the angle between the tangent vectors at mutual points of these curves is constant [12].*

Theorem 2.4. *Let the Frenet frames of the Bertrand curves (α,β) be $\{\boldsymbol{T},\boldsymbol{N},\boldsymbol{B}\}$ and $\{\boldsymbol{T}^*,\boldsymbol{N}^*,\boldsymbol{B}^*\}$, respectively. So, especially if $\boldsymbol{N} = \boldsymbol{N}^*$ is chosen, the relationship between these Frenet frames is*

$$\begin{bmatrix} \boldsymbol{T}^* \\ \boldsymbol{N}^* \\ \boldsymbol{B}^* \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{bmatrix} \tag{2.1}$$

where θ is the angle between the tangent vectors of Bertrand curves (α,β) [12].

2.1. Spinors

Spinors form a vector space equipped with a linear group representation of the spin group. Here this vector space is usually over the complex numbers. Cartan [4] introduced the spinors geometrically over the complex numbers as follows. Let the vector $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{C}^3$ be an isotropic vector $({x_1}^2 + {x_2}^2 + {x_3}^2 = 0)$ and the space $\mathbb{C}^3$ be the 3-dimensional complex vector space. A two-dimensional surface in the space $\mathbb{C}^2$ is formed by the set of isotropic vectors in the vector space $\mathbb{C}^3$. If this two-dimensional surface is parameterized by coordinates γ_1 and γ_2, then $x_1 = {\gamma_1}^2 - {\gamma_2}^2$, $x_2 = i({\gamma_1}^2 + {\gamma_2}^2)$,

$x_3 = -2\gamma_1\gamma_2$ is obtained. So, the equations $\gamma_1 = \pm\sqrt{\frac{x_1 - ix_2}{2}}$, $\gamma_2 = \pm\sqrt{\frac{-x_1 - ix_2}{2}}$ are obtained. Each isotropic vector in the complex vector space $\mathbb{C}^3$ corresponds to two vectors, $(\gamma_1,\ \gamma_2)$ and $(-\gamma_1,\ -\gamma_2)$ in the space $\mathbb{C}^2$. Conversely; both vectors so given in space $\mathbb{C}^2$ correspond to the same isotropic vector $\boldsymbol{x}$. Moreover, Cartan expressed that the two-dimensional complex vectors $\gamma = (\gamma_1,\ \gamma_2)$ described in this way are called spinors

$$\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}.$$

Using Cartan's study [4], Torres del Castillo and Barrales [6] matched the isotropic vector $\boldsymbol{a} + i\boldsymbol{b} \in \mathbb{C}^3$ with spinor $\gamma = (\gamma_1, \gamma_2)$ where $\boldsymbol{a},\ \boldsymbol{b} \in \mathbb{R}^3$. So, considering the Pauli matrices they created the matrices σ and gave

$$\begin{aligned} \boldsymbol{a} + i\boldsymbol{b} &= \gamma^t \sigma \gamma, \\ \boldsymbol{c} &= -\hat{\gamma}^t \sigma \gamma \end{aligned} \tag{2.2}$$

where $\boldsymbol{a} + i\boldsymbol{b}$ is the isotropic vector in the space $\mathbb{C}^3$, $\boldsymbol{c} \in \mathbb{R}^3$ and the mate $\hat{\gamma}$ of the spinor γ is

$$\hat{\gamma} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\overline{\gamma} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \overline{\gamma_1} \\ \overline{\gamma_2} \end{pmatrix} = \begin{pmatrix} -\overline{\gamma_2} \\ \overline{\gamma_1} \end{pmatrix}.$$

When all necessary operations are considered, it is seen that the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ have the same length $\|\boldsymbol{a}\| = \|\boldsymbol{b}\| = \|\boldsymbol{c}\| = \overline{\gamma}^t\gamma$ and these vectors are mutually orthogonal. The correspondence between spinors and orthogonal bases given by equation (2.2) is two to one; the spinors γ and $-\gamma$ correspond to the same ordered orthonormal basis $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$, with $\langle \boldsymbol{a} \times \boldsymbol{b}, \boldsymbol{c} \rangle > 0$. It is important to notice that the ordered triads $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$, $\{\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{a}\}$ and $\{\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b}\}$ correspond to different spinors [6]. Moreover, the following proposition can be given.

Proposition 2.5. *Let two arbitrary spinors be γ and ψ. Then, the following statements hold;*

$$\begin{aligned} &i) && \overline{\psi^t \sigma \gamma} = -\hat{\psi}^t \sigma \hat{\gamma} \\ &ii) && \widehat{\lambda\psi + \mu\gamma} = \overline{\lambda}\hat{\psi} + \overline{\mu}\hat{\gamma} \\ &iii) && \hat{\hat{\gamma}} = -\gamma \\ &iv) && \psi^t \sigma \gamma = \gamma^t \sigma \psi \end{aligned}$$

where λ, $\mu \in \mathbb{C}$ [6].

Now, let a curve parameterized by arc-length be $\alpha : I \to \mathbb{E}^3$, $(I \subseteq \mathbb{R})$. So, $\|\alpha'(s)\| = 1$ where s is the arc-length parameter of the curve (α). Moreover, consider the Frenet vectors of this curve as $\{\boldsymbol{N}, \boldsymbol{B}, \boldsymbol{T}\}$ and the spinor ζ represents the Frenet vector triad $\{\boldsymbol{N}, \boldsymbol{B}, \boldsymbol{T}\}$. Thus, from Eq (2.2) the following equations can be obtained

$$\begin{aligned} \boldsymbol{N} + i\boldsymbol{B} &= \zeta^t \sigma \zeta = ({\zeta_1}^2 - {\zeta_2}^2,\ i({\zeta_1}^2 + {\zeta_2}^2),\ -2\zeta_1\zeta_2), \\ \boldsymbol{T} &= -\hat{\zeta}^t \sigma \zeta = (\zeta_1\overline{\zeta_2} + \overline{\zeta_1}\zeta_2,\ i(\zeta_1\overline{\zeta_2} - \overline{\zeta_1}\zeta_2),\ |\zeta_1|^2 - |\zeta_2|^2) \end{aligned} \tag{2.3}$$

with $\overline{\zeta}^t\zeta = 1$ [6]. Moreover, the following theorem can be given.

Theorem 2.6. *If the spinor ζ with two complex components represents the triad $\{\boldsymbol{N}, \boldsymbol{B}, \boldsymbol{T}\}$ of a curve (α) parameterized by its arc-length s, the Frenet equations are equivalent to the single spinor equation*

$$\frac{d\zeta}{ds} = \frac{1}{2}(-i\tau\zeta + \kappa\hat{\zeta})$$

where κ and τ denote the curvature and torsion of the curve (α), respectively [6].

Before I get to the main section, I have to emphasize that: While Torres del Castillo and Barrales matched a spinor with Frenet frame $\{\boldsymbol{N}, \boldsymbol{B}, \boldsymbol{T}\}$ of an unit speed curve $\alpha : I \to \mathbb{E}^3$, they separated Frenet vectors as $\boldsymbol{N} + i\boldsymbol{B}$ and $\boldsymbol{T}$ where $\boldsymbol{N} + i\boldsymbol{B}$ is an isotropic vector. Unlike Torres del Castillo and Barrales, in this paper, the Frenet frame $\{\boldsymbol{B}, \boldsymbol{T}, \boldsymbol{N}\}$ corresponds to a different spinor and this Frenet frame is separated as $\boldsymbol{B} + i\boldsymbol{T}$ and $\boldsymbol{N}$ where $\boldsymbol{B} + i\boldsymbol{T}$ is an isotropic vector.

3. Main theorems and proofs

Now let the Frenet vectors on the point $\alpha(s)$ of the regular and unit speed curve $\alpha : I \subseteq \mathbb{R} \to \mathbb{E}^3$ be $\{\boldsymbol{B}, \boldsymbol{T}, \boldsymbol{N}\}$. Moreover, the Frenet vectors $\{\boldsymbol{B}, \boldsymbol{T}, \boldsymbol{N}\}$ correspond to a spinor ξ. So, the spinor equations of this Frenet frame can be written by

$$\boldsymbol{B} + i\boldsymbol{T} = \xi^t \sigma \xi \tag{3.1}$$

and

$$\boldsymbol{N} = -\hat{\xi}^t \sigma \xi \tag{3.2}$$

where $\bar{\xi}^t \xi = 1$. So, the following theorems can be given.

Theorem 3.1. *Let ξ be the spinor corresponding to the Frenet vectors $\{\boldsymbol{B}, \boldsymbol{T}, \boldsymbol{N}\}$ of the unit speed and regular curve $\alpha : I \to \mathbb{E}^3$. So, the Frenet equations are equivalent to the single spinor equation*

$$\frac{d\xi}{ds} = \frac{(\tau - i\kappa)}{2}\hat{\xi} \tag{3.3}$$

where τ and κ denote the torsion and curvature of the curve (α) and $\hat{\xi}$ is the mate of spinor ξ, respectively.

Proof. If the derivative of the Eq (3.1) with respect to arc-length parameter s of the curve (α) is considered, one can write

$$\frac{d\left(\boldsymbol{B} + i\boldsymbol{T}\right)}{ds} = \frac{d\xi^t}{ds}\sigma\xi + \xi^t\sigma\frac{d\xi}{ds}.$$

So, since $\left\{\xi, \hat{\xi}\right\}$ is a basis for spinors with two complex components in the space $\mathbb{C}^2$, the spinor ${d\xi}/{ds}$ can be written ${d\xi}/{ds} = f\xi + g\hat{\xi}$ where f and g are arbitrary complex valued functions and $\hat{\xi}$ is the mate of spinor ξ. If Proposition 2.5 is considered, one can obtain $f = 0,\ g = \frac{\tau - i\kappa}{2}$. Thus, $\frac{d\xi}{ds} = \frac{(\tau - i\kappa)}{2}\hat{\xi}$. □

Now, in addition to reference [6], I give the components of the vector $\boldsymbol{B} + i\boldsymbol{T}$, separately, with the following theorem.

Theorem 3.2. *Let a spinor ξ represent the Frenet frame $\{\boldsymbol{B}, \boldsymbol{T}, \boldsymbol{N}\}$ of the regular and unit speed curve (α). So, the spinor equations of Frenet vectors $\boldsymbol{T}$ and $\boldsymbol{B}$ from the complex vector $\boldsymbol{B} + i\boldsymbol{T}$ can be written as*

$$\boldsymbol{T} = -\tfrac{i}{2}\left(\xi^t\sigma\xi + \hat{\xi}^t\sigma\hat{\xi}\right),$$
$$\boldsymbol{B} = \tfrac{1}{2}(\xi^t\sigma\xi - \hat{\xi}^t\sigma\hat{\xi}).$$

Proof. Let a spinor ξ represent the Frenet vectors $\{\boldsymbol{B},\boldsymbol{T},\boldsymbol{N}\}$ of the curve (α). If Eq (3.1) is considered, $\boldsymbol{B} = \operatorname{Re}(\xi^t\sigma\xi)$ and $\boldsymbol{T} = \operatorname{Im}(\xi^t\sigma\xi)$ hold. Moreover, from Proposition 2.5 one can write $\boldsymbol{T} = -\frac{i}{2}(\xi^t\sigma\xi + \hat{\xi}^t\sigma\hat{\xi})$, $\boldsymbol{B} = \frac{1}{2}\left(\xi^t\sigma\xi - \hat{\xi}^t\sigma\hat{\xi}\right)$. □

Corollary 3.3. *If the spinor ξ represents the Frenet frame $\{\boldsymbol{B},\boldsymbol{T},\boldsymbol{N}\}$ of the curve (α), then the spinor components of Frenet vectors can be written as*

$$\begin{aligned}
\boldsymbol{T} &= -\tfrac{i}{2}\left({\xi_1}^2 - {\xi_2}^2 + \overline{\xi_2}^2 - \overline{\xi_1}^2, i\left({\xi_1}^2 + {\xi_2}^2 + \overline{\xi_1}^2 + \overline{\xi_2}^2\right), 2\left(\overline{\xi_1\xi_2} - \xi_1\xi_2\right)\right),\\
\boldsymbol{N} &= (\xi_1\overline{\xi_2} + \overline{\xi_1}\xi_2, i(\xi_1\overline{\xi_2} - \overline{\xi_1}\xi_2), |\xi_1|^2 - |\xi_2|^2),\\
\boldsymbol{B} &= \tfrac{1}{2}\left({\xi_1}^2 - {\xi_2}^2 + \overline{\xi_1}^2 - \overline{\xi_2}^2, i\left({\xi_1}^2 + {\xi_2}^2 - \overline{\xi_1}^2 - \overline{\xi_2}^2\right), -2\left(\xi_1\xi_2 + \overline{\xi_1\xi_2}\right)\right).
\end{aligned}$$

On the other hand, let any curve (β) which is not parameterized by arc length be considered and the Frenet vectors of this curve be $\{\boldsymbol{B}^*, \boldsymbol{T}^*, \boldsymbol{N}^*\}$. Moreover, let the different spinor ϕ represent the Frenet vectors of the curve (β). So, similar to the Eqs (3.1) and (3.2), one can write

$$\begin{aligned}
\boldsymbol{B}^* + i\boldsymbol{T}^* &= \phi^t\sigma\phi,\\
\boldsymbol{N}^* &= -\widehat{\phi}^t\sigma\phi.
\end{aligned}$$

Thus, the spinor equations of the Frenet vectors of the curve (β) similar to Corollary 3.3 can be written by components as

$$\begin{aligned}
\boldsymbol{T}^* &= -\tfrac{i}{2}\left({\phi_1}^2 - {\phi_2}^2 + \overline{\phi_2}^2 - \overline{\phi_1}^2, i\left({\phi_1}^2 + {\phi_2}^2 + \overline{\phi_1}^2 + \overline{\phi_2}^2\right), 2\left(\overline{\phi_1\phi_2} - \phi_1\phi_2\right)\right),\\
\boldsymbol{N}^* &= (\phi_1\overline{\phi_2} + \overline{\phi_1}\phi_2, i(\phi_1\overline{\phi_2} - \overline{\phi_1}\phi_2), |\phi_1|^2 - |\phi_2|^2),\\
\boldsymbol{B}^* &= \tfrac{1}{2}\left({\phi_1}^2 - {\phi_2}^2 + \overline{\phi_1}^2 - \overline{\phi_2}^2, i\left({\phi_1}^2 + {\phi_2}^2 - \overline{\phi_1}^2 - \overline{\phi_2}^2\right), -2\left(\phi_1\phi_2 + \overline{\phi_1\phi_2}\right)\right).
\end{aligned}$$

So, one can give the following theorem without proof. The proof of this theorem can be easily done similar to Theorem 3.1.

Theorem 3.4. *Let the Frenet vectors of the curve (β) which is not parameterized by arc length be $\{\boldsymbol{B}^*, \boldsymbol{T}^*, \boldsymbol{N}^*\}$ and the spinor corresponding to this curve be ϕ. So, the Frenet equation of this curve in terms of a single spinor equation is written as*

$$\frac{d\phi}{dt} = \left\|\frac{d\beta}{dt}\right\| \frac{(\tau^* - i\kappa^*)}{2}\hat{\phi}$$

where t is any arbitrary parameter of the curve (β).

Here, if s is considered arc-length parameter of the curve (β), $\left\|\frac{d\beta}{dt}\right\| = \frac{ds}{dt}$ holds. So, one can write

$$\frac{d\phi}{ds} = \frac{(\tau^* - i\kappa^*)}{2}\hat{\phi}.$$

Now, the curves α and β mentioned above are considered Bertrand curves. Let the spinors ξ and ϕ represent the Bertrand curves α and β, respectively. Moreover, the Frenet vectors of the curve α are $\{\boldsymbol{B},\boldsymbol{T},\boldsymbol{N}\}$ and the Frenet vectors of the curve β are $\{\boldsymbol{B}^*,\boldsymbol{T}^*,\boldsymbol{N}^*\}$. So, the geometric interpretation of angles between spinors can be done with the following theorems.

Theorem 3.5. *Let the curves $\alpha,\beta : \mathrm{I} \to \mathbb{E}^3$ be Bertrand curves and the curves α and β correspond to the spinors ξ and ϕ, respectively. Moreover, the Frenet vectors of the curves α and β shall be $\{\boldsymbol{B},\boldsymbol{T},\boldsymbol{N}\}$ and $\{\boldsymbol{B}^*,\boldsymbol{T}^*,\boldsymbol{N}^*\}$, respectively. So, the relationship between spinors of corresponding Bertrand curves is*

$$\phi = \pm e^{-\frac{i\theta}{2}}\xi. \tag{3.4}$$

Proof. Let the spinors ξ and ϕ represent the Frenet vectors $\{\boldsymbol{B},\boldsymbol{T},\boldsymbol{N}\}$ and $\{\boldsymbol{B}^*,\boldsymbol{T}^*,\boldsymbol{N}^*\}$ of Bertrand curves $\alpha,\beta : \mathrm{I} \to \mathbb{E}^3$, respectively. So, if the relationship between Bertrand curves in the Eq (2.1) is used, one can obtain

$$\begin{aligned}\boldsymbol{B}^* + i\boldsymbol{T}^* &= \sin\theta\boldsymbol{T} + cos\theta B + i\cos\theta\boldsymbol{T} - i\sin\theta\boldsymbol{B}\\ &= (\cos\theta - i\sin\theta)(\boldsymbol{B} + i\boldsymbol{T})\\ &= e^{-i\theta}(\boldsymbol{B} + i\boldsymbol{T})\end{aligned}$$

where θ is the angle between the tangent vectors of the Bertrand curves (α,β). From here it appears that the angle between the $\boldsymbol{B} + i\boldsymbol{T}$ and $\boldsymbol{B}^* + i\boldsymbol{T}^*$ vectors is also θ. So, one can write

$$\phi^t\sigma\phi = e^{-i\theta}(\xi^t\sigma\xi)$$

and finally

$$\begin{aligned}\phi_1 &= \pm e^{-\frac{i\theta}{2}}\xi_1,\\ \phi_2 &= \pm e^{-\frac{i\theta}{2}}\xi_2.\end{aligned}$$

It is known from Section 2.2 that the spinors γ and $-\gamma$ correspond to the same ordered orthonormal basis. Namely, the spinors ϕ and $-\phi$ correspond to the vector $\boldsymbol{B}^* + i\boldsymbol{T}^*$, while the spinors ξ and $-\xi$ correspond to the vector $\boldsymbol{B} + i\boldsymbol{T}$. So, one can write

$$\phi = \pm e^{-\frac{i\theta}{2}}\xi.$$

□

Theorem 3.6. (Main Theorem): *Let the spinors ξ and ϕ represent the Bertrand curves $\alpha,\beta : \mathrm{I} \to \mathbb{E}^3$, respectively. So, if the angle between the tangent vectors of these curves is θ, then the angle between the spinors ξ and ϕ is $\frac{\theta}{2}$.*

The following corollaries of the main theorem can be given.

Corollary 3.7. *Let the spinors ξ and ϕ represent the Bertrand curves $\alpha,\beta : \mathrm{I} \to \mathbb{E}^3$, respectively. Then, the spinor ξ returns to the spinor ϕ, while the vector $\widehat{\xi}$ makes a reverse rotation to the vector $\widehat{\phi}$. Moreover, the rotation angle is also $\frac{\theta}{2}$,*

$$\hat{\phi} = \pm e^{\frac{i\theta}{2}}\hat{\xi}.$$

Corollary 3.8. *Let the curves $\alpha,\beta : \mathrm{I} \to \mathbb{E}^3$ be Bertrand curves and the curves α and β correspond to the spinors ξ and ϕ, respectively. Thus, the angle between the derivatives of spinors ξ and ϕ is also $\frac{\theta}{2}$.*

Corollary 3.9. *Let the curves (α,β) be Bertrand curves and the spinors ξ and ϕ represent these curves, respectively. Thus, the derivative of spinor ϕ which represent the curve (β) with respect to the curvatures of curve (α) can be rewritten as*

$$\frac{d\phi}{dt} = e^{-i\theta}\left(\frac{\tau - i\kappa}{2}\right)\hat{\phi}.$$

4. Conclusions

Spinors can be considered as concrete objects using a set of Cartesian coordinates. For example, spinors can be formed from Pauli spin matrices corresponding to coordinate axes in three-dimensional Euclidean space. By using matrices with these $2x2$ dimensional complex elements, spinors, which are column matrices, are created. In this case, the spin group is isomorphic to the group of $2x2$ unitary matrices with determinant one, which naturally sits inside the matrix algebra. This group acts by conjugation on the real vector space spanned by the Pauli matrices themselves, realizing it as a group of rotations among them, but it also acts on the column vectors (that is, the spinors). So, in this study, a different geometric interpretation of spinors has been created by using Pauli matrices and spinors. A different (spinor) representation of Bertrand curves, one of the most important curve pairs in differential geometry, has been obtained. Thus, this study in geometry brings a new perspective as an interesting geometric interpretation of spinors used in many fields of science.

Conflict of interest

The author declares no conflict of interest.

References

1. Y. Balci, T. Erisir, M. A. Gungor, Hyperbolic spinor Darboux equations of spacelike curves in Minkowski 3-space, *J. Chungcheong Math. Soc.*, **28** (2015), 525–535.
2. J. Bertrand, Memoire sur la théorie des courbes a double courbure, *J. Math. Pure. Appl.*, **15** (1850), 332–350.
3. J. F. Burke, Bertrand curves associated with a pair of curves, *Mathematics Magazine*, **34** (1960), 60–62.
4. É. Cartan, *The theory of spinors*, New York: Dover Publications, 1966.
5. Y. M. Cheng, C. C. Lin, On the generalized Bertrand curves in Euclidean n-spaces, *Note Mat.*, **29** (2009), 33–39.
6. G. F. T. del Castillo, G. S. Barrales, Spinor formulation of the differential geometry of curves, *Rev. Colombiana Mat.*, **38** (2004), 27–34.
7. R. Delanghe, F. Sommen, V. Soucek, *Clifford algebra and spinor-valued functions: A function theory for the Dirac operator*, New York: Dover Publications, 1966.
8. P. A. M. Dirac, The quantum theory of the electron, *P. Roy. Soc. A*, **117** (1928), 610–624.
9. F. N. Ekmekci, K. Ilarslan, On Bertrand curves and their characterization, *Differ. Geom. Dyn. Syst.*, **3** (2001), 17–24.
10. T. Erisir, M. A. Gungor, M. Tosun, Geometry of the hyperbolic spinors corresponding to alternative frame, *Adv. Appl. Clifford Algebr.*, **25** (2015), 799–810.
11. T. Erisir, N. C. Kardag, Spinor representations of Involute Evolute curves in $\mathbb{E}^3$, *Fundam. J. Math. Appl.*, **2** (2019), 148–155.

12. H. H. Hacisalihoglu, *Differential geometry*, Ankara University, Faculty of Science, Ankara, 1996.
13. G. Juvet, Opérateurs de Dirac et équations de Maxwell, Oxford Scholarship Online, 2016. Available from: 10.1093/acprof:oso/9780198782926.001.0001.
14. Z. Ketenci, T. Erisir, M. A. Gungor, A construction of hyperbolic spinors according to Frenet frame in Minkowski space, *J. Dyn. Syst. Geom. Theor.*, **13** (2015), 179–193.
15. I. Kişi, M. Tosun, Spinor Darboux equations of curves in Euclidean 3-space, *Math. Morav.*, **19** (2015), 87–93.
16. L. D. Landau, E. M. Lifshitz, *Quantum mechanics: Non-relativistic theory*, Oxford: Pergamon Press, 1977.
17. P. Lounesto, *Clifford algebras and spinors*, Cambridge University Press, 2001.
18. P. Lounesto, Crumeyrolle's bivectors and spinors, In: *Clifford algebras and spinor structures*, Dordrecht: Springer, 1995, 137–166.
19. P. Lucas, J. A. Ortega-Yagües, Bertrand curves in the three-dimensional sphere, *J. Geom. Phys.*, **62** (2012), 1903–1914.
20. W. Pauli, Zur Quantenmechanik des Magnetischen Elektrons, *Zeitschrift für Physik*, **43** (1927), 601–623.
21. F. Sauter, Lösung der Diracschen Gleichungen ohne Spezialisierung der Diracschen Operatoren, *Zeitschrift für Physik*, **63** (1930), 803–814.
22. D. Unal, I. Kisi, M. Tosun, Spinor Bishop equation of curves in Euclidean 3-space, *Adv. Appl. Clifford Algebr.*, **23** (2013), 757–765.
23. J. Vaz, R. da Rocha, An introduction to Clifford algebras and spinors, Oxford Scholarship Online, 2016. Available from: 10.1093/acprof:oso/9780198782926.001.0001.
24. A. Wachter, *Relativistic quantum mechanics*, Dordrecht: Springer, 2011.

AIMS Press

© 2021 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)