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Research article

Some results for the family of univalent functions related with Limaçon domain

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Abstract: The investigation of univalent functions is one of the fundamental ideas of Geometric function theory (GFT). However, the class of these functions cannot be investigated as a whole for some particular kind of problems. As a result, the study of its subclasses has been receiving numerous attentions. In this direction, subfamilies of the class of univalent functions that map the open unit disc onto the domain bounded by limaçon of Pascal were recently introduced in the literature. Due to the several applications of this domain in Mathematics, Statistics (hypothesis testing problem) and Engineering (rotary fluid processing machines such as pumps, compressors, motors and engines.), continuous investigation of these classes are of interest in this article. To this end, the family of functions for which $\frac{\varsigma f'(\varsigma)}{f(\varsigma)}$ and $\frac{(\varsigma f'(\varsigma))'}{f'(\varsigma)}$ map open unit disc onto region bounded by limaçon are studied. Coefficients bounds, Fekete Szegö inequalities and the bounds of the third Hankel determinants are derived. Furthermore, the sharp radius for which the classes are linked to each other and to the notable subclasses of univalent functions are found. Finally, the idea of subordination is utilized to obtain some results for functions belonging to these classes.

Keywords: univalent functions; Schwarz functions; Limaçon domain; subordination; Hankel determinan
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1. Introduction

One of the most fascinating areas of Complex analysis is the study of geometric characterization of univalent functions in the open unit disc U. Because of the challenging problem in studying the class $S_{(1-1)}$ of univalent functions in U as a whole, several subclasses of it emerged. The most studied of these are the classes C_{CV}, S_{ST}, $C_{CV}(\beta)$ and $S_{ST}(\beta)$ $(0 \le \beta < 1)$ of convex functions, starlike functions and, convex and starlike functions of order β, respectively. Since the image domains of U plays a significant role in their geometric characterization, various subclasses of $S_{(1-1)}$ have been receiving attention in different directions and perspectives (see [4, 6, 20, 23, 28, 32, 33, 36–38]). For this reason, Ma and Minda [17] gave a unified treatment of both S_{ST} and C_{CV}. They considered the class Ψ of analytic univalent functions $\psi(\varsigma)$ with $\mathrm{Re}\psi(\varsigma) > 0$ and for which $\psi(U)$ is symmetric with respect to the real axis and starlike with respect to $\psi(0)$ such that $\psi'(0) > 0$. They initiated the following classes of functions that generalized and unified many renowned subclasses of $S_{(1-1)}$:

$$S^*_{ST}(\psi) = \left\{ f \in \mathcal{A} \colon \frac{\varsigma f'(\varsigma)}{f(\varsigma)} \prec \psi(\varsigma) \right\}$$

and

$$C_{CV}(\psi) = \left\{ f \in \mathcal{A} \colon \frac{(\varsigma f'(\varsigma))'}{f'(\varsigma)} \prec \psi(\varsigma) \right\},$$

where $\mathcal{A}$ is the class of analytic functions $f(\varsigma)$ of the form

$$f(\varsigma) = \varsigma + \sum_{n=2}^{\infty} \delta_n \varsigma^n. \tag{1.1}$$

If

$$\psi(\varsigma) = 1 + \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{\varsigma}}{1-\sqrt{\varsigma}}\right)^2,$$

then $C_{CV}(\psi) = UCV$ is the Goodman class of uniformly convex functions [8, 30], which was later modified and examined by Kanas and Wisniowska [13,14]. Similarly, $ST_{hpl}(s) = S^*\left(\frac{1}{(1-\varsigma)^s}\right)$, $CV_{hpl}(s) = C\left(\frac{1}{(1-\varsigma)^s}\right)$ $(0 < s \le 1)$, are made-known by Kanas and Ebadian [15, 16], respectively. These consist of functions $f \in \mathcal{A}$ such that $\varsigma f'(\varsigma)/f(\varsigma)$ and $(\varsigma f'(\varsigma))'/f'(\varsigma)$ lie in the domain bounded by the right branch of a hyperbola

$$H(s) = \left\{ \sigma e^{i\theta} \in \mathbb{C} \colon \sigma = \frac{1}{(2\cos\frac{\theta}{s})^s}, \ |\theta| < \frac{\pi s}{2} \right\}.$$

More special families of Ma and Minda classes can be found in [3, 9, 10, 24–26, 31, 34, 39].

Recently, Kanas et al [18] introduced novel subclasses $ST_{\mathcal{L}}(s)$ and $CV_{\mathcal{L}}(s)$ of S_{ST} and C_{CV}, respectively. Geometrically, they consist of functions $f(\varsigma) \in \mathcal{A}$ such that $\varsigma f'(\varsigma)/f(\varsigma)$ and $(\varsigma f'(\varsigma))'/f'(\varsigma)$ lie in the region bounded by the limaçon defined as

$$\partial\mathbb{L}_s(U) = \left\{ u + iv \colon [(u-1)^2 + v^2 - s^4]^2 = 4s^2[(u-1+s^2)^2 + v^2] \right\}, \ s \in [-1, 1] - \{0\} \tag{1.2}$$

as shown in Figure 1 for different values of s. $s = 0.35, 0.5, 0.6, 0.71, 0.75$ and 1 corresponds to blue, red, green, gray, yellow and black. Some novel properties of these classes were derived in [18].

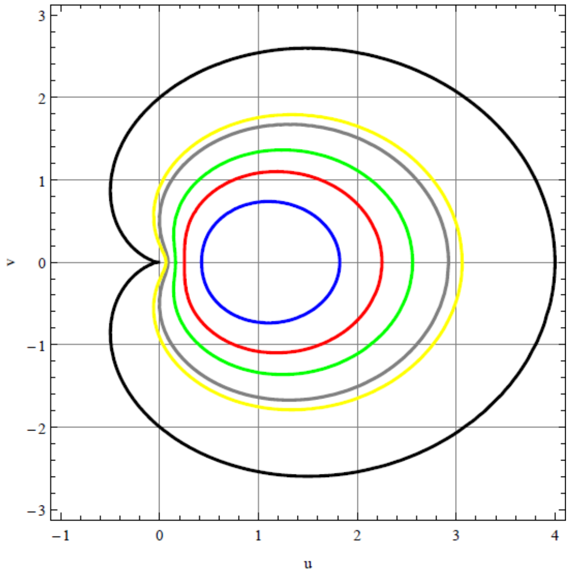

Figure 1. Image representing $\partial\mathbb{L}_s(U)$ for different values of s.

Motivated by this present work and other aforementioned articles, the goal in this paper is to continue with the investigation of some interesting properties of the classes $ST_{\mathcal{L}}(s)$ and $CV_{\mathcal{L}}(s)$. To this end, the sharp bounds of the Hankel determinant, subordination conditions as well as some radius results for these novel classes are investigated.

2. Materials and method

To put our investigations in a clear perspective, some preliminaries and definitions are presented as follows:

Denoted by $\mathcal{W}$ is the class of analytic functions

$$w(\varsigma) = \sum_{n=1}^{\infty} w_n \varsigma^n, \quad \varsigma \in U \tag{2.1}$$

such that $w(0) = 0$ and $|w(\varsigma)| < 1$. These functions are known as Schwarz functions. If $f(\varsigma)$ and $g(\varsigma)$ are analytic functions in U, then $f(\varsigma)$ is subordinate to $g(\varsigma)$ (written as $f(\varsigma) \prec g(\varsigma)$) if there exists a Schwarz function $w(\varsigma) \in \mathcal{W}$ such that $f(\varsigma) = g(w(\varsigma)),\ \varsigma \in U$.

Janowski [12] introduced the class $P(\mathbb{A}_{\eth}, \mathbb{B}_{\eth}),\ -1 \leq \mathbb{B}_{\eth} < \mathbb{A}_{\eth} \leq 1$ of functions $p(\varsigma)$ satisfying the subordination condition

$$p(\varsigma) \prec \frac{1 + \mathbb{A}_{\eth}\varsigma}{1 + \mathbb{B}_{\eth}\varsigma},$$

or equivalently, satisfying the inequality

$$\left| p(\varsigma) - \frac{1 - \mathbb{A}_{\eth}\mathbb{B}_{\eth} r^2}{1 - \mathbb{B}_{\eth}{}^2 r^2} \right| \leq \frac{(\mathbb{A}_{\eth} - \mathbb{B}_{\eth}) r}{1 - \mathbb{B}_{\eth}{}^2 r^2}, \quad |\varsigma| \leq r\ (0 < r < 1). \tag{2.2}$$

As a special cases, $P(1, -1) \equiv P$ and $P(1 - 2\beta, -1) \equiv P(\beta)\,(0 \leq \beta < 1)$ are the classes of functions of positive real part and that whose real part is greater than β, respectively (see [7]).

Definition 2.1. *Noonan and Thomas [22] defined for $q \geq 1, n \geq 1$, the qth Hankel determinant of $f(\varsigma) \in S_{1-1}$ of the form* (1.1) *as follows*:

$$\mathcal{H}_q(n) = \begin{vmatrix} \delta_n & \delta_{n+1} & \dots & \delta_{n+q-1} \\ \delta_{n+1} & \delta_{n+2} & \dots & \delta_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ \delta_{n+q-1} & \delta_{n+q-2} & \dots & \delta_{n+2q-2} \end{vmatrix} \tag{2.3}$$

This determinant has been studied by many researchers. In particular Babalola [2] obtained the sharp bounds of $\mathcal{H}_3(1)$ for the classes S_{ST} and C_{CV}. By this definition, $\mathcal{H}_3(1)$ is given as:

$$\begin{aligned} \mathcal{H}_3(1) &= \begin{vmatrix} \delta_1 & \delta_2 & \delta_3 \\ \delta_2 & \delta_3 & \delta_4 \\ \delta_3 & \delta_4 & \delta_5 \end{vmatrix} \\ &= \delta_3(\delta_2\delta_4 - \delta_3^2) - \delta_4(\delta_4 - \delta_2\delta_3) + \delta_5(\delta_3 - \delta_2^2), \quad \delta_1 = 1, \end{aligned}$$

and the by triangle inequality,

$$|\mathcal{H}_3(1)| \leq |\delta_3|\,|\delta_2\delta_4 - \delta_3^2| + |\delta_4|\,|\delta_4 - \delta_2\delta_3| + |\delta_5|\,|\delta_3 - \delta_2^2|. \tag{2.4}$$

Clearly, one can see that $\mathcal{H}_2(1) = |\delta_3 - \delta_2^2|$ is a particular instance of the well-known Fekete Szegö functional $|\delta_3 - \mu\delta_2^2|$, where μ is a real number.

Definition 2.2. *[18] Let $p(\varsigma) = 1 + \sum_{n=1}^{\infty} c_n\varsigma^n$. Then $p \in P(\mathbb{L}_s)$ if and only if*

$$p(\varsigma) \prec (1 + s\varsigma)^2, \quad 0 < s \leq \frac{1}{\sqrt{2}}, \ \varsigma \in U,$$

or equivalently, if $p(\varsigma)$ satisfies the inequality

$$|p(\varsigma) - 1| < 1 - (1 - s)^2.$$

Demonstrated in [18], was the inclusion relation

$$\{w \in \mathbb{C} \colon |w - 1| < 1 - (1 - s)^2\} \subset \mathbb{L}_s(U) \subset \{w \in \mathbb{C} \colon |w - 1| < (1 + s)^2 - 1\}. \tag{2.5}$$

It is worthy of note that the function $\mathbb{L}_s(\varsigma) = (1 + s\varsigma)^2$ is the analytic characterization of $\mathbb{L}_s(U)$ given by (1.2). Also, $\mathbb{L}_s(\varsigma)$ is starlike and convex univalent in U for $0 < s \leq \frac{1}{2}$. Furthermore, $\mathbb{L}_s(\varsigma) \in P(\beta)$, where $\beta = (1 - s)^2, \ 0 < s \leq \frac{1}{2}$.

Definition 2.3. *Let $f \in \mathcal{A}$. Then $f \in ST_{\mathcal{L}}(s)$ if and only if*

$$\frac{\varsigma f'(\varsigma)}{f(\varsigma)} \in P(\mathbb{L}_s), \quad 0 < s \leq \frac{1}{\sqrt{2}}.$$

Also, $f \in CV_{\mathcal{L}}(s)$ if and only if

$$zf'(\varsigma) \in ST_{\mathcal{L}}(s), \quad 0 < s \leq \frac{1}{\sqrt{2}}.$$

Moreover, the integral representation of functions $f \in ST_{\mathcal{L}}(s)$ is given as

$$f(\varsigma) = \varsigma \exp\left(\int_0^{\varsigma} \frac{p(t)-1}{t} dt\right), \quad p \in P(\mathbb{L}_s),$$

while that of $g \in CV_{\mathcal{L}}(s)$ is given as

$$g(\varsigma) = \int_0^{\varsigma} \frac{f(t)}{t} dt, \quad f \in ST_{\mathcal{L}}(s).$$

Furthermore, the extremal functions for each of the classes are given by

$$\begin{aligned} \Psi_{s,n}(\varsigma) &= \varsigma \exp\left(\int_0^{z} \frac{\mathbb{L}_s(t^n)-1}{t} dt\right), \quad \Psi_{s,n}(\varsigma) \in ST_{\mathcal{L}}(s) \\ &= \varsigma \exp\left(\frac{2s}{n}\varsigma^n + \frac{s^2}{2n}\varsigma^{2n}\right), \quad n = 1,2,3,\ldots \\ &= \varsigma + \frac{2s}{n}\varsigma^{n+1} + \frac{(n+4)s^2}{2n^2}\varsigma^{2n+1} + \ldots . \end{aligned} \tag{2.6}$$

and for $K_{s,n}(\varsigma) \in CV_{\mathcal{L}}(s)$,

$$\begin{aligned} K_{s,n}(\varsigma) &= \int_0^{\varsigma} \frac{\Psi_{s,n}(t)}{t} dt, \quad \Psi_{s,n}(\varsigma) \in ST_{\mathcal{L}}(s), \quad n = 1,2,3,\ldots \\ &= \varsigma + \frac{2s}{n(n+1)}\varsigma^{n+1} + \ldots . \end{aligned} \tag{2.7}$$

3. A set of lemmas

Lemma 3.1. *[1] If $w \in \mathcal{W}$ is of the form* (2.1)*, then for a real number σ,*

$$|w_2 - \sigma w_1^2| \le \begin{cases} -\sigma, & \text{for } \sigma \le -1, \\ 1, & \text{for } -1 \le \sigma \le 1, \\ \sigma & \text{for } \sigma \ge 1. \end{cases}$$

When $\sigma < -1$ or $\sigma > 1$, equality holds if and only if $w(\varsigma) = \varsigma$ or one of its rotations. If $-1 < \sigma < 1$, then equality holds if and only if $w(\varsigma) = \varsigma^2$ or one of its rotations. Equality holds for $\sigma = -1$ if and only if $w(\varsigma) = \frac{\varsigma(x+\varsigma)}{1+x\varsigma}$ $(0 \le x \le 1)$ or one of its rotations while for $\sigma = 1$, equality holds if and only if $w(\varsigma) = -\frac{\varsigma(x+\varsigma)}{1+x\varsigma}$ $(0 \le x \le 1)$ or one of its rotations.

Also, the sharp upper bound above can be improved as follows when $-1 \le \sigma \le 1$:

$$|w_2 - \sigma w_1^2| + (1+\sigma)|w_1|^2 \le 1 \ \ (-1 < \sigma \le 0)$$

and

$$|w_2 - \sigma w_1^2| + (1-\sigma)|w_1|^2 \le 1 \ \ (0 < \sigma < 1).$$

Lemma 3.2. *[16] If $w \in \mathcal{W}$ is of the form* (2.1)*, then for some complex numbers ξ and η such that $|\xi| \leq 1$ and $|\eta| \leq 1$,*

$$w_2 = \xi(1 - w_1^2)$$

and

$$w_3 = (1 - w_1^2)(1 - |\xi|^2)\eta - w_1(1 - w_1^2)\xi^2.$$

Lemma 3.3. *[19, Theorem 3.4h, p. 132] Let $q(\varsigma)$ be univalent in U and let θ and φ be analytic in a domain D containing $q(U)$ with $\varphi(\omega) \neq 0$, when $\omega \in q(U)$. Set*
$Q(\varsigma) = \varsigma q'(\varsigma) \cdot \varphi(q(\varsigma))$, $h(\varsigma) = \theta(q(\varsigma)) + Q(\varsigma)$, and suppose that either

(i) $h(\varsigma)$ is convex, or $Q(\varsigma)$ is starlike,
(ii)

$$\mathrm{Re}\frac{\varsigma h'(\varsigma)}{Q(\varsigma)} = \mathrm{Re}\left(\frac{\theta'(q(\varsigma))}{\varphi(q(\varsigma))} + \frac{\varsigma Q'(\varsigma)}{Q(\varsigma)}\right) > 0.$$

If $p(\varsigma)$ is analytic in U with $p(0) = q(0)$, $p(U) \subset D$ and

$$\theta(p(\varsigma)) + \varsigma p'(\varsigma)\varphi(p(\varsigma)) \prec \theta(q(\varsigma)) + \varsigma q'(\varsigma)\varphi(q(\varsigma)) = h(\varsigma), \tag{3.1}$$

then $p(\varsigma) \prec q(\varsigma)$, and $q(\varsigma)$ is the best dominant in the sense that $p \prec t \Rightarrow q \prec t$ for all t.

Lemma 3.4. *[29] Let $h(\varsigma) = 1 + \sum_{n=1}^{\infty} c_n \varsigma^n$, $G(\varsigma) = 1 + \sum_{n=1}^{\infty} d_n \varsigma^n$ and $h(\varsigma) \prec G(\varsigma)$. If $G(\varsigma)$ is univalent in U and $G(U)$ is convex, then $|c_n| \leq |d_1|$, for all $n \geq 1$.*

Lemma 3.5. *[11] Let $w \in \mathcal{W}$. If $|w(\varsigma)|$ attains its maximum value on the circle $|\varsigma| = r$ at a point $\varsigma_0 \in U$, then we have $\varsigma_0 w'(\varsigma_0) = k w(\varsigma_0)$, where $k \geq 1$.*

Throughout this work $f(\varsigma)$ is taken to be of the form (1.1) while $w(\varsigma)$ is of the form (2.1). In the next sections, the main results are presented.

4. Coefficient results

In this section, we assume $0 < s \leq \frac{1}{2}$. First, we establish a few auxiliary results whose applications will be needed hereafter.

Lemma 4.1. *Let $f \in ST_{\mathcal{L}}(s)$. Then*

$$|\delta_n| \leq \frac{(2s)_{n-1}}{(n-1)!}, \quad n \geq 2. \tag{4.1}$$

Proof. From the definition of $f \in ST_{\mathcal{L}}(s)$, we have

$$\frac{\varsigma f'(\varsigma)}{f(\varsigma)} = p(\varsigma), \quad p \in P(\mathbb{L}_s), \tag{4.2}$$

where

$$p(\varsigma) := 1 + c_1\varsigma + c_2\varsigma^2 + c_3\varsigma^3 + \ldots. \tag{4.3}$$

Comparing the coefficients of ς^n in (4.2), it follows that

$$(n-1)\delta_n = c_{n-1} + \delta_2 c_{n-2} + \delta_3 c_{n-3} + \cdots + \delta_{n-1} c_1. \tag{4.4}$$

It is obvious from Lemma 3.4 and the fact that $\mathbb{L}_s(\varsigma)$ is convex for $0 < s \leq \frac{1}{2}$ that

$$|c_n| \leq 2s, \quad n \geq 1. \tag{4.5}$$

Using this result in (4.4), we obtain

$$|\delta_n| \leq \frac{2s}{n-1} \sum_{m=1}^{n-1} |\delta_m|, \quad \delta_1 = 1,\ n \geq 2. \tag{4.6}$$

We need to show (4.1) by Mathematical induction. For this reason, assume (4.1) is true and proceed to prove

$$|\delta_{n+1}| \leq \frac{(2s)_n}{(n)!}, \quad n \geq 2.$$

From (4.6),

$$\begin{aligned} |\delta_2| =& |c_1| \leq (2s)_1 , \\ |\delta_3| \leq& \frac{2s}{2}(1 + |\delta_2|) \leq \frac{(2s)_2}{2!}, \\ |\delta_4| \leq& \frac{2s}{2}(1 + |\delta_2| + |\delta_3|) \leq \frac{(2s)_3}{3!}, \end{aligned}$$

and finally,

$$\begin{aligned} |\delta_{n+1}| \leq& \frac{2s}{n}\left(1 + (2s)_1 + \frac{(2s)_2}{2!} + \frac{(2s)_3}{3!} + \cdots + \frac{(2s)_{n-1}}{(n-1)!}\right) \\ =& \frac{(2s)_n}{n!}. \end{aligned}$$

Therefore,

$$|\delta_{n+1}| \leq \frac{(2s)_n}{(n)!}, \quad n \geq 2.$$

Hence, by Mathematical induction, we have the desired result. □

In view of Theorem 4.1 and the definition of functions in $CV_{\mathcal{L}}(s)$, we are led to the following result.

Lemma 4.2. *Let $f \in CV_{\mathcal{L}}(s)$. Then*

$$|\delta_n| \leq \frac{(2s)_{n-1}}{(n)!}, \quad n \geq 2.$$

Lemma 4.3. *Let $f \in ST_{\mathcal{L}}(s)$. Then*

$$|\delta_2\delta_4 - \delta_3^2| \leq \frac{19s^4}{12}.$$

The bound $\frac{19s^4}{12}$ is sharp for the function

$$\Psi_{s,1}(\varsigma) = \varsigma + 2s\varsigma^2 + \frac{5}{2}s^2\varsigma^3 + \frac{7}{3}s^3\varsigma^4 + \ldots . \tag{4.7}$$

Proof. For $f \in \mathcal{ST}_{\mathcal{L}}(s)$,

$$\frac{\varsigma f'(\varsigma)}{f(\varsigma)} = (1 + sw(\varsigma))^2, \tag{4.8}$$

where $w \in \mathcal{W}$. Comparing coefficients of ς, ς^2 and ς^3 in (4.8), we arrive at

$$\delta_2 = 2sw_1, \quad \delta_3 = s\left(w_2 + \frac{5}{2}sw_1^2\right) \quad \text{and} \quad \delta_4 = \frac{2}{3}sw_3 + \frac{8}{3}s^2w_1w_2 + \frac{7}{3}s^3w_1^3. \tag{4.9}$$

By Lemma 3.1, we obtain

$$|\delta_2\delta_4 - \delta_3^2| = \frac{4s^2}{3}\left|w_1(1-w_1^2)(1-|\xi|^2)\eta - (1-w_1^2)(3-w_1^2)\xi^2 + \frac{1}{4}sw_1^2\xi(1-w_1^2) - \frac{19}{16}s^2w_1^4\right|.$$

Let $x = w_1$, $\xi = y$ with $0 \leq x \leq 1$ and $|y| \leq 1$. Then the triangle inequality gives

$$|\delta_2\delta_4 - \delta_3^2| \leq \mathcal{F}(x, |y|),$$

where

$$\mathcal{F}(x, |y|) = \frac{4s^2}{3}\left(x(1-x^2)(1-|y|^2) + (1-x^2)(3-x^2)|y|^2 + \frac{1}{4}sx^2|y|(1-x^2) + \frac{19}{16}s^2x^4\right),$$

and

$$\frac{\partial F}{\partial |y|} = \frac{4s^2}{3}\left(2(1-x^2)(3-x^2-x)|y| + \frac{1}{4}sx^2(1-x^2)\right) > 0.$$

This means that $\mathcal{F}(x, |y|)$ is increasing on the interval $[0, 1]$. So,

$$\begin{aligned}\mathcal{F}(x, |y|) \leq& \frac{4s^2}{3}\left((1-x^2)(12 + x^2(s-4)) + \frac{19}{16}s^2x^4\right)\\ :=& \mathcal{F}(x),\end{aligned}$$

where

$$\mathcal{F}'(x) = 2x(s+8) + 19s^2x^3 > 0,$$

which implies that $\mathcal{F}(x)$ is an increasing function of x on $[0, 1]$. Consequently,

$$|\delta_2\delta_4 - \delta_3^2| \leq \frac{4s^2}{3}\mathcal{F}(1) = \frac{19s^4}{12}.$$

□

Lemma 4.4. *Let $f \in \mathcal{CV}_{\mathcal{L}}(s)$. Then*

$$|\delta_2\delta_4 - \delta_3^2| \leq \frac{s^2}{9}.$$

The bound $\frac{s^2}{9}$ is sharp for the function

$$K_{s,2}(\varsigma) = \varsigma + \frac{s}{3}\varsigma^3 + \frac{3s^2}{20}\varsigma^5 + \dots. \tag{4.10}$$

Proof. From the definition of $f \in CV_{\mathcal{L}}(s)$ and (4.9), it is easy to see that

$$|\delta_2\delta_4 - \delta_3^2| = \left|\frac{1}{6}s^2 w_1 w_3 + \frac{1}{9}s^3 w_1^2 w_2 - \frac{1}{9}s^2 w_2^2 - \frac{1}{9}s^4 w_1^4\right|.$$

The rest of the proof follows as in Theorem 4.3. □

Lemma 4.5. *Let $f \in ST_{\mathcal{L}}(s)$. Then*

$$|\delta_2\delta_3 - \delta_4| \leq \frac{2s}{3}.$$

The bound $\frac{2s}{3}$ is best possible for the function

$$\Psi_{s,3}(\varsigma) = \varsigma + \frac{2}{3}s\varsigma^4 + \frac{7}{18}s^2\varsigma^7 + \dots . \tag{4.11}$$

Proof. From (4.9), a computation gives

$$\delta_2\delta_3 - \delta_4 = 2s^2 w_1\left(w_2 - \frac{7}{6}sw_1^2\right) - \frac{8}{3}s^2 w_1\left(w_2 - \frac{15}{8}sw_1^2\right) - \frac{2}{3}sw_3.$$

Employing Lemma 3.2, we write the expression for w_3, and applying the triangle inequality together with Lemma 3.1, we obtain

$$|\delta_2\delta_3 - \delta_4| \leq \frac{14s^2}{3} + \frac{2s}{3}\left[(1-x^2)(1-|y|^2) + x(1-x^2)|y|^2\right], \tag{4.12}$$

where we have taken $w_1 = x$, $\xi = y$ with $0 \leq x \leq 1$ and $|y| \leq 1$. Let $\mathcal{H}(x,|y|)$ represents the right side of (4.12). Then

$$\frac{\partial\mathcal{H}(x,|y|)}{\partial|y|} = -\frac{4s}{3}(1-x^2)(1-x) \leq 0.$$

Thus,

$$\mathcal{H}(x,|y|) \leq \mathcal{H}(x,0) := \mathcal{H}(x),$$

where

$$\mathcal{H}(x) = \frac{14s^2 x}{3} + \frac{2(1-x^2)s}{3} \quad \text{and} \quad \mathcal{H}'(x) = \frac{2s}{3}(7s - 2x).$$

It is clear that $\mathcal{H}(x)$ attains its maximum value at $x = \frac{7s}{2}$. Thus, $\mathcal{H}(x) \leq \mathcal{H}\left(\frac{7s}{2}\right) = \frac{2s}{3}$. Consequently,

$$|\delta_2\delta_3 - \delta_4| \leq \frac{2s}{3}.$$

□

Lemma 4.6. *Let $f \in CV_{\mathcal{L}}(s)$. Then*

$$|\delta_2\delta_3 - \delta_4| \leq \frac{s}{6}.$$

This bound cannot be improved since the function

$$K_{s,3}(\varsigma) = \varsigma + \frac{1}{6}s\varsigma^4 + \frac{1}{18}s^2\varsigma^7 + \dots \tag{4.13}$$

attains the equality.

Proof. Using the definition of $f \in CV_{\mathcal{L}}(s)$ and (4.9), we find

$$\delta_2\delta_3 - \delta_4 = \frac{1}{4}s^3w_1^3 - \frac{1}{3}s^2w_1w_2 - \frac{1}{6}sw_3.$$

Let $w_1 = x\,(0 \le x < 1)$ and $\xi = y$ with $|y| \le 1$. Then applying Lemma 3.2 and following the procedure of proof as in Theorem 4.5, we arrive at the desired result. □

5. Fekete Szegö inequality for the classes $ST_{\mathcal{L}}(s)$ and $CV_{\mathcal{L}}(s)$

Lemma 5.1. *Let $f \in ST_{\mathcal{L}}(s)$. Then for a real number μ,*

$$|\delta_3 - \mu\delta_2^2| \le \begin{cases} \frac{s^2(5-8\mu)}{2}, & \text{for } \mu \le \frac{5s-2}{8s}, \\ s, & \text{for } \frac{5s-2}{8s} \le \mu \le \frac{5s+2}{8s}, \\ \frac{s^2(8\mu-5)}{2}, & \text{for } \mu \ge \frac{5s+2}{8s}. \end{cases}$$

It is asserted also that

$$|\delta_3 - \mu\delta_2^2| + \left(\mu - \frac{5s-2}{8s}\right)|\delta_2|^2 \le s\,, \quad \frac{5s+2}{8s} < \mu \le \frac{5}{8}$$

and

$$|\delta_3 - \mu\delta_2^2| - \left(\mu - \frac{5s+2}{8s}\right)|\delta_2|^2 \le s\,, \quad \frac{5}{8} < \mu < \frac{5s+2}{8s}.$$

These inequalities are sharp for the functions

$$\begin{cases} \overline{\lambda}\Psi_{s,1}(\lambda\varsigma), & \text{for } \mu \in (-\infty, \frac{5s-2}{8s}) \cup (\frac{5s+2}{8s}, \infty), \\ \overline{\lambda}\Psi_{s,2}(\lambda\varsigma), & \text{for } \frac{5s-2}{8s} \le \mu \le \frac{5s+2}{8s}, \\ \overline{\lambda}\mathcal{P}_x(\lambda\varsigma), & \text{for } \mu = \frac{5s-2}{8s}, \\ \overline{\lambda}Q_x(\lambda\varsigma), & \text{for } \mu = \frac{5s+2}{8s}, \end{cases}$$

where $|\lambda| = 1$ and

$$\frac{\varsigma\mathcal{P}_x'(\varsigma)}{\mathcal{P}_x(\varsigma)} = \mathbb{L}_s\left(\frac{\varsigma(x+\varsigma)}{1+x\varsigma}\right), \quad \frac{\varsigma Q_x'(\varsigma)}{Q_x(\varsigma)} = \mathbb{L}_s\left(-\frac{\varsigma(x+\varsigma)}{1+x\varsigma}\right), \quad 0 \le x \le 1.$$

Proof. From (4.9), we have

$$\left|\delta_3 - \delta_2^2\right| = s\left|w_2 - \frac{s(8\mu-5)}{2}w_1^2\right|.$$

Then using Lemma 3.1 with $\sigma = \frac{s(8\mu-5)}{2}$, we obtain the required result. □

For $\mu = 1$ in Theorem 5.1, we deduce the following sharp result.

Corollary 5.1. *Let $f \in ST_{\mathcal{L}}(s)$. Then*

$$|\delta_3 - \delta_2^2| \le s.$$

Lemma 5.2. *Let $f \in CV_{\mathcal{L}}(s)$. Then for a real number μ,*

$$|\delta_3 - \mu\delta_2^2| \leq \begin{cases} \frac{s^2(5-6\mu)}{6}, & \text{for } \mu \leq \frac{5s-2}{6s}, \\ \frac{s}{3}, & \text{for } \frac{5s-2}{6s} \leq \mu \leq \frac{5s+2}{6s}, \\ \frac{s^2(6\mu-5)}{6}, & \text{for } \mu \geq \frac{5s+2}{6s}. \end{cases}$$

It is asserted also that

$$|\delta_3 - \mu\delta_2^2| + \left(\mu - \frac{5s-2}{6s}\right)|\delta_2|^2 \leq \frac{s}{3}, \quad \frac{5s-2}{6s} < \mu \leq \frac{5}{6}$$

and

$$|\delta_3 - \mu\delta_2^2| - \left(\mu - \frac{5s+2}{6s}\right)|\delta_2|^2 \leq \frac{s}{3}, \quad \frac{5}{6} < \mu < \frac{5s+2}{6s}.$$

These inequalities are sharp for the functions

$$\begin{cases} \overline{\lambda}K_{s,1}(\lambda\varsigma), & \text{for } \mu \in (-\infty, \frac{5s-2}{6s}) \cup (\frac{5s+2}{6s}, \infty), \\ \overline{\lambda}K_{s,2}(\lambda\varsigma), & \text{for } \frac{5s-2}{6s} \leq \mu \leq \frac{5s+2}{6s}, \\ \overline{\lambda}\mathcal{P}_x(\lambda\varsigma), & \text{for } \mu = \frac{5s-2}{6s}, \\ \overline{\lambda}\mathcal{Q}_x(\lambda\varsigma), & \text{for } \mu = \frac{5s+2}{6s}, \end{cases}$$

where $|\lambda| = 1$ and

$$\frac{(\varsigma\mathcal{P}_x'(\varsigma))'}{\mathcal{P}_x'(\varsigma)} = \mathbb{L}_s\left(\frac{\varsigma(x+\varsigma)}{1+x\varsigma}\right), \quad \frac{(\varsigma\mathcal{Q}_x'(\varsigma))'}{\mathcal{Q}_x'(\varsigma)} = \mathbb{L}_s\left(-\frac{\varsigma(x+\varsigma)}{1+x\varsigma}\right), \quad 0 \leq x \leq 1.$$

Proof. Using the definition of $CV_{\mathcal{L}}(s)$ and (4.9), we get

$$\left|\delta_3 - \delta_2^2\right| = \frac{s}{3}\left|w_2 - \frac{s(6\mu-5)}{2}w_1^2\right|.$$

Then using Lemma 3.1 with $\sigma = \frac{s(6\mu-5)}{2}$, we obtain the desired result. □

For $\mu = 1$ in Theorem 5.2, we deduce the following sharp result.

Corollary 5.2. *Let $f \in CV_{\mathcal{L}}(s)$. Then*

$$|\delta_3 - \delta_2^2| \leq \frac{s}{3}.$$

Theorem 5.3. *Let $f \in ST_{\mathcal{L}}(s)$. Then*

$$|\mathcal{H}_3(1)| \leq \frac{s^2}{36}(2s+1)(57s^3 + 12s^2 + 46s + 34)$$

Proof. The proof is immediate from (2.4), Lemma 4.1, Lemma 4.3, Lemma 4.5 and Corollary 5.1. □

Theorem 5.4. *Let $f \in CV_{\mathcal{L}}(s)$. Then*

$$|\mathcal{H}_3(1)| \leq \frac{s^2}{540}(2s+1)(12s^2 + 6s + 33)$$

Proof. The proof is straightforward from (2.4), Lemma 4.2, Lemma 4.4, Lemma 4.6 and Corollary 5.2. □

6. $ST_{\mathcal{L}}(s)$ and $CV_{\mathcal{L}}(s)$ radii

Theorem 6.1. *The $CV_{\mathcal{L}}(s)$-radius for the class $S_{ST}(\beta)$ (where $\beta = (1-s)^2$) is given by*

$$R_1 = \begin{cases} \frac{1}{2+2s-s^2+\sqrt{s^4-4s^3+6s^2-4s+7}}, & \text{for } s \neq \frac{\sqrt{2}-1}{\sqrt{2}}, \\ \frac{1}{5}, & \text{for } s = \frac{\sqrt{2}-1}{\sqrt{2}}. \end{cases} \tag{6.1}$$

This radius is sharp for the functions given by

$$f_0(\varsigma) = \begin{cases} \frac{z}{(1-z)^{2s(2-s)}}, & \text{for } s \neq \frac{\sqrt{2}-1}{\sqrt{2}}, \\ \frac{z}{1-z}, & \text{for } s = \frac{\sqrt{2}-1}{\sqrt{2}}. \end{cases} \tag{6.2}$$

Proof. Let $f \in S_{ST}(\beta)$. Then

$$\frac{\varsigma f'(\varsigma)}{f(\varsigma)} \in P(\beta), \quad \beta = (1-s)^2.$$

This means that there exists $w \in \mathcal{W}$ such that

$$\frac{\varsigma f'(\varsigma)}{f(\varsigma)} = 1 + \frac{2(1-\beta)w(\varsigma)}{1-w(\varsigma)},$$

Let

$$\frac{\varsigma f'(\varsigma)}{f(\varsigma)} = p(\varsigma). \tag{6.3}$$

Then by Schwarz lemma,

$$|p(\varsigma) - 1| \leq \frac{2(1-\beta)r}{1-r}. \tag{6.4}$$

It follows from logarithmic differentiation of (6.3) that

$$\left|\frac{(\varsigma f'(\varsigma))'}{f'(\varsigma)} - 1\right| \leq |p(\varsigma) - 1| + \left|\frac{\varsigma p'(\varsigma)}{p(\varsigma)}\right|.$$

It is known from [27] that for $p \in P(\beta)$,

$$\left|\frac{\varsigma p'(\varsigma)}{p(\varsigma)}\right| \leq \frac{2(1-\beta)r}{(1-r)(1+(1-2\beta)r)}.$$

Using this result together with (6.4), we write

$$\left|\frac{(\varsigma f'(\varsigma))'}{f'(\varsigma)} - 1\right| \leq \frac{2r(1-\beta)(2+(1-2\beta)r)}{(1-r)(1+(1-2\beta)r)}.$$

We need to show that

$$\left|\frac{(\varsigma f'(\varsigma))'}{f'(\varsigma)} - 1\right| \leq 1-\beta.$$

However, it holds if

$$3(1-2\beta)r^2 + 2(2+\beta)r - 1 \leq 0.$$

Let $\mathcal{T}(r) = 3(1-2\beta)r^2 + 2(2+\beta)r - 1$. Then $\mathcal{T}(0) = -1 < 0$ and $T(1) = 6 - 4\beta > 0$ such that $\mathcal{T}(0)\mathcal{T}(1) < 0$. Thus, there exists $R_1 \in [0,1]$ such that

$$3(1-2\beta)R_1^2 + 2(2+\beta)R_1 - 1 = 0. \tag{6.5}$$

Therefore, $3(1-2\beta)r^2 + 2(2+\beta)r - 1 \leq 0$ for all $r < R_1$ and R_1 is the smallest roots of (6.5) given by (6.1).

For sharpness, we consider the functions $f_0(\varsigma)$ defined by (6.2). At the point $z = R_1$, we have

$$\left|\frac{(\varsigma f_0'(\varsigma))'}{f_0'(\varsigma)} - 1\right| = 1 - (1-s)^2.$$

□

Theorem 6.2. *Let $f \in ST_{\mathcal{L}}(s)$. Then $f \in C_{CV}$ for all z in the disc $|\varsigma| < R_2$, where R_2 is the positive roots of the equation*

$$s^3x^5 - 3s^2x^4 + (3-s^2)sx^3 - (1-3s^2)x^2 - (3s+2)x + 1 = 0. \tag{6.6}$$

Proof. For $f \in ST_{\mathcal{L}}(s)$,

$$\frac{\varsigma f'(\varsigma)}{f(\varsigma)} = (1 + sw(\varsigma))^2,$$

where $w \in \mathcal{W}$. Therefore

$$\begin{aligned}\mathrm{Re}\frac{(\varsigma f'(\varsigma))'}{f'(\varsigma)} &\geq \mathrm{Re}(1+sw(\varsigma))^2 - 2r\left|\frac{w'(\varsigma)}{1+sw(\varsigma)}\right| \\ &\geq (1-sr)^2 - \frac{2r(1-|w(\varsigma)|^2)}{(1-r^2)(1-sr)},\end{aligned}$$

where we have used the extension of Schwarz lemma (see [21]). Thus,

$$\mathrm{Re}\frac{(\varsigma f'(\varsigma))'}{f'(\varsigma)} \geq \frac{(1-sr)^3(1-r^2) - 2r}{(1-sr)(1-r^2)} > 0$$

if $(1-sr)^3(1-r^2) - 2r > 0$. Let $\mathcal{T}(r) = (1-sr)^3(1-r^2) - 2r$. Then $\mathcal{T}(0) = 1 > 0$ and $\mathcal{T}(1) = -2 < 0$ with $\mathcal{T}(0)\mathcal{T}(1) < 0$. Therefore, there exists $R_2 \in [0,1]$ such that $(1-sR_2)^3(1-R_2^2) - 2R_2 = 0$. Hence, $(1-sr)^3(1-r^2) - 2r > 0$ holds for all $r < R_2$, where R_2 is the smallest positive roots of (6.6). □

Theorem 6.3. *Let $p \in P$. Then $p \in P(\mathbb{L}_s)$ for all z in the disc*

$$|\varsigma| < \frac{2s - s^2}{2 + 2s - s^2}, \quad 0 < s \leq \frac{\sqrt{2}}{2}. \tag{6.7}$$

Proof. Let $p \in P$. Then

$$\left|p(\varsigma) - \frac{1+r^2}{1-r^2}\right| < \frac{2r}{1-r^2}, \quad r \in (0,1).$$

We want to prove that

$$|p(\varsigma) - 1| \leq 1 - (1-s)^2.$$

Now,

$$
\begin{aligned}
|p(\varsigma)-1| \le & \left|p(\varsigma)-\frac{1+r^2}{1-r^2}\right|+\frac{2r^2}{1-r^2}\\
\le & \frac{2r}{1-r^2}+\frac{2r^2}{1-r^2}\\
= & \frac{2r}{1-r}\\
< & 1-(1-s)^2
\end{aligned}
$$

if (6.7) is satisfied. To show that the radius cannot be improved, we consider the function

$$p_0(\varsigma)=\frac{1+\varsigma}{1-\varsigma}.$$

Then for $\varsigma=\frac{2s-s^2}{2+2s-s^2}$,

$$\left|p_0(\varsigma)-1\right|=\left|\frac{2\varsigma}{1-\varsigma}\right|=2s-s^2,$$

which shows that equality is attained for (6.7). □

Corollary 6.1. *The $ST_{\mathcal{L}}(s)$-radius and $CV_{\mathcal{L}}(s)$-radius for the classes of starlike and convex functions are given by* (6.7).

7. Some properties of the function $\mathbb{L}_s(\varsigma)$

Theorem 7.1. *If $p(\varsigma)$ is analytic in U with $p(0)=1$ and satisfies the condition*

$$\operatorname{Re}\left(\frac{\varsigma p'(\varsigma)}{p(\varsigma)}\right)<\frac{2s}{s-1}, \tag{7.1}$$

or

$$\operatorname{Re}\left(\frac{\varsigma p'(\varsigma)}{p(\varsigma)}\right)>\frac{2s}{s+1}, \tag{7.2}$$

then $p(\varsigma)\prec(1+s\varsigma)^2$ for $-1<s<0$.

Proof. Let $p(\varsigma)$ be defined by

$$p(\varsigma)=(1+sw(\varsigma))^2 \tag{7.3}$$

Clearly, $w(\varsigma)$ is analytic in U with $w(0)=0$. To prove our result, it is required to show that $|w(\varsigma)|<1$ for all $\varsigma\in U$. From (7.3), a simple calculation gives

$$\frac{\varsigma p'(\varsigma)}{p(\varsigma)}=\frac{2s\varsigma w'(\varsigma)}{1+sw(\varsigma)}.$$

Suppose there exists a point $\varsigma_0\in U$ such that

$$\max_{|\varsigma|\le|\varsigma_0|}|w(\varsigma)|=|w(\varsigma_0)|=1.$$

Then by Lemma 3.5, $w(\varsigma_0) = e^{i\theta}$ and $\varsigma_0 w'(\varsigma_0) = kw(\varsigma_0)$. Thus,

$$\begin{aligned}\mathrm{Re}\left(\frac{\varsigma_0 p'(\varsigma_0)}{p(\varsigma_0)}\right) =& \mathrm{Re}\left(\frac{2s\varsigma_0 w'(\varsigma_0)}{1+sw(\varsigma_0)}\right)\\ =& 2k\left(1-\mathrm{Re}\left(\frac{1}{1+se^{i\theta}}\right)\right)\\ >& -2k\left(\frac{s}{1-s}\right)\\ \geq& \frac{2s}{s-1}.\end{aligned}$$

This contradicts (7.1). Therefore, there exists no $\varsigma_0 \in U$ such that $|w(\varsigma_0)| = 1$. Thus $|w(\varsigma)| < 1$ in U, so that $p(\varsigma) \prec (1+s\varsigma)^2$ for $-1 < s < 0$.

Similarly,

$$\begin{aligned}\mathrm{Re}\left(\frac{\varsigma_0 p'(\varsigma_0)}{p(\varsigma_0)}\right) =& 2k\left(1-\mathrm{Re}\left(\frac{1}{1+se^{i\theta}}\right)\right)\\ <& 2k\left(\frac{s}{1+s}\right)\\ \leq& \frac{2s}{s+1},\end{aligned}$$

which contradicts the assumption (7.2). Hence, the proof is completed. □

Following the discussion demonstrated by Sharma et al in [35] for Theorem 3, we present the following results.

Theorem 7.2. *Let* $-1 < \mathbb{B}_{\eth} < \mathbb{A}_{\eth} \leq 1$, $0 < s \leq \frac{1}{\sqrt{2}}$ *and* $p(\varsigma) = \frac{1+\mathbb{A}_{\eth}\varsigma}{1+\mathbb{B}_{\eth}\varsigma}$. *Then* $p \in P(\mathbb{L}_s)$ *if and only if*

$$1-2s+s^2 \leq \frac{1-\mathbb{A}_{\eth}}{1-\mathbb{B}_{\eth}} \leq \frac{1+\mathbb{A}_{\eth}}{1+\mathbb{B}_{\eth}} \leq 1+2s-s^2 \tag{7.4}$$

or, equivalently, if and only if

$$\mathbb{A}_{\eth} \leq \begin{cases} 2s-s^2+(1-s)^2\mathbb{B}_{\eth}, & \text{for } \mathbb{B}_{\eth}(\mathbb{B}_{\eth}-\mathbb{A}_{\eth}) \leq 0,\\ 2s-s^2+(1+2s-s^2)\mathbb{B}_{\eth}, & \text{for } \mathbb{B}_{\eth}(\mathbb{B}_{\eth}-\mathbb{A}_{\eth}) \geq 0. \end{cases} \tag{7.5}$$

Proof. The proof follows the techniques presented in [35, Theorem 3] □

For $\mathbb{B}_{\eth} = 0, \mathbb{A}_{\eth} = 0$ and $\mathbb{B}_{\eth} = -\mathbb{A}_{\eth}$, we give the following consequences of Theorem 7.2.

Corollary 7.1.

(i) $p(\varsigma) = 1+\mathbb{A}_{\eth}\varsigma \in P(\mathbb{L}_s) \Longleftrightarrow 0 < \mathbb{A}_{\eth} \leq 2s-s^2$.
(ii) $p(\varsigma) = 1/(1+\mathbb{B}_{\eth}\varsigma) \in P(\mathbb{L}_s) \Longleftrightarrow (s^2-2s)/(1+2s-s^2) \leq \mathbb{B}_{\eth} < 0$.
(iii) $p(\varsigma) = (1+\mathbb{A}_{\eth}\varsigma)/(1-\mathbb{A}_{\eth}\varsigma) \in P(\mathbb{L}_s) \Longleftrightarrow 0 < \mathbb{A}_{\eth} \leq (2s-s^2)/(2+2s-s^2)$.

Corollary 7.2. *Let* $-1 < \mathbb{B}_\daleth < \mathbb{A}_\daleth \leq 1$ *and consider*

$$\frac{\varsigma f'(\varsigma)}{f(\varsigma)} = \frac{1+\mathbb{A}_\daleth\varsigma}{1+\mathbb{B}_\daleth\varsigma} \quad \text{and} \quad \frac{(\varsigma f'(\varsigma))'}{f'(\varsigma)} = \frac{1+\mathbb{A}_\daleth\varsigma}{1+\mathbb{B}_\daleth\varsigma}.$$

Then $f \in ST_{\mathcal{L}}(s)$ *and* $f \in CV_{\mathcal{L}}(s)$*, respectively if and only if conditions* (7.4) *or* (7.5) *is satisfied*

Applying Corollary 7.1 along with the integral representation for the classes $ST_{\mathcal{L}}(s)$ and $CV_{\mathcal{L}}(s)$, respectively, we present the following examples.

Example 7.3.

(i) For $0 < \mathbb{A}_\daleth \leq 2s - s^2$,

$$f_1(\varsigma) = \varsigma \exp(\mathbb{A}_\daleth\varsigma) \in ST_{\mathcal{L}}(s) \quad \text{and} \quad f_2(\varsigma) = \frac{\exp(\mathbb{A}_\daleth\varsigma)-1}{\mathbb{A}_\daleth} \in CV_{\mathcal{L}}(s).$$

(ii) For $\frac{s^2-2s}{1+2s-s^2} \leq \mathbb{B}_\daleth < 0$,

$$f_3(\varsigma) = \frac{\varsigma}{1+\mathbb{B}_\daleth\varsigma} \in ST_{\mathcal{L}}(s) \quad \text{and} \quad f_4(\varsigma) = \frac{1}{\mathbb{B}_\daleth}\log(1+\mathbb{B}_\daleth\varsigma) \in CV_{\mathcal{L}}(s).$$

(iii) For $0 < \mathbb{A}_\daleth \leq \frac{2s-s^2}{2+2s-s^2}$,

$$f_5(\varsigma) = \frac{\varsigma}{(1-\mathbb{A}_\daleth\varsigma)^2} \in ST_{\mathcal{L}}(s) \quad \text{and} \quad f_6(\varsigma) = \frac{\varsigma}{1-\mathbb{A}_\daleth\varsigma} \in CV_{\mathcal{L}}(s).$$

8. Sufficient conditions and related results

Theorem 8.1. *Let* $-1 \leq \mathbb{B}_\daleth < \mathbb{A}_\daleth \leq 1,\ 0 < s \leq \frac{1}{2}$ *and* $p(\varsigma)$ *be analytic in* U *with* $p(0) = 1$ *such that*

$$1 + \rho\varsigma p'(\varsigma) \prec \frac{1+\mathbb{A}_\daleth\varsigma}{1+\mathbb{B}_\daleth\varsigma} \ \ (\rho \in \mathbb{R}\backslash\{0\}, \varsigma \in U). \tag{8.1}$$

If

$$|\rho| \geq \frac{\mathbb{A}_\daleth - \mathbb{B}_\daleth}{2s(1-s)(1-|\mathbb{B}_\daleth|)}, \tag{8.2}$$

then $p \in P(\mathbb{L}_s)$.

Proof. Let $q(\varsigma) = (1+s\varsigma)^2,\ 0 < s \leq \frac{1}{2}$. Then $q(\varsigma)$ is convex univalent in U. Consider the functions $\phi(\omega) = \rho$ and $\theta(\omega) = 1$. These functions are both analytic in a domain containing $q(U)$ with $\phi(\omega) \neq 0$. A computation shows that

$$Q(\varsigma) = \rho\varsigma q'(\varsigma) = 2\rho s\varsigma(1+s\varsigma) \quad \text{and} \quad h(\varsigma) = 1 + \rho\varsigma q'(\varsigma) = 1 + 2\rho s\varsigma(1+s\varsigma).$$

Further,

$$\operatorname{Re}\frac{\varsigma Q'(\varsigma)}{Q(\varsigma)} \geq \frac{1-2s}{1-s} > 0$$

and

$$\operatorname{Re}\frac{\varsigma h'(\varsigma)}{Q(\varsigma)} = \operatorname{Re}\frac{\varsigma Q'(\varsigma)}{Q(\varsigma)} > 0.$$

Using Lemma 3.3, the subordination condition

$$1 + \rho\varsigma p'(\varsigma) \prec 1 + \rho\varsigma q'(\varsigma)$$

implies $p(\varsigma) \prec q(\varsigma)$. To complete the proof, it suffices to prove that the circular disc 2.2 is contained in the region bounded by the curve $h(e^{i\theta})$ ($\theta \in [0, 2\pi)$). To this end, we must show that

$$\left| h(e^{i\theta}) - \frac{1 - \mathbb{A}_{\supset}\mathbb{B}_{\supset}}{1 - \mathbb{B}_{\supset}^2} \right| \geq \frac{\mathbb{A}_{\supset} - \mathbb{B}_{\supset}}{1 - \mathbb{B}_{\supset}^2}.$$

Now,

$$\begin{aligned} \left| h(e^{i\theta}) - \frac{1 - \mathbb{A}_{\supset}\mathbb{B}_{\supset}}{1 - \mathbb{B}_{\supset}^2} \right| =& \left| 2\rho s e^{i\theta}(1 + s e^{i\theta}) + \frac{\mathbb{B}_{\supset}(\mathbb{A}_{\supset} - \mathbb{B}_{\supset})}{1 - \mathbb{B}_{\supset}^2} \right| \\ \geq& 2s|\rho|(1 - s) - \frac{|\mathbb{B}_{\supset}|(\mathbb{A}_{\supset} - \mathbb{B}_{\supset})}{1 - |\mathbb{B}_{\supset}|^2}. \end{aligned}$$

Thus,

$$2s|\rho|(1 - s) - \frac{|\mathbb{B}_{\supset}|(\mathbb{A}_{\supset} - \mathbb{B}_{\supset})}{1 - |\mathbb{B}_{\supset}|^2} \geq \frac{\mathbb{A}_{\supset} - \mathbb{B}_{\supset}}{1 - |\mathbb{B}_{\supset}|^2}$$

if (8.2) is satisfied. □

Theorem 8.2. *Let* $0 < s \leq \frac{1}{2}$ *and* $p(\varsigma)$ *be analytic in* U *with* $p(0) = 1$ *such that*

$$1 + \rho\varsigma p'(\varsigma) \prec (1 + s\varsigma)^2 \quad (\rho \in \mathbb{R}\backslash\{0\}, \varsigma \in U). \tag{8.3}$$

If

$$|\rho| \geq \frac{2 + s}{2(1 - s)}, \tag{8.4}$$

then $p \in P(\mathbb{L}_s)$.

Proof. Following the same arguments as in the proof of Theorem 8.1, we arrive at where to show that

$$(1 + s\varsigma)^2 \prec 1 + 2\rho s\varsigma(1 + s\varsigma) := h(\varsigma).$$

To achieve this, it is enough to show that the domain bounded by the limaçon is inside the region bounded by the curve $h(e^{i\theta})$ ($\theta \in [0, 2\pi)$). As a result, we need to find ρ for which

$$|h(e^{i\theta}) - 1| \geq (1 + s)^2 - 1.$$

Now,

$$\begin{aligned} |h(e^{i\theta}) - 1| =& 2|\rho|s|1 + s e^{i\theta}| \\ \geq& 2|\rho|s(1 - s) \\ \geq& 2s + s^2 \end{aligned}$$

if

$$|\rho| \geq \frac{2 + s}{2(1 - s)}. \tag{8.5}$$

□

Theorem 8.3. *Let* $-1 \leq \mathbb{B}_{\eth} < \mathbb{A}_{\eth} \leq 1,\ 0 < s \leq \frac{1}{\sqrt{2}}$ *and* $p(\varsigma)$ *be analytic in* U *with* $p(0) = 1$ *such that*

$$1 + \rho\varsigma p'(\varsigma) \prec (1 + s\varsigma)^2 \ \ (\rho \in \mathbb{R}\backslash\{0\}, \varsigma \in U).$$

If

$$|\rho| \geq \frac{(2s + s^2)(1 + |\mathbb{B}_{\eth}|)^2}{\mathbb{A}_{\eth} - \mathbb{B}_{\eth}}, \tag{8.6}$$

then

$$p(\varsigma) \prec \frac{1 + \mathbb{A}_{\eth}\varsigma}{1 + \mathbb{B}_{\eth}\varsigma}.$$

Proof. Let $q(\varsigma) = \frac{1+\mathbb{A}_{\eth}\varsigma}{1+\mathbb{B}_{\eth}\varsigma}$. We have that $q(\varsigma)$ is convex univalent in U. Therefore, following the method of proof in Theorem 8.1, we arrive at where to show that

$$(1 + s\varsigma)^2 \prec 1 + \rho\frac{(\mathbb{A}_{\eth} - \mathbb{B}_{\eth})\varsigma}{(1 + \mathbb{B}_{\eth}\varsigma)^2} := h(\varsigma).$$

For this, we need to establish that the region bounded by the limaçon lies inside the domain bounded by the curve $h(e^{i\theta})$ $(\theta \in [0, 2\pi)$. A simple observation of (2.5) suggests it suffices to show

$$|h(e^{i\theta}) - 1| \geq (1 + s)^2 - 1.$$

Now,

$$\begin{aligned}|h(e^{i\theta}) - 1| =&|\rho|\frac{\mathbb{A}_{\eth} - \mathbb{B}_{\eth}}{|1 + \mathbb{B}_{\eth}\varsigma|^2}\\ \geq&|\rho|\frac{\mathbb{A}_{\eth} - \mathbb{B}_{\eth}}{(1 + |\mathbb{B}_{\eth}|)^2}.\end{aligned}$$

But

$$\frac{\mathbb{A}_{\eth} - \mathbb{B}_{\eth}}{(1 + |\mathbb{B}_{\eth}|)^2} \geq (1 + s)^2 - 1$$

provided (8.6) holds. This completes the proof. □

Theorem 8.4. *Let* $0 < s \leq \frac{1}{\sqrt{2}}$ *and* $p(\varsigma)$ *be analytic in* U *with* $p(0) = 1$ *such that*

$$1 + \rho\frac{\varsigma p'(\varsigma)}{p(\varsigma)} \prec (1 + s\varsigma)^2 \ \ (\rho \in \mathbb{R}\backslash\{0\}, \varsigma \in U). \tag{8.7}$$

If

$$|\rho| \geq \frac{(2 - s)(1 + s)}{2}, \tag{8.8}$$

then

$$p(\varsigma) \prec (1 + s\varsigma)^2.$$

Proof. Let $q(\varsigma) = (1+s\varsigma)^2$. Then $q(\varsigma)$ is convex univalent in U. The function $\phi(\omega) = \rho/\omega$ and $\theta(\omega) = 1$ are analytic in the domain containing $q(U)$. Set

$$Q(\varsigma) = \varsigma q'(\varsigma)\phi(q(\varsigma)) = \rho\frac{\varsigma q'(\varsigma)}{q(\varsigma)} = 2\rho\frac{s\varsigma}{1 + s\varsigma}$$

and

$$h(\varsigma) = \theta(q(\varsigma)) + Q(\varsigma) = 1 + \rho\frac{\varsigma q'(\varsigma)}{q(\varsigma)} = 1 + 2\rho\frac{s\varsigma}{1 + s\varsigma}.$$

Then

$$\mathrm{Re}\frac{\varsigma h'(\varsigma)}{Q(\varsigma)} = \mathrm{Re}\frac{\varsigma Q'(\varsigma)}{Q(\varsigma)} > \frac{1}{1+s} > 0.$$

From Lemma 3.3, the differential subordination

$$1 + \rho\frac{\varsigma p'(\varsigma)}{p(\varsigma)} \prec 1 + \rho\frac{\varsigma q'(\varsigma)}{q(\varsigma)}, \quad \varsigma \in U$$

implies $p(\varsigma) \prec q(\varsigma)$. To finalize the proof, we need to prove

$$(1 + s\varsigma)^2 \prec 1 + 2\rho\frac{s\varsigma}{1 + s\varsigma}, \tag{8.9}$$

which is equivalent to showing

$$\mathbb{L}_s(U) \subset h(U).$$

It is easy to see that the transformation $h(\varsigma) = 1 + \rho\frac{\varsigma q'(\varsigma)}{q(\varsigma)}$ maps U onto the disc $\mathcal{D}(a, r)$, where

$$a = \frac{1 - (1 + 2|\rho|)s^2}{1 - s^2} < 1 \quad \text{and} \quad r = \frac{2|\rho|s}{1 - s^2}.$$

Therefore (8.9) holds if and only if

$$1 - \frac{1 - (1 + 2|\rho|)s^2}{1 - s^2} < \frac{2|\rho|s}{1 - s^2},$$

which implies

$$|\rho| > \frac{(1 + s)(2 - s)}{2}.$$

□

We choose to omit the proof of the next theorem since it follows the same argument as in Theorem 8.3.

Theorem 8.5. *Let* $-1 \leq \mathbb{B}_{\mathfrak{d}} < \mathbb{A}_{\mathfrak{d}} \leq 1$, $0 < s \leq \frac{1}{\sqrt{2}}$ *and* $p(\varsigma)$ *be analytic in* U *with* $p(0) = 1$ *such that*

$$1 + \rho\frac{\varsigma p'(\varsigma)}{p(\varsigma)} \prec (1 + s\varsigma)^2 \;\; (\rho \in \mathbb{R}\backslash\{0\}, \varsigma \in U).$$

If

$$|\rho| \geq \frac{(1 + |\mathbb{A}_{\mathfrak{d}}|)(1 + |\mathbb{B}_{\mathfrak{d}}|)(2s + s^2)}{\mathbb{A}_{\mathfrak{d}} - \mathbb{B}_{\mathfrak{d}}}, \tag{8.10}$$

then

$$p(\varsigma) \prec \frac{1 + \mathbb{A}_{\mathfrak{d}}\varsigma}{1 + \mathbb{B}_{\mathfrak{d}}\varsigma} \quad \varsigma \in U.$$

Remark 8.1.

(i) If we put $p(\varsigma) = \varsigma f'(\varsigma)/f(\varsigma)$ and $p(\varsigma) = (\varsigma f'(\varsigma))'/f'(\varsigma)$ in Theorem 8.1-8.5, we obtain the conditions on ρ for which the respective subordination conditions (8.1), (8.3) and (8.7) imply $f \in ST_{\mathcal{L}}(s)$ and $f \in CV_{\mathcal{L}}(s)$.

(ii) We note that our condition $0 < s \leq \frac{1}{2}$ cannot be relaxed in Theorem 8.1 and Theorem 8.2. Otherwise, starlikeness of Q will not be achieved. As such, the proof of the theorems will be extremely difficult to obtain via Lemma 3.3.

9. Conclusion

The Ma and Minda classes of functions are the comprehensive generalization of the classes S_{ST} and C_{CV}. These classes are vital in (GFT) because of their importance in science and technology. To this end, continuous studies of their subfamily, which are related to Limaçon domain were investigated. Coefficients bounds, Fekete Szegö inequality as well as the upper bounds of the third Hankel determinants for these subclasses were derived. Finally, the techniques of differential subordination were also used to obtain some restrictions for which analytic functions belonged to these families. In addition, to have more new theorems under present examinations, new generalization and applications can be explored with some positive and novel outcomes in various fields of science, especially, in applied mathematics. These new surveys will be presented in future research work being processed by authors of the present paper.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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