Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(4): 3432–3454.
DOI:10.3934/math.2021205
Received: 24 October 2020
Accepted: 08 January 2021
Published: 20 January 2021

Research article

Drift coefficient inversion problem of Kolmogorov-type equation

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Abstract: Kolmogorov-type equations often appear in stochastic analysis and have important applications in financial derivatives pricing, stochastic control and other fields. In this paper, we consider an inverse problem of reconstructing drift coefficient in a Kolmogorov-type equation. Being different from other works, the unknown drift coefficient is related to both temporal and spatial variables, which makes theoretical analysis rather difficult. Until now, documents dealt with evolutional inverse drift problems are quite few. Inspired by the Rothe's idea, we introduce a new time semi-discrete scheme to find the optimal solution at each time layer. Then we construct an approximate solution of the unknown drift coefficient and strictly analyze its convergence. After establishing the necessary conditions for the limit minimizer, we prove the uniqueness and stability of the global optimal solution.

Keywords: Kolmogorov-type equation; inverse problem; drift coefficient; optimal control; uniqueness
Mathematics Subject Classification: 35R30, 49J20

1. Introduction

Kolmogorov-type equation is a kind of important partial differential equation in stochastic analysis, which has significant applications in financial mathematics, stochastic control and other fields. Let's consider a one-dimensional process X_t given by the stochastic differential equation

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t, \tag{1.1}$$

where dW_t is increment of standard Brownian motion satisfies

$$\mathbb{E}(dW_t) = 0, \quad \mathrm{Var}(dW_t) = dt,$$

and a, b are some smooth real valued functions. Suppose $\mathcal{P}_{s,t}(x,y)$ be the following conditional expectation

$$\mathcal{P}_{s,t}(x,y) = \mathbb{E}(\delta_y(X_t)|\, X_s = y)$$

for $s < t$. Then, by using the famous Itô and Feynman-Kac formula (see [15, 21]), we deduce that $\mathcal{P}_{s,t}(x,y)$ satisfies the following Kolmogorov-type equation:

$$\frac{\partial}{\partial s}\mathcal{P}_{s,t}(x,y) + a(s,x)\frac{\partial}{\partial x}\mathcal{P}_{s,t}(x,y) + \frac{1}{2}b^2(s,x)\frac{\partial^2}{\partial x^2}\mathcal{P}_{s,t}(x,y) = 0. \tag{1.2}$$

The parameter a in (1.1) is called the drift coefficient, while b is called the diffusion coefficient or volatility coefficient.

In this paper, we are interested in the Kolmogorov-type inverse problem. Let $Q = I \times (0,T]$, $I = (0,l)$. We consider the following mathematical model:

$$Lu = u_t - u_{xx} + p(x,t)u_x = 0, \qquad (x,t) \in Q, \tag{1.3}$$

$$u(x,0) = \varphi(x), \qquad x \in \bar{I}, \tag{1.4}$$

$$u(0,t) = u(l,t) = 0, \qquad t \in (0,T], \tag{1.5}$$

where $p(x,t)$ is an unknown coefficient to be identified and $\varphi(x)$ is a given non-trivial smooth function which satisfies

$$\varphi \geq 0, \;\; \varphi \not\equiv 0; \quad \varphi \in C^{2,\alpha}(\bar{I}),$$

for some $\alpha > 0$. Assume that an additional condition is given as follows:

$$u(x,t) = g(x,t), \quad (x,t) \in Q, \tag{1.6}$$

where $g(x,t)$ is a given function which may contain measurement error. We would like to determine the function pair (u,g) simultaneously from (1.3)–(1.5)/(1.6).

The Eq (1.3) belongs to the type of Kolmogorov equation. In fact, by using suitable variable substitution one can easily arrive at (1.3). The Eq (1.2) is a Cauchy problem on unbounded domain, while Eqs (1.3)–(1.5) is an initial boundary value problem on bounded domain. But that is not the point either. If the range of process X_t is bounded, then (1.2) will be transformed into (1.3)–(1.5).

Drift coefficient or drift rate is an important parameter in the process of stock price change. In financial markets, the stock price movement is determined by the expected drift rate and Brownian motion. The stochastic characteristics of volatility, expected drift rate and Brownian motion determine that the motion process of stock price is full of randomness and uncertainty. Simply speaking, Itô process divides the motion process of stock price into two independent processes: the drift term and the volatility term. The drift term can be understood as the expected rate of return of the stock price, while the volatility term is used to measure the variability of variables.

In practical application, it is quite difficult to determine the drift coefficient in advance. In the past, people usually get a rough estimate of drift rate by experiences. The disadvantages of this method are obvious and will bring about large errors. It is undoubtedly of great theoretical and practical significance to calibrate the drift coefficient by indirect means.

In this paper, we investigate the inverse problem of identifying a drift coefficient for the Kolmogorov-type equation from the knowledge of observation data. Since the unknown drift coefficients are related to both temporal and spatial variables, the additional data are given in the whole region. It seems that such problem is trivial. If the extra condition $g(x,t)$ is given accurately, one can easily obtain the unknown function $p(x,t)$ from the Eq (1.3):

$$p(x,t) = \frac{g_{xx} - g_t}{g_x}. \tag{1.7}$$

But in practice, the formula (1.7) cannot be applicable. First of all, it is difficult to measure such a huge amount of data in practice. So $g(x,t)$ is only given approximately and may contain errors (see Section 2). Secondly, for the noisy measurement data, the small errors will lead to great changes in the solution (see [12, 16, 24, 27, 28]). Finally, it is difficult to avoid the case of $g_x = 0$ in (1.7). Therefore, some regularization technique should be adopted to overcome the ill-posedness of the inverse problem.

Inverse coefficient problems for parabolic equations are well studied in the literature. In [18, 19], the inverse problem of identifying the implied volatility in the Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(S)S^2\frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0, \qquad (S,t) \in [0,\infty) \times [0,T),$$

has been studied by using an optimal control framework. The existence and uniqueness of the minimizer are proved rigorously and an iterative algorithm is designed to obtain the numerical solution. In [2], a general inverse source problem is studied carefully and the global uniqueness of the solution is obtained. Optimization method is applied to identify the convection coefficient in [11], where the mathematical model is a Cauchy problem of convection-diffusion equations. The inverse radiative coefficient problem has been widely investigated in [5, 6, 16, 23, 26, 34], where the temperature distribution function $u(x,t)$ satisfies the following heat conduction equation

$$u_t - \Delta u + p(x)u = 0, \quad (x,t) \in Q.$$

In [6, 26], the determination of $p(x)$ is studied by the contraction mapping principle and Hölder space method, respectively. Moreover, inverse problems on the purely time dependent case, i.e., $p = p(t)$, can be found in [3, 4, 7, 8]. Numerical treatments for general cases $p = p(x,t,u)$ can be found in [29–31].

For general parabolic equations

$$\partial_t u + \mathcal{A}u + \sigma(x)u = f(x,t), \quad (x,t) \in Q,$$

the uniqueness and stability of determining $\sigma(x)$ is obtained in [17], where $\mathcal{A}$ ia a general elliptic differential operator. In [5, 23, 34], the optimization method is applied to stabilize the inverse problem, and the existence of minimizer and the convergence of approximate solution are proved rigorously. The inverse problem of simultaneously reconstructing the initial value and the radiation coefficient $p(x)$ is investigated in [33]. For the general case that the unknown coefficient(s) depend(s) on both spatial and temporal variables, we suggest that readers refer to references, e.g., in [9, 10, 20, 22, 27].

In this paper, we would like to discuss the inverse problem mainly from the mathematical analysis angle. Particularly, we focus on the uniqueness and stability of the minimizer of the optimal control problem. Local uniqueness of convection coefficient is obtained in [11], where the observation time T should not be too large. This defect is improved in this paper. We obtain the global uniqueness and stability of the minimizer, which is extremely important in numerical calculation. To the best of our knowledge, this work is the first one concerning global uniqueness and stability of optimal solution in inverse coefficient problem for Kolmogorov-type equations.

This paper is organized as follows. In Section 2, we introduce an optimal control method and obtain the semi-discrete approximate solution $p^h(x,t)$. Meanwhile, the necessary condition which must be satisfied by the minimizer is deduced. In Section 3, we establish the uniform estimates for the approximate solution $p^h(x,t)$ and use them to discuss the asymptotic behavior. The necessary condition of limit minimizer $p(x,t)$ is derived in Section 4. Finally, in Section 5 the global uniqueness and stability results are proved rigorously.

2. Optimal control problem

The observation data $g(x,t)$ is just an artificial solution that may contain errors. As mentioned in Section 1, it is impossible to measure the information $u(x,t)$ at every position x and every time t. Let's consider the following discrete grid. Assume that the domain $\bar{Q} = [0,l]\times[0,T]$ is divided into a $M\times N$ mesh with the spatial step size $\Delta x = \frac{l}{M}$ in the x-direction and the time step size $h = \frac{T}{N}$, respectively (*without loss of generality, we assume that the grid nodes are equidistant*). Grid points (x_i, t_n) are defined by

$$x_i = i\Delta x, \qquad i = 0,1,2,\cdots,M,$$

$$t_n = nh, \qquad n = 0,1,2,\cdots,N,$$

in which M and N are two positive integers. In practice, one can only give the observations on a finite number of discrete points as follows:

$$u(x_i,t_n) = g(x_i,t_n), \quad i = 1,2,\cdots,M-1; \;\; n = 1,2,\cdots,N; \;\; (x_i,t_n)\in\bar{Q}.$$

The observation function (1.6) is actually obtained by some interpolation techniques from discrete data $u(x_i,t_n)$, e.g., triangular element interpolation. Therefore, it is absurd to try to get $p(x,t)$ directly from $u(x_i,t_n)$.

In this paper, we attempt to reconstruct $p(x,t)$ by time semi-discrete scheme. This idea is widely used in studying the existence of solutions of parabolic equations, e.g., the famous Rothe method (see [32]). The essence of this method is to do difference with time variable and use the theory of elliptic equation to solve the difference equation and estimate its solution, then construct the approximate solution and complete the limit process to the true solution. Similar to the Rothe method, we first introduce the time semi-discrete scheme, i.e., we find $p(x,t_n)$ step by step, where $t_n = nh$, $n = 0,1,\cdots,N$. If $p(x,t_0),\cdots,p(x,t_{n-1})$ have been identified, then from the extra condition $u(x,t_n) = g(x,t_n)$, we find $p(x,t_n)$ such that

$$J_n(p(x,t_n)) = \inf_{p\in A} J_n(p),$$

where A is an admissible set and $J_n(p)$ is a control function. Based on the obtained $p(x,t_n)$, for any h we construct an approximate function $p^h(x,t)$ defined as follows:

$$p^h(x,t) = \begin{cases} p(x,t_n), & t = t_n, \\ \mathit{linear}, & t_{n-1}\le t\le t_n. \end{cases}$$

Finally, considering the limit of $p^h(x,t)$ as $h \to 0$, we get $p(x,t)$.

We first introduce the following Sobolev spaces (see [1]). Denote

$$L^p(Q) = \left\{ u : \quad \int_Q |u|^p dxdt < +\infty \right\},$$

$$L^\infty(Q) = \left\{ u : \quad \text{esssup}_Q |u| < +\infty \right\},$$

and

$$W_p^{m,k}(Q) = \{ u : \quad D^\alpha u,\ D_t^r u \in L^p(Q),\ \text{for any } |\alpha| \le m \text{ and } r \le k \}.$$

The corresponding norms are given by

$$\|u\|_{L^p(Q)} = \left(\int_Q |u|^p dxdt \right)^{\frac{1}{p}},$$

$$\|u\|_{L^\infty(Q)} = \text{esssup}_Q |u|,$$

and

$$\|u\|_{W_p^{m,k}(Q)} = \sum_{|\alpha| \le m} \|D^\alpha u\|_{L^p(Q)} + \sum_{r \le k} \|D_t^r u\|_{L^p(Q)},$$

respectively. As $p = 2$, $W_p^{m,k}(Q)$ is abbreviated as $H^{m,k}(Q)$.

For $0 < \alpha < 1$, we introduce the Hölder semi-norm

$$[u]_{\alpha,\alpha/2;Q} = \sup_{M,N \in Q, M \ne N} \frac{|u(M) - u(N)|}{|d(M,N)|^\alpha},$$

where $d(M,N)$ is defined as follows:

$$d(M,N) = \left(|x-y|^2 + |t-s| \right)^{1/2}$$

for any two points $M(x,t),\ N(y,s) \in Q$. Denote $C^{\alpha,\alpha/2}(\overline{Q})$ to represent the function set on Q satisfies $[u]_{\alpha,\alpha/2;Q} < +\infty$ and define the following norm:

$$|u|_{\alpha,\alpha/2;Q} = |u|_{0;Q} + [u]_{\alpha,\alpha/2;Q}.$$

For any non-negative integer k, the function space $C^{2k+\alpha,k+\alpha/2}(\overline{Q})$ and the corresponding norm can be defined analogously.

For the direct problems (1.3)–(1.5), the Schauder theory for parabolic equations [13, 14, 25] guarantees that there is a unique solution, $u(x,t) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q})$, for any positive coefficient $p \in C^{\alpha,\frac{\alpha}{2}}(\bar{Q})$.

Suppose that the function $g(x,t)$ satisfies the following condition

$$\|g\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q})} \le C, \quad \max_{0 \le t \le T} \|g(\cdot,t)\|_{H^1(I)} \le C, \tag{2.1}$$

where C is a constant. Hereafter, we use $C > 0$ to denote generic constants which are independent of h and may change line by line.

To reconstruct the unknown coefficient, we will introduce the following time semi-discrete cost functional and time semi-discrete optimal control problem.

Set

$$0 = t_0 < t_1 < \cdots < t_n = T$$

be a partition of the interval $[0, T]$ with $t_n = nh$ and $h = \frac{T}{N}$. Let

$$A = \left\{ p(x) | \underline{p} \le p(x) \le \overline{p},\ p \in H^1(I) \right\} \tag{2.2}$$

be the admissible set, where $\underline{p}$ and $\overline{p}$ are two given positive constants.

Beginning with a given function $p_0(x) \in A$ with

$$p_0(x) \in W^{1,\infty}(I),$$

we introduce the following sequence of optimal control problem Q_n:

Problem Qn: Assume that $p_0, p_1, \cdots, p_{n-1} \in A$ are known, find a $p_n \in A$ such that

$$J_n(p_n) = \inf_{p \in A} J_n(p), \tag{2.3}$$

where $J_n(p)$ is the cost functional

$$J_n(p) = \frac{\sigma}{2}\left(\frac{1}{h}\|p - p_{n-1}\|^2_{L^2(I)} + \|\nabla p\|^2_{L^2(I)}\right) + \frac{1}{2h}\|u(\cdot, t_n; p) - g(\cdot, t_n)\|^2_{L^2(I)}, \tag{2.4}$$

$u(x, t; p)$ is the solution of (1.3)–(1.5) in $[0, t_n]$ corresponding to the coefficient

$$\tilde{p} = \begin{cases} \frac{t-t_{n-1}}{h} p(x) + \frac{t_n - t}{h} p_{n-1}(x), & t_{n-1} \le t \le t_n, \\ \frac{t-t_{k-1}}{h} p_k(x) + \frac{t_k - t}{h} p_{k-1}(x), & t_{k-1} \le t \le t_k, 1 \le k \le n-1, \end{cases} \tag{2.5}$$

and $\sigma > 0$ is a regularization parameter.

Theorem 2.1. There exists a $p_n \in A$ such that

$$J_n(p_n) = \inf_{p \in A} J_n(p).$$

The proof of this theorem is similar to that in [11].

Such a p_n is called an optimal control of problem Q_n. From theorem 2.1, the functions $p_0, p_1, \cdots, p_N \in A$ are well defined when $p_0 \in A$ is known. For $(x, t) \in \bar{Q}$, let

$$p^h(x, t) = \frac{t - t_{n-1}}{h} p_n(x) + \frac{t_n - t}{h} p_{n-1}(x), \qquad t_{n-1} \le t \le t_n,\ \ n = 1, \cdots, N.$$

which is called the discrete reconstruction of the unknown coefficient. Then recovering $p(x, t)$ is reduced to investigating the behavior of the discrete reconstruction $p^h(x, t)$ as $h \to 0$.

Now we derive the necessary condition for the optimal control problem Q_n as follows:

Theorem 2.2. Assume that $p_0 \in A$ is given. Let $p_n \in A$ be an optimal control of problem Q_n, $n = 1, \cdots, N$ and $u^h(x,t)$ be the solution of (1.3)–(1.5) in $[0,T]$ corresponding to the coefficient $\tilde{p} = p^h(x,t)$. Then for any $\omega \in A$, we have

$$\sigma \int_0^l \left[\frac{p_n - p_{n-1}}{h}(\omega - p_n) + \nabla p_n \cdot \nabla(\omega - p_n) \right] dx + \frac{1}{h} \int_{t_{n-1}}^{t_n} \int_0^l \frac{t - t_{n-1}}{h}(p_n - \omega) u_x^h v^h dx dt \geq 0, \tag{2.6}$$

where $v^h(x,t)$ satisfies the following equation:

$$\begin{cases} -v_t^h - v_{xx}^h - (p^h(x,t)v^h)_x = 0, & (x,t) \in (0,l) \times [t_{n-1}, t_n], \\ v^h(0,t) = v^h(l,t) = 0, \\ v^h(x,t_n) = u^h(x,t_n) - g(x,t_n). \end{cases} \tag{2.7}$$

Proof. Let $p_n \in A$ be an optimal control of problem Q_n. Note that A is a convex set, for any $\omega \in A$,

$$p^\lambda = (1-\lambda)p_n + \lambda\omega \in A, \qquad \lambda \in [0,1].$$

Hence for any $\omega \in A$, the function $j(\lambda) = J_n(p^\lambda)$ is well defined and reaches its minimum at $\lambda = 0$. Then we have

$$j'(0) = \frac{d}{d\lambda} J_n(p^\lambda) \Big|_{\lambda=0} \geq 0,$$

i.e., for any $\omega \in A$,

$$\frac{d}{d\lambda} \int_o^l \left[\sigma\left(\frac{|p^\lambda(x) - p_{n-1}(x)|^2}{h} + |\nabla p^\lambda(x)|^2\right) + \frac{1}{h}|u(x,t_n;p^\lambda) - g(x,t_n)|^2 \right] dx \Big|_{\lambda=0} \geq 0, \tag{2.8}$$

where $u(x,t;p^\lambda)$ is the solution of (1.3)–(1.5) corresponding to

$$\tilde{p} = \begin{cases} \frac{t-t_{n-1}}{h} p^\lambda(x) + \frac{t_n - t}{h} p_{n-1}(x), & t_{n-1} \leq t \leq t_n, \\ \frac{t-t_{k-1}}{h} p_k(x) + \frac{t_k - t}{h} p_{k-1}(x), & t_{k-1} \leq t \leq t_k, 1 \leq k \leq n-1. \end{cases}$$

Set

$$\xi(x,t) = \frac{du(x,t;p^\lambda)}{d\lambda} \Big|_{\lambda=0}.$$

We derive from (2.8) the following inequality

$$\sigma \int_0^l \left[\frac{p_n - p_{n-1}}{h}(\omega - p_n) + \nabla p_n \cdot \nabla(\omega - p_n) \right] dx + \frac{1}{h} \int_0^l (u^h(x,t_n) - g(x,t_n))\xi(x,t_n) dx \geq 0. \tag{2.9}$$

By direct differentiation to λ on both side of (1.3)–(1.5) in which p is replaced by p^λ, it can be seen that $\xi(x,t)$ is the solution of the following initial-boundary value problem of parabolic equation:

$$\begin{cases} L\xi = \xi_t - \xi_{xx} + p^h(x,t)\xi_x = \frac{t-t_{n-1}}{h}(p_n - \omega)u_x^h, & (x,t) \in I \times [t_{n-1}, t_n], \\ \xi(0,t) = \xi(l,t) = 0, & (x,t) \in \partial I \times (t_{n-1}, t_n], \\ \xi(x,t_{n-1}) = 0, & x \in I. \end{cases} \tag{2.10}$$

Suppose $v^h(x,t)$ is the solution to the following problem:

$$\begin{cases} L^*v^h = -v_t^h - v_{xx}^h - (p^h(x,t)v^h)_x = 0, & (x,t) \in (0,l)\times[t_{n-1},t_n], \\ v^h(0,t) = v^h(l,t) = 0, \\ v^h(x,t_n) = u^h(x,t_n) - g(x,t_n), \end{cases} \tag{2.11}$$

where L^* is the adjoint operator of the operator L.

From (2.10), (2.11) and the Green formula, we have

$$\begin{aligned} 0 &= \int_{t_{n-1}}^{t_n}\int_0^l \xi L^* v^h dxdt \\ &= -\int_0^l \xi(x,t_n)[u^h(x,t_n) - g(x,t_n)]dx + \int_{t_{n-1}}^{t_n}\int_0^l v^h L\xi dxdt \\ &= -\int_0^l \xi(x,t_n)[u^h(x,t_n) - g(x,t_n)]dx + \int_{t_{n-1}}^{t_n}\int_0^l \frac{t-t_{n-1}}{h}(p_n-\omega)u_x^h v^h dxdt. \end{aligned} \tag{2.12}$$

Combining (2.9) and (2.12), we get

$$\sigma\int_0^l \left[\frac{p_n-p_{n-1}}{h}(\omega-p_n) + \nabla p_n\cdot\nabla(\omega-p_n)\right]dx + \frac{1}{h}\int_{t_{n-1}}^{t_n}\int_0^l \frac{t-t_{n-1}}{h}(p_n-\omega)u_x^h v^h dxdt \ge 0,$$

for any $\omega \in A$.

This completes the proof of theorem 2.2. □

3. Uniform estimates

We will derive some uniform estimates for the sequence of discrete optimal controls $p_0, p_1, \cdots, p_N$ and the discrete reconstruction of unknown coefficient $p^h(x,t)$ as $h \to 0$.

In this paper, C will be denoted different constants which is independent of parameters h and σ.

Lemma 3.1. Let $u^h(x,t)$ be the solution to the following problem:

$$u_t - u_{xx} + \tilde{p}(x,t)u_x = 0, \qquad (x,t)\in Q, \tag{3.1}$$

$$u(x,0) = \varphi(x), \qquad x\in I, \tag{3.2}$$

$$u(0,t) = u(l,t) = 0, \qquad t\in(0,T], \tag{3.3}$$

with $\tilde{p}(x,t) = p^h(x,t)$. Then there exists a constant C, such that

$$\|u^h\|_{L^\infty(Q)} + \int_0^T\int_0^l (\left|u_t^h\right|^2 + \left|u_{xx}^h\right|^2)dxdt + \max_{0\le t\le T}\int_0^l \left|u_x^h\right|^2 dx \le C. \tag{3.4}$$

The proof of this lemma is standard (see [13]). For the sake of completeness, we give a brief proof of Lemma 1.

Proof. Multiplying u_t^h on both sides of Eq (3.1) and integrating by parts, we have

$$\int_0^t\int_0^l \left(u_t^h\right)^2 dxdt - \int_0^t\int_0^l u_t^h u_{xx}^h dxdt + \int_0^t\int_0^l u_t^h p^h u_x^h dxdt$$

$$
\begin{aligned}
&= \int_0^t\int_0^l \left(u_t^h\right)^2 dxdt + \int_0^t\int_0^l u_{tx}^h u_x^h dxdt + \int_0^t\int_0^l u_t^h p^h u_x^h dxdt \\
&= \int_0^t\int_0^l \left(u_t^h\right)^2 dxdt + \frac{1}{2}\int_0^t\int_0^l \left[\left(u_x^h\right)^2\right]_t dxdt + \int_0^t\int_0^l u_t^h p^h u_x^h dxdt \\
&= \int_0^t\int_0^l \left(u_t^h\right)^2 dxdt + \frac{1}{2}\int_0^l \left(u_x^h(\cdot,t)\right)^2 dx - \frac{1}{2}\int_0^l \varphi_x^2 dx + \int_0^t\int_0^l u_t^h p^h u_x^h dxdt \\
&= 0.
\end{aligned}
$$

Using the bound of p^h and the Cauchy inequality, we obtain

$$
\begin{aligned}
&\int_0^t\int_0^l \left(u_t^h\right)^2 dxdt + \frac{1}{2}\int_0^l \left(u_x^h(\cdot,t)\right)^2 dx \\
\le\ & \overline{p}\int_0^t\int_0^l \left|u_t^h\right|\cdot\left|u_x^h\right| dxdt + \frac{1}{2}\int_0^l \varphi_x^2 dx \\
\le\ & \frac{1}{2}\int_0^t\int_0^l \left(u_t^h\right)^2 dxdt + \frac{\overline{p}^2}{2}\int_0^t\int_0^l \left|u_x^h\right|^2 dxdt + \frac{1}{2}\int_0^l \varphi_x^2 dx,
\end{aligned}
$$

i.e.,

$$
\int_0^t\int_0^l \left(u_t^h\right)^2 dxdt + \int_0^l \left(u_x^h(\cdot,t)\right)^2 dx \le \overline{p}^2\int_0^t\int_0^l \left|u_x^h\right|^2 dxdt + \int_0^l \varphi_x^2 dx.
$$

Then, using the Gronwall inequality, one can easily get

$$
\int_0^T\int_0^l \left(u_t^h\right)^2 dxdt + \max_{0\le t\le T}\int_0^l \left|u_x^h\right|^2 dx \le \mathrm{e}^{\overline{p}^2 T}\int_0^l \varphi_x^2 dx \le C.
$$

The rest of (3.4) can be proved similarly. □

Lemma 3.2. Let $p^\lambda = (1-\lambda)p_n + \lambda p_{n-1}, 0 \le \lambda \le 1$ and $u^\lambda(x,t) = u(x,t,p^\lambda)$ be the solution of (3.1)–(3.3) in $[0, t_n]$ with

$$
\tilde{p} = \begin{cases} \frac{t-t_{n-1}}{h} p^\lambda(x) + \frac{t_n - t}{h} p_{n-1}(x), & t_{n-1} \le t \le t_n, \\ \frac{t-t_{k-1}}{h} p_k(x) + \frac{t_k - t}{h} p_{k-1}(x), & t_{k-1} \le t \le t_k, 1 \le k \le n-1. \end{cases} \tag{3.5}
$$

Then there exists a constant C, such that

$$
\|u^\lambda\|_{L^\infty(I\times[t_{n-1},t_n])} + \max_{t_{n-1}\le t\le t_n}\int_0^l \left|u_x^\lambda\right|^2 dx \le C, \tag{3.6}
$$

and

$$
\int_{t_{n-1}}^{t_n}\int_0^l (\left|u_t^\lambda\right|^2 + \left|u_{xx}^\lambda\right|^2) dxdt + \max_{0\le t\le T}\int_0^l \left|u_x^\lambda\right|^2 dx \le C\int_{t_{n-1}}^{t_n}\int_0^l \left|u_x^h\right|^2 dxdt. \tag{3.7}
$$

Proof. Because the proof of (3.6) is similar to that of (3.4), here we only prove (3.7).

Let $\omega = u^\lambda - u^h$. Then from (3.1)–(3.3), we can verify that ω satisfies

$$
\omega_t - \omega_{xx} + \tilde{p}(x,t)\omega_x = (p^h - \tilde{p})u_x^h, \qquad (x,t) \in (0,l)\times[t_{n-1},t_n], \tag{3.8}
$$

$$
\omega(0,t) = \omega(l,t) = 0, \qquad t \in (t_{n-1}, t_n], \tag{3.9}
$$

$$\omega(x,t_{n-1}) = 0, \qquad x \in I. \tag{3.10}$$

Multiplying Eq (3.8) both sides with ω_{xx} and integrating over $(0,l)\times[t_{n-1},t]$, $t\in[t_{n-1},t_n]$, we obtain that

$$\begin{aligned}
&\int_{t_{n-1}}^{t}\int_0^l \omega_{xx}^2 dxdt + \frac{1}{2}\int_{t_{n-1}}^{t}\int_0^l (\omega_x^2)_t dxdt\\
&= \int_{t_{n-1}}^{t}\int_0^l \tilde{p}\omega_x\omega_{xx} dxdt + \int_{t_{n-1}}^{t}\int_0^l (\tilde{p}-p^h)u_x^h\omega_{xx} dxdt\\
&\le \int_{t_{n-1}}^{t}\int_0^l \left(\frac{1}{4}|\omega_{xx}|^2 + C|\tilde{p}|^2|\omega_x|^2\right) dxdt + \int_{t_{n-1}}^{t}\int_0^l \left(\frac{1}{4}|\omega_{xx}|^2 + C|\tilde{p}-p^h|^2\left|u_x^h\right|^2\right) dxdt\\
&= \frac{1}{2}\int_{t_{n-1}}^{t}\int_0^l |\omega_{xx}|^2 dxdt + C\int_{t_{n-1}}^{t}\int_0^l |\tilde{p}|^2|\omega_x|^2 dxdt + C\int_{t_{n-1}}^{t}\int_0^l |\tilde{p}-p^h|^2\left|u_x^h\right|^2 dxdt,
\end{aligned} \tag{3.11}$$

where we have used the ε-Cauchy inequality.

Noting the boundedness of $\tilde{p}$ and p^h, we get

$$\frac{1}{2}\int_{t_{n-1}}^{t}\int_0^l \omega_{xx}^2 dxdt + \frac{1}{2}\int_0^l \omega_x^2(\cdot,t)dx \le C\int_{t_{n-1}}^{t}\int_0^l \omega_x^2 dxdt + C\int_{t_{n-1}}^{t}\int_0^l \left|u_x^h\right|^2 dxdt. \tag{3.12}$$

Using the Gronwall inequality, we obtain

$$\int_{t_{n-1}}^{t}\int_0^l \omega_{xx}^2 dxdt + \max_{t_{n-1}\le t\le t_n}\int_0^l \omega_x^2 dx \le C\int_{t_{n-1}}^{t}\int_0^l \left|u_x^h\right|^2 dxdt. \tag{3.13}$$

This and (3.4) give the results (3.7).

This completes the proof of Lemma 3.2. □

Remark 3.1: We would like to give the specific form of Gronwall inequality, and use it to show the detailed proof of inequality (3.13).

Gronwall inequality: *Let $G(\tau)\ge 0$ be a continuous differentiable function on $[0,T]$ which satisfies $G(0)=0$. If*

$$\frac{dG(\tau)}{d\tau} \le kG(\tau) + F(\tau),$$

where $F(\tau)\ge 0$ is a non-decreasing integrable function on $[0,T]$, and $k>0$ is a constant, then we have

$$\frac{dG(\tau)}{d\tau} \le \mathrm{e}^{k\tau}F(\tau),$$

and

$$G(\tau) \le k^{-1}(\mathrm{e}^{k\tau}-1)F(\tau).$$

Letting

$$G(t) = \int_{t_{n-1}}^{t}\int_0^l \omega_x^2 dxdt, \quad F(t) = 2C\int_{t_{n-1}}^{t}\int_0^l \left|u_x^h\right|^2 dxdt,$$

then

$$\frac{dG(t)}{dt}=\int_0^l \omega_x^2(\cdot,t)dx.$$

Due to $\int_{t_{n-1}}^t\int_0^l \omega_{xx}^2 dxdt \ge 0$, we have from (3.12)

$$\frac{dG(t)}{dt}\le 2CG(t)+F(\tau).$$

So, using the Gronwall inequality above, we get

$$G(t)\le (2C)^{-1}(\mathrm{e}^{2Ct-1})F(t)\le (2C)^{-1}(\mathrm{e}^{2CT}-1)F(t)\le CF(t).$$

Substituting the above inequality to (3.12), we get (3.13).

Lemma 3.3. Let $v^\lambda(x,t)$ be the solution to the following problem

$$\begin{cases} -v_t^\lambda - v_{xx}^\lambda-(\tilde{p}(x,t)v^\lambda)_x=0, & (x,t)\in(0,l)\times[t_{n-1},t_n],\\ v^\lambda(0,t)=v^\lambda(l,t)=0, \\ v^\lambda(x,t_n)=u^\lambda(x,t_n)-g(x,t_n), \end{cases} \tag{3.14}$$

where $\tilde{p}(x,t)$ is defined by (3.5). Then there exists a constant C, such that

$$\|v^\lambda\|_{L^\infty((0,l)\times[t_{n-1},t_n])}\le C. \tag{3.15}$$

Proof. We claim $v^\lambda\in H^{1,1}((0,l)\times[t_{n-1},t_n])$.

Multiplying Eq (3.14) both sides with v^λ and integrating over $(0,l)\times[t,t_n]$, $t\in[t_{n-1},t_n)$, we get

$$-\int_t^{t_n}\int_0^l v_t^\lambda v^\lambda dxdt-\int_t^{t_n}\int_0^l v_{xx}^\lambda v^\lambda dxdt-\int_t^{t_n}\int_0^l(\tilde{p}v^\lambda)_x v^\lambda dxdt=0.$$

Integrating by parts and using the boundary conditions, we have

$$\begin{aligned}
&\frac{1}{2}\int_0^l |v^\lambda|^2dx+\int_t^{t_n}\int_0^l |v_x^\lambda|^2dxdt\\
=\ & -\int_t^{t_n}\int_0^l(\tilde{p}v^\lambda)v_x^\lambda dxdt+\frac{1}{2}\int_t^{t_n}\int_0^l |u^\lambda(\cdot,t_n)-g(\cdot,t_n)|^2dxdt\\
\le\ & \frac{1}{2}\int_t^{t_n}\int_0^l |v_x^\lambda|^2dxdt+\frac{1}{2}\int_t^{t_n}\int_0^l \tilde{p}^2|v^\lambda|^2dxdt+\int_t^{t_n}\int_0^l(|u^\lambda(\cdot,t_n)|^2+|g(\cdot,t_n)|^2)dxdt\\
\le\ & \frac{1}{2}\int_t^{t_n}\int_0^l |v_x^\lambda|^2dxdt+\frac{\overline{p}^2}{2}\int_t^{t_n}\int_0^l |v^\lambda|^2dxdt+C,
\end{aligned}$$

where we have used the bound of $\tilde{p}$, lemmas 3.1, 3.2, and the Cauchy inequality. Then we get

$$\frac{1}{2}\int_0^l |v^\lambda|^2dx+\frac{1}{2}\int_t^{t_n}\int_0^l |v_x^\lambda|^2dxdt\le\frac{\overline{p}^2}{2}\int_t^{t_n}\int_0^l |v^\lambda|^2dxdt+C.$$

Using the Gronwall inequality, we get $\int_{t_{n-1}}^{t_n}\int_0^l |v_x^\lambda|^2dxdt\le C$. Similarly, we can prove $\int_{t_{n-1}}^{t_n}\int_0^l |v_t^\lambda|^2dxdt\le C$. So, $v^\lambda\in H^{1,1}((0,l)\times[t_{n-1},t_n])$ and the conclusion can be obtained immediately from the embedding theorem. □

Theorem 3.4. Let $p_n \in A$ be an optimal control of problem Q_n. Then there exists a constant C, such that

$$\sum_{n=1}^{N}\int_0^l \frac{|p_n - p_{n-1}|^2}{h}dx + \max_{1\le n\le N}\int_0^l |\nabla p_n|^2 dx \le C. \tag{3.16}$$

Proof. As p_n is the minimizer of J_n, we have

$$J_n(p_n) \le J_n(p_{n-1}). \tag{3.17}$$

From (3.17) one can derive

$$\sigma\int_0^l\left(\frac{|p_n - p_{n-1}|^2}{h} + |\nabla p_n|^2 - |\nabla p_{n-1}|^2\right)dx \le \frac{1}{h}\int_0^l [|u(x,t_n;p_{n-1}) - g(x,t_n)|^2 - |u(x,t_n;p_n) - g(x,t_n)|^2]dx, \tag{3.18}$$

where $u(x, t_n; p_{n-1})$ is the solution of (3.1)–(3.3) corresponding to the coefficient

$$\tilde{p} = \begin{cases} p_{n-1}(x), & t_{n-1} \le t \le t_n, \\ \frac{t-t_{k-1}}{h}p_k(x) + \frac{t_k - t}{h}p_{k-1}(x), & t_{k-1} \le t \le t_k, 1 \le k \le n-1. \end{cases}$$

Summing up (3.18) from $n = 1$ to k, we have

$$\begin{aligned}&\sigma\sum_{n=1}^{k}\int_0^l \frac{|p_n - p_{n-1}|^2}{h}dx + \sigma\int_0^l |\nabla p_k|^2 dx \\ &\le \sigma\int_0^l |\nabla p_0|^2 dx + \sum_{n=1}^{k}\frac{1}{h}\int_0^l [|u(x,t_n;p_{n-1}) - g(x,t_n)|^2 - |u(x,t_n;p_n) - g(x,t_n)|^2]dx. \end{aligned}\tag{3.19}$$

From the definition of p^λ and u^λ in lemma 3.2, we obtain

$$\begin{aligned}|u(x,t_n;p_{n-1}) - g(x,t_n)|^2 - |u(x,t_n;p_n) - g(x,t_n)|^2 &= \int_0^1 \frac{d|u^\lambda(x,t_n) - g(x,t_n)|^2}{d\lambda}d\lambda \\ &= 2\int_0^1 (u^\lambda(x,t_n) - g(x,t_n))\frac{du^\lambda(x,t_n)}{d\lambda}d\lambda.\end{aligned}$$

By the same argument used in theorem 2.2, we deduce that

$$\int_0^l (u^\lambda(x,t_n) - g(x,t_n))\frac{du^\lambda(x,t_n)}{d\lambda}dx = \int_{t_{n-1}}^{t_n}\int_0^l \frac{t - t_{n-1}}{h}(p_n - p_{n-1})u_x^\lambda v^\lambda dxdt. \tag{3.20}$$

Hence the right side of (3.18) can be estimated by

$$\begin{aligned}&\frac{1}{h}\int_0^l [|u(x,t_n;p_{n-1}) - g(x,t_n)|^2 - |u(x,t_n;p_n) - g(x,t_n)|^2]dx \\ &= \frac{2}{h}\int_0^1\int_0^l\int_{t_{n-1}}^{t_n}\frac{t - t_{n-1}}{h}(p_n - p_{n-1})u_x^\lambda v^\lambda dtdxd\lambda \\ &\le 2\int_0^1 d\lambda\int_0^l\int_{t_{n-1}}^{t_n}\frac{|p_n - p_{n-1}|}{h}\left|u_x^\lambda\right|\left|v^\lambda\right| dtdx\end{aligned}$$

$$\leq \frac{\sigma}{2}\int_0^l\int_{t_{n-1}}^{t_n}\frac{|p_n-p_{n-1}|^2}{h^2}dtdx+\frac{2}{\sigma}\int_0^1 d\lambda\int_0^l\int_{t_{n-1}}^{t_n}\left|u_x^\lambda\right|^2\left|v^\lambda\right|^2 dtdx$$

$$\leq \frac{\sigma}{2}\int_0^l\frac{|p_n-p_{n-1}|^2}{h}dx+\frac{2}{\sigma}\max_{0\leq\lambda\leq1}\int_0^l\int_{t_{n-1}}^{t_n}\left|u_x^\lambda\right|^2\left|v^\lambda\right|^2 dtdx. \tag{3.21}$$

Combining (3.19) and (3.21) and using lemmas 3.1, 3.2, 3.3, we get

$$\begin{aligned}&\sum_{n=1}^k\int_0^l\frac{|p_n-p_{n-1}|^2}{h}dx+\int_0^l|\nabla p_k|^2dx\\ \leq\ & C\int_0^l|\nabla p_0|^2dx+C\max_{0\leq\lambda\leq1}\int_0^l\int_{t_{n-1}}^{t_n}\left|u_x^\lambda\right|^2\left|v^\lambda\right|^2 dtdx\\ \leq\ & C,\end{aligned}$$

for $1\leq k\leq N$.

This completes the proof of theorem 3.4. □

Based on the theorem 3.4 one can easily obtain the following theorem.

Theorem 3.5. For p^h we have the following estimate

$$\int_0^T\int_0^l\left|p_t^h\right|^2dxdt+\max_{0\leq t\leq T}\int_0^l\left|p_x^h\right|^2dx\leq C. \tag{3.22}$$

Theorem 3.6. There exists a constant C, such that

$$\|p^h\|_{C^{\frac{1}{2},\frac{1}{4}}(\bar{Q})}\leq C. \tag{3.23}$$

Proof. From the definition of the admissible set and the estimates in theorem 3.4 and 3.5, we have

$$\max_{0\leq t\leq T}\|p^h(x,t)\|_{H^1(0,l)}\leq C.$$

Using Soblev's embedding theorem, there exists a constant C such that

$$\left|p^h(x,t)-p^h(y,t)\right|\leq C|x-y|^{\frac{1}{2}},$$

for any $t\in[0,T]$.

To obtain the t-Hölder estimate for the function $p^h(x,t)$, we assume that for any given points $(x,s),(x,t)\in Q$, without loss of generality, the rectangle

$$D=\{(\xi,\tau)|x\leq\xi\leq x+\sqrt{t-s},s\leq\tau\leq t\}\subset Q.$$

Then we have

$$\begin{aligned}\int_s^t\int_x^{x+\sqrt{t-s}}p_\tau^h(\xi,\tau)d\xi d\tau &= \int_x^{x+\sqrt{t-s}}(p^h(\xi,t)-p^h(\xi,s))d\xi\\ &= (p^h(\hat{x},t)-p^h(\hat{x},s))\sqrt{t-s},\end{aligned}$$

where $\hat{x} = x + \theta\sqrt{t-s}, 0 \le \theta \le 1$.

By theorem 3.5, we derive that

$$\begin{aligned}
|p^h(\hat{x},t) - p^h(\hat{x},s)| &= (t-s)^{-\frac{1}{2}} \int_s^t \int_x^{x+\sqrt{t-s}} p^h_\tau(\xi,\tau)d\xi d\tau \\
&\le (t-s)^{-\frac{1}{2}} \left(\int_s^t \int_x^{x+\sqrt{t-s}} d\xi d\tau\right)^{\frac{1}{2}} \left(\int_0^T \int_0^l \left|p^h_t\right|^2 dxdt\right)^{\frac{1}{2}} \\
&\le (t-s)^{\frac{3}{4}-\frac{1}{2}} \left(\int_0^T \int_0^l \left|p^h_t\right|^2 dxdt\right)^{\frac{1}{2}} \\
&\le C(t-s)^{\frac{1}{4}}.
\end{aligned}$$

Then we have

$$\begin{aligned}
|p^h(x,t) - p^h(x,s)| &\le |p^h(x,t) - p^h(\hat{x},t)| + |p^h(\hat{x},t) - p^h(\hat{x},s)| + |p^h(\hat{x},s) - p^h(x,s)| \\
&\le C(t-s)^{\frac{1}{4}}.
\end{aligned}$$

This completes the proof of the theorem 3.6. □

Theorem 3.7. Let $v^h(x,t)$ be the solution of Eq (2.7). Then there exists a constant C which is independent of h, such that

$$\int_{t_{n-1}}^{t_n} \int_0^l \left|v^h_t\right|^2 dxdt + \max_{t_{n-1}\le t\le t_n} \int_0^l \left|v^h_x\right|^2 dx \le C, \tag{3.24}$$

and

$$\|v^h\|_{C^{\frac{1}{2},\frac{1}{4}}([0,l]\times[t_{n-1},t_n])} \le C. \tag{3.25}$$

The proof of (3.24) is standard. The estimate (3.25) can be derived by the same argument used in theorem 3.6.

4. Necessary condition

In this section, we will discuss the limiting behavior of the discrete reconstruction $p^h(x,t)$ of the unknown coefficient as $h \to 0$.

Let

$$\tilde{A} = \{p(x,t) | \underline{p} \le p(x,t) \le \overline{p}, p \in H^1(Q) \cap L^\infty([0,T], H^1(0,l))\},$$

and

$$\bar{p}^h(x,t) = \begin{cases} p_0(x), & t = t_0, \\ p_k(x), & t_{k-1} \le t \le t_k, 1 \le k \le n. \end{cases}$$

From the estimates in theorem 3.4, 3.5, and 3.6 we have the following convergence results.

Theorem 4.1. There exists a subsequence of $p^h(x,t)$ and a function $p \in \tilde{A}$, such that

$$\begin{array}{rll} p^h \to & p, & \textit{weakly in } H^1(Q), \\ p^h \to & p, & \textit{in } C(\bar{Q}), \\ \bar{p}^h \to & p, & \textit{in } L^2(Q), \\ \nabla \bar{p}^h \to & \nabla p, & \textit{weakly in } L^2(Q). \end{array} \tag{4.1}$$

Proof. From the definition of $\bar{p}^h(x,t)$, we have

$$\max_{0\le t\le T} \int_0^l |\nabla \bar{p}^h|^2 dx \le C.$$

So we only need to prove that p^h and $\bar{p}^h$ converge to the same function.

From theorem 3.4, we obtain

$$\begin{aligned} \int_0^T \int_0^l |p^h - \bar{p}^h|^2 dxdt &= \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \int_0^l \frac{(t_n - t)^2}{h^2} (p_n - p_{n-1})^2 dxdt \\ &= \frac{h^2}{3} \sum_{n=1}^{N} \int_0^l \frac{|p_n - p_{n-1}|^2}{h} dx \\ &\le Ch^2. \end{aligned}$$

This implies the result. □

Function $p(x,t) = \lim\limits_{h\to 0} p^h(x,t)$ is the reconstruction of the unknown coefficient. We call it the limiting optimal control of our problem. Now we derive the necessary condition for $p(x,t)$.

Theorem 4.2. Let $p(x,t)$ be the limiting optimal control and $u(x,t)$ be the solution to the following problem:

$$Lu = u_t - u_{xx} + p(x,t)u_x = 0, \qquad (x,t) \in Q, \tag{4.2}$$

$$u(x,0) = \varphi(x), \qquad x \in I, \tag{4.3}$$

$$u(0,t) = u(l,t) = 0, \qquad t \in (0.T]. \tag{4.4}$$

Then, for any $\omega \in \tilde{A}$, we have

$$\int_Q \left[p_t(\omega - p) + \nabla p \cdot \nabla(\omega - p) + \frac{1}{2\sigma}(u - g)u_x(p - \omega) \right] dxdt \ge 0. \tag{4.5}$$

Proof. Note that $\|p^h\|_{C^{\frac{1}{2},\frac{1}{4}}(Q)} \le C$. Without loss of generality, we can assume $\alpha \le \frac{1}{2}$. Then Schauder's theory guarantees that

$$\|u^h\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q)} \le C, \tag{4.6}$$

and

$$u^h \to u \qquad \text{in } C^{2,1}(\bar{Q}).$$

Firstly we prove (4.5) for $\omega \in \tilde{A} \bigcap C^\infty(\bar{Q})$.

Let ω be a function in $\tilde{A} \bigcap C^\infty(\bar{Q})$, then $\omega(x, t_n) \in A, n = 1, \cdots, N$. Thus from the necessary condition (2.6) of optimal control problem Q_n, we get

$$\int_0^l \left[\frac{p_n(x) - p_{n-1}(x)}{h} (\omega(x, t_n) - p_n(x)) + \nabla p_n(x) \cdot \nabla(\omega(x, t_n) - p_n(x)) \right. \\ \left. + \frac{1}{\sigma h} \int_{t_{n-1}}^{t_n} \frac{t - t_{n-1}}{h} u_x^h(x, t) v^h(x, t) dt (p_n(x) - \omega(x, t_n)) \right] dx \geq 0, \tag{4.7}$$

$1 \leq n \leq N$, and $v^h(x, t)$ be the solution of (2.7). From the definition of p^h and $\bar{p}^h$, it follows that

$$\int_0^l \int_{t_{n-1}}^{t_n} \left[p_t^h(x, t)(\omega(x, t_n) - \bar{p}^h(x, t)) + \nabla \bar{p}^h(x, t) \cdot \nabla(\omega(x, t_n) - \bar{p}^h(x, t)) \right. \\ \left. + \frac{t - t_{n-1}}{\sigma h} u_x^h(x, t) v^h(x, t)(\bar{p}^h(x, t) - \omega(x, t_n)) \right] dt dx \geq 0.$$

Therefore, we obtain

$$\int_0^l \int_{t_{n-1}}^{t_n} \left[p_t^h(x, t)(\omega(x, t) - \bar{p}^h(x, t)) + \nabla \bar{p}^h(x, t) \cdot \nabla(\omega(x, t) - \bar{p}^h(x, t)) \right. \\ \left. + \frac{t - t_{n-1}}{\sigma h} u_x^h(x, t) v^h(x, t)(\bar{p}^h(x, t) - \omega(x, t)) \right] dt dx \geq E_n, \tag{4.8}$$

where

$$E_n = \int_0^l \int_{t_{n-1}}^{t_n} \left[p_t^h(x, t)(\omega(x, t) - \omega(x, t_n)) + \nabla \bar{p}^h(x, t) \cdot \nabla(\omega(x, t) - \omega(x, t_n)) \right. \\ \left. + \frac{t - t_{n-1}}{\sigma h} u_x^h(x, t) v^h(x, t)(\omega(x, t_n) - \omega(x, t)) \right] dt dx.$$

Let

$$\begin{aligned} I_1 &= \int_0^l \int_{t_{n-1}}^{t_n} p_t^h(x, t)(\omega(x, t) - \omega(x, t_n)) dt dx, \\ I_2 &= \int_0^l \int_{t_{n-1}}^{t_n} \nabla \bar{p}^h(x, t) \cdot \nabla(\omega(x, t) - \omega(x, t_n)) dt dx, \\ I_3 &= \int_0^l \int_{t_{n-1}}^{t_n} \frac{t - t_{n-1}}{\sigma h} u_x^h(x, t) v^h(x, t)(\omega(x, t_n) - \omega(x, t))] dt dx. \end{aligned}$$

Thus, we have

$$E_n = I_1 + I_2 + I_3.$$

By the smoothness of ω, there exists a constant C which is independent of h, such that

$$|\omega(x, t) - \omega(x, t_n)| = |\omega_t(x, t_n - \theta(t_n - t))(t_n - t)| \leq Ch,$$

where $0 \le \theta \le 1$. Then, for the item I_1, we have the following estimate

$$\begin{aligned}
|I_1| &\le Ch \int_0^l \int_{t_{n-1}}^{t_n} \left| p_t^h(x,t) \right| dtdx \\
&\le Ch \left(\int_0^l \int_{t_{n-1}}^{t_n} \left| p_t^h(x,t) \right|^2 dtdx + \int_0^l \int_{t_{n-1}}^{t_n} dtdx \right) \\
&\le Ch \int_0^l \int_{t_{n-1}}^{t_n} \left| p_t^h(x,t) \right|^2 dtdx + Ch^2.
\end{aligned}$$

So, for I_2 we have a similar estimate by the same argument of I_1,

$$|I_2| \le Ch^2 \int_0^l |\nabla p_n|^2 dx + Ch^2.$$

For item I_3, by lemma 3.1 and theorem 3.7, we have the following estimate

$$\begin{aligned}
|I_3| &\le \frac{Ch}{\sigma} \int_0^l \int_{t_{n-1}}^{t_n} \frac{t - t_{n-1}}{h} u_x^h v^h dtdx \\
&\le \frac{Ch}{\sigma} \int_0^l \int_{t_{n-1}}^{t_n} \frac{t - t_{n-1}}{h} dtdx \\
&\le Ch^2.
\end{aligned}$$

Therefore, for E_n, we derive that

$$|E_n| \le Ch^2 + Ch \int_0^l \int_{t_{n-1}}^{t_n} \left| p_t^h(x,t) \right|^2 dtdx + Ch^2 \int_0^l |\nabla p_n|^2 dx.$$

One can easily see that

$$\sum_{n=1}^{N} |E_n| \le Ch. \tag{4.9}$$

From (4.8) and (4.9), we have

$$\begin{aligned}
\sigma \int_0^l \int_0^T &\left[p_t^h(x,t)(\omega(x,t) - \bar{p}^h(x,t)) + \nabla \bar{p}^h(x,t) \cdot \nabla(\omega(x,t) - \bar{p}^h(x,t)) \right] dtdx \\
&+ \sum_{n=1}^{N} \int_0^l \int_{t_{n-1}}^{t_n} \frac{t - t_{n-1}}{h} u_x^h(x,t) v^h(x,t)(\bar{p}^h(x,t) - \omega(x,t)) dtdx \ge -Ch. \qquad (4.10)
\end{aligned}$$

By noting that

$$v^h(x,t_n) = u^h(x,t_n) - g(x,t_n),$$

we obtain

$$\begin{aligned}
|v^h(x,t) - (u^h(x,t) - g(x,t))| &\le |v^h(x,t) - v^h(x,t_n)| + |(u^h(x,t_n) - g(x,t_n)) - (u^h(x,t) - g(x,t))| \\
&\le Ch^{\frac{\alpha}{2}},
\end{aligned}$$

where we have used (2.1), (4.6) and theorem 3.7.

Hence

$$\left|\sum_{n=1}^{N}\int_0^l\int_{t_{n-1}}^{t_n}\frac{t-t_{n-1}}{h}u_x^h(x,t)\left[v^h(x,t)-(u^h(x,t)-g(x,t))\right](\bar{p}^h(x,t)-\omega(x,t_n))dtdx\right|\leq Ch^{\frac{\alpha}{2}}. \quad (4.11)$$

From (4.10) and (4.11), we obtain that

$$\sigma\int_0^l\int_0^T\left[p_t^h(x,t)(\omega(x,t)-\bar{p}^h(x,t))+\nabla\bar{p}^h(x,t)\cdot\nabla(\omega(x,t)-\bar{p}^h(x,t))\right]dtdx$$
$$+\sum_{n=1}^{N}\int_0^l\int_{t_{n-1}}^{t_n}\frac{t-t_{n-1}}{h}(u^h(x,t)-g(x,t))u_x^h(x,t)(\bar{p}^h(x,t)-\omega(x,t))dtdx\geq -Ch^{\frac{\alpha}{2}}. \quad (4.12)$$

Note that

$$\int_{t_{n-1}}^{t_n}\left(\frac{t-t_{n-1}}{h}-\frac{1}{2}\right)dt=0.$$

By the direct calculation, we have

$$\int_{t_{n-1}}^{t_n}\frac{t-t_{n-1}}{h}f^h(t)dt-\frac{1}{2}\int_{t_{n-1}}^{t_n}f^h(t)dt=\int_{t_{n-1}}^{t_n}\left(\frac{t-t_{n-1}}{h}-\frac{1}{2}\right)(f^h(t)-f^h(t_n))dt, \quad (4.13)$$

where

$$f^h(t)=\int_0^l(u^h(x,t)-g(x,t))u_x^h(x,t)(\bar{p}^h(x,t)-\omega(x,t))dx.$$

It can be seen that

$$f^h(t)\in C^{\frac{\alpha}{2}}[t_{n-1},t_n].$$

Hence we have, from (4.13)

$$\left|\int_{t_{n-1}}^{t_n}\frac{t-t_{n-1}}{h}f^h(t)dt-\frac{1}{2}\int_{t_{n-1}}^{t_n}f^h(t)dt\right|\leq\|f^h\|_{C^{\frac{\alpha}{2}}[t_{n-1},t_n]}h^{1+\frac{\alpha}{2}}. \quad (4.14)$$

From (4.14) we obtain that

$$\left|\int_0^l\int_{t_{n-1}}^{t_n}\frac{t-t_{n-1}}{h}(u^h(x,t)-g(x,t))u_x^h(x,t)(\bar{p}^h(x,t)-\omega(x,t))dtdx\right.$$
$$\left.-\frac{1}{2}\int_0^l\int_{t_{n-1}}^{t_n}(u^h(x,t)-g(x,t))u_x^h(x,t)(\bar{p}^h(x,t)-\omega(x,t))dtdx\right|\leq Ch^{1+\frac{\alpha}{2}}. \quad (4.15)$$

Summing up (4.15) from $n=1$ to N, we have

$$\left|\sum_{n=1}^{N}\int_0^l\int_{t_{n-1}}^{t_n}\frac{t-t_{n-1}}{h}(u^h(x,t)-g(x,t))u_x^h(x,t)(\bar{p}^h(x,t)-\omega(x,t))dtdx\right.$$
$$\left.-\frac{1}{2}\int_Q(u^h(x,t)-g(x,t))u_x^h(x,t)(\bar{p}^h(x,t)-\omega(x,t))dtdx\right|\leq Ch^{\frac{\alpha}{2}}. \quad (4.16)$$

Combining (4.12) and (4.16), one can easily obtain

$$\int_Q \left[p_t^h(\omega - \bar{p}^h) + \nabla \bar{p}^h \cdot \nabla(\omega - \bar{p}^h) + \frac{1}{2\sigma}(u^h - g)u_x^h(\bar{p}^h - \omega) \right] dtdx \geq -Ch^{\frac{\alpha}{2}}. \tag{4.17}$$

Letting $h \to 0$, we deduce that, from theorem 4.1 and (4.17)

$$\int_Q \left[p_t(\omega - p) + \nabla p \cdot \nabla \omega + \frac{1}{2\sigma}(u - g)u_x(p - \omega) \right] dxdt - \limsup_{h \to 0} \int_Q |\nabla \bar{p}^h|^2 dtdx \geq 0. \tag{4.18}$$

By the property of weak convergence, we have

$$\liminf_{h \to 0} \int_Q |\nabla \bar{p}^h|^2 dxdt \geq \int_Q |\nabla p|^2 dxdt.$$

Then from (4.18), we deduce that

$$\int_Q \left[p_t(\omega - p) + \nabla p \cdot \nabla(\omega - p) + \frac{1}{2\sigma}(u - g)u_x(p - \omega) \right] dxdt \geq 0, \tag{4.19}$$

for any $\omega \in \tilde{A} \bigcap C^\infty(\bar{Q})$.

The necessary condition (4.19) remains true for any $\omega \in \tilde{A}$ by the approximation argument.

This completes the proof of theorem 4.2.

□

Corollary 4.3. If $\varphi(x) \in H^1(I)$, then necessary condition (4.5) remains true for any $\omega \in \tilde{A}$.

Corollary 4.4. Let $p(x, t)$ be the limiting optimal control and $u(x, t)$ be the solution of (4.2)–(4.4). Then, for any $\omega \in \tilde{A}$, we have

$$\int_0^s \int_0^l \left[p_t(\omega - p) + \nabla p \cdot \nabla(\omega - p) + \frac{1}{2\sigma}(u - g)u_x(p - \omega) \right] dxdt \geq 0, \tag{4.20}$$

for any $s \in [0, T]$.

Proof. Let $\delta > 0$ and $\eta_\delta \in C^1[0, T]$ be a cut-off function such that

$$\eta_\delta(t) = \begin{cases} 1, & 0 \leq t \leq s, \\ 0, & s + \delta \leq t \leq T. \end{cases}$$

Note that $\tilde{A}$ is a convex set, for any $\omega \in \tilde{A}$,

$$\tilde{\omega} = p + \eta_\delta(\omega - p) \in \tilde{A}.$$

From theorem 4.2 we have

$$\int_Q \left[p_t(\tilde{\omega} - p) + \nabla p \cdot \nabla(\tilde{\omega} - p) + \frac{1}{2\sigma}(u - g)u_x(p - \tilde{\omega}) \right] dxdt \geq 0.$$

Hence

$$\int_Q \left[p_t(\omega - p) + \nabla p \cdot \nabla(\omega - p) + \frac{1}{2\sigma}(u - g)u_x(p - \omega) \right] \eta_\delta(t) dxdt \geq 0.$$

Letting $\delta \to 0$, we obtain the result. □

5. Uniqueness and stability

We will derive the uniqueness and stability of limiting optimal controls in the sense of L^2 norm. In this part to prove the uniqueness and stability of the limiting optimal control $p(x,t)$, we establish the following estimate firstly.

Theorem 5.1. Suppose that $p_0(x), \bar{p}_0(x), g(x,t), \bar{g}(x,t)$ are given functions, $p_0, \bar{p}_0 \in A \bigcap W^{1,\infty}(Q)$ and $g, \bar{g}$ satisfy condition (2.1). Let $p(x,t), \bar{p}(x,t)$ be the limiting optimal controls corresponding to $(p_0, g), (\bar{p}_0, \bar{g})$ respectively. Then there exists a constant C such that

$$||p-\bar{p}||_{L^\infty([0,T],L^2(I))} + ||\nabla(p-\bar{p})||_{L^2(Q)} \le \frac{C}{\sigma}(||g-\bar{g}||_{L^2(Q)} + ||p_0-\bar{p}_0||_{L^2(I)}). \tag{5.1}$$

Proof. Let $u(x,t), \bar{u}(x,t)$ be the solution of (4.2)–(4.4) corresponding to $p(x,t), \bar{p}(x,t)$ respectively. It can be easily verified that $W = u - \bar{u}$ satisfies the following equation

$$W_t - W_{xx} + p(x,t)W_x = (\bar{p}-p)\bar{u}_x, \qquad (x,t) \in Q, \tag{5.2}$$

$$W(x,0) = 0, \qquad x \in I, \tag{5.3}$$

$$W(0,t) = W(l,t) = 0, \qquad t \in (0,T]. \tag{5.4}$$

Utilizing the standard energy estimate for parabolic equations, we deduce that

$$||W||_{L^2(Q_s)} + ||W_x||_{L^2(Q_s)} \le C||\bar{p}-p||_{L^2(Q_s)}, \tag{5.5}$$

where $s \in [0,T], Q_s = (0,l) \times [0,s]$.

From (4.20), for any $s \in [0,T]$, we have

$$\int_0^s \int_0^l \left[p_t(\bar{p}-p) + \nabla p \cdot \nabla(\bar{p}-p) + \frac{1}{2\sigma}(u-g)u_x(p-\bar{p}) \right] dxdt \ge 0, \tag{5.6}$$

and

$$\int_0^s \int_0^l \left[\bar{p}_t(p-\bar{p}) + \nabla \bar{p} \cdot \nabla(p-\bar{p}) + \frac{1}{2\sigma}(\bar{u}-\bar{g})\bar{u}_x(\bar{p}-p) \right] dxdt \ge 0. \tag{5.7}$$

Hence

$$\int_0^s \int_0^l \left[(p-\bar{p})(p-\bar{p})_t + |\nabla(p-\bar{p})|^2 \right] dxdt \le \frac{1}{2\sigma} \int_0^s \int_0^l (p-\bar{p})[(u-g)u_x - (\bar{u}-\bar{g})\bar{u}_x] dxdt. \tag{5.8}$$

Therefore, we obtain

$$\begin{aligned}
&\sigma \int_0^l (p-\bar{p})^2(x,s)dx + 2\sigma \int_0^s \int_0^l |\nabla(p-\bar{p})|^2 dxdt \\
&\le \sigma \int_0^l (p_0-\bar{p}_0)^2 dx + \int_0^s \int_0^l (p-\bar{p})^2 dxdt + \int_0^s \int_0^l [(u-g)u_x - (\bar{u}-\bar{g})\bar{u}_x]^2 dxdt.
\end{aligned} \tag{5.9}$$

Note that

$$[(u-g)u_x - (\bar{u}-\bar{g})\bar{u}_x]^2 = [uu_x - gu_x - \bar{u}\bar{u}_x + \bar{g}\bar{u}_x]^2$$

$$
\begin{aligned}
&= [(uu_x - \bar{u}\bar{u}_x) + (\bar{g}\bar{u}_x - gu_x)]^2 \\
&\le C[(uu_x - \bar{u}\bar{u}_x)^2 + (\bar{g}\bar{u}_x - gu_x)^2] \\
&\le C[(u - \bar{u})^2 u_x^2 + \bar{u}^2(u_x - \bar{u}_x)^2 + (\bar{g} - g)^2\bar{u}_x^2 + g^2(\bar{u}_x - u_x)^2] \\
&\le C[(g - \bar{g})^2 + (u - \bar{u})^2 + (u_x - \bar{u}_x)^2].
\end{aligned}
$$

Hence

$$
\int_0^s \int_0^l [(u - g)u_x - (\bar{u} - \bar{g})\bar{u}_x]^2 dxdt \le C(\|g - \bar{g}\|^2_{L^2(Q_s)} + \|u - \bar{u}\|^2_{L^2(Q_s)} + \|(u - \bar{u})_x\|^2_{L^2(Q_s)}). \tag{5.10}
$$

From (5.5), (5.9), and (5.10), we have

$$
\begin{aligned}
&\sigma \int_0^l (p - \bar{p})^2(x, s)dx + 2\sigma \int_0^s \int_0^l |\nabla(p - \bar{p})|^2 dxdt \\
&\qquad \le \sigma \int_0^l (p_0 - \bar{p}_0)^2 dx + C \int_0^s \int_0^l (p - \bar{p})^2 dxdt + C \int_0^s \int_0^l (g - \bar{g})^2 dxdt.
\end{aligned} \tag{5.11}
$$

By Gronwall's inequality, we obtain that

$$
\int_0^l (p - \bar{p})^2(x, s)dx + \int_0^s \int_0^l |\nabla(p - \bar{p})|^2 dxdt \le \frac{C}{\sigma}\left(\int_0^l (p_0 - \bar{p}_0)^2 dx + \int_0^s \int_0^l (g - \bar{g})^2 dxdt\right),
$$

for any $s \in [0, T]$.

This completes the proof of theorem 5.1. □

Corollary 5.2. Based on the uniqueness of the solution $p(x, t)$, by theorem 4.1, we get that the whole sequence $p^h(x, t)$ defined by linear interpolation of minimizers $p_n(x)$ of optimal control problems $Q_n(n = 0, 1, \cdots, N)$ converge to the recovered coefficient $p(x, t)$ uniformly on the domain $\bar{Q}$.

6. Conclusions

In this paper, we reconstruct the drift coefficient $p(x, t)$ for the following Kolmogorov-type equation

$$
u_t - u_{xx} + p(x, t)u_x = 0,
$$

where u satisfies the homogeneous Dirichlet boundary condition. In an optimal control framework, the problem is transformed to a sequence of inverse problems Q_n ($n = 0, 1, \cdots, N$). Then the unknown coefficient is purely space dependent. We establish the existence and the necessary condition of the minimizer $p_n(x)$ for problem Q_n. From $p_n(x)$ and the using linear interpolation, we construct the discrete reconstruction $p^h(x, t)$. After careful analysis, it is found that there exists a subsequence of $p^h(x, t)$ converging to a function $p(x, t)$, and the corresponding necessary condition of $p(x, t)$ is also deduced. In the end, the uniqueness and stability of $p(x, t)$ are obtained in the sense of L^2 norm, which indicates that the procedure of recovering $p(x, t)$ from a given function $g(x, t)$ is stable.

In the current paper, we focus on the analysis of drift coefficient, but have no special requirements for diffusion coefficient or volatility coefficient. In the real market, the volatility coefficient is not always positive, but only non-negative. In such case, our mathematical model will be transformed into a degenerate Kolmogorov-type equation. Of course, this type of equation is more difficult than ordinary one, which is also our research direction in the future.

Acknowledgements

The work is partially supported by the National Natural Science Foundation of China (Grant Numbers: 11961042, 61663018), and Foundation of A Hundred Youth Talents Training Program of Lanzhou Jiaotong University.

Conflict of interest

The authors declare no conflict of interest.

References

1. R. A. Adams, *Sobolev spaces,* Academic Press, New York, 1975.
2. I. Bushuyev, Global uniqueness for inverse problems with final observation, *Inverse Probl.,* **11** (1995), L11–L16.
3. J. R. Cannon, Y. Lin, S. Xu, Numerical procedure for the determination of an unknown coefficient in semilinear parabolic partial differential equations, *Inverse Probl.,* **10** (1994), 227–243.
4. J. R. Cannon, Y. Lin, An inverse problem of finding a parameter in a semilinear heat equation, *J. Math. Anal. Appl.,* **145** (1990), 470–484.
5. Q. Chen, J. J. Liu, Solving an inverse parabolic problem by optimization from final measurement data, *J. Comput. Appl. Math.,* **193** (2006), 183–203.
6. M. Choulli, M. Yamamoto, Generic well-posedness of an inverse parabolic problem - the Hölder space approach, *Inverse Probl.,* **12** (1996), 195–205.
7. M. Dehghan, Identification of a time-dependent coefficient in a partial differential equation subject to an extra measurement, *Numer. Meth. Part. Diff. Equ.,* **21** (2005), 611–622.
8. M. Dehghan, Determination of a control function in three-dimensional parabolic equations, *Math. Comput. Simul.,* **61** (2003), 89–100.
9. Z. C. Deng, L. Yang, J. N. Yu, G. W. Luo, Identifying the radiative coefficient of an evolutional type heat conduction equation by optimization method, *J. Math. Anal. Appl.,* **362** (2010), 210–223.
10. Z. C. Deng, J. N. Yu, L. Yang, Optimization method for an evolutional type inverse heat conduction problem, *J. Phys. A: Math. Theor.,* **41** (2008), 035201.
11. Z. C. Deng, J. N. Yu, L. Yang, Identifying the coefficient of first-order in parabolic equation from final measurement data, *Math. Comput. Simul.,* **77** (2008), 421–435.
12. H. W. Engl, M. Hanke, A. Neubauer, *Regularization of Inverse Problems,* Dordrecht: Kluwer Academic Publishers, 1996.
13. L. C. Evans, *Partial Differential Equations,* American Math Society, 1998.
14. A. Friedman, *Partial Differential Equations of Parabolic Type,* Englewood Cliffs, NJ: Prentice-Hall, 1964.
15. G. L. Gong, *Stochastic differential equation and its application,* Tsinghua University Press, Beijing, 2008.
16. V. Isakov, *Inverse Problems for Partial Differential Equations,* Springer, New York, 1998.

17. V. Isakov, Inverse parabolic problems with the final overdetermination, *Comm. Pure Appl. Math.,* **44** (1991), 185–209.
18. L. S. Jiang, Y. S. Tao, Identifying the volatility of underlying assets from option prices, *Inverse Probl.,* **17** (2001), 137–155.
19. L. S. Jiang, Q. H. Chen, L. J. Wang, J. E. Zhang, A new well-posed algorithm to recover implied local volatility, *Quant. Finance,* **3** (2003), 451–457.
20. L. S. Jiang, B. J. Bian, Identifying the principal coefficient of parabolic equations with non-divergent form, *J. Phys.: Conf. Ser.,* **12** (2005), 58–65.
21. L. S. Jiang, C. L. Xu, X. M. Ren, S. H. Li, *Mathematical model and case analysis of financial derivatives pricing,* Higher Education Press, Beijing, 2008.
22. B. Kaltenbacher, M. V. Klibanov, An inverse problem for a nonlinear parabolic equation with applications in population dynamics and magnetics, *SIAM J. Math. Anal.,* **39** (2008), 1863–1889.
23. Y. L. Keung, J. Zou, Numerical identifications of parameters in parabolic systems, *Inverse Probl.,* **14** (1998), 83–100.
24. A. Kirsch, *An Introduction to the Mathematical Theory of Inverse Problem,* Springer, New York, 1999.
25. O. Ladyzenskaya, V. Solonnikov, N. Ural'Ceva, *Linear and Quasilinear Equations of Parabolic Type,* Providence, KI: American Mathematical Society, 1968.
26. W. Rundell, The determination of a parabolic equation from initial and final data, *Proc. Am. Math. Soc.,* **99** (1987), 637–642.
27. A. A. Samarskii, P. N. Vabishchevich, *Numerical Methods for Solving Inverse Problems of Mathematical Physics,* Berlin, Germany, 2007.
28. A. Tikhonov, V. Arsenin, *Solutions of Ill-posed Problems,* Geology Press, Beijing, 1979.
29. D. Trucu, D. B. Ingham, D. Lesnic, Inverse temperature-dependent perfusion coefficient reconstruction, *Int. J. Non-Linear Mech.,* **45** (2010), 542–549.
30. D. Trucu, D. B. Ingham, D. Lesnic, Reconstruction of the space- and time-dependent blood perfusion coefficient in bio-heat transfer, *Heat Transfer Eng.,* **32** (2011), 800–810.
31. D. Trucu, D. B. Ingham, D. Lesnic, Space-dependent perfusion coefficient identification in the transient bio-heat equation, *J. Eng. Math.,* **67** (2010), 307–315.
32. Z. Q. Wu, J. X. Yin, C. P. Wang, *Introduction to elliptic and parabolic equations,* Science Press, Beijing, 2003.
33. M. Yamamoto, J. Zou, Simultaneous reconstruction of the initial temperature and heat radiative coefficient, *Inverse Probl.,* **17** (2001), 1181–1202.
34. L. Yang, J. N. Yu, Z. C. Deng, An inverse problem of identifying the coefficient of parabolic equation, *Appl. Math. Modell.,* **32** (2008), 1984–1995.

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