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Research article

Solutions of multiplicative ordinary differential equations via the multiplicative differential transform method

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Abstract: In this study, fundamental definitions and theorems of the Multiplicative Differential Transform Method (MDTM) are given. First and second order multiplicative initial value problems are numerically solved with the help of MDTM.

Keywords: multiplicative derivative; multiplicative ordinary differential equation; multiplicative differential transform method; multiplicative power series
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1. Introduction

Classical analysis was described by Gottfried Leibniz and Isaac Newton in the 17th century. Classical analysis consists of concepts such as limit, derivative, integral and series. Since the basis of these concepts is based on addition and subtraction, this analysis is also referred to as additive analysis.

Volterra type analysis was defined by Vito Volterra in 1887 as an alternative to classical analysis [14]. After the definition of Volterra analysis, some new studies were performed between 1967 and 1970 by Michael Grossman and Robert Katz. As a result of these studies, new analyses called geometric analysis, bigeometric analysis and anageometric analysis have been defined [12]. Some basic definitions and concepts are given about this new analysis, which is also called non-Newtonian analysis [12].

Geometric analysis, which is one of the non-Newtonian types of analysis, was first expressed by Dick Stanley as a multiplicative analysis [9]. Addition and subtraction in classical analysis corresponded to multiplication and division in geometric analysis. For this reason geometric analysis is called multiplicative calculus. In the following years, some studies on multiplicative analysis were done by Duff Campell [7]. Then, in 2008, fundamental concepts of multiplicative analysis were defined and some applications were given by Bashirov [2].

Some studies in recent years [2, 3] have proven that the concept of multiplicative analysis, which has emerged as an alternative to classical analysis and offers a different perspective to problems encountered in science and engineering, has developed quite rapidly [3]. In recent years [1, 4–6, 8, 10, 11, 15–19], using the basic concepts of multiplicative analysis some work was done on multiplicative ordinary differential equations.

In this study, a new transform method is introduced, namely one-dimensional multiplicative differential transform method (MDTM) and solutions of some multiplicative differential equations are investigated with the help of this method.

2. Multiplicative derivatives and multiplicative integrals

In this section, we will give some basic definitions and properties of the multiplicative derivative theory which can be found in [2,3,5,9].

Definition 2.1. Let $f : R \to R^{+}$ be a positive function. The *multiplicative derivative* of the function f is given by:

$$\frac{d^{*}f}{dx}(x) = f^{*}(x) = \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{1/h}. \tag{2.1}$$

Assuming that f is a positive function and using properties of the classical derivative, the multiplicative derivative can be written as

$$\begin{aligned} f^{*}(x) &= \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{1/h} \\ &= \lim_{h\to 0}\left[1+\frac{f(x+h)-f(x)}{f(x)}\right]^{\left(\frac{f(x)}{f(x+h)-f(x)}\frac{f(x+h)-f(x)}{h}\frac{1}{f(x)}\right)} \\ &= \exp\left(\frac{f'(x)}{f(x)}\right) \\ f^{*}(x) &= \exp(\ln\circ f)'(x) \end{aligned} \tag{2.2}$$

for $(\ln\circ f)(x) = \ln[f(x)]$.

Definition 2.2. If the multiplicative derivative f^{*} as a function also has a multiplicative derivative, then multiplicative derivative of f^{*} is called second order multiplicative derivative of f and it is represented by f^{**}. Similarly, we can define n^{th} order multiplicative derivative of f with the notation $f^{*(n)}$. With n times repetition of the multiplicative differentiation operation, a positive f function has an n^{th} order multiplicative derivative at the point x which is defined as

$$f^{*(n)}(x) = \exp(\ln\circ f)^{(n)}(x). \tag{2.3}$$

Theorem 2.1. *If a positive function f is differentiable with the multiplicative derivative at the point x, then it is differentiable in the classical sense and the relation between these two derivatives can be shown as*

$$f'(x) = f(x)\ln f^{*}(x). \tag{2.4}$$

Theorem 2.2. *Let f and g be differentiable with the multiplicative derivative. If c is an arbitrary constant, then the functions $c \cdot f$, $f \cdot g$, $f + g$, f/g, f^g have multiplicative derivatives given by*

$$\begin{aligned} &\textbf{\textit{1)}} \quad (c \cdot f)^* (x) = f^* (x), \\ &\textbf{\textit{2)}} \quad (f \cdot g)^* (x) = f^* (x) \cdot g^* (x), \\ &\textbf{\textit{3)}} \quad (f + g)^* (x) = f^* (x)^{\frac{f(x)}{f(x)+g(x)}} g^* (x)^{\frac{g(x)}{f(x)+g(x)}}, \\ &\textbf{\textit{4)}} \quad (f/g)^* (x) = f^* (x) / g^* (x), \\ &\textbf{\textit{5)}} \quad (f^g)^* (x) = f^* (x)^{g(x)} f (x)^{g'(x)}. \end{aligned} \tag{2.5}$$

Theorem 2.3. *Let g be differentiable in the multiplicative sense, h be differentiable in the classical sense. If*

$$f (x) = (g \circ h) (x),$$

then, it follows that

$$f^* (x) = [g^* (h (x))]^{h'(x)}. \tag{2.6}$$

Definition 2.3. A multiplicative integral is also defined in [2] for positive bounded functions and if f is Riemann integrable on $[a, b]$, then

$$* \int_a^b f (x)^{dx} = \exp\left[\int_a^b (\ln f (x))\, dx\right] = e^{\int_a^b [\ln f(x)] dx}. \tag{2.7}$$

This multiplicative integral has the properties:

$$\begin{aligned} &\textbf{1)} \quad * \int_a^b \left[f (x)^k\right]^{dx} = * \left[\int_a^b (f (x))^{dx}\right]^k, \qquad \text{for } k \in \mathbb{R}, \\ &\textbf{2)} \quad * \int_a^b [f (x) g (x)]^{dx} = * \int_a^b [f (x)]^{dx} * \int_a^b [g (x)]^{dx}, \\ &\textbf{3)} \quad * \int_a^b \left[\frac{f(x)}{g(x)}\right]^{dx} = \frac{* \int_a^b [f(x)]^{dx}}{* \int_a^b [g(x)]^{dx}}, \\ &\textbf{4)} \quad * \int_a^b f (x)^{dx} = * \int_a^c f (x)^{dx} * \int_c^b f (x)^{dx}, \qquad \text{for } a \le c \le b. \end{aligned} \tag{2.8}$$

3. One dimensional multiplicative differential transform method

Multiplicative linear differential equations can be defined as in [16, 17] by

$$\left(y^{*(n)}\right)^{a_n(x)} \left(y^{*(n-1)}\right)^{a_{n-1}(x)} \cdots (y^{**})^{a_2(x)} (y^*)^{a_1(x)} (y)^{a_0(x)} = f (x) \tag{3.1}$$

Here, $f(x)$ is a positive definite function. Multiplicative Taylor Theorem is defined as below [2, 19].

$$f (x + h) = \prod_{m=0}^{n} \left[f^{*(m)} (x)\right]^{h^m/m!} \left[f^{*(n+1)} (x + \theta h)\right]^{h^{n+1}/(n+1)!}, \text{ for some } \theta \in (0, 1) \tag{3.2}$$

Definition 3.1. Let $f(x)$ be a multiplicative analytic function with one variable. Then, for given x_0 and for $k \in \mathbb{N}_0$, the multiplicative differential transform function $F^*(k)$ of function $f(x)$ is defined as

$$F^*(k) = \left[f^{*(k)}(x_0)\right]^{1/k!}. \tag{3.3}$$

The inverse multiplicative differential transform function of $F^*(k)$ is written as

$$f(x) = \prod_{k=0}^{\infty} \left[F^*(k)\right]^{(x-x_0)^k}, \left(f(x) = \prod_{k=0}^{\infty} \left[f^{*(k)}(x_0)\right]^{(x-x_0)^k/k!}\right). \tag{3.4}$$

Here, if we take $x_0 = 0$, then

$$f(x) = \prod_{k=0}^{\infty} \left[F^*(k)\right]^{x^k}. \tag{3.5}$$

The equality

$$f(x) = \prod_{k=0}^{\infty} \left[f^{*(k)}(0)\right]^{x^k/k!} \tag{3.6}$$

is called the Taylor series with one variable. The product

$$\prod_{k=n+1}^{\infty} \left[F^*(k)\right]^{x^k} \tag{3.7}$$

of remaining terms gives the truncation error and the n^{th} degree Taylor polynomial

$$f(x) \approx \prod_{k=0}^{n} \left[F^*(k)\right]^{x^k} \tag{3.8}$$

gives an approximation of $f(x)$ for x near 0.

3.1. Fundamental theorems of multiplicative differential transform with one variable

Lemma 3.1. *If $f(x) = c$ where c is a positive constant, then for any x_0 the multiplicative differential transform of f at x_0 is given by*

$$F^*(k) = \begin{cases} c, & \text{if} \quad k = 0, \\ 1, & \text{if} \quad k \geq 1. \end{cases} \tag{3.9}$$

Theorem 3.1. *If $\lambda \in \mathbb{R}$ and $y(x)$ is a multiplicative analytic function, then multiplicative differential transform of function*

$$f(x) = \left[y(x)\right]^{\lambda} \tag{3.10}$$

is

$$F^*(k) = \left[Y^*(k)\right]^{\lambda}. \tag{3.11}$$

Theorem 3.2. *Let $y(x)$ and $z(x)$ be two multiplicative analytic functions. Then the multiplicative differential transform of function $f(x) = y(x)z(x)$ is*

$$F^*(k) = Y^*(k)\, Z^*(k). \tag{3.12}$$

Proof.

$$F^{*}(k)=\left[f^{*(k)}\left(x_{0}\right)\right]^{1/k!}=\left[y^{*(k)}\left(x_{0}\right)z^{*(k)}\left(x_{0}\right)\right]^{1/k!}=Y^{*}\left(k\right)Z^{*}\left(k\right).$$

□

Theorem 3.3. *Let $y(x)$ and $z(x)$ be two multiplicative analytic functions. Then the multiplicative differential transform of function $f(x)=y(x)/z(x)$ is*

$$F^{*}(k)=\frac{Y^{*}\left(k\right)}{Z^{*}\left(k\right)}. \tag{3.13}$$

Theorem 3.4. *If $y(x)$ is a multiplicative analytic function, then the multiplicative differential transform of function $f(x)=y^{*}(x)$ is*

$$F^{*}(k)=\left[Y^{*}\left(k+1\right)\right]^{(k+1)}. \tag{3.14}$$

Proof.

$$\begin{aligned}
F^{*}(k) &= \left[f^{*(k)}\left(x_{0}\right)\right]^{1/k!}=\left[\left(y^{*}\right)^{*(k)}\left(x_{0}\right)\right]^{1/k!}\\
&= \left[y^{*(k+1)}\left(x_{0}\right)\right]^{1/k!}=\left[y^{*(k+1)}\left(x_{0}\right)\right]^{(k+1)/(k+1)!}\\
&= \left[\left(y^{*(k+1)}\left(x_{0}\right)\right)^{1/(k+1)!}\right]^{k+1}=\left[Y^{*}\left(k+1\right)\right]^{(k+1)}.
\end{aligned}$$

□

Theorem 3.5. *If $y(x)$ is a multiplicative analytic function, then the multiplicative differential transform of function $f(x)=y^{**}(x)$ is*

$$F^{*}(k)=\left[Y^{*}\left(k+2\right)\right]^{(k+1)(k+2)}. \tag{3.15}$$

Proof.

$$\begin{aligned}
F^{*}(k) &= \left[f^{*(k)}\left(x_{0}\right)\right]^{1/k!}=\left[\left(y^{**}\right)^{*(k)}\left(x_{0}\right)\right]^{1/k!}\\
&= \left[y^{*(k+2)}\left(x_{0}\right)\right]^{1/k!}=\left[y^{*(k+2)}\left(x_{0}\right)\right]^{[(k+1)(k+2)]/(k+2)!}\\
&= \left[\left(y^{*(k+2)}\left(x_{0}\right)\right)^{1/(k+2)!}\right]^{(k+1)(k+2)}=\left[Y^{*}\left(k+2\right)\right]^{(k+1)(k+2)}.
\end{aligned}$$

□

Theorem 3.6. *If $y(x)$ is a multiplicative analytic function, then the multiplicative differential transform of function $f(x)=y^{*(n)}(x)$ is*

$$F^{*}(k)=\left[Y^{*}\left(k+n\right)\right]^{(k+n)!/k!}. \tag{3.16}$$

Theorem 3.7. *Let $y(x)$ and $z(x)$ be two multiplicative analytic functions. Then the multiplicative differential transform of function $f_{2}\left(x\right)=z\left(x\right)^{\ln y(x)}=\exp\left\{\ln y\left(x\right)\ln z\left(x\right)\right\}$ is*

$$F_{2}^{*}(k)=\prod_{r=0}^{k}\exp\left\{\ln Y^{*}\left(r\right)\ln Z^{*}\left(k-r\right)\right\}. \tag{3.17}$$

Proof.

$$
\begin{aligned}
F_2^*(k) = & \left[f_2^{*(k)}(x_0)\right]^{1/k!} = \left\{\left[\left(z(x)^{\ln y(x)}\right)^{*(k)}\right]_{x=x_0}\right\}^{1/k!} = \left\{\left[\left(z(x)^{\ln y(x)}\right)^{*k}\right]\right\}^{1/k!}\Bigg|_{x=x_0} \\
= & \left\{\left[z^{*(k)}(x)\right]^{\ln y(x)}\left[z^{*(k-1)}(x)\right]^{\binom{k}{1}\ln y^*(x)}\cdots\left[z^{*(k-r)}(x)\right]^{\binom{k}{r}\ln y^{*(r)}(x)}\cdots[z(x)]^{\ln y^{*(k)}(x)}\right\}^{1/k!}\Bigg|_{x=x_0} \\
= & \left\{\left[z^{*(k)}(x_0)\right]^{\ln y(x_0)}\left[z^{*(k-1)}(x)\right]^{\binom{k}{1}\ln y^*(x_0)}\cdots\left[z^{*(k-r)}(x_0)\right]^{\binom{k}{r}\ln y^{*(r)}(x_0)}\cdots[z(x_0)]^{\ln y^{*(k)}(x_0)}\right\}^{1/k!} \\
= & \left\{\prod_{r=0}^{k}\left[z^{*(k-r)}(x_0)\right]^{\binom{k}{r}\ln y^{*(r)}(x_0)}\right\}^{1/k!} \\
= & \left\{\prod_{r=0}^{k}\left[z^{*(k-r)}(x_0)\right]^{\frac{k!}{(k-r)!r!}\ln y^{*(r)}(x_0)}\right\}^{1/k!} \\
= & \left\{\prod_{r=0}^{k}\left[\left(z^{*(k-r)}(x_0)\right)^{1/(k-r)!}\right]^{\frac{1}{r!}\ln y^{*(r)}(x_0)}\right\} \\
= & \left\{\prod_{r=0}^{k}\left[\left(z^{*(k-r)}(x_0)\right)^{1/(k-r)!}\right]^{\ln\left(y^{*(r)}(x_0)\right)^{1/r!}}\right\} \\
= & \prod_{r=0}^{k} Z^*(k-r)^{\ln Y^*(r)} \\
= & \prod_{r=0}^{k}\exp\{\ln Y^*(r)\ln Z^*(k-r)\}.
\end{aligned}
$$

□

Theorem 3.8. *Let $y_1(x)$, $y_2(x)$, and $y_3(x)$ be multiplicative analytic functions. Then the multiplicative differential transform of function*

$$f_3(x) = \exp\left[\ln y_1(x)\ln y_2(x)\ln y_3(x)\right] \tag{3.18}$$

is

$$F_3^*(k) = \prod_{r_2=0}^{k}\prod_{r_1=0}^{r_2}\exp\left[\ln Y_1^*(r_1)\ln Y_2^*(r_2-r_1)\ln Y_3^*(k-r_2)\right]. \tag{3.19}$$

Proof. Let's define $f_2(x) = \exp\{\ln y_1(x)\ln y_2(x)\}$. From the theorem above for the function $f_2(x) = \exp\{\ln y_1(x)\ln y_2(x)\}$, we have

$$F_2^*(k) = \prod_{r_1=0}^{k}\exp\left[\ln Y_1^*(r_1)\ln Y_2^*(k-r_1)\right].$$

Using this we can write that

$$f_3(x) = \exp\{[\ln y_1(x)\ln y_2(x)]\ln y_3(x)\}$$

$$\begin{aligned}
f_3(x) &= \exp\{\ln[\exp\{\ln y_1(x)\ln y_2(x)\}]\ln y_3(x)\}\\
F_3^*(k) &= \prod_{r_2=0}^{k}\exp\left[\ln F_2^*(r_2)\ln Y_3^*(k-r_2)\right]\\
&= \prod_{r_2=0}^{k}\exp\left\{\ln\left(\prod_{r_1=0}^{r_2}\exp\left[\ln Y_1^*(r_1)\ln Y_2^*(r_2-r_1)\right]\right)\ln Y_3^*(k-r_2)\right\}\\
&= \prod_{r_2=0}^{k}\exp\left\{\left(\sum_{r_1=0}^{r_2}\ln\exp\left[\ln Y_1^*(r_1)\ln Y_2^*(r_2-r_1)\right]\right)\ln Y_3^*(k-r_2)\right\}\\
&= \prod_{r_2=0}^{k}\exp\left\{\left(\sum_{r_1=0}^{r_2}\left[\ln Y_1^*(r_1)\ln Y_2^*(r_2-r_1)\right]\right)\ln Y_3^*(k-r_2)\right\}\\
&= \prod_{r_2=0}^{k}\exp\sum_{r_1=0}^{r_2}\left[\ln Y_1^*(r_1)\ln Y_2^*(r_2-r_1)\ln Y_3^*(k-r_2)\right]\\
F_3^*(k) &= \prod_{r_2=0}^{k}\prod_{r_1=0}^{r_2}\exp\left[\ln Y_1^*(r_1)\ln Y_2^*(r_2-r_1)\ln Y_3^*(k-r_2)\right].
\end{aligned}$$

□

Theorem 3.9. *Let $y_1(x)$, $y_2(x)$, …, $y_n(x)$ be multiplicative analytic functions. Then the multiplicative differential transform of function*

$$f_n(x) = \exp\left[\ln y_1(x)\ln y_2(x)\ldots\ln y_n(x)\right] \tag{3.20}$$

is

$$F_n^*(k) = \prod_{r_{n-1}=0}^{k}\prod_{r_{n-2}=0}^{r_{n-1}}\ldots\prod_{r_2=0}^{r_3}\prod_{r_1=0}^{r_2}\exp\left[\ln Y_1^*(r_1)\ln Y_2^*(r_2-r_1)\ldots\ln Y_n^*(k-r_{n-1})\right]. \tag{3.21}$$

Proof. We will proof this by using induction. In the theorems above we showed that it is true for $n = 2$ and $n = 3$.
Now, let's assume it is true for $n-1$ and show that it is true for n. Thus assume for

$$f_{n-1}(x) = \exp\left[\ln y_1(x)\ln y_2(x)\ldots\ln y_{n-1}(x)\right]. \tag{3.22}$$

we have

$$F_{n-1}^*(k) = \prod_{r_{n-2}=0}^{k}\prod_{r_{n-3}=0}^{r_{n-2}}\cdots\prod_{r_2=0}^{r_3}\prod_{r_1=0}^{r_2}\exp\left[\ln Y_1^*(r_1)\ln Y_2^*(r_2-r_1)\ldots\ln Y_{n-1}^*(k-r_{n-2})\right]. \tag{3.23}$$

For the function

$$\begin{aligned}
f_n(x) &= \exp\left[\ln y_1(x)\ln y_2(x)\ldots\ln y_{n-1}(x)\ln y_n(x)\right]\\
&= \exp\left[\{\ln y_1(x)\ln y_2(x)\ldots\ln y_{n-1}(x)\}\ln y_n(x)\right]
\end{aligned}$$

$$= \exp\{\ln[\exp\{\ln y_1(x)\ln y_2(x)\ldots\ln y_{n-1}(x)\}]\ln y_n(x)\}$$

$$f_n(x) = \exp\{\ln[f_{n-1}(x)]\ln y_n(x)\}$$

we can write

$$\begin{aligned}
F_n^*(k) &= \prod_{r_{n-1}=0}^{k}\exp\left[\ln F_{n-1}^*(r_{n-1})\ln Y_n^*(k-r_{n-1})\right]\\
&= \prod_{r_{n-1}=0}^{k}\exp\left\{\ln\left(\prod_{r_{n-2}=0}^{r_{n-1}}\cdots\prod_{r_1=0}^{r_2}\exp\left[\ln Y_1^*(r_1)\ldots\ln Y_{n-1}^*(r_{n-1}-r_{n-2})\right]\right)\ln Y_n^*(k-r_{n-1})\right\}\\
&= \prod_{r_{n-1}=0}^{k}\exp\left\{\sum_{r_{n-2}=0}^{r_{n-1}}\cdots\sum_{r_1=0}^{r_2}\ln\exp\left[\ln Y_1^*(r_1)\cdots\ln Y_{n-1}^*(r_{n-1}-r_{n-2})\right]\ln Y_n^*(k-r_{n-1})\right\}\\
F_n^*(k) &= \prod_{r_{n-1}=0}^{k}\prod_{r_{n-2}=0}^{r_{n-1}}\cdots\prod_{r_1=0}^{r_2}\exp\left[\ln Y_1^*(r_1)\ldots\ln Y_{n-1}^*(k-r_{n-2})\ln Y_n^*(k-r_{n-1})\right].
\end{aligned}$$

And this proves the theorem. □

Theorem 3.10. *The multiplicative differential transform of function* $f(x) = \exp\{x^m\}$ *at* $x_0 = 0$ *is*

$$F^*(k) = \delta^*(k-m) = \begin{cases} e, & \text{for } k = m \\ 1, & \text{for } k \neq m \end{cases} \tag{3.24}$$

Proof. For $k < m$, we have

$$\begin{aligned}
F^*(k) &= \left[f^{*(k)}(x)\right]^{1/k!}\Big|_{x=0} = \left[(\exp\{x^m\})^{*(k)}\right]^{1/k!}\Big|_{x=0}\\
&= \left[\exp\left\{m(m-1)\ldots(m-k+1)x^{m-k}\right\}\right]^{1/k!}\Big|_{x=0} = [\exp\{0\}]^{1/k!} = 1.
\end{aligned}$$

For $k = m$, we have

$$F^*(k) = F^*(m) = \left[f^{*(m)}(x)\right]^{1/m!}\Big|_{x=0} = [\exp\{m!\}]^{1/m!}\Big|_{x=0} = \exp\{m!(1/m!)\}\Big|_{x=0} = e.$$

For $k > m$, $\exists h \in \mathbb{Z}^+$such that $k = m + h$, and

$$\begin{aligned}
F^*(k) &= \left[f^{*(k)}(x)\right]^{1/k!}\Big|_{x=0} = \left[f^{*(m+h)}(x)\right]^{1/(m+h)!}\Big|_{x=0}\\
&= \left[\left\{f^{*(m)}(x)\right\}^{*(h)}\right]^{1/(m+h)!}\Bigg|_{x=0} = \left[\{\exp(m!)\}^{*(h)}\right]^{1/(m+h)!} = 1.
\end{aligned}$$

□

Theorem 3.11. *The multiplicative differential transform of function $f(x) = \exp\left\{e^{\lambda x}\right\}$ at $x_0 = 0$ is*

$$F^*(k) = \exp\left\{\lambda^k/k!\right\}. \tag{3.25}$$

Proof.

$$\begin{aligned} F^*(k) &= \left[f^{*(k)}(x)\right]^{1/k!}\Big|_{x=0} = \left[\exp\left\{\lambda^k e^{\lambda x}\right\}\right]^{1/k!}\Big|_{x=0} \\ &= \left[\exp\left\{e^{\lambda x}\right\}\right]^{\lambda^k/k!}\Big|_{x=0} = \left[\exp\left\{e^{0}\right\}\right]^{\lambda^k/k!} = \exp\left\{\lambda^k/k!\right\}. \end{aligned}$$

□

Theorem 3.12. *The multiplicative differential transform of function $f(x) = \exp\left\{a^{\lambda x}\right\}$ at $x_0 = 0$ is*

$$F^*(k) = \exp\left\{(\lambda \ln a)^k/k!\right\}. \tag{3.26}$$

Proof.

$$\begin{aligned} F^*(k) &= \left[f^{*(k)}(x)\right]^{1/k!}\Big|_{x=0} = \left[\exp\left\{(\lambda \ln a)^k a^{\lambda x}\right\}\right]^{1/k!}\Big|_{x=0} \\ &= \left[\exp\left\{(\lambda \ln a)^k\right\}\right]^{1/k!} = \exp\left\{(\lambda \ln a)^k/k!\right\}. \end{aligned}$$

□

Theorem 3.13. *The multiplicative differential transform of function $f(x) = \exp\{\sin(\omega x + \alpha)\}$ at $x_0 = 0$ is*

$$F^*(k) = \exp\left\{\frac{\omega^k}{k!}\sin\left(\frac{\pi}{2}k + \alpha\right)\right\}. \tag{3.27}$$

Proof.

$$\begin{aligned} F^*(k) &= \left[f^{*(k)}(x)\right]^{1/k!}\Big|_{x=0} = \left[\exp\left\{\omega^k \sin\left(\frac{\pi}{2}k + \omega x + \alpha\right)\right\}\right]^{1/k!}\Bigg|_{x=0} \\ &= \exp\left\{\frac{\omega^k}{k!}\sin\left(\frac{\pi}{2}k + \alpha\right)\right\}. \end{aligned}$$

□

Theorem 3.14. *The multiplicative differential transform of function $f(x) = \exp\{\cos(\omega x + \alpha)\}$ at $x_0 = 0$ is*

$$F^*(k) = \exp\left\{\frac{\omega^k}{k!}\cos\left(\frac{\pi}{2}k + \alpha\right)\right\}. \tag{3.28}$$

Proof.

$$\begin{aligned} F^*(k) &= \left[f^{*(k)}(x)\right]^{1/k!}\Big|_{x=0} = \left[\exp\left\{\omega^k \cos\left(\frac{\pi}{2}k + \omega x + \alpha\right)\right\}\right]^{1/k!}\Bigg|_{x=0} \\ &= \exp\left\{\frac{\omega^k}{k!}\cos\left(\frac{\pi}{2}k + \alpha\right)\right\}. \end{aligned}$$

□

4. Approximate solution of the Cauchy problem for first order linear multiplicative differential equation by the multiplicative differential transform method

Here, we will give a theorem and its application about approximate solution of the Cauchy problem for first order linear multiplicative differential equations by the multiplicative differential transform method.

Theorem 4.1. *Suppose the Cauchy problem for first order linear multiplicative differential equation*

$$(y^{*})\, y^{\ln p(x)} = q(x)\,,\ y(0) = y_0 \tag{4.1}$$

is given. Here $p(x) > 0$. Then the solution of this problem is obtained from the recurrence relations

$$Y^{*}(k+1) = \exp\left\{-\frac{1}{k+1}\left[\sum_{r=0}^{k}\left[\ln Y^{*}(r)\ln P^{*}(k-r)\right] - \ln\left\{Q^{*}(k)\right\}\right]\right\}, Y^{*}(0) = y_0. \tag{4.2}$$

Here the multiplicative differential transforms of $y(x), p(x)$ and $q(x)$ are $Y^{*}(k), P^{*}(k)$ and $Q^{*}(k)$, respectively.

Proof. Taking the multiplicative differential transform of both sides of the equation

$$\begin{aligned}
Y^{*}(k+1)^{(k+1)}\prod_{r=0}^{k}\exp\left[\ln Y^{*}(r)\ln P^{*}(k-r)\right] &= Q^{*}(k)\,,\\
\ln\left\{\left[Y^{*}(k+1)\right]^{(k+1)}\prod_{r=0}^{k}\exp\left[\ln Y^{*}(r)\ln P^{*}(k-r)\right]\right\} &= \ln\left\{Q^{*}(k)\right\},\\
\ln\left[Y^{*}(k+1)\right]^{(k+1)} + \ln\left\{\prod_{r=0}^{k}\exp\left[\ln Y^{*}(r)\ln P^{*}(k-r)\right]\right\} &= \ln\left\{Q^{*}(k)\right\},\\
(k+1)\ln\left[Y^{*}(k+1)\right] + \ln\exp\sum_{r=0}^{k}\left[\ln Y^{*}(r)\ln P^{*}(k-r)\right] &= \ln\left\{Q^{*}(k)\right\}.
\end{aligned}$$

Thus we have

$$\ln\left[Y^{*}(k+1)\right] = -\frac{1}{k+1}\left[\sum_{r=0}^{k}\left[\ln Y^{*}(r)\ln P^{*}(k-r)\right] - \ln\left\{Q^{*}(k)\right\}\right].$$

As a result, we get the recurrence relation

$$Y^{*}(k+1) = \exp\left\{-\frac{1}{k+1}\left[\sum_{r=0}^{k}\left[\ln Y^{*}(r)\ln P^{*}(k-r)\right] - \ln\left\{Q^{*}(k)\right\}\right]\right\}.$$

And, applying the multiplicative differential transform to $y(0) = y_0$ we have

$$Y^{*}(0) = y_0. \tag{4.3}$$

□

***Example* 4.1.** Suppose the Cauchy problem

$$(y^*)\, y^{\ln e^{4x}} = e^{2x}, y(0) = e^{3/2} \tag{4.4}$$

is given near $x = 0$. Let's solve it by multiplicative differential transform method. First, the function $p(x) = e^{4x}$ has multiplicative differential transform

$$\begin{aligned}
P^*(1) &= \left[p^{*(1)}(x)\right]^{1/1!}\Big|_{x=0} = \left[e^{\left(\ln e^{4x}\right)'}\right]^{1/1!}\Big|_{x=0} = e^4 \\
P^*(0) &= \left[p(0)\right]^{1/0!}\Big|_{x=0} = e^0 = 1 \\
P^*(k) &= \left[p^{*(k)}(0)\right]^{1/k!}\Big|_{x=0} = 1^{1/k!} = 1, \quad k = 2, 3, \ldots
\end{aligned}$$

The second function $q(x) = e^{2x}$ has multiplicative differential transform

$$\begin{aligned}
Q^*(1) &= \left[q^*(0)\right]^{1/1!} = e^2 \\
Q^*(k) &= 1, \text{ for } k = 0 \text{ or } k = 2, 3, \ldots
\end{aligned}$$

To sum up

$$P^*(k) = \begin{cases} 1, & \text{for } k \neq 1, \\ e^4, & \text{for } k = 1, \end{cases} \tag{4.5}$$

and

$$Q^*(k) = \begin{cases} 1, & \text{for } k \neq 1, \\ e^2, & \text{for } k = 1 \end{cases}. \tag{4.6}$$

Also we will use $Y^*(0) = y(0) = e^{3/2}$. Thus we must solve the recurrence relations

$$\begin{aligned}
Y^*(k+1) &= \exp\left\{-\frac{1}{k+1}\left[\sum_{r=0}^{k}\left[\ln Y^*(r)\ln P^*(k-r)\right] - \ln\{Q^*(k)\}\right]\right\}, \\
Y^*(0) &= e^{3/2}.
\end{aligned}$$

For $k = 0$ we have

$$\begin{aligned}
Y^*(1) &= \exp\left\{-\frac{1}{0+1}\left[\sum_{r=0}^{0}\left[\ln Y^*(r)\ln P^*(-r)\right] - \ln\{Q^*(0)\}\right]\right\} \\
&= \exp\left\{-\left(\left[\ln Y^*(0)\ln P^*(0)\right] - \ln\{Q^*(0)\}\right)\right\} \\
&= \exp\left\{-\left[\frac{3}{2}\cdot 0\right] + 0\right\} \\
Y^*(1) &= 1.
\end{aligned} \tag{4.7}$$

For $k = 1$ we have

$$
\begin{aligned}
Y^*(2) &= \exp\left\{-\frac{1}{2}\left[\sum_{r=0}^{1}[\ln Y^*(r)\ln P^*(1-r)] - \ln\{Q^*(1)\}\right]\right\}\\
&= \exp\left\{-\frac{1}{2}[\ln Y^*(0)\ln P^*(1) + \ln Y^*(1)\ln P^*(0) - \ln\{Q^*(1)\}]\right\}\\
&= \exp\left\{-\frac{1}{2}\left[\ln e^{3/2}\ln e^4 + \ln 1\ln 1 - \ln e^2\right]\right\}\\
&= \exp(-2)\\
Y^*(2) &= e^{-2}. \qquad (4.8)
\end{aligned}
$$

Thus an approximate solution of order two is

$$
\begin{aligned}
y(x) &\cong \prod_{k=0}^{2}[Y^*(k)]^{x^k} = Y^*(0)[Y^*(1)]^x[Y^*(2)]^{x^2} \qquad (4.9)\\
y(x) &\cong e^{3/2}\cdot\left(e^{-2}\right)^{x^2} \qquad (4.10)
\end{aligned}
$$

5. Approximate solution of the Cauchy problem for second order linear multiplicative differential equation by the multiplicative differential transform method

Theorem 5.1. *Suppose the Cauchy problem for second order linear multiplicative differential equation*

$$(y^{**})(y^*)^{\gamma_1}y^{\gamma_0} = f(x),\ y(0) = y_0,\ y^*(0) = y_1 \qquad (5.1)$$

is given near $x = 0$. Here $\gamma_0 = \ln a_0$, $\gamma_1 = \ln a_1$ and the multiplicative differential transforms of $y(x), a_0(x)$ and $a_1(x)$ are $Y^*(k)$, $A_0^*(k)$ and $A_1^*(k)$, respectively. Solution of this problem is obtained by the recurrence relations below:

$$
\begin{aligned}
Y^*(k+2) &= \exp\left\{\frac{\ln F^*(k) - \sum_{r=0}^{k}\left[(r+1)\ln Y^*(r+1)\ln A_1^*(k-r) + \ln Y^*(r)\ln A_0^*(k-r)\right]}{(k+2)(k+1)}\right\}, \qquad (5.2)\\
Y^*(0) &= y_0,\ Y^*(1) = y_1. \qquad (5.3)
\end{aligned}
$$

Proof. Using the equalities $\gamma_1 = \ln a_1$, $\gamma_0 = \ln a_0$, the equation above can be written as

$$f(x) = (y^{**})(y^*)^{\ln a_1}y^{\ln a_0}$$

Taking the multiplicative differential transform of both sides of the equation

$$
\begin{aligned}
F^*(k) &= Y^*(k+2)^{(k+2)(k+1)} \prod_{r=0}^{k} \exp\left\{\ln Y^*(r+1)^{(r+1)} \ln A_1^*(k-r)\right\} \prod_{r=0}^{k} \exp\left\{\ln Y^*(r) \ln A_0^*(k-r)\right\} \\
&= Y^*(k+2)^{(k+2)(k+1)} \exp \sum_{r=0}^{k} \left\{\ln Y^*(r+1)^{(r+1)} \ln A_1^*(k-r)\right\} \exp \sum_{r=0}^{k} \left\{\ln Y^*(r) \ln A_0^*(k-r)\right\} \\
&= Y^*(k+2)^{(k+2)(k+1)} \exp\left(\sum_{r=0}^{k} \left\{\ln Y^*(r+1)^{(r+1)} \ln A_1^*(k-r)\right\} + \sum_{r=0}^{k} \left\{\ln Y^*(r) \ln A_0^*(k-r)\right\} \right) \\
&= Y^*(k+2)^{(k+2)(k+1)} \exp\left(\sum_{r=0}^{k} \left\{\ln Y^*(r+1)^{(r+1)} \ln A_1^*(k-r) + \ln Y^*(r) \ln A_0^*(k-r)\right\} \right) \\
&= Y^*(k+2)^{(k+2)(k+1)} \prod_{r=0}^{k} \exp\left\{\ln Y^*(r+1)^{(r+1)} \ln A_1^*(k-r) + \ln Y^*(r) \ln A_0^*(k-r)\right\}
\end{aligned}
$$

Taking logarithm of both sides, we can write

$$
\begin{aligned}
&\ln F^*(k) \\
&= \ln\left[Y^*(k+2)^{(k+2)(k+1)} \prod_{r=0}^{k} \exp\left\{\ln Y^*(r+1)^{(r+1)} \ln A_1^*(k-r) + \ln Y^*(r) \ln A_0^*(k-r)\right\} \right] \\
&= (k+2)(k+1) \ln\left[Y^*(k+2)\right] + \ln\left[\prod_{r=0}^{k} \exp\left\{\ln Y^*(r+1)^{(r+1)} \ln A_1^*(k-r) + \ln Y^*(r) \ln A_0^*(k-r)\right\} \right] \\
&= (k+2)(k+1) \ln\left[Y^*(k+2)\right] + \ln \exp \textstyle\sum_{r=0}^{k} \left\{\ln Y^*(r+1)^{(r+1)} \ln A_1^*(k-r) + \ln Y^*(r) \ln A_0^*(k-r)\right\} \\
&= (k+2)(k+1) \ln\left[Y^*(k+2)\right] + \textstyle\sum_{r=0}^{k} \left\{\ln Y^*(r+1)^{(r+1)} \ln A_1^*(k-r) + \ln Y^*(r) \ln A_0^*(k-r)\right\}
\end{aligned}
$$

Thus we have

$$
\ln Y^*(k+2) = \frac{\ln F^*(k) - \sum_{r=0}^{k} \left[(r+1) \ln Y^*(r+1) \ln A_1^*(k-r) + \ln Y^*(r) \ln A_0^*(k-r)\right]}{(k+2)(k+1)},
$$

and we get the recurrence relation

$$
Y^*(k+2) = \exp\left\{ \frac{\ln F^*(k) - \sum_{r=0}^{k} \left[(r+1) \ln Y^*(r+1) \ln A_1^*(k-r) + \ln Y^*(r) \ln A_0^*(k-r)\right]}{(k+2)(k+1)} \right\}.
$$

For instance for $k = 0$ we have

$$
Y^*(2) = \exp\left\{ \frac{1}{2} \left[\ln F^*(0) - \ln Y^*(1) \ln A_1^*(0) - \ln Y^*(0) \ln A_0^*(0)\right] \right\}
$$

and for $k = 1$ we have

$$
Y^*(3) \;=\; \exp\left\{ \frac{1}{6} \left[\ln F^*(1) - \sum_{r=0}^{1} \left\{(r+1) \ln Y^*(r+1) \ln A_1^*(1-r) + \ln Y^*(r) \ln A_0^*(1-r)\right\} \right] \right\}
$$

$$
\begin{aligned}
Y^{*}(3) &= \exp\left\{\frac{1}{6}\left[\ln F^{*}(1) - \ln Y^{*}(1)\ln A_{1}^{*}(1) - \ln Y^{*}(0)\ln A_{0}^{*}(1)\right.\right.\\
&\quad \left.\left. -2\ln Y^{*}(2)\ln A_{1}^{*}(0) - \ln Y^{*}(1)\ln A_{0}^{*}(0)\right]\right\}
\end{aligned}
$$

And, using multiplicative differential transform to $y(0) = y_0$, $y^{*}(0) = y_1$ we have

$$
\begin{aligned}
Y^{*}(0) &= y_0,\\
Y^{*}(1) &= y_1.
\end{aligned}
$$

□

***Example* 5.1.** Suppose the Cauchy problem

$$
\left(y^{**}\right)\left(y^{*}\right)^{3} y^{2} = 1,\ y(0) = e,\ y^{*}(0) = e^{-1} \tag{5.4}
$$

is given. Multiplicative differential transforms of $a_0 = e^2$ and $a_1 = e^3$ are

$$
A_{0}^{*}(k) = \begin{cases} e^{2}, & \text{for } k = 0,\\ 1, & \text{for } k > 0, \end{cases} \quad \text{and } A_{1}^{*}(k) = \begin{cases} e^{3}, & \text{for } k = 0,\\ 1, & \text{for } k > 0, \end{cases} \tag{5.5}
$$

respectively. Hence

$$
\ln A_{0}^{*}(k) = \begin{cases} 2, & \text{for } k = 0,\\ 0, & \text{for } k > 0, \end{cases} \quad \text{and } \ln A_{1}^{*}(k) = \begin{cases} 3, & \text{for } k = 0,\\ 0, & \text{for } k > 0, \end{cases} \tag{5.6}
$$

So, for the sum in the recurrence (5.2), namely

$$
Y^{*}(k+2) = \exp\left\{\frac{\ln F^{*}(k) - \sum_{r=0}^{k}\left[(r+1)\ln Y^{*}(r+1)\ln A_{1}^{*}(k-r) + \ln Y^{*}(r)\ln A_{0}^{*}(k-r)\right]}{(k+2)(k+1)}\right\}
$$

we have

$$
\ln A_{1}^{*}(k-r) = 0 \text{ and } \ln A_{0}^{*}(k-r) = 0
$$

for $r < k$. Moreover, differential transform function of $f(x) = 1$ is $F^{*}(k) = 1$ so $\ln F^{*}(k) = 0$. Thus, we write

$$
\begin{aligned}
Y^{*}(k+2) &= \exp\left\{\frac{-\sum_{r=0}^{k}\left[(r+1)\ln Y^{*}(r+1)\ln A_{1}^{*}(k-r) + \ln Y^{*}(r)\ln A_{0}^{*}(k-r)\right]}{(k+2)(k+1)}\right\}\\
&= \exp\left\{\left[(k+1)\ln Y^{*}(k+1)\ln A_{1}^{*}(0) + \ln Y^{*}(k)\ln A_{0}^{*}(0)\right]\cdot\frac{-1}{(k+2)(k+1)}\right\}\\
&= \exp\left\{\left[3(k+1)\ln Y^{*}(k+1) + 2\ln Y^{*}(k)\right]\cdot\frac{-1}{(k+2)(k+1)}\right\}\\
&= \exp\left\{\left[\ln\left(Y^{*}(k+1)\right)^{3(k+1)} + \ln\left(Y^{*}(k)\right)^{2}\right]\cdot\frac{-1}{(k+2)(k+1)}\right\}
\end{aligned}
$$

$$= \exp\left\{\ln\left[(Y^*(k+1))^{3(k+1)}(Y^*(k))^2\right]\cdot\frac{-1}{(k+2)(k+1)}\right\}$$

$$= \left(\exp\left\{\ln\left[(Y^*(k+1))^{3(k+1)}(Y^*(k))^2\right]\right\}\right)^{-1/[(k+2)(k+1)]},$$

and we get the recurrence relation

$$Y^*(k+2) = \left\{[Y^*(k+1)]^{3(k+1)}[Y^*(k)]^2\right\}^{-1/[(k+2)(k+1)]}. \tag{5.7}$$

From the conditions $y(0) = e$, $y^*(0) = e^{-1}$ we have

$$\begin{aligned} Y^*(0) &= e, \\ Y^*(1) &= e^{-1} \end{aligned}$$

respectively. Hence solution will be obtained from the recurrence relations

$$\begin{aligned} Y^*(k+2) &= \left\{[Y^*(k+1)]^{3(k+1)}[Y^*(k)]^2\right\}^{-1/[(k+2)(k+1)]}, \\ Y^*(0) &= e, \\ Y^*(1) &= e^{-1}. \end{aligned} \tag{5.8}$$

For $k = 0$

$$\begin{aligned} Y^*(2) &= \left\{[Y^*(1)]^3[Y^*(0)]^2\right\}^{-1/2} = \left(e^{-3}e^2\right)^{-1/2} \\ Y^*(2) &= e^{1/2}, \end{aligned} \tag{5.9}$$

For $k = 1$

$$\begin{aligned} Y^*(3) &= \left\{[Y^*(2)]^{3\cdot 2}[Y^*(1)]^2\right\}^{-1/(3\cdot 2)} = \left(e^3e^{-2}\right)^{-1/6} \\ Y^*(3) &= e^{-1/6}. \end{aligned} \tag{5.10}$$

Thus an approximate solution of order three is

$$\begin{aligned} y(x) &\cong \prod_{k=0}^{3}[Y^*(k)]^{x^k} = Y^*(0)[Y^*(1)]^x[Y^*(2)]^{x^2}[Y^*(3)]^{x^3} \\ &\cong e\left(e^{-1}\right)^x\left(e^{1/2}\right)^{x^2}\left(e^{-1/6}\right)^{x^3} \\ y(x) &\cong \exp\left(1 - x + \frac{x^2}{2} - \frac{x3}{6}\right). \end{aligned} \tag{5.11}$$

6. Conclusions

In this study, we present the multiplicative differential transform method (MDTM) to find the approximate numerical solution of multiplicative ordinary differential equations. Then, we apply this new method to some multiplicative ordinary differential equations. It is observed that the MDTM is an effective method for multiplicative ordinary differential equations. Multiplicative power series solutions are obtained with the MDTM. The number of terms in the solution is increased to improve the accuracy of the obtained approximate solution. In some examples, the series solution obtained by the help of the MDTM can be written as an exact solution.

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Conflict of interest

The authors declare that they have no competing interests regarding the publication of this manuscript.

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