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Research article

Solving the system of nonlinear integral equations via rational contractions

Kumara Swamy Kalla[1], **Sumati Kumari Panda**[1], **Thabet Abdeljawad**[2,3,4,*] **and Aiman Mukheimer**[2]

[1] Department of Mathematics, GMR Institute of Technology, Rajam – 532 127, Andhra Pradesh, India
[2] Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, Saudi Arabia
[3] Department of Medical Research, China Medical University, Taichung, Taiwan
[4] Department of Computer Science and Information Engineering, Asia University, Taichung, Taiwan

* **Correspondence:** Email: tabdeljawad@psu.edu.sa.

Abstract: In this paper, some coupled coincidence point theorems for two mappings established using rational type contractions in the setting of partially ordered $\mathscr{G}-$metric spaces. Moreover, supporting examples are provided to strengthen our obtained results. By considering $\mathscr{G}-$metric space, we propose a fairly simple solution for a system of nonlinear integral equations by using fixed point technique.

Keywords: partially ordered $\mathscr{G}$-metric space; mixed weakly g-monotone property; coupled coincidence point; coupled fixed point; nonlinear integral equations
Mathematics Subject Classification: 54H25

1. Introduction

The new era of fixed point theory associated with metrics is now ineluctably associated with medical biological sciences, abstract terminology, space analysis and epidemiological data mining through engineering. This were often persisted by extending metric fixed point theory to a profusion of literature from computational engineering, fluid mechanics, and medical science. Fixed point theory has made a brief appearance as its own literature in the analysis of metric spaces, as keep referring to many other mathematical groups. Popular uses of metric fixed point theory involve defining and/or generalizing the various metric spaces and the notion of contractions. These extensions are also rendered with the intended consequence of a deeper comprehension of the geometric properties of Banach spaces, set theory, and non-expensive mappings.

A qualitative principle that concerns seeking conditions on the set $\mathscr{M}$ structure and choosing a mapping on $\mathscr{M}$ to get a fixed point is generally referred to as a fixed point theorem. The fixed point theory framework falls from the larger field of nonlinear functional analysis. Many of the natural sciences and engineering physical questions are usually developed in the form of numerical and analytical equations. Fixed-point assumptions find potential advantages in proving the existence of the solutions of some differential and integral equations which occur in the analysis of heat and mass transfer problems, chemical and electro-chemical processes, fluid dynamics, molecular physics and in many other fields. In 2006, Mustafa and Sims [17] introduced the concept of $\mathscr{G}$-metric space as a generalization of metric space.

It is calculated that investigators get their fresh outputs from engineering mathematics and/or its applications from $\sim 60\%$. For example, *non-linear integral equations*: it has been commonly utilized both in engineering and technology streams of all kinds. These are also appealing to researchers because of the simplicity of using non-linear integral equations and/or their implementations for approximation/numerical/data analysis strategies. In bio-medical sciences, evolution, database technology and computational systems, the steady flow of non-linear integral equations will create fresh avenues in broad directions. Non-linear integral equations are gradually becoming methods for different aspects of hydrodynamics, cognitive science, respectively (see for example [20–36]). The impetus of this work is to prove coupled coincidence point theorems for two mappings via rational type contractions satisfying mixed g-monotone property which are the generalizations of theorems of Chouhan and Richa Sharma [5] and extensions of some other existed results.

The basic definitions and propositions which are used to derive our main results are given below and also note that $\mathscr{G}$-metric and g-monotone property are denoted by $\vartheta-$metric and $\mathscr{B}$-monotone property respectively through out this paper.

Definition 1.1. ([17]) Let $\mathscr{M}$ be a set wihch is nonempty and $\vartheta : \mathscr{M} \times \mathscr{M} \times \mathscr{M} \to \mathbb{R}$ such that

(ϑ_1) $\vartheta(a, b, c) \geq 0$ for all $a, b, c \in \mathscr{M}$ with $\vartheta(a, b, c) = 0$ if $a = b = c$;
(ϑ_2) $\vartheta(a, a, b) > 0$ for all $a, b \in \mathscr{M}$ with $a \neq b$;
(ϑ_3) $\vartheta(a, a, b) \leq \vartheta(a, b, c)$ for all $a, b, c \in \mathscr{M}$ with $c \neq b$;
(ϑ_4) $\vartheta(a, b, c) = \vartheta(a, c, b) = \vartheta(b, a, c) = \vartheta(c, a, b) = \vartheta(b, c, a) = \vartheta(c, b, a)$ for all $a, b, c \in \mathscr{M}$;
(ϑ_5) $\vartheta(a, b, c) \leq \vartheta(a, d, d) + \vartheta(d, b, c)$ for all $a, b, c, d \in \mathscr{M}$.

Then the pair $(\mathscr{M}, \vartheta)$ is called a ϑ-metric space with ϑ-metric ϑ on $\mathscr{M}$. Axioms (ϑ_4) and (ϑ_5) are referred to as the *symmetry* and the *rectangle inequality* (of ϑ) respectively.

If $(\mathscr{M}, \leq)$ is partially ordered set in the definition of ϑ-metric space, then $(\mathscr{M}, \vartheta, \leq)$ is partially ordered ϑ-metric space.
Given a ϑ-metric space $(\mathscr{M}, \vartheta)$, define

$$\rho_\vartheta(a, b) = \vartheta(a, b, b) + \vartheta(a, a, b) \text{ for all } a, b, c \in \mathscr{M}. \tag{1.1}$$

Then it is seen in [17] that ϑ is a metric on $\mathscr{M}$, and that the family of all ϑ-balls $\{B_\vartheta(a, r) : a \in \mathscr{M}, r > 0\}$ is the base topology, called the ϑ-metric topology $\tau(\vartheta)$ on $\mathscr{M}$, where $B_\vartheta(a, r) = \{b \in \mathscr{M} : \vartheta(a, b, b) < r\}$. Further, it was shown that the ϑ-metric topology coincides with the metric topology induced by the metric ϑ, which allows us to readily transform many concepts from metric spaces into ϑ-metric space.

Definition 1.2. ([17]) If a sequence $\langle a_\kappa \rangle_{n=1}^{\infty}$ in $(\mathscr{M}, \vartheta)$ converges to an element $p \in \mathscr{M}$ in the ϑ-metric topology $\tau(\vartheta)$, then $\langle a_\kappa \rangle_{n=1}^{\infty}$ is called ϑ-convergent with limit p.

Proposition 1.1. ([17]) If $(\mathscr{M},\vartheta)$ is a ϑ-metric space, then following are equivalent.

(i) $\langle a_\kappa\rangle_{\kappa=1}^{\infty}$ is ϑ-convergent to a;
(ii) $\vartheta(a_\kappa,a_\kappa,a)\to 0$ as $\kappa\to\infty$;
(iii) $\vartheta(a_\kappa,a,a)\to 0$ as $\kappa\to\infty$;
(iv) $\vartheta(a_\kappa,a_\eta,a)\to 0$ as $\kappa,\eta\to\infty$.

Definition 1.3. ([17]) A sequence $\langle a_\kappa\rangle$ in $(\mathscr{M},\vartheta)$is called ϑ-Cauchy if for every $\sigma>0$ there exists a positive integer $\mathscr{N}$ such that $\vartheta(a_\kappa,a_\eta,a_\zeta)<\sigma$ for all $\kappa,\eta,\zeta\geq\mathscr{N}$.

Proposition 1.2. ([18]) If $(\mathscr{M},\vartheta)$ is a ϑ-metric space,then $\langle a_\kappa\rangle$ is ϑ-Cauchy if and only if for every $\sigma>0$, there exists a positive integer $\mathscr{N}$ such that $\vartheta(a_\kappa,a_\eta,a_\eta)<\sigma$ for all $\eta,\kappa\geq\mathscr{N}$.

Proposition 1.3. ([17]) Every ϑ-convergent sequence in $(\mathscr{M},\vartheta)$ is ϑ-Cauchy.

Definition 1.4. ([17]) If every ϑ-Cauchy sequence in $\mathscr{M}$ converges in $\mathscr{M}$, then $(\mathscr{M},\vartheta)$ is called ϑ-complete.

Proposition 1.4. ([17]) If $(\mathscr{M},\vartheta)$is a ϑ-metric space and $\mathscr{B}$ is a self-map on $\mathscr{M}$, then $\mathscr{B}$ is ϑ-continuous at a point $a\in\mathscr{M}$ iff the sequence$\langle\mathscr{B}a_\kappa\rangle$ converges to $\mathscr{B}a$ whenever $\langle a_\kappa\rangle$ converges to a.

Proposition 1.5. ([17]) The ϑ-metric $\vartheta(a,b,c)$ is continuous jointly in all the variables a,b and c.

Proposition 1.6. ([17]) If $(\mathscr{M},\vartheta)$ is a ϑ-metric space, then

(i) if $\vartheta(a,b,c)=0$ then $a=b=c$;
(ii) $\vartheta(a,b,c)\leq\vartheta(a,a,b)+\vartheta(a,a,c)$;
(iii) $\vartheta(a,b,b)\leq 2\vartheta(b,a,a)$;
(iv) $\vartheta(a,b,c)\leq\vartheta(a,x,c)+\vartheta(x,b,c)$ for all $a,b,c,x\in\mathscr{M}$.

Definition 1.5. ([6]) Let $(\mathscr{M},\vartheta)$ be a ϑ-metric space and $\mathscr{X}:\mathscr{M}\times\mathscr{M}\to\mathscr{M}$ be a mapping on $\mathscr{M}\times\mathscr{M}$. Then $\mathscr{X}$ is called continuous if the sequence $\langle\mathscr{X}(a_n,b_n)\rangle$ converge to $\mathscr{X}(a,b)$ whenever the sequences $\langle a_n\rangle$ and $\langle b_n\rangle$ are converge to a and b respectively.

Definition 1.6. ([11]) Let $\mathscr{M}$ be a set which is nonempty and $\mathscr{X}:\mathscr{M}\times\mathscr{M}\to\mathscr{M}$, $\mathscr{B}:\mathscr{M}\to\mathscr{M}$ be two mappings. Then $\mathscr{X}$ is said to be commute with $\mathscr{B}$ if $\mathscr{X}(\mathscr{B}a,\mathscr{B}b)=\mathscr{B}(\mathscr{X}(a,b))$.

Definition 1.7. ([11]) Let $\mathscr{M}$ be a set which is nonempty set and $\mathscr{X}:\mathscr{M}\times\mathscr{M}\to\mathscr{M}$, $\mathscr{B}:\mathscr{M}\to\mathscr{M}$ be two mappings. Then $\mathscr{X}$ is said to have *mixed $\mathscr{B}$-monotone property* if

$$
\begin{gathered}
a_1,a_2\in\mathscr{M},\mathscr{B}a_1\leq\mathscr{B}a_2\Rightarrow\mathscr{X}(a_1,b)\leq\mathscr{X}(a_2,b)\\
\text{and}\\
b_1,b_2\in\mathscr{M},\mathscr{B}b_1\leq\mathscr{B}b_2\Rightarrow\mathscr{X}(b_1,a)\geq\mathscr{X}(b_2,a)\\
\text{for all } a,b\in\mathscr{M}.
\end{gathered}
$$

If $\mathscr{B}$ is an identity mapping in the above deinition, then $\mathscr{X}$ has mixed monotone property.

Definition 1.8. ([2]) If $\mathscr{M}$ is a set which is nonempty and $\mathscr{X}:\mathscr{M}\times\mathscr{M}\to\mathscr{M}$ is a mapping such that$\mathscr{X}(a,b)=a$ and $\mathscr{X}(b,a)=b$, then $(a,b)\in\mathscr{M}\times\mathscr{M}$ is called coupled fixed point of $\mathscr{X}$.

Definition 1.9. ([17]) If $\mathscr{M}$ is a set which is nonempty set and $\mathscr{X} : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$, $\mathscr{B} : \mathscr{M} \to \mathscr{M}$ are two mappings such that $\mathscr{X}(a, b) = \mathscr{B}a$ and $\mathscr{X}(b, a) = \mathscr{B}b$, then $(a, b) \in \mathscr{M} \times \mathscr{M}$ is called coupled coincidence point of $\mathscr{X}$ and $\mathscr{B}$.

If $\mathscr{B}$ is an identity mapping in the above definition, then (a, b) is called coupled fixed point of a mapping $\mathscr{X}$.

2. Main results

Theorem 2.1. *Let $(\mathscr{M}, \vartheta, \leq)$ be a partially ordered complete ϑ-metric space and $\mathscr{X} : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$, $\mathscr{B} : \mathscr{M} \to \mathscr{M}$ be two continuous mappings such that $\mathscr{X}$ has mixed $\mathscr{B}$-monotone property and*

(i) there exist $\alpha, \beta, \gamma \in [0, 1)$ and $L \geq 0$ with $8\alpha + \beta + \gamma < 1$ such that

$$\begin{aligned}
&\vartheta(\mathscr{X}(x, y), \mathscr{X}(u, v), \mathscr{X}(w, z)) \\
&\leq \alpha \frac{\vartheta(\mathscr{B}, \mathscr{X}(x,y), \mathscr{X}(y,x))\vartheta(\mathscr{B}u, \mathscr{X}(u,v), \mathscr{X}(u,v))\vartheta(\mathscr{B}w, \mathscr{X}(w,z), \mathscr{X}(w,z))}{[\vartheta(\mathscr{B}x, \mathscr{B}u, \mathscr{B}w)]^2} + \beta\vartheta(\mathscr{B}x, \mathscr{B}u, \mathscr{B}w), \\
&\quad + \gamma\vartheta(\mathscr{B}y, \mathscr{B}v, \mathscr{B}z) + Lmin\Big\{\vartheta(\mathscr{B}x, \mathscr{X}(u, v), \mathscr{X}(w, z)), \vartheta(\mathscr{B}u, \mathscr{X}(x, y), \mathscr{X}(w, z)), \\
&\qquad \vartheta(\mathscr{B}w, \mathscr{X}(x, y), \mathscr{X}(u, v)), \vartheta(\mathscr{B}x, \mathscr{X}(x, y), \mathscr{X}(x, y)), \\
&\qquad \vartheta(\mathscr{B}u, \mathscr{X}(u, v), \mathscr{X}(u, v)), \vartheta(\mathscr{B}w, \mathscr{X}(w, z), \mathscr{X}(w, z))\Big\} \\
&\qquad \textit{for all } x, y, u, v, w, z \in \mathscr{M} \textit{ with } \mathscr{B}x \geq \mathscr{B}u \geq \mathscr{B}w \textit{ and } \mathscr{B}y \leq \mathscr{B}v \leq \mathscr{B}z;
\end{aligned} \tag{2.1}$$

(ii) $\mathscr{X}(\mathscr{M} \times \mathscr{M}) \subseteq \mathscr{B}(\mathscr{M})$;

(iii) $\mathscr{B}$ commutes with $\mathscr{X}$.

If there exist $x_0, y_0 \in \mathscr{M}$ such that $\mathscr{B}x_0 \leq \mathscr{X}(x_0, y_0)$ and $\mathscr{B}y_0 \geq \mathscr{X}(y_0, x_0)$, then $\mathscr{X}$ and $\mathscr{B}$ have a coupled coincidence point in $\mathscr{M} \times \mathscr{M}$.

Proof. Let x_0 and y_0 be any two elements in $\mathscr{M}$ such that $\mathscr{B}x_0 \leq \mathscr{X}(x_0, y_0)$ and $\mathscr{B}y_0 \geq \mathscr{X}(y_0, x_0)$. Since $\mathscr{X}(\mathscr{M} \times \mathscr{M}) \subseteq \mathscr{B}(\mathscr{M})$, we construct two sequences $\langle x_\kappa \rangle_{\kappa=1}^{\infty}$ and $\langle y_\kappa \rangle_{\kappa=1}^{\infty}$ in $\mathscr{M}$ as follows:

$$\mathscr{B}x_{\kappa+1} = \mathscr{X}(x_\kappa, y_\kappa) \text{ and } \mathscr{B}y_{\kappa+1} = \mathscr{X}(y_\kappa, x_\kappa) \text{ for } \kappa \in \mathbb{N}.$$

Since $\mathscr{X}$ has mixed $\mathscr{B}$-monotone property, we have

$$\begin{aligned}
\mathscr{B}x_\kappa &= \mathscr{X}(x_{\kappa-1}, y_{\kappa-1}) \leq \mathscr{X}(x_\kappa, y_{\kappa-1}) \leq \mathscr{X}(x_\kappa, y_\kappa) = \mathscr{B}x_{\kappa+1} \\
\text{and } \mathscr{B}y_{\kappa+1} &= \mathscr{X}(y_\kappa, x_\kappa) \leq \mathscr{X}(y_{\kappa-1}, x_\kappa) \leq \mathscr{X}(y_{\kappa-1}, x_{\kappa-1}) = \mathscr{B}y_\kappa.
\end{aligned}$$

Now using (2.1) with $x = x_\kappa$, $y = y_\kappa$, $u = w = x_{\kappa-1}$ and $v = z = y_{\kappa-1}$, we get

$$\begin{aligned}
&\vartheta(\mathscr{B}x_{\kappa+1}, \mathscr{B}x_\kappa, \mathscr{B}x_\kappa) = \vartheta(\mathscr{X}(x_\kappa, y_\kappa), \mathscr{X}(x_{\kappa-1}, y_{\kappa-1}), \mathscr{X}(x_{\kappa-1}, y_{\kappa-1})) \\
&\leq \alpha \frac{\vartheta(\mathscr{B}x_\kappa, \mathscr{X}(x_\kappa,y_\kappa), \mathscr{X}(x_\kappa,y_\kappa))\vartheta(\mathscr{B}x_{\kappa-1}, \mathscr{X}(x_{\kappa-1},y_{\kappa-1}), \mathscr{X}(x_{\kappa-1},y_{\kappa-1}))\vartheta(\mathscr{B}x_{\kappa-1}, \mathscr{X}(x_{\kappa-1},y_{\kappa-1}), \mathscr{X}(x_{\kappa-1},y_{\kappa-1}))}{[\vartheta(\mathscr{B}x_\kappa, \mathscr{B}x_{\kappa-1}, \mathscr{B}x_{\kappa-1})]^2} \\
&\quad + \beta\vartheta(\mathscr{B}x_\kappa, \mathscr{B}x_{\kappa-1}, \mathscr{B}x_{\kappa-1}) + \gamma\vartheta(\mathscr{B}y_\kappa, \mathscr{B}y_{\kappa-1}, \mathscr{B}y_{\kappa-1}) \\
&\quad + Lmin\Big\{\vartheta(\mathscr{B}x_\kappa, \mathscr{X}(x_{\kappa-1}, y_{\kappa-1}), \mathscr{X}(x_{\kappa-1}, y_{\kappa-1})), \vartheta(\mathscr{B}x_{\kappa-1}, \mathscr{X}(x_\kappa, y_\kappa), \mathscr{X}(x_{\kappa-1}, y_{\kappa-1})),
\end{aligned}$$

$$\vartheta(\mathscr{B}x_{\kappa-1}, \mathscr{X}(x_\kappa, y_\kappa), \mathscr{X}(x_{\kappa-1}, y_{\kappa-1})), \vartheta(\mathscr{B}x_\kappa, \mathscr{X}(x_\kappa, y_\kappa), \mathscr{X}(y_\kappa, x_\kappa)),$$

$$\vartheta(\mathscr{B}x_{\kappa-1}, \mathscr{X}(x_{\kappa-1}, y_{\kappa-1}), \mathscr{X}(x_{\kappa-1}, y_{\kappa-1})), \vartheta(\mathscr{B}x_{\kappa-1}, \mathscr{X}(x_{\kappa-1}, y_{\kappa-1}), \mathscr{X}(x_{\kappa-1}, y_{\kappa-1}))\Big\}$$

$$\begin{aligned}
=&\alpha \tfrac{\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa+1},\mathscr{B}x_{\kappa+1})\vartheta(\mathscr{B}x_{\kappa-1},\mathscr{B}x_\kappa,\mathscr{B}x_\kappa)\vartheta(\mathscr{B}x_{\kappa-1},\mathscr{B}x_\kappa,\mathscr{B}x_\kappa)}{[\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1})]^2} \\
&+ \beta\vartheta(\mathscr{B}x_\kappa, \mathscr{B}x_{\kappa-1}, \mathscr{B}x_{\kappa-1}) + \gamma\vartheta(\mathscr{B}y_\kappa, \mathscr{B}y_{\kappa-1}, \mathscr{B}y_{\kappa-1}) \\
&+ Lmin\Big\{\vartheta(\mathscr{B}x_\kappa, \mathscr{B}x_\kappa, \mathscr{B}x_\kappa), \vartheta(\mathscr{B}x_{\kappa-1}, \mathscr{B}x_{\kappa+1}, \mathscr{B}x_\kappa), \\
&\vartheta(\mathscr{B}x_{\kappa-1}, \mathscr{B}x_{\kappa+1}, \mathscr{B}x_\kappa)), \vartheta(\mathscr{B}x_\kappa, \mathscr{B}x_{\kappa+1}, \mathscr{B}x_{\kappa+1}), \\
&\vartheta(\mathscr{B}x_{\kappa-1}, \mathscr{B}x_\kappa, \mathscr{B}x_\kappa), \vartheta(\mathscr{B}x_{\kappa-1}, \mathscr{B}x_\kappa, \mathscr{B}x_\kappa)\Big\} \\
\leq& 8\alpha \tfrac{\vartheta(\mathscr{B}x_{\kappa+1},\mathscr{B}x_\kappa,,\mathscr{B}x_\kappa)\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1})\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1})}{[\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1})]^2} \\
&+ \beta\vartheta(\mathscr{B}x_\kappa, \mathscr{B}x_{\kappa-1}, \mathscr{B}x_{\kappa-1}) + \gamma\vartheta(\mathscr{B}y_\kappa, \mathscr{B}y_{\kappa-1}, \mathscr{B}y_{\kappa-1})
\end{aligned}$$

so that

$$\vartheta(\mathscr{B}x_{\kappa+1}, \mathscr{B}x_\kappa, \mathscr{B}x_\kappa) \leq \frac{1}{(1-8\alpha)}[\beta\vartheta(\mathscr{B}x_\kappa, \mathscr{B}x_{\kappa-1}, \mathscr{B}x_{\kappa-1}) + \gamma\vartheta(\mathscr{B}y_\kappa, \mathscr{B}y_{\kappa-1}, \mathscr{B}y_{\kappa-1})] \tag{2.2}$$

Similarly,

$$\vartheta(\mathscr{B}y_{\kappa+1}, \mathscr{B}y_\kappa, \mathscr{B}y_\kappa) \leq \frac{1}{(1-8\alpha)}[\beta\vartheta(\mathscr{B}y_\kappa, \mathscr{B}y_{\kappa-1}, \mathscr{B}y_{\kappa-1}) + \gamma\vartheta(\mathscr{B}x_\kappa, \mathscr{B}x_{\kappa-1}, \mathscr{B}x_{\kappa-1})] \tag{2.3}$$

Adding (2.2) and (2.3), we have

$$\begin{aligned}
&\vartheta(\mathscr{B}x_{\kappa+1}, \mathscr{B}x_\kappa, \mathscr{B}x_\kappa) + \vartheta(\mathscr{B}y_{\kappa+1}, \mathscr{B}y_\kappa, \mathscr{B}y_\kappa) \\
&\qquad \leq \frac{\beta+\gamma}{1-8\alpha}[\vartheta(\mathscr{B}x_\kappa, \mathscr{B}x_{\kappa-1}, \mathscr{B}x_{\kappa-1}) + \vartheta(\mathscr{B}y_\kappa, \mathscr{B}y_{\kappa-1}, \mathscr{B}y_{\kappa-1})].
\end{aligned} \tag{2.4}$$

Let $\Delta_\kappa = \vartheta(\mathscr{B}x_{\kappa+1}, \mathscr{B}x_\kappa, \mathscr{B}x_\kappa) + \vartheta(\mathscr{B}y_{\kappa+1}, \mathscr{B}y_\kappa, \mathscr{B}y_\kappa)$ and $c = \dfrac{\beta+\gamma}{1-8\alpha}$
where $0 \leq c < 1$ inview of choice of α,β and γ.
Now the inequality (2.4) becomes as follows
$\Delta_\kappa \leq c.\Delta_{\kappa-1}$ for $\kappa \in \mathbb{N}$
Consequently $\Delta_\kappa \leq c\Delta_{\kappa-1} \leq c^2\Delta_{\kappa-2} \leq ... \leq c^\kappa\Delta_0$
If $\Delta_0 = 0$, we have $\vartheta(\mathscr{B}x_1, \mathscr{B}x_0, \mathscr{B}x_0) + \vartheta(\mathscr{B}y_1, \mathscr{B}y_0, \mathscr{B}y_0) = 0$
or $\vartheta(\mathscr{X}(x_0, y_0), \mathscr{B}x_0, \mathscr{B}x_0) + \vartheta(\mathscr{X}(y_0, x_0), \mathscr{B}y_0, \mathscr{B}y_0) = 0$ which implies that $\mathscr{X}(x_0, y_0) = \mathscr{B}x_0$ and $\mathscr{X}(y_0, x_0) = \mathscr{B}y_0$.
That is, (x_0, y_0) is a coupled coincidence point of $\mathscr{X}$ and $\mathscr{B}$.
Suppose that $\Delta_0 > 0$.
Now by applying rectangle inequality of ϑ-metric repeatedly and using inequality $\Delta_\kappa \leq c^\kappa\Delta_0$, we have

$$\begin{aligned}
\vartheta(\mathscr{B}x_\eta, \mathscr{B}x_\kappa, \mathscr{B}x_\kappa) + \vartheta(\mathscr{B}y_\eta, \mathscr{B}y_\kappa, \mathscr{B}y_\kappa) \leq& [\vartheta(\mathscr{B}x_{\kappa+1}, \mathscr{B}x_\kappa, \mathscr{B}x_\kappa) + \vartheta(\mathscr{B}x_\eta, \mathscr{B}x_{\kappa+1}, \mathscr{B}x_{\kappa+1})] \\
&+ [\vartheta(\mathscr{B}y_{\kappa+1}, \mathscr{B}y_\kappa, \mathscr{B}y_\kappa) + \vartheta(\mathscr{B}y_\eta, \mathscr{B}y_{\kappa+1}, \mathscr{B}y_{\kappa+1})] \\
\leq& [\vartheta(\mathscr{B}x_{\kappa+1}, \mathscr{B}x_\kappa, \mathscr{B}x_\kappa) + \vartheta(\mathscr{B}x_{\kappa+2}, \mathscr{B}x_{\kappa+1}, \mathscr{B}x_{\kappa+1}) + \vartheta(\mathscr{B}x_\eta, \mathscr{B}x_{\kappa+2}, \mathscr{B}x_{\kappa+2})]
\end{aligned}$$

$$[\vartheta(\mathscr{B}y_{\kappa+1}, \mathscr{B}y_{\kappa}, \mathscr{B}y_{\kappa}) + \vartheta(\mathscr{B}y_{\kappa+2}, \mathscr{B}y_{\kappa+1}, \mathscr{B}y_{\kappa+1}) + \vartheta(\mathscr{B}y_{\eta}, \mathscr{B}y_{\kappa+2}, \mathscr{B}y_{\kappa+2})]$$
$$\vdots$$
$$\begin{aligned}
&\leq[\vartheta(\mathscr{B}x_{\kappa+1}, \mathscr{B}x_{\kappa}, \mathscr{B}x_{\kappa}) + \vartheta(\mathscr{B}x_{\kappa+2}, \mathscr{B}x_{\kappa+1}, \mathscr{B}x_{\kappa+1}) + \cdots + \vartheta(\mathscr{B}x_{\eta}, \mathscr{B}x_{\eta-1}, \mathscr{B}x_{\eta-1})]\\
&\quad[\vartheta(\mathscr{B}y_{\kappa+1}, \mathscr{B}y_{\kappa}, \mathscr{B}y_{\kappa}) + \vartheta(\mathscr{B}y_{\kappa+2}, \mathscr{B}y_{\kappa+1}, \mathscr{B}y_{\kappa+1}) + \cdots + \vartheta(\mathscr{B}y_{\eta}, \mathscr{B}y_{\eta-1}, \mathscr{B}y_{\eta-1})]\\
&=\Delta_{\kappa} + \Delta_{\kappa+1} + \Delta_{\kappa+2} + \cdots + \Delta_{\eta-1}\\
&\leq[c^{\kappa} + c^{\kappa+1} + \cdots + c^{\eta-1}]\Delta_0\\
&\leq c^{\kappa} \cdot \frac{1}{1-c}\Delta_0 \text{ for } \eta > \kappa
\end{aligned}$$

or

$$\vartheta(\mathscr{B}x_{\eta}, \mathscr{B}x_{\kappa}, \mathscr{B}x_{\kappa}) + \vartheta(\mathscr{B}y_{\eta}, \mathscr{B}y_{\kappa}, \mathscr{B}y_{\kappa}) \leq c^{\kappa} \cdot \frac{1}{1-c}\Delta_0 \text{ for } \eta > \kappa. \tag{2.5}$$

Since $0 \leq c < 1, c^{\kappa} \to 0$ as $\kappa \to \infty$.
Now applying limit as $\kappa \to \infty$ with $\eta > \kappa$ in the inequality (2.5), we have
$\vartheta(\mathscr{B}x_{\eta}, \mathscr{B}x_{\kappa}, \mathscr{B}x_{\kappa}) + \vartheta(\mathscr{B}y_{\eta}, \mathscr{B}y_{\kappa}, \mathscr{B}y_{\kappa}) \leq 0$ which follows that $\langle x_{\kappa}\rangle_{\kappa=1}^{\infty}$ and $\langle y_{\kappa}\rangle_{\kappa=1}^{\infty}$ are cauchy sequences in $\mathscr{M}$.
Since $(\mathscr{M}, \vartheta)$ is a partially ordered complete ϑ-metric space, there exist $p, q \in \mathscr{M}$ such that $x_{\kappa} \to p$ and $y_{\kappa} \to q$.
Now we prove that (p, q) is a coupled coincidence point of $\mathscr{X}$ and $\mathscr{B}$.
Since $\mathscr{X}$ commutes with $\mathscr{B}$, we have

$$\mathscr{X}(\mathscr{B}x_{\kappa}, \mathscr{B}y_{\kappa}) = \mathscr{B}(\mathscr{X}(x_{\kappa}, y_{\kappa})) = \mathscr{B}(\mathscr{B}x_{\kappa+1})$$
$$\text{and}$$
$$\mathscr{X}(\mathscr{B}y_{\kappa}, \mathscr{B}x_{\kappa}) = \mathscr{B}(\mathscr{X}(y_{\kappa}, x_{\kappa})) = \mathscr{B}(\mathscr{B}y_{\kappa+1})$$

Since $\mathscr{X}$ and $\mathscr{B}$ are continuous, we have

$$\lim_{\kappa\to\infty} \mathscr{X}(\mathscr{B}x_{\kappa}, \mathscr{B}y_{\kappa}) = \lim_{\kappa\to\infty} \mathscr{B}(\mathscr{B}x_{\kappa+1}) = \mathscr{B}p$$
$$\text{and}$$
$$\lim_{\kappa\to\infty} \mathscr{X}(\mathscr{B}y_{\kappa}, \mathscr{B}x_{\kappa}) = \lim_{\kappa\to\infty} \mathscr{B}(\mathscr{B}y_{\kappa+1}) = \mathscr{B}q$$

Snice ϑ is continuous in all its variables, we have

$$\vartheta(\mathscr{X}(p, q), \mathscr{B}p, \mathscr{B}p) = \vartheta(\lim_{\kappa\to\infty} \mathscr{X}(\mathscr{B}x_{\kappa}, \mathscr{B}y_{\kappa}), \mathscr{B}p, \mathscr{B}p)) = \vartheta(\mathscr{B}p, \mathscr{B}p, \mathscr{B}p)$$

so that

$$\vartheta(\mathscr{X}(p, q), \mathscr{B}p, \mathscr{B}p) = 0$$

which implies that $\mathscr{X}(p, q) = \mathscr{B}p$.
Similarly, it can be proved that $\mathscr{X}(q, p) = \mathscr{B}q$.
Hence (p, q) is a coupled coincidence point of $\mathscr{X}$ and $\mathscr{B}$. □

Remark 2.1. (i.) If we assume $\mathscr{B}$ is an identity mapping and $\gamma = 0$ in the Theorem 2.1, then we get Theorem 3.1 in the results of the Chouhan and Richa Sharma [5];

(ii.) If we take, $\alpha = 0$ and $\beta = \alpha$, $\gamma = \beta$ in the Theorem 2.1, we get Theorem 3.1 in the results of Chandok et al. [3];

(iii.) By taking $\gamma = 0$ and $L = 0$, we get Theorem 2.1 in the results of Chakrabarti [4].

That is Theorem 2.1 is generalization and extension of above three results.

The following is the example to illusrative Theorem 2.1.

Example 2.1. Let $\mathscr{M} = [0,1]$, with ϑ-metric $\vartheta(x,y,z) = \begin{cases} 0, & x = y = z, \\ max\{x,y,z\}, & \text{otherwise}. \end{cases}$

Define partial order on X as $x \geq y$ for any $x, y \in \mathscr{M}$. Then$(\mathscr{M}, \vartheta, \leq)$ is a partially ordered ϑ-complete.

Define $\mathscr{X} : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ by $\mathscr{X}(x,y) = \begin{cases} \frac{x}{3(y^2+2)}, & x \geq y \\ 0 & \text{otherwise} \end{cases}$

and $\mathscr{B} : \mathscr{M} \to \mathscr{M}$ by $\mathscr{B}x = \frac{x}{4}$.

Then clearly $\mathscr{X}$, $\mathscr{B}$ are continuous and $\mathscr{X}$ satisfies mixed $\mathscr{B}$-monotone property. We show that $\mathscr{X}$ satisfies the inequality (2.1) with $\alpha = \beta = \gamma = \frac{1}{32}$ so that $0 \leq 8\alpha + \beta + \gamma < 1$ and for any $L \geq 0$.
Let $x, y, z, u, v, w \in \mathscr{M}$ be such that $x \geq u \geq w$ and $y \leq v \leq z$.

We discuss four cases:

Case (i): If $x \geq y, u \geq v$ and $w \geq z$ then we have $\mathscr{X}(x,y) = \frac{x}{3(y^2+2)}$, $\mathscr{X}(u,v) = \frac{u}{3(v^2+2)}$ and $\mathscr{X}(w,z) = \frac{w}{3(z^2+2)}$

$$
\begin{aligned}
\vartheta(\mathscr{X}(x,y), \mathscr{X}(u,v), \mathscr{X}(w,z)) &= max\left\{\frac{x}{3(y^2+2)}, \frac{u}{3(v^2+2)}, \frac{w}{3(z^2+2)}\right\} \\
&= \frac{x}{3(y^2+2)} \\
&\leq \frac{1}{32}[\frac{16wux}{x^2} + \frac{x}{4} + \frac{u}{4}] \\
&= \frac{1}{32}\frac{\vartheta(\mathscr{B}x, \mathscr{X}(x,y), \mathscr{X}(y,x))\vartheta(\mathscr{B}u, \mathscr{X}(u,v), \mathscr{X}(u,v))\vartheta(\mathscr{B}w, \mathscr{X}(w,z), \mathscr{X}(w,z))}{[\vartheta(\mathscr{B}x, \mathscr{B}u, \mathscr{B}w)]^2} \\
&\quad + \frac{1}{32}\vartheta(\mathscr{B}x, \mathscr{B}u, \mathscr{B}w) + \frac{1}{32}\vartheta(\mathscr{B}y, \mathscr{B}v, \mathscr{B}z) \\
&\leq \frac{1}{32}\frac{\vartheta(\mathscr{B}x, \mathscr{X}(x,y), \mathscr{X}(y,x))\vartheta(\mathscr{B}u, \mathscr{X}(u,v), \mathscr{X}(u,v))\vartheta(\mathscr{B}w, \mathscr{X}(w,z), \mathscr{X}(w,z))}{[\vartheta(\mathscr{B}x, \mathscr{B}u, \mathscr{B}w)]^2} \\
&\quad + \frac{1}{32}\vartheta(\mathscr{B}x, \mathscr{B}u, \mathscr{B}w) + \frac{1}{32}\vartheta(\mathscr{B}y, \mathscr{B}v, \mathscr{B}z) \\
&\quad + Lmin\Big\{\vartheta(\mathscr{B}x, \mathscr{X}(u,v), \mathscr{X}(w,z)), \vartheta(\mathscr{B}u, \mathscr{X}(x,y), \mathscr{X}(w,z)), \\
&\qquad \vartheta(\mathscr{B}w, \mathscr{X}(x,y), \mathscr{X}(u,v)), \vartheta(\mathscr{B}x, \mathscr{X}(x,y), \mathscr{X}(x,y)), \\
&\qquad \vartheta(\mathscr{B}u, \mathscr{X}(u,v), \mathscr{X}(u,v)), \vartheta(\mathscr{B}w, \mathscr{X}(w,z), \mathscr{X}(w,z)\Big\}
\end{aligned}
$$

Case (ii): If $x \geq y, u \geq v$ and $w < z$ then we have $\mathscr{X}(x,y) = \frac{x}{3(y^2+2)}$, $\mathscr{X}(u,v) = \frac{u}{3(v^2+2)}$ and $\mathscr{X}(w,z) = 0$

$$
\begin{aligned}
\vartheta(\mathscr{X}(x,y),\mathscr{X}(u,v),\mathscr{X}(w,z)) &= max\left\{\frac{x}{3(y^2+2)},\frac{u}{3(v^2+2)},0\right\}\\
&=\frac{x}{3(y^2+2)}\\
&\le\frac{1}{32}[\frac{16wux}{x^2}+\frac{x}{4}+\frac{u}{4}]\\
&\le\frac{1}{32}\frac{\vartheta(\mathscr{B}x,\mathscr{X}(x,y),\mathscr{X}(y,x))\vartheta(\mathscr{B}u,\mathscr{X}(u,v),\mathscr{X}(u,v))\vartheta(\mathscr{B}w,\mathscr{X}(w,z),\mathscr{X}(w,z))}{[\vartheta(\mathscr{B}x,\mathscr{B}u,\mathscr{B}w)]^2}\\
&\quad+\frac{1}{32}\vartheta(\mathscr{B}x,\mathscr{B}u,\mathscr{B}w)+\frac{1}{32}\vartheta(\mathscr{B}y,\mathscr{B}v,\mathscr{B}z)\\
&\quad+Lmin\Big\{\vartheta(\mathscr{B}x,\mathscr{X}(u,v),\mathscr{X}(w,z)),\vartheta(\mathscr{B}u,\mathscr{X}(x,y),\mathscr{X}(w,z)),\\
&\qquad\vartheta(\mathscr{B}w,\mathscr{X}(x,y),\mathscr{X}(u,v)),\vartheta(\mathscr{B}x,\mathscr{X}(x,y),\mathscr{X}(x,y)),\\
&\qquad\vartheta(\mathscr{B}u,\mathscr{X}(u,v),\mathscr{X}(u,v)),\vartheta(\mathscr{B}w,\mathscr{X}(w,z),\mathscr{X}(w,z))\Big\}
\end{aligned}
$$

Case (iii): If $x \ge y, u < v$ and $w < z$, then we have $\mathscr{X}(x,y) = \frac{x}{3(y^2+2)}$, $\mathscr{X}(u,v) = 0$ and $\mathscr{X}(w,z) = 0$

$$
\begin{aligned}
\vartheta(\mathscr{X}(x,y),\mathscr{X}(u,v),\mathscr{X}(w,z)) &= max\left\{\frac{x}{3(y^2+2)},0,0\right\}\\
&=\frac{x}{3(y^2+2)}\\
&\le\frac{1}{32}[\frac{16wux}{x^2}+\frac{x}{4}+\frac{u}{4}]\\
&\le\frac{1}{32}\frac{\vartheta(\mathscr{B}x,\mathscr{X}(x,y),\mathscr{X}(y,x))\vartheta(\mathscr{B}u,\mathscr{X}(u,v),\mathscr{X}(u,v))\vartheta(\mathscr{B}w,\mathscr{X}(w,z),\mathscr{X}(w,z))}{[\vartheta(\mathscr{B}x,\mathscr{B}u,\mathscr{B}w)]^2}\\
&\quad+\frac{1}{32}\vartheta(\mathscr{B}x,\mathscr{B}u,\mathscr{B}w)+\frac{1}{32}\vartheta(\mathscr{B}y,\mathscr{B}v,\mathscr{B}z)\\
&\quad+Lmin\Big\{\vartheta(\mathscr{B}x,\mathscr{X}(u,v),\mathscr{X}(w,z)),\vartheta(\mathscr{B}u,\mathscr{X}(x,y),\mathscr{X}(w,z)),\\
&\qquad\vartheta(\mathscr{B}w,\mathscr{X}(x,y),\mathscr{X}(u,v)),\vartheta(\mathscr{B}x,\mathscr{X}(x,y),\mathscr{X}(x,y)),\\
&\qquad\vartheta(\mathscr{B}u,\mathscr{X}(u,v),\mathscr{X}(u,v)),\vartheta(\mathscr{B}w,\mathscr{X}(w,z),\mathscr{X}(w,z))\Big\}
\end{aligned}
$$

Case (iv): If $x < y, u < v$ and $w < z$, then $\mathscr{X}(x,y) = \frac{x}{3(y^2+2)}$, $\mathscr{X}(u,v) = 0$ and $\mathscr{X}(w,z) = 0$, it follows that

$$
\begin{aligned}
\vartheta(\mathscr{X}(x,y),\mathscr{X}(u,v),\mathscr{X}(w,z)) &\le \frac{1}{32}\frac{\vartheta(\mathscr{B}x,\mathscr{X}(x,y),\mathscr{X}(y,x))\vartheta(\mathscr{B}u,\mathscr{X}(u,v),\mathscr{X}(u,v))\vartheta(\mathscr{B}w,\mathscr{X}(w,z),\mathscr{X}(w,z))}{[\vartheta(\mathscr{B}x,\mathscr{B}u,\mathscr{B}w)]^2}\\
&\quad+\frac{1}{32}\vartheta(\mathscr{B}x,\mathscr{B}u,\mathscr{B}w)+\frac{1}{32}\vartheta(\mathscr{B}y,\mathscr{B}v,\mathscr{B}z)\\
&\quad+Lmin\Big\{\vartheta(\mathscr{B}x,\mathscr{X}(u,v),\mathscr{X}(w,z)),\vartheta(\mathscr{B}u,\mathscr{X}(x,y),\mathscr{X}(w,z)),\\
&\qquad\vartheta(\mathscr{B}w,\mathscr{X}(x,y),\mathscr{X}(u,v)),\vartheta(\mathscr{B}x,\mathscr{X}(x,y),\mathscr{X}(x,y)),
\end{aligned}
$$

$$\vartheta(\mathscr{B}u, \mathscr{X}(u,v), \mathscr{X}(u,v)), \vartheta(\mathscr{B}w, \mathscr{X}(w,z), \mathscr{X}(w,z))\Big\}$$

In similar manner, the cases $x < y, u \geq v, w \geq z$; $x < y, u < v, w \geq z$ and all others can be handled. Thus $\mathscr{X}$ and $\mathscr{B}$ satify all the conditions of Theorem 2.1 and also note that $(0,0)$ is a coupled coincidence point of $\mathscr{X}$ and $\mathscr{B}$.

Theorem 2.2. *Let $(\mathscr{M}, \vartheta, \leq)$ be a partially ordered complete ϑ-metric space and $\mathscr{X} : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$, $\mathscr{B} : \mathscr{M} \to \mathscr{M}$ be two continuous mappings such that $\mathscr{X}$ has mixed $\mathscr{B}$-monotone property and*

(i) there exist $\alpha, \beta, \gamma \in [0, 1)$ and $L \geq 0$ with $2\alpha + \beta + \gamma < 1$ such that

$$\begin{aligned}
&\vartheta(\mathscr{X}(x,y), \mathscr{X}(u,v), \mathscr{X}(w,z)) \qquad (2.6)\\
&\leq \alpha \frac{\vartheta(\mathscr{B}x,\mathscr{X}(x,y),\mathscr{X}(y,x))[1+\vartheta(\mathscr{B}u,\mathscr{X}(u,v),\mathscr{X}(u,v))][1+\vartheta(\mathscr{B}w,\mathscr{X}(w,z),\mathscr{X}(w,z))]}{[1+2\vartheta(\mathscr{B}x,\mathscr{B}u,\mathscr{B}w)]^2} + \beta\vartheta(\mathscr{B}x, \mathscr{B}u, \mathscr{B}w),\\
&+ \gamma\vartheta(\mathscr{B}y, \mathscr{B}v, \mathscr{B}z) + Lmin\{\vartheta(\mathscr{B}x, \mathscr{X}(u,v), \mathscr{X}(w,z)), \vartheta(\mathscr{B}u, \mathscr{X}(x,y), \mathscr{X}(w,z)),\\
&\vartheta(\mathscr{B}x, \mathscr{X}(x,y), \mathscr{X}(x,y)), \vartheta(\mathscr{B}u, \mathscr{X}(u,v), \mathscr{X}(u,v))\}\\
&\text{for all } x, y, u, v, w, z \in \mathscr{M} \text{ with } \mathscr{B}x \geq \mathscr{B}u \geq \mathscr{B}w \text{ and } \mathscr{B}y \leq \mathscr{B}v \leq \mathscr{B}z;
\end{aligned}$$

(ii) $\mathscr{X}(\mathscr{M} \times \mathscr{M}) \subseteq g(\mathscr{M})$;

(iii) $\mathscr{B}$ commutes with $\mathscr{X}$.

If there exist $x_0, y_0 \in \mathscr{M}$ such that $\mathscr{B}x_0 \leq \mathscr{X}(x_0, y_0)$ and $\mathscr{B}y_0 \geq \mathscr{X}(y_0, x_0)$, then $\mathscr{X}$ and $\mathscr{B}$ have a coupled coincidence point in $\mathscr{M} \times \mathscr{M}$.

Proof. Let x_0 and y_0 be any two elements in $\mathscr{M}$ such that $\mathscr{B}x_0 \leq \mathscr{X}(x_0, y_0)$ and $\mathscr{B}y_0 \geq \mathscr{X}(y_0, x_0)$. Since $\mathscr{X}(\mathscr{M} \times \mathscr{M}) \subseteq \mathscr{B}(\mathscr{M})$, we constuct two sequences $\langle x_\kappa \rangle_{\kappa=1}^{\infty}$ and $\langle y_\kappa \rangle_{\kappa=1}^{\infty}$ in $\mathscr{M}$ as follows:

$$\mathscr{B}x_{\kappa+1} = \mathscr{X}(x_\kappa, y_\kappa) \text{ and } \mathscr{B}y_{\kappa+1} = \mathscr{X}(y_\kappa, x_\kappa) \text{ for } \kappa \in \mathbb{N}.$$

Since $\mathscr{X}$ has mixed $\mathscr{B}$-monotone property, we have

$$\begin{aligned}
\mathscr{B}x_\kappa &= \mathscr{X}(x_{\kappa-1}, y_{\kappa-1}) \leq \mathscr{X}(x_\kappa, y_{\kappa-1}) \leq \mathscr{X}(x_\kappa, y_\kappa) = \mathscr{B}x_{\kappa+1}\\
\text{and } \mathscr{B}y_{\kappa+1} &= \mathscr{X}(y_\kappa, x_\kappa) \leq \mathscr{X}(y_{\kappa-1}, x_\kappa) \leq \mathscr{X}(y_{\kappa-1}, x_{\kappa-1}) = \mathscr{B}y_\kappa.
\end{aligned}$$

Now using (2.1) with $x = x_\kappa$,$y = y_\kappa$, $u = w = x_{\kappa-1}$ and $v = z = y_{\kappa-1}$, we get

$$\begin{aligned}
&\vartheta(\mathscr{B}x_{\kappa+1}, \mathscr{B}x_\kappa, \mathscr{B}x_\kappa) = \vartheta(\mathscr{X}(x_\kappa, y_\kappa), \mathscr{X}(x_{\kappa-1}, y_{\kappa-1}), \mathscr{X}(x_{\kappa-1}, y_{\kappa-1}))\\
&\leq \alpha \frac{\vartheta(\mathscr{B}x_\kappa,\mathscr{X}(x_\kappa,y_\kappa),\mathscr{X}(x_\kappa,y_\kappa))[1+\vartheta(\mathscr{B}x_{\kappa-1},\mathscr{X}(x_{\kappa-1},y_{\kappa-1}),\mathscr{X}(x_{\kappa-1},y_{\kappa-1}))][1+\vartheta(\mathscr{B}x_{\kappa-1},\mathscr{X}(x_{\kappa-1},y_{\kappa-1}),\mathscr{X}(x_{\kappa-1},y_{\kappa-1}))]}{[1+2\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1})]^2}\\
&\quad + \beta\vartheta(\mathscr{B}x_\kappa, \mathscr{B}x_{\kappa-1}, \mathscr{B}x_{\kappa-1}) + \gamma\vartheta(\mathscr{B}y_\kappa, \mathscr{B}y_{\kappa-1}, \mathscr{B}y_{\kappa-1})\\
&\quad + Lmin\{\vartheta(\mathscr{B}x_\kappa, \mathscr{X}(x_{\kappa-1}, y_{\kappa-1}), \mathscr{X}(x_{\kappa-1}, y_{\kappa-1})), \vartheta(\mathscr{B}x_{\kappa-1}, \mathscr{X}(x_\kappa, y_\kappa), \mathscr{X}(x_{\kappa-1}, y_{\kappa-1})),\\
&\quad \vartheta(\mathscr{B}x_\kappa, \mathscr{X}(x_\kappa, y_\kappa), \mathscr{X}(y_\kappa, x_\kappa)), \vartheta(\mathscr{B}x_{\kappa-1}, \mathscr{X}(x_{\kappa-1}, y_{\kappa-1}), \mathscr{X}(x_{\kappa-1}, y_{\kappa-1}))\}\\
&= \alpha \frac{\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa+1},\mathscr{B}x_{\kappa+1})[1+\vartheta(\mathscr{B}x_{\kappa-1},\mathscr{B}x_\kappa,\mathscr{B}x_\kappa)][1+\vartheta(\mathscr{B}x_{\kappa-1},\mathscr{B}x_\kappa,\mathscr{B}x_\kappa)]}{[1+2\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1})]^2}\\
&\quad + \beta\vartheta(\mathscr{B}x_\kappa, \mathscr{B}x_{\kappa-1}, \mathscr{B}x_{\kappa-1}) + \gamma\vartheta(\mathscr{B}y_\kappa, \mathscr{B}y_{\kappa-1}, \mathscr{B}y_{\kappa-1})\\
&\quad + Lmin\{\vartheta(\mathscr{B}x_\kappa, \mathscr{B}x_\kappa, \mathscr{B}x_\kappa), \vartheta(\mathscr{B}x_{\kappa-1}, \mathscr{B}x_{\kappa+1}, \mathscr{B}x_\kappa),\\
&\quad \vartheta(\mathscr{B}x_\kappa, \mathscr{B}x_{\kappa+1}, \mathscr{B}x_{\kappa+1}), \vartheta(\mathscr{B}x_{\kappa-1}, \mathscr{B}x_\kappa, \mathscr{B}x_\kappa)\}
\end{aligned}$$

$$\leq \alpha \frac{2\vartheta(\mathscr{B}x_{\kappa+1},\mathscr{B}x_\kappa,\mathscr{B}x_\kappa)[1+2\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1})][1+2\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1})]}{[1+2\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1})]^2}$$
$$+\beta\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1})+\gamma\vartheta(\mathscr{B}y_\kappa,\mathscr{B}y_{\kappa-1},\mathscr{B}y_{\kappa-1})$$

so that

$$\vartheta(\mathscr{B}x_{\kappa+1},\mathscr{B}x_\kappa,\mathscr{B}x_\kappa) \leq \frac{1}{(1-2\alpha)}[\beta\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1})+\gamma\vartheta(\mathscr{B}y_\kappa,\mathscr{B}y_{\kappa-1},\mathscr{B}y_{\kappa-1})] \tag{2.7}$$

Similarly,

$$\vartheta(\mathscr{B}y_{\kappa+1},\mathscr{B}y_\kappa,\mathscr{B}y_\kappa) \leq \frac{1}{(1-2\alpha)}[\beta\vartheta(\mathscr{B}y_\kappa,\mathscr{B}y_{\kappa-1},\mathscr{B}y_{\kappa-1})+\gamma\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1})] \tag{2.8}$$

Adding (2.7) and (2.8), we have

$$\begin{aligned}\vartheta(\mathscr{B}x_{\kappa+1},\mathscr{B}x_\kappa,\mathscr{B}x_\kappa)+\vartheta(\mathscr{B}y_{\kappa+1},\mathscr{B}y_\kappa,\mathscr{B}y_\kappa)& \\ \leq \frac{\beta+\gamma}{1-2\alpha}[\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1})+\vartheta(\mathscr{B}y_\kappa,\mathscr{B}y_{\kappa-1},\mathscr{B}y_{\kappa-1})].&\end{aligned} \tag{2.9}$$

Let $\Delta_\kappa = \vartheta(\mathscr{B}x_{\kappa+1},\mathscr{B}x_\kappa,\mathscr{B}x_\kappa)+\vartheta(\mathscr{B}y_{\kappa+1},\mathscr{B}y_\kappa,\mathscr{B}y_\kappa)$ and $c = \dfrac{\beta+\gamma}{1-2\alpha}$ where $0 \leq c < 1$ inview of choice of α,β and γ.
Now the inequality (2.9) becomes as follows
$\Delta_\kappa \leq c.\Delta_{\kappa-1}$ for $n \in \mathbb{N}$.
Consequently $\Delta_\kappa \leq c\Delta_{\kappa-1} \leq c^2\Delta_{\kappa-2} \leq ... \leq c^\kappa\Delta_0$.
If $\Delta_0 = 0$, we have $\vartheta(\mathscr{B}x_1,\mathscr{B}x_0,\mathscr{B}x_0)+\vartheta(\mathscr{B}y_1,\mathscr{B}y_0,\mathscr{B}y_0) = 0$
or $\vartheta(\mathscr{X}(x_0,y_0),\mathscr{B}x_0,\mathscr{B}x_0)+\vartheta(\mathscr{X}(y_0,x_0),\mathscr{B}y_0,\mathscr{B}y_0) = 0$ which implies that $\mathscr{X}(x_0,y_0) = \mathscr{B}x_0$ and $\mathscr{X}(y_0,x_0) = \mathscr{B}y_0$.
That is, (x_0,y_0) is a coupled coincidence point of $\mathscr{X}$ and $\mathscr{B}$.
Suppose that $\Delta_0 > 0$.
Now using repeated application of rectangle inequality of ϑ-metric and inequality $\Delta_\kappa \leq c^\kappa\Delta_0$, we have

$$\begin{aligned}\vartheta(\mathscr{B}x_\eta,\mathscr{B}x_\kappa,\mathscr{B}x_\kappa)+\vartheta(\mathscr{B}y_\eta,\mathscr{B}y_\kappa,\mathscr{B}y_\kappa) &\leq [\vartheta(\mathscr{B}x_{\kappa+1},\mathscr{B}x_\kappa,\mathscr{B}x_\kappa)+\vartheta(\mathscr{B}x_\eta,\mathscr{B}x_{\kappa+1},\mathscr{B}x_{\kappa+1})]\\ &\quad+[\vartheta(\mathscr{B}y_{\kappa+1},\mathscr{B}y_\kappa,\mathscr{B}y_\kappa)+\vartheta(\mathscr{B}y_\eta,\mathscr{B}y_{\kappa+1},\mathscr{B}y_{\kappa+1})]\\ &\leq[\vartheta(\mathscr{B}x_{\kappa+1},\mathscr{B}x_\kappa,\mathscr{B}x_\kappa)+\vartheta(\mathscr{B}x_{\kappa+2},\mathscr{B}x_{\kappa+1},\mathscr{B}x_{\kappa+1})+\vartheta(\mathscr{B}x_\eta,\mathscr{B}x_{\kappa+2},\mathscr{B}x_{\kappa+2})]\\ &\quad[\vartheta(\mathscr{B}y_{\kappa+1},\mathscr{B}y_\kappa,\mathscr{B}y_\kappa)+\vartheta(\mathscr{B}y_{\kappa+2},\mathscr{B}y_{\kappa+1},\mathscr{B}y_{\kappa+1})+\vartheta(\mathscr{B}y_\eta,\mathscr{B}y_{\kappa+2},\mathscr{B}y_{\kappa+2})]\\ &\vdots\\ &\leq[\vartheta(\mathscr{B}x_{\kappa+1},\mathscr{B}x_\kappa,\mathscr{B}x_\kappa)+\vartheta(\mathscr{B}x_{\kappa+2},\mathscr{B}x_{\kappa+1},\mathscr{B}x_{\kappa+1})+\cdots+\vartheta(\mathscr{B}x_\eta,\mathscr{B}x_{\eta-1},\mathscr{B}x_{\eta-1})]\\ &\quad[\vartheta(\mathscr{B}y_{\kappa+1},\mathscr{B}y_\kappa,\mathscr{B}y_\kappa)+\vartheta(\mathscr{B}y_{\kappa+2},\mathscr{B}y_{\kappa+1},\mathscr{B}y_{\kappa+1})+\cdots+\vartheta(\mathscr{B}y_\eta,\mathscr{B}y_{\eta-1},\mathscr{B}y_{\eta-1})]\\ &=\Delta_\kappa+\Delta_{\kappa+1}+\Delta_{\kappa+2}+\cdots+\Delta_{\eta-1}\\ &\leq[c^\kappa+c^{\kappa+1}+\cdots+c^{\eta-1}]\Delta_0\\ &\leq c^\kappa\cdot\frac{1}{1-c}\Delta_0 \text{ for } \eta>\kappa\end{aligned}$$

or

$$\vartheta(\mathscr{B}x_\eta,\mathscr{B}x_\kappa,\mathscr{B}x_\kappa)+\vartheta(\mathscr{B}y_\eta,\mathscr{B}y_\kappa,\mathscr{B}y_\kappa) \leq c^\kappa\cdot\frac{1}{1-c}\Delta_0 \text{ for } \eta>\kappa. \tag{2.10}$$

Since $0 \leq c < 1, c^{\kappa} \to 0$ as $\kappa \to \infty$.
Now applying limit as $\kappa \to \infty$ with $\eta > \kappa$ in the inequality (2.10), we have
$\vartheta(\mathscr{B}x_\eta, \mathscr{B}x_\kappa, \mathscr{B}x_\kappa) + \vartheta(\mathscr{B}y_\eta, \mathscr{B}y_\kappa, \mathscr{B}y_\kappa) \leq 0$ which follows that $\langle x_\kappa \rangle_{\kappa=1}^{\infty}$ and $\langle y_\kappa \rangle_{\kappa=1}^{\infty}$ are ϑ−cauchy sequences in $\mathscr{M}$.
Since $(\mathscr{M}, \vartheta)$ is a partially ordered ϑ-complete, there exist $a, b \in \mathscr{M}$ such that $x_\kappa \to a$ and $y_\kappa \to b$.
Now we prove that (a, b) is a coupled coincidence point of $\mathscr{X}$ and $\mathscr{B}$.
Since $\mathscr{X}$ commutes with $\mathscr{B}$, we have

$$\mathscr{X}(\mathscr{B}x_\kappa, \mathscr{B}y_\kappa) = \mathscr{B}(\mathscr{X}(x_\kappa, y_\kappa)) = \mathscr{B}(\mathscr{B}x_{\kappa+1})$$
and
$$\mathscr{X}(\mathscr{B}y_\kappa, \mathscr{B}x_\kappa) = \mathscr{B}(\mathscr{X}(y_\kappa, x_\kappa)) = \mathscr{B}(\mathscr{B}y_{\kappa+1})$$

Since $\mathscr{X}$ and $\mathscr{B}$ are continuous, we have

$$\lim_{\kappa\to\infty} \mathscr{X}(\mathscr{B}x_\kappa, \mathscr{B}y_\kappa) = \lim_{\kappa\to\infty} \mathscr{B}(\mathscr{B}x_{\kappa+1}) = \mathscr{B}a$$
and
$$\lim_{\kappa\to\infty} \mathscr{X}(\mathscr{B}y_\kappa, \mathscr{B}x_\kappa) = \lim_{\kappa\to\infty} \mathscr{B}(\mathscr{B}y_{\kappa+1}) = \mathscr{B}b$$

Since ϑ is continuous in all its variables, we have

$$\vartheta(\mathscr{X}(a, b), \mathscr{B}a, \mathscr{B}b) = \vartheta(\lim_{\kappa\to\infty} \mathscr{X}(\mathscr{B}x_\kappa, \mathscr{B}y_\kappa), \mathscr{B}a, \mathscr{B}a) = \vartheta(\mathscr{B}a, \mathscr{B}a, \mathscr{B}a)$$

so that
$$\vartheta(\mathscr{X}(a, b), \mathscr{B}b, \mathscr{B}b) = 0$$

which implies that $\mathscr{X}(a, b) = \mathscr{B}a$.
Similarly, it can be verified that $\mathscr{X}(b, a) = \mathscr{B}b$.
Thus(a, b) is a coupled coincidence point of $\mathscr{X}$ and $\mathscr{B}$ in $\mathscr{M} \times \mathscr{M}$. □

Remark 2.2. If we take $\mathscr{B}$ is an identity mapping and $\gamma = 0$ in the Theorem 2.2, we get Theorem 3.1 in the results of the Chouhan and Richa Sharma [5].

Theorem 2.3. *Let $(\mathscr{M}, \vartheta, \leq)$ be a partially ordered complete ϑ-metric space and $\mathscr{X} : \mathscr{M}\times\mathscr{M} \to \mathscr{M}$, $\mathscr{B} : \mathscr{M} \to \mathscr{M}$ be two continuous mappings such that $\mathscr{X}$ has mixed $\mathscr{B}$-monotone property and*

(i) there exist non negative real numbers $\Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5, \Psi_6, \Psi_7, \Psi_8$ and Ψ_9 with $0 \leq \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 + \Psi_5 + \Psi_6 + \Psi_7 + \Psi_8 + \Psi_9 < 1$ such that

$$\begin{aligned} &\vartheta(\mathscr{X}(x, y), \mathscr{X}(u, v), \mathscr{X}(w, z)) \qquad (2.11)\\ \leq &\Psi_1 \frac{\vartheta(\mathscr{B}x, \mathscr{B}u, \mathscr{B}w) + \vartheta(\mathscr{B}y, \mathscr{B}v, \mathscr{B}z)}{2} + \Psi_2 \frac{\vartheta(\mathscr{X}(x, y), \mathscr{X}(u, v), \mathscr{X}(w, z)) \cdot \vartheta(\mathscr{B}x, \mathscr{B}u, \mathscr{B}w)}{1 + \vartheta(\mathscr{B}x, \mathscr{B}u, \mathscr{B}w) + \vartheta(\mathscr{B}y, \mathscr{B}v, \mathscr{B}z)},\\ &+\Psi_3 \frac{\vartheta(\mathscr{X}(x, y), \mathscr{X}(u, v), \mathscr{X}(w, z)) \cdot \vartheta(\mathscr{B}y, \mathscr{B}v, \mathscr{B}z)}{1 + \vartheta(\mathscr{B}x, \mathscr{B}u, \mathscr{B}w) + \vartheta(\mathscr{B}y, \mathscr{B}v, \mathscr{B}z)} + \Psi_4 \frac{\vartheta(\mathscr{B}x, \mathscr{B}x, \mathscr{X}(x, y)) \cdot \vartheta(\mathscr{B}x, \mathscr{B}u, \mathscr{B}w)}{1 + \vartheta(\mathscr{B}x, \mathscr{B}u, \mathscr{B}w) + \vartheta(\mathscr{B}y, \mathscr{B}v, \mathscr{B}z)},\\ &+ \Psi_5 \frac{\vartheta(\mathscr{B}x, \mathscr{B}x, \mathscr{X}(x, y)) \cdot \vartheta(\mathscr{B}y, \mathscr{B}v, \mathscr{B}z)}{1 + \vartheta(\mathscr{B}x, \mathscr{B}u, \mathscr{B}w) + \vartheta(\mathscr{B}y, \mathscr{B}v, \mathscr{B}z)} + \Psi_6 \frac{\vartheta(\mathscr{B}u, \mathscr{B}u, \mathscr{X}(u, v)) \cdot \vartheta(\mathscr{B}x, \mathscr{B}u, \mathscr{B}w)}{1 + \vartheta(\mathscr{B}x, \mathscr{B}u, \mathscr{B}w) + \vartheta(\mathscr{B}y, \mathscr{B}v, \mathscr{B}z)}, \end{aligned}$$

$$+ \Psi_7 \frac{\vartheta(\mathscr{B}u, \mathscr{B}u, \mathscr{X}(u,v)) \cdot \vartheta(\mathscr{B}y, \mathscr{B}v, \mathscr{B}z)}{1 + \vartheta(\mathscr{B}x, \mathscr{B}u, \mathscr{B}w) + \vartheta(\mathscr{B}y, \mathscr{B}v, \mathscr{B}z)} + \Psi_8 \frac{\vartheta(\mathscr{B}w, \mathscr{B}w, \mathscr{X}(w,z)) \cdot \vartheta(\mathscr{B}x, \mathscr{B}u, \mathscr{B}w)}{1 + \vartheta(\mathscr{B}x, \mathscr{B}u, \mathscr{B}w) + \vartheta(\mathscr{B}y, \mathscr{B}v, \mathscr{B}z)},$$
$$+ \Psi_9 \frac{\vartheta(\mathscr{B}w, \mathscr{B}w, \mathscr{X}(w,z)) \cdot \vartheta(\mathscr{B}y, \mathscr{B}v, \mathscr{B}z)}{1 + \vartheta(\mathscr{B}x, \mathscr{B}u, \mathscr{B}w) + \vartheta(\mathscr{B}y, \mathscr{B}v, \mathscr{B}z)}$$

for all $x, y, u, v, w, z \in \mathscr{M}$ with $\mathscr{B}x \geq \mathscr{B}u \geq \mathscr{B}w$ and $\mathscr{B}y \leq \mathscr{B}v \leq \mathscr{B}z$;

(ii) $\mathscr{X}(\mathscr{M} \times \mathscr{M}) \subseteq \mathscr{B}(\mathscr{M})$;
(iii) $\mathscr{B}$ *commutes with* $\mathscr{X}$.

If there exist $x_0, y_0 \in \mathscr{M}$ such that $\mathscr{B}x_0 \leq \mathscr{X}(x_0, y_0)$ and $\mathscr{B}y_0 \geq \mathscr{X}(y_0, x_0)$, then $\mathscr{X}$ and $\mathscr{B}$ have a coupled coincidence point in $\mathscr{M} \times \mathscr{M}$.

Proof. Let x_0 and y_0 be any two elements in $\mathscr{M}$ such that $\mathscr{B}x_0 \leq \mathscr{X}(x_0, y_0)$ and $\mathscr{B}y_0 \leq \mathscr{X}(y_0, x_0)$. Since $\mathscr{X}(\mathscr{M} \times \mathscr{M}) \subseteq g(\mathscr{M})$, we constuct two sequences $\langle x_\kappa \rangle_{\kappa=1}^{\infty}$ and $\langle y_\kappa \rangle_{\kappa=1}^{\infty}$ in $\mathscr{M}$ as follows:

$$\mathscr{B}x_{\kappa+1} = \mathscr{X}(x_\kappa, y_\kappa) \text{ and } \mathscr{B}y_{\kappa+1} = \mathscr{X}(y_\kappa, x_\kappa) \text{ for } \kappa \in \mathbb{N}.$$

Since $\mathscr{X}$ has mixed $\mathscr{B}$-monotone property, we have

$$\mathscr{B}x_\kappa = \mathscr{X}(x_{\kappa-1}, y_{\kappa-1}) \leq \mathscr{X}(x_\kappa, y_{\kappa-1}) \leq \mathscr{X}(x_\kappa, y_\kappa) = \mathscr{B}x_{\kappa+1}$$
$$\text{and } \mathscr{B}y_{\kappa+1} = \mathscr{X}(y_\kappa, x_\kappa) \leq \mathscr{X}(y_{\kappa-1}, x_\kappa) \leq \mathscr{X}(y_{\kappa-1}, x_{\kappa-1}) = \mathscr{B}y_\kappa.$$

Now using (2.11) with $x = x_\kappa$, $y = y_\kappa$, $u = w = x_{\kappa-1}$ and $v = z = y_{\kappa-1}$, we get

$$\begin{aligned}
\vartheta(\mathscr{B}x_{\kappa+1}, \mathscr{B}x_\kappa, \mathscr{B}x_\kappa) &= \vartheta(\mathscr{X}(x_\kappa, y_\kappa), \mathscr{X}(x_{\kappa-1}, y_{\kappa-1}), \mathscr{X}(x_{\kappa-1}, y_{\kappa-1})) \\
&\leq \Psi_1 \frac{\vartheta(\mathscr{B}x_\kappa, \mathscr{B}x_{\kappa-1}, \mathscr{B}x_{\kappa-1}) + \vartheta(\mathscr{B}y_\kappa, \mathscr{B}y_{\kappa-1}, \mathscr{B}y_{\kappa-1})}{2} \\
&\quad + \Psi_2 \frac{\vartheta(\mathscr{X}(x_\kappa, y_\kappa), \mathscr{X}(x_{\kappa-1}, y_{\kappa-1}), \mathscr{X}(x_{\kappa-1}, y_{\kappa-1})) \cdot \vartheta(\mathscr{B}x_\kappa, \mathscr{B}x_{\kappa-1}, \mathscr{B}x_{\kappa-1})}{1 + \vartheta(\mathscr{B}x_\kappa, \mathscr{B}x_{\kappa-1}, \mathscr{B}x_{\kappa-1}) + \vartheta(\mathscr{B}y_\kappa, \mathscr{B}y_{\kappa-1}, \mathscr{B}y_{\kappa-1})}, \\
&\quad + \Psi_3 \frac{\vartheta(\mathscr{X}(x_\kappa, y_\kappa), \mathscr{X}(x_{\kappa-1}, y_{\kappa-1}), \mathscr{X}(x_{\kappa-1}, y_{\kappa-1})) \cdot \vartheta(\mathscr{B}y_\kappa, \mathscr{B}y_{\kappa-1}, \mathscr{B}y_{\kappa-1})}{1 + \vartheta(\mathscr{B}x_\kappa, \mathscr{B}x_{\kappa-1}, \mathscr{B}x_{\kappa-1}) + \vartheta(\mathscr{B}y_\kappa, \mathscr{B}y_{\kappa-1}, \mathscr{B}y_{\kappa-1})} \\
&\quad + \Psi_4 \frac{\vartheta(\mathscr{B}x_\kappa, \mathscr{B}x_\kappa, \mathscr{X}(x_\kappa, y_\kappa)) \cdot \vartheta(\mathscr{B}x_\kappa, \mathscr{B}x_{\kappa-1}, \mathscr{B}x_{\kappa-1})}{1 + \vartheta(\mathscr{B}x_\kappa, \mathscr{B}x_{\kappa-1}, \mathscr{B}x_{\kappa-1}) + \vartheta(\mathscr{B}y_\kappa, \mathscr{B}y_{\kappa-1}, \mathscr{B}y_{\kappa-1})}, \\
&\quad + \Psi_5 \frac{\vartheta(\mathscr{B}x_\kappa, \mathscr{B}x_\kappa, \mathscr{X}(x_\kappa, y_\kappa)) \cdot \vartheta(\mathscr{B}y_\kappa, \mathscr{B}y_{\kappa-1}, \mathscr{B}y_{\kappa-1})}{1 + \vartheta(\mathscr{B}x_\kappa, \mathscr{B}x_{\kappa-1}, \mathscr{B}x_{\kappa-1}) + \vartheta(\mathscr{B}y_\kappa, \mathscr{B}y_{\kappa-1}, \mathscr{B}y_{\kappa-1})} \\
&\quad + \Psi_6 \frac{\vartheta(\mathscr{B}x_{\kappa-1}, \mathscr{B}x_{\kappa-1}, \mathscr{X}(x_{\kappa-1}, y_{\kappa-1})) \cdot \vartheta(\mathscr{B}x_\kappa, \mathscr{B}x_{\kappa-1}, \mathscr{B}x_{\kappa-1})}{1 + \vartheta(\mathscr{B}x_\kappa, \mathscr{B}x_{\kappa-1}, \mathscr{B}x_{\kappa-1}) + \vartheta(\mathscr{B}y_\kappa, \mathscr{B}y_{\kappa-1}, \mathscr{B}y_{\kappa-1})}, \\
&\quad + \Psi_7 \frac{\vartheta(\mathscr{B}x_{\kappa-1}, \mathscr{B}x_{\kappa-1}, \mathscr{X}(x_{\kappa-1}, y_{\kappa-1})) \cdot \vartheta(\mathscr{B}y_\kappa, \mathscr{B}y_{\kappa-1}, \mathscr{B}y_{\kappa-1})}{1 + \vartheta(\mathscr{B}x_\kappa, \mathscr{B}x_{\kappa-1}, \mathscr{B}x_{\kappa-1}) + \vartheta(\mathscr{B}y_\kappa, \mathscr{B}y_{\kappa-1}, \mathscr{B}y_{\kappa-1})} \\
&\quad + \Psi_8 \frac{\vartheta(\mathscr{B}x_{\kappa-1}, \mathscr{B}x_{\kappa-1}, \mathscr{X}(x_{\kappa-1}, y_{\kappa-1})) \cdot \vartheta(\mathscr{B}x_\kappa, \mathscr{B}x_{\kappa-1}, \mathscr{B}x_{\kappa-1})}{1 + \vartheta(\mathscr{B}x_\kappa, \mathscr{B}x_{\kappa-1}, \mathscr{B}x_{\kappa-1}) + \vartheta(\mathscr{B}y_\kappa, \mathscr{B}y_{\kappa-1}, \mathscr{B}y_{\kappa-1})}, \\
&\quad + \Psi_9 \frac{\vartheta(\mathscr{B}x_{\kappa-1}, \mathscr{B}x_{\kappa-1}, \mathscr{X}(x_{\kappa-1}, y_{\kappa-1})) \cdot \vartheta(\mathscr{B}y_\kappa, \mathscr{B}y_{\kappa-1}, \mathscr{B}y_{\kappa-1})}{1 + \vartheta(\mathscr{B}x_\kappa, \mathscr{B}x_{\kappa-1}, \mathscr{B}x_{\kappa-1}) + \vartheta(\mathscr{B}y_\kappa, \mathscr{B}y_{\kappa-1}, \mathscr{B}y_{\kappa-1})}
\end{aligned}$$

$$= \Psi_1 \frac{\vartheta(\mathscr{B}x_\kappa, \mathscr{B}x_{\kappa-1}, \mathscr{B}x_{\kappa-1}) + \vartheta(\mathscr{B}y_\kappa, \mathscr{B}y_{\kappa-1}, \mathscr{B}y_{\kappa-1})}{2}$$

$$+\Psi_2\frac{\vartheta(\mathscr{B}x_{\kappa+1},\mathscr{B}x_\kappa,\mathscr{B}x_\kappa)\cdot\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1})}{1+\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1})+\vartheta(\mathscr{B}y_\kappa,\mathscr{B}y_{\kappa-1},\mathscr{B}y_{n-1})},$$
$$+\Psi_3\frac{\vartheta(\mathscr{B}x_{\kappa+1},\mathscr{B}x_\kappa,\mathscr{B}x_\kappa)\cdot\vartheta(\mathscr{B}y_\kappa,\mathscr{B}y_{\kappa-1},\mathscr{B}y_{\kappa-1})}{1+\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1})+\vartheta(\mathscr{B}y_\kappa,\mathscr{B}y_{\kappa-1},\mathscr{B}y_{\kappa-1})}$$
$$+\Psi_4\frac{\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa+1})\cdot\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1})}{1+\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1})+\vartheta(\mathscr{B}y_\kappa,\mathscr{B}y_{\kappa-1},\mathscr{B}y_{\kappa-1})},$$
$$+\Psi_5\frac{\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa+1})\cdot\vartheta(\mathscr{B}y_\kappa,\mathscr{B}y_{\kappa-1},\mathscr{B}y_{\kappa-1})}{1+\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1})+\vartheta(\mathscr{B}y_\kappa,\mathscr{B}y_{\kappa-1},\mathscr{B}y_{\kappa-1})}$$
$$+\Psi_6\frac{\vartheta(\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1},\mathscr{B}x_\kappa)\cdot\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1})}{1+\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1})+\vartheta(\mathscr{B}y_\kappa,\mathscr{B}y_{\kappa-1},\mathscr{B}y_{\kappa-1})},$$
$$+\Psi_7\frac{\vartheta(\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1},\mathscr{B}x_\kappa)\cdot\vartheta(\mathscr{B}y_\kappa,\mathscr{B}y_{\kappa-1},\mathscr{B}y_{\kappa-1})}{1+\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1})+\vartheta(\mathscr{B}y_\kappa,\mathscr{B}y_{\kappa-1},\mathscr{B}y_{\kappa-1})}$$
$$+\Psi_8\frac{\vartheta(\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1},\mathscr{B}x_\kappa)\cdot\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1})}{1+\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1})+\vartheta(\mathscr{B}y_\kappa,\mathscr{B}y_{\kappa-1},\mathscr{B}y_{\kappa-1})},$$
$$+\Psi_9\frac{\vartheta(\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1},\mathscr{B}x_\kappa)\cdot\vartheta(\mathscr{B}y_\kappa,\mathscr{B}y_{\kappa-1},\mathscr{B}y_{\kappa-1})}{1+\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1})+\vartheta(\mathscr{B}y_\kappa,\mathscr{B}y_{\kappa-1},\mathscr{B}y_{\kappa-1})}$$

$$\begin{aligned}\leq&\Psi_1\frac{\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1})+\vartheta(\mathscr{B}y_\kappa,\mathscr{B}y_{\kappa-1},\mathscr{B}y_{\kappa-1})}{2}\\&+\Psi_2\vartheta(\mathscr{B}x_{\kappa+1},\mathscr{B}x_\kappa,\mathscr{B}x_\kappa)+\Psi_3\vartheta(\mathscr{B}x_{\kappa+1},\mathscr{B}x_\kappa,\mathscr{B}x_\kappa)+\Psi_4\vartheta(\mathscr{B}x_{\kappa+1},\mathscr{B}x_\kappa,\mathscr{B}x_\kappa)\\&+\Psi_5\vartheta(\mathscr{B}x_{\kappa+1},\mathscr{B}x_\kappa,\mathscr{B}x_\kappa)+\Psi_6\vartheta(\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1},\mathscr{B}x_\kappa)+\Psi_7\vartheta(\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1},\mathscr{B}x_\kappa)\\&+\Psi_8\vartheta(\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1},\mathscr{B}x_\kappa)+\Psi_9\vartheta(\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1},\mathscr{B}x_\kappa)\end{aligned}$$

so that

$$\begin{aligned}\vartheta(\mathscr{B}x_{\kappa+1},\mathscr{B}x_\kappa,\mathscr{B}x_\kappa)\leq&\frac{\frac{\Psi_1}{2}+\Psi_6+\Psi_7+\Psi_8+\Psi_9}{[1-(\Psi_2+\Psi_3+\Psi_4+\Psi_5)]}\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1})\\&+\frac{\Psi_1}{2[1-(\Psi_2+\Psi_3+\Psi_4+\Psi_5)]}\vartheta(\mathscr{B}y_\kappa,\mathscr{B}y_{\kappa-1},\mathscr{B}y_{\kappa-1})\end{aligned}\tag{2.12}$$

Similarly,

$$\begin{aligned}\vartheta(\mathscr{B}y_{\kappa+1},\mathscr{B}y_\kappa,\mathscr{B}y_\kappa)\leq&\frac{\frac{\Psi_1}{2}+\Psi_6+\Psi_7+\Psi_8+\Psi_9}{[1-(\Psi_2+\Psi_3+\Psi_4+\Psi_5)]}\vartheta(\mathscr{B}y_\kappa,\mathscr{B}y_{\kappa-1},\mathscr{B}y_{\kappa-1})\\&+\frac{\Psi_1}{2[1-(\Psi_2+\Psi_3+\Psi_4+\Psi_5)]}\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1})\end{aligned}\tag{2.13}$$

Adding (2.12) and (2.13), we have

$$\begin{aligned}&\vartheta(\mathscr{B}x_{\kappa+1},\mathscr{B}x_\kappa,\mathscr{B}x_\kappa)+\vartheta(\mathscr{B}y_{\kappa+1},\mathscr{B}y_\kappa,\mathscr{B}y_\kappa)\\&\quad\leq\frac{\Psi_1+\Psi_6+\Psi_7+\Psi_8+\Psi_9}{1-(\Psi_2+\Psi_3+\Psi_4+\Psi_5)}[\vartheta(\mathscr{B}x_\kappa,\mathscr{B}x_{\kappa-1},\mathscr{B}x_{\kappa-1})+\vartheta(\mathscr{B}y_\kappa,\mathscr{B}y_{\kappa-1},\mathscr{B}y_{\kappa-1})].\end{aligned}\tag{2.14}$$

Let $\Delta_\kappa=\vartheta(\mathscr{B}x_{\kappa+1},\mathscr{B}x_\kappa,\mathscr{B}x_\kappa)+\vartheta(\mathscr{B}y_{\kappa+1},\mathscr{B}y_\kappa,\mathscr{B}y_\kappa)$ and $c=\dfrac{\Psi_1+\Psi_6+\Psi_7+\Psi_8+\Psi_9}{1-(\Psi_2+\Psi_3+\Psi_4+\Psi_5)}$ where $0\leq c<1$ inview of choice of $\Psi_1,\Psi_2,\Psi_3,\Psi_4,\Psi_5,\Psi_6,\Psi_7,\Psi_8$ and Ψ_9.

Now the inequality (2.14) becomes as follows
$\Delta_\kappa \leq c.\Delta_{\kappa-1}$ for $n \in \mathbb{N}$,
Consequently $\Delta_\kappa \leq c\Delta_{\kappa-1} \leq c^2\Delta_{\kappa-2} \leq ... \leq c^\kappa\Delta_0$.
If $\Delta_0 = 0$, we have $\vartheta(\mathscr{B}x_1, \mathscr{B}x_0, \mathscr{B}x_0) + \vartheta(\mathscr{B}y_1, \mathscr{B}y_0, \mathscr{B}y_0) = 0$,
or $\vartheta(\mathscr{X}(x_0, y_0), \mathscr{B}x_0, \mathscr{B}x_0) + \vartheta(\mathscr{X}(y_0, x_0), \mathscr{B}y_0, \mathscr{B}y_0) = 0$ which implies that $\mathscr{X}(x_0, y_0) = \mathscr{B}x_0$ and $\mathscr{X}(y_0, x_0) = \mathscr{B}y_0$.
That is, (x_0, y_0) is a coupled coincidence point of $\mathscr{X}$ and $\mathscr{B}$.
Suppose that $\Delta_0 > 0$.
Now using rectangle inequality of ϑ-metric repeatedly and inequality $\Delta_\kappa \leq c^\kappa\Delta_0$, we have

$$\begin{aligned}
&\vartheta(\mathscr{B}x_\eta, \mathscr{B}x_\kappa, \mathscr{B}x_\kappa) + \vartheta(\mathscr{B}y_\eta, \mathscr{B}y_\kappa, \mathscr{B}y_\kappa) \leq [\vartheta(\mathscr{B}x_{\kappa+1}, \mathscr{B}x_\kappa, \mathscr{B}x_\kappa) + \vartheta(\mathscr{B}x_\eta, \mathscr{B}x_{\kappa+1}, \mathscr{B}x_{\kappa+1})] \\
&\qquad\qquad + [\vartheta(\mathscr{B}y_{\kappa+1}, \mathscr{B}y_\kappa, \mathscr{B}y_\kappa) + \vartheta(\mathscr{B}y_\eta, \mathscr{B}y_{\kappa+1}, \mathscr{B}y_{\kappa+1})] \\
&\leq [\vartheta(\mathscr{B}x_{\kappa+1}, \mathscr{B}x_\kappa, \mathscr{B}x_\kappa) + \vartheta(\mathscr{B}x_{\kappa+2}, \mathscr{B}x_{\kappa+1}, \mathscr{B}x_{\kappa+1}) + \vartheta(\mathscr{B}x_\eta, \mathscr{B}x_{\kappa+2}, \mathscr{B}x_{\kappa+2})] \\
&\quad [\vartheta(\mathscr{B}y_{\kappa+1}, \mathscr{B}y_\kappa, \mathscr{B}y_\kappa) + \vartheta(\mathscr{B}y_{\kappa+2}, \mathscr{B}y_{\kappa+1}, \mathscr{B}y_{\kappa+1}) + \vartheta(\mathscr{B}y_\eta, \mathscr{B}y_{\kappa+2}, \mathscr{B}y_{\kappa+2})] \\
&\qquad \vdots \\
&\leq [\vartheta(\mathscr{B}x_{\kappa+1}, \mathscr{B}x_\kappa, \mathscr{B}x_\kappa) + \vartheta(\mathscr{B}x_{\kappa+2}, \mathscr{B}x_{\kappa+1}, \mathscr{B}x_{\kappa+1}) + \cdots + \vartheta(\mathscr{B}x_\eta, \mathscr{B}x_{\eta-1}, \mathscr{B}x_{\eta-1})] \\
&\quad [\vartheta(\mathscr{B}y_{\kappa+1}, \mathscr{B}y_\kappa, \mathscr{B}y_\kappa) + \vartheta(\mathscr{B}y_{\kappa+2}, \mathscr{B}y_{\kappa+1}, \mathscr{B}y_{\kappa+1}) + \cdots + \vartheta(\mathscr{B}y_\eta, \mathscr{B}y_{\eta-1}, \mathscr{B}y_{\eta-1})] \\
&= \Delta_\kappa + \Delta_{\kappa+1} + \Delta_{\kappa+2} + \cdots + \Delta_{\eta-1} \\
&\leq [c^\kappa + c^{\kappa+1} + \cdots + c^{\eta-1}]\Delta_0 \\
&\leq c^\kappa \cdot \frac{1}{1-c}\Delta_0 \text{ for } \eta > \kappa
\end{aligned}$$

or

$$\vartheta(\mathscr{B}x_\eta, \mathscr{B}x_\kappa, \mathscr{B}x_\kappa) + \vartheta(\mathscr{B}y_\eta, \mathscr{B}y_\kappa, \mathscr{B}y_\kappa) \leq c^\kappa \cdot \frac{1}{1-c}\Delta_0 \text{ for } \eta > \kappa. \tag{2.15}$$

Since $0 \leq c < 1$, $c^\kappa \to 0$ as $\kappa \to \infty$.
Now applying limit as $\kappa \to \infty$ with $\eta > \kappa$ in the inequality (2.15), we have
$\vartheta(\mathscr{B}x_\eta, \mathscr{B}x_\kappa, \mathscr{B}x_\kappa) + \vartheta(\mathscr{B}y_\eta, \mathscr{B}y_\kappa, \mathscr{B}y_\kappa) \leq 0$ which follows that $\langle x_\kappa \rangle_{\kappa=1}^\infty$ and $\langle y_\kappa \rangle_{\kappa=1}^\infty$ are $\vartheta-$cauchy sequences in $\mathscr{M}$.
Since $(\mathscr{M}, \vartheta)$ is a partially ordered complete ϑ-metric space, there exist $r, s \in \mathscr{M}$ such that $x_\kappa \to r$ and $y_\kappa \to s$.
Now we prove that (r, s) is a coupled coincidence point of $\mathscr{X}$ and $\mathscr{B}$.
Since $\mathscr{X}$ commutes with $\mathscr{B}$, we have

$$\mathscr{X}(\mathscr{B}x_\kappa, \mathscr{B}y_\kappa) = \mathscr{B}(\mathscr{X}(x_\kappa, y_\kappa)) = \mathscr{B}(\mathscr{B}x_{\kappa+1})$$
$$\text{and}$$
$$\mathscr{X}(\mathscr{B}y_\kappa, \mathscr{B}x_\kappa) = \mathscr{B}(\mathscr{X}(y_\kappa, x_\kappa)) = \mathscr{B}(\mathscr{B}y_{\kappa+1})$$

Since $\mathscr{X}$ and $\mathscr{B}$ are continuous, we have

$$\lim_{\kappa\to\infty} \mathscr{X}(\mathscr{B}x_\kappa, \mathscr{B}y_\kappa) = \lim_{\kappa\to\infty} \mathscr{B}(\mathscr{B}x_{\kappa+1}) = \mathscr{B}r$$
$$\text{and}$$

$$\lim_{\kappa\to\infty}\mathscr{X}(\mathscr{B}y_\kappa,\mathscr{B}x_\kappa)=\lim_{\kappa\to\infty}\mathscr{B}(\mathscr{B}y_{\kappa+1})=\mathscr{B}s$$

Since ϑ is continuous in all its variables, we have

$$\vartheta(\mathscr{X}(r,s),\mathscr{B}r,\mathscr{B}s)=\vartheta(\lim_{\kappa\to\infty}\mathscr{X}(\mathscr{B}x_\kappa,\mathscr{B}y_\kappa),\mathscr{B}r,\mathscr{B}r)=\vartheta(\mathscr{B}r,\mathscr{B}r,\mathscr{B}r),$$

so that

$$\vartheta(\mathscr{X}(r,s),\mathscr{B}r,\mathscr{B}s)=0$$

which implies that $\mathscr{X}(r,s)=\mathscr{B}r$.
Similarly, it can be proved that $\mathscr{X}(s,r)=\mathscr{B}s$.
Hence (s,r) is a coupled coincidence point of $\mathscr{X}$ and $\mathscr{B}$. □

Taking $\alpha=0$, $L=0$ and $\mathscr{B}$ is an identity mapping in Theorem 2.1, we get

Corollary 2.1. Let $(\mathscr{M},\vartheta,\leq)$ be a partially ordered complete ϑ-metric space and $\mathscr{X}:\mathscr{M}\times\mathscr{M}\to\mathscr{M}$ be a continuous mapping such that $\mathscr{X}$ has mixed monotone property and there exist $\beta,\gamma\in[0,1)$ with $\beta+\gamma<1$ such that

$$\vartheta(\mathscr{X}(x,y),\mathscr{X}(u,v),\mathscr{X}(w,z))\leq\beta\vartheta(x,u,w)+\gamma\vartheta(y,v,z). \tag{2.16}$$

If there exist $x_0,y_0\in\mathscr{M}$ such that $x_0\leq\mathscr{X}(x_0,y_0)$ and $y_0\geq\mathscr{X}(y_0,x_0)$, then $\mathscr{X}$ has a coupled fixed point in $\mathscr{M}\times\mathscr{M}$.

3. An application to system of nonlinear integral equations

Consider the following system of nonlinear integral equations:

$$f(s)=q(s)+\int_0^a\lambda(s,t)[\mathscr{X}_1(t,f(t))+\mathscr{X}_2(t,g(t))]dt, \tag{3.1}$$
$$g(s)=q(s)+\int_0^a\lambda(s,t)[\mathscr{X}_1(t,g(t))+\mathscr{X}_2(t,f(t))]dt,$$
$$s\in[0,L],L>0.$$

Let $\mathscr{M}=C([0,L],\mathbb{R})$ be the class of all real valued continuous functions on $[0,L]$.
Define

$$\vartheta(a,b,c)=sup\{|a(s)-b(s)|\,/\,s\in[0,L]\}x+sup\{|b(s)-c(s)|\,/\,s\in[0,L]\}$$
$$+sup\{|c(s)-a(s)|\,/\,s\in[0,L]\}$$

and the partial ordered relation on $\mathscr{M}$ as

$$a\leq b\Leftrightarrow a(s)\leq b(s)\text{ for all }a,b\in\mathscr{M}\text{ and }s\in[0,L]. \tag{3.2}$$

Then $(\mathscr{M},\vartheta,\leq)$ is a partially ordered complete ϑ-metric space. We make the following assumptions:

(a) The mappings $\mathscr{X}_1:[0,L]\times\mathbb{R}\to\mathbb{R}$, $\mathscr{X}_2:[0,L]\times\mathbb{R}\to\mathbb{R}$, $q:[0,L]\to\mathbb{R}$ and $\lambda:[0,L]\times\mathbb{R}\to[0,\infty)$ are continuous;

(b) there exist $c > 0$ and $\beta, \gamma \in [0, 1)$ with $\beta + \gamma < 1$ such that

$$0 \leq \mathscr{X}_1(s, b) - \mathscr{X}_1(s, a) \leq c\beta(b - a)$$
$$0 \leq \mathscr{X}_2(s, a) - \mathscr{X}_2(s, b) \leq c\gamma(b - a)$$

for all $a, b \in \mathbb{R}$ with $b \geq a$ and $s \in [0, L]$;

(c) $c \sup\{\int_0^L \lambda(s, t)dt : s \in [0, L]\} < 1$;

(d) there exists u_0 and v_0 in $\mathscr{M}$ such that

$$u_0(s) \geq q(s) + \int_0^L \lambda(s, t)[\mathscr{X}_1(t, u_0(t)) + \mathscr{X}_2(t, v_0(t))]dt,$$
$$v_0(s) \leq q(s) + \int_0^L \lambda(s, t)[\mathscr{X}_1(t, v_0(t)) + \mathscr{X}_2(t, u_0(t))]dt.$$

Then the system (3.1) has a solution in $\mathscr{M} \times \mathscr{M}$ where $\mathscr{M} = C([0, L], \mathbb{R})$. To achieve this, define $\mathscr{X} : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ as

$$\mathscr{X}(f, g)(s) = q(s) + \int_0^L \lambda(s, t)[\mathscr{X}_1(t, f(t)) + \mathscr{X}_2(t, g(t))]dt \text{ for all } f, g \in \mathscr{M} \text{ and } s \in [0, L]. \tag{3.3}$$

Using condition (b), it can be shown that $\mathscr{X}$ has mixed monotone property. Now for $x, y, u, v, w, z \in \mathscr{M}$ with $x \geq u \geq w$, $y \leq v \leq z$,

$$\begin{aligned}
&\vartheta(\mathscr{X}(x, y), \mathscr{X}(u, v), \mathscr{X}(w, z)) = \sup\{|\mathscr{X}(x, y)(s) - \mathscr{X}(u, v)(s)| \,/ s \in [0, L]\} \\
&\quad + \sup\{|\mathscr{X}(u, v)(s) - \mathscr{X}(w, z)(s)| \,/ s \in [0, L]\} + \sup\{|\mathscr{X}(w, z)(s) - \mathscr{X}(x, y)(s)| \,/ s \in [0, L]\} \\
&= \sup\left\{\left|\int_0^L \lambda(s, t)[\mathscr{X}_1(t, x(t)) + \mathscr{X}_2(t, y(t))]dt - \int_0^L \lambda(s, t)[\mathscr{X}_1(t, u(t)) + \mathscr{X}_2(t, v(t))]dt\right| / s \in [0, L]\right\} \\
&\quad + \sup\left\{\left|\int_0^L \lambda(s, t)[\mathscr{X}_1(t, u(t)) + \mathscr{X}_2(t, v(t))]dt - \int_0^L \lambda(s, t)[\mathscr{X}_1(t, w(t)) + \mathscr{X}_2(t, z(t))]dt\right| / s \in [0, L]\right\} \\
&\quad + \sup\left\{\left|\int_0^L \lambda(s, t)[\mathscr{X}_1(t, w(t)) + \mathscr{X}_2(t, z(t))]dt - \int_0^L \lambda(s, t)[\mathscr{X}_1(t, x(t)) + \mathscr{X}_2(t, y(t))]dt\right| / s \in [0, L]\right\}
\end{aligned}$$

$$\begin{aligned}
&\leq \sup\left\{\left|\int_0^L [\mathscr{X}_1(t, x(t)) - \mathscr{X}_1(t, u(t))]\right| |\lambda(s, t)| \, dt / s \in [0, L]\right\} \\
&\quad + \sup\left|\int_0^L [\mathscr{X}_2(t, y(t)) - \mathscr{X}_2(t, v(t))]\right| |\lambda(s, t)| \, dt / s \in [0, L]\} \\
&\quad + \sup\left\{\left|\int_0^L [\mathscr{X}_1(t, u(t)) - \mathscr{X}_1(t, w(t))]\right| |\lambda(s, t)| \, dt / s \in [0, L]\right\} \\
&\quad + \sup\left|\int_0^L [\mathscr{X}_2(t, v(t)) - \mathscr{X}_2(t, z(t))]\right| |\lambda(s, t)| \, dt / s \in [0, L]\} \\
&\quad + \sup\left\{\left|\int_0^L [\mathscr{X}_1(t, w(t)) - \mathscr{X}_1(t, x(t))]\right| |\lambda(s, t)| \, dt / s \in [0, L]\right\}
\end{aligned}$$

$$+\sup\left|\int_0^L [\mathscr{X}_2(t,z(t))-\mathscr{X}_2(t,y(t))]\right||\lambda(s,t)|\,dt/s\in[0,L]\}$$

$$\le c\beta\sup\{\int_0^L |x(t)-u(t)|\,|\lambda(s,t)|\,dt/s\in[0,L]\}+c\gamma\sup\{\int_0^L |y(t)-v(t)|\,|\lambda(s,t)|\,dt/s\in[0,L]\}$$
$$+c\beta\sup\{\int_0^L |u(t)-w(t)|\,|\lambda(s,t)|\,dt/s\in[0,L]\}+c\gamma\sup\{\int_0^L |v(t)-z(t)|\,|\lambda(s,t)|\,dt/s\in[0,L]\}$$
$$+c\beta\sup\{\int_0^L |w(t)-x(t)|\,|\lambda(s,t)|\,dt/s\in[0,L]\}+c\gamma\sup\{\int_0^L |z(t)-y(t)|\,|\lambda(s,t)|\,dt/s\in[0,L]\}$$

$$\le\beta[\sup\{|x(s)-u(s)|\,/s\in[0,L]\}+\sup\{|u(s)-w(s)|\,/s\in[0,L]\}$$
$$+\sup\{|w(s)-x(s)|\,/s\in[0,L]\}]\cdot c\ \sup\{\int_0^L |\lambda(s,t)|\,dt/s\in[0,L]\}$$
$$+\gamma[\sup\{|y(s)-v(s)|\,/s\in[0,L]\}+\sup\{|v(s)-z(s)|\,/s\in[0,L]\}$$
$$+\sup\{|z(s)-y(s)|\,/s\in[0,L]\}]\cdot c\ \sup\{\int_0^L |\lambda(s,t)|\,dt/s\in[0,L]\}$$
$$\le\beta[\sup\{|x(s)-u(s)|\,/s\in[0,L]\}+\sup\{|u(s)-w(s)|\,/s\in[0,L]\}+\sup\{|w(s)-x(s)|\,/s\in[0,L]]$$
$$+\gamma\sup\{|y(s)-v(s)|\,/s\in[0,L]\}+\sup\{|v(s)-z(s)|\,/s\in[0,L]\}+\sup\{|z(s)-y(s)|\,/s\in[0,L]]$$
$$=\beta\vartheta(x,u,w)+\gamma\vartheta(y,v,z)$$

So that

$$\vartheta(\mathscr{X}(x,y),\mathscr{X}(u,v),\mathscr{X}(w,z))\le\beta\vartheta(x,u,w)+\gamma\vartheta(y,v,z)$$

Hence all the conditions of Corollary 2.1 are satisfied. Therefore, $\mathscr{X}$ has a coupled fixed point in $\mathscr{M}\times\mathscr{M}$. In other words, the system (3.1) of nonlinear integral equations has a solution in $\mathscr{M}\times\mathscr{M}$. The aforesaid application is illustrated by the following example:

Example 3.1. Let $\mathscr{M}=C([0,1],\mathbb{R})$, Now consider the integral equation in $\mathscr{M}$ as

$$\mathscr{X}(f,g)(s)=\frac{s^3+7}{4}+\int_0^1\frac{t^2}{24(s+3)}[f(t)+\frac{2}{g(t)+3}]dt. \tag{3.4}$$

Then clearly the above equation is in the form of following equation:

$$\mathscr{X}(f,g)(s)=q(s)+\int_0^L\lambda(s,t)[\mathscr{X}_1(t,f(t))+\mathscr{X}_2(t,g(t))]dt \text{ for all } f,g\in\mathscr{M} \text{ and } s\in[0,L],$$

where $q(s)=\frac{s^3+7}{4}$, $\lambda(s,t)=\frac{t^2}{24(s+3)}$, $\mathscr{X}_1(t,s)=s$, $\mathscr{X}_2(t,s)=\frac{2}{s+3}$ and $L=1$.
That is, (3.4) is a special case of (3.3) in Theorem 3.1.

Here it is easy to verify that the functions $q(s),\lambda(s,t),\mathscr{X}_1(t,s)$ and $\mathscr{X}_2(t,s)$ are continuous. Moreover, there exist $c=9$, $\beta=\frac{1}{3}$ and $\gamma=\frac{1}{2}$ with $\beta+\gamma<1$ such that

$$0\le\mathscr{X}_1(s,b)-\mathscr{X}_1(s,a)\le c\beta(b-a)$$

$$0 \leq \mathscr{X}_2(s,a) - \mathscr{X}_2(s,b) \leq c\gamma(b-a)$$

for all $a, b \in \mathbb{R}$ with $b \geq a$ and $s \in [0,1]$.
and

$$\begin{aligned} c\sup\{\int_0^L \lambda(s,t)dt : s \in [0,L]\} =& 9\sup\{\int_0^1 \frac{t^2}{24(s+3)}dt : s \in [0,1]\}. \\ =& 9\sup\{\frac{1}{72(s+3)} : s \in [0,1]\} < 1. \end{aligned}$$

Thus the conditions (a), (b) and (c) of Theorem 3.1 are satisfied.

Now consider $u_0(s) = 1$ and $v_0(s) = 1$. Then we have

$$\begin{aligned} q(s) + \int_0^1 \lambda(s,t)[\mathscr{X}_1(t, v_0(t)) + \mathscr{X}_2(t, u_0(t))]dt &= \frac{s^3+7}{4} + \int_0^1 \frac{t^2}{24(s+3)}[1 + \frac{2}{4}]dt \\ &= \frac{s^3+7}{4} + \frac{1}{48(s+3)} \geq 1 \end{aligned}$$

That is, $v_0 \leq \mathscr{X}(v_0, u_0)$. Similarly, it can be shown that $u_0 \geq \mathscr{X}(u_0, v_0)$.

Thus all the conditions of Theorem 3.1 are satisfied. It follows that the integral Eq (3.4) has a solution in $\mathscr{M} \times \mathscr{M}$ with $\mathscr{M} = C([0,1], \mathbb{R})$.

4. Conclusions

Some coupled coincidence point theorems for two mappings established using rational type contractions in the setting of partially ordered $\mathscr{G}-$metric spaces. By considering $\mathscr{G}-$metric space, we propose a fairly simple solution for a system of nonlinear integral equations by using fixed point technique. Moreover, supporting example (exact solution) is provided to strengthen our obtained results.

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Conflict of interest

The authors declare that they have no competing interests.

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