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Research article

Neutral differential equations with noncanonical operator: Oscillation behavior of solutions

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Abstract: The objective of this work is to study the oscillatory behavior of neutral differential equations with several delays. By using both Riccati substitution technique and comparison with delay equations of first-order, we establish new oscillation criteria. Our new criteria are simplifying and complementing some related results that have been published in the literature. Moreover, some examples are given to show the applicability of our results.

Keywords: differential equations; even-order; neutral delay; noncanonical operator; oscillation criteria
Mathematics Subject Classification: 34C10, 34K11

1. Introduction

In this work, we consider the even-order neutral differential equation with several delays

$$\left(r(l)\left(v^{(n-1)}(l)\right)^{\alpha}\right)' + \sum_{i=1}^{k} q_i(l) f(u(g_i(l))) = 0, \tag{1.1}$$

where $l \geq l_0$, $v(l) = u(l) + p(l) u(\tau(l))$, $n \geq 4$ is an even integer, $\alpha \in Q^{+}_{odd} := \{a/b : a, b \in \mathbb{Z}^{+} \text{ are odd}\}$ and the following conditions are fulfilled:

(**i**) r is a differentiable real-valued function and p, τ, q_i are continuous real-valued functions on $[l_0, \infty)$;

(ii) $r'(l) \geq 0$, $p(l) \in [0, p_0]$, p_0 is a constant, $\tau(l) \leq l$, and $\lim_{l\to\infty} \tau(l) = \infty$;
(iii) $g_i \in C([l_0, \infty), \mathbb{R})$, $g_i(l) \leq l$, $g_i'(l) > 0$ and $\lim_{l\to\infty} g_i(l) = \infty$;
(iv) $f \in C(\mathbb{R}, \mathbb{R})$, $f(u) \geq \varrho u^\beta$ for $u \neq 0$, ϱ is a positive constant, β is a ratio of odd positive integers; and

$$\int_{l_0}^{\infty} r^{-1/\alpha}(s)\,\mathrm{d}s < \infty. \tag{1.2}$$

The function $u \in C([l_u, \infty))$ with $l_u \geq l_0$, is said to be a solution of (1.1) if u has the property $v \in C^{n-1}[l_u, \infty)$, $r\left(v^{(n-1)}\right)^\alpha \in C^1[l_u, \infty)$, and satisfies (1.1) on $[l_u, \infty)$. We consider only those solutions u of (1.1) which satisfy $\sup\{|u(l)| : l \geq l\} > 0$, for all $l \geq l_u$. As usual, a solution of (1.1) is called oscillatory if it has arbitrarily large zeros, otherwise, it is called nonoscillatory.

In numerical models of various physical, organic, and intrinsic phenomena, differential equations (even of the fourth order) are usually experienced. In particular, there are many applications of the delay differential equation, for example, in elasticity problems, structural deformation principles, or soil settlement; see [23, 24].

The oscillation and nonoscillation of higher-order functional differential equations have concerned many authors, see [2–33]. A broad range of methods have been used to investigate the properties of solutions to various groups of equations. As a matter of fact, equation (1.1) (i.e., half-linear/Emden-Fowler differential equation) arises in a variety of real-world problems such as in the study of p-Laplace equations non-Newtonian fluid theory, the turbulent flow of a polytrophic gas in a porous medium, and so forth; see the following papers for more details [5–7].

Agarwal et al. [2] and Zhang et al. [29] investigated the oscillatory behavior of a higher-order differential equation

$$\left(r(l)\left(u^{(n-1)}(l)\right)^\alpha\right)' + q(l)\,u^\beta(\tau(l)) = 0, \tag{1.3}$$

and considered the both cases (1.2) and

$$\int_{l_0}^{\infty} r^{-1/\alpha}(s)\,\mathrm{d}s = \infty. \tag{1.4}$$

In particular, assuming that $\tau(l) < l$, $\alpha \geq \beta$ and (1.2) holds, the results obtained by Zhang et al. [29] ensure that every solution u of (1.3) is either oscillatory or satisfies $\lim_{l\to\infty} u(l) = 0$.
Meng and Xu [16] established oscillation criteria for even-order neutral differential equations

$$\left(a(l)\left|w^{(n-1)}(l)\right|^{\alpha-1} w(l)\right)' + q(l)\,f(u(\sigma(l))) = 0, \tag{1.5}$$

where $w(l) = (l) + p(l)\,u(l-\tau)$, $a'(l) \geq 0$, $f(u)/|u|^{\alpha-1}u \geq k > 0$, k is a constant and (1.4) holds.
Baculikova et al. [3] considered the equation

$$\left[r(l)\left(u^{(n-1)}(l)\right)^\alpha\right]' + q(l)\,f(u(\tau(l))) = 0$$

and proved this equation is oscillatory if the first-order equation

$$y'(l) + q(l)f\left(\frac{\delta\tau^{n-1}(l)}{(n-1)!\,r^{\frac{1}{\alpha}}(\tau(l))}\right) f\left(y^{\frac{1}{\alpha}}(\tau(l))\right) = 0$$

is oscillatory when (1.4) holds.
Moaaz et al. [21] investigated the oscillatory behavior of the equation

$$(a(l)(v^{(n-1)}(l))^{\alpha})' + f(l, u(\sigma(l))) = 0,$$

where $0 \le p(l) \le p_0 < \infty$, $|f(l,u)| \ge q(l)|u|^{\alpha}$ and under the condition (1.4).

In this work, based on the Riccati substitution technique and comparison with delay equations of first-order, we obtain new sufficient conditions for oscillation of (1.1). Unlike most of the previous related works, we are interested in studying (1.1) in the noncanonical case (1.2). Examples illustrating our new results are also given.

The following lemmas are needed in the proofs of our results:

Lemma 1.1. *[1] Let $\psi \in C^n([l_0,\infty), \mathbb{R}^+)$, $\psi^{(n)}$ be of fixed sign and not identically zero on a subray of $[l_0,\infty)$, and $\psi^{(n-1)}\psi^{(n)} \le 0$ for $l \ge l_1 \in [l_0,\infty)$. If $\lim_{l\to\infty} \psi(l) \neq 0$, then*

$$\psi \ge \frac{\lambda}{(n-1)!} l^{n-1} \left|\psi^{(n-1)}\right|,$$

for every $\lambda \in (0,1)$ and $l \ge l_\lambda \in [l_1,\infty)$.

Lemma 1.2. *[20] Assume that $s \ge 0$, $B \ge 0$ and $A > 0$. Then*

$$Bs - As^{(\alpha+1)/\alpha} \le \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^{\alpha}}.$$

Lemma 1.3. *[11, Lemma 1.1] Assume that $f \in C^m([l_0,\infty),(0,\infty))$ and $f^{(m)}$ is eventually of one sign for all large l. Then, there exists a nonnegative integer $h \le m$, with $m+h$ even for $f^{(m)} \ge 0$, or $m+h$ odd for $f^{(m)} \le 0$ such that*

$$h > 0 \text{ yields } f^{(k)}(l) > 0 \text{ for } k = 0,1,...,h-1,$$

and

$$h \le m-1 \text{ yields } (-1)^{h+k} f^{(k)}(l) > 0 \text{ for } k = h, h+1, ..., m-1,$$

eventually.

Lemma 1.4. *[11] If $\varphi \in C^m([l_0,\infty),(0,\infty))$, $\varphi^{(k)}(l) > 0$, $k = 0,1,...,m$ and $\varphi^{(m+1)}(l) < 0$. Then,*

$$\varphi(l) \ge \frac{\lambda}{m} l\varphi'(l),$$

for every $\lambda \in (0,1)$ eventually.

2. Main results

Let us define the following:

$$\eta(l) := \begin{cases} c_1^{\beta-\alpha} & \text{if } \alpha \ge \beta; \\ c_2\, \delta_0^{\beta-\alpha}(l) & \text{if } \alpha < \beta, \end{cases}$$

$$\mu(l) := \begin{cases} c_3^{\beta-\alpha} & \text{if } \alpha \geq \beta; \\ \left(\frac{c_4}{(n-3)!}\int_l^\infty (\varrho - l)^{n-3}\,\delta_0(\varrho)\mathrm{d}\varrho\right)^{\beta-\alpha} & \text{if } \alpha < \beta, \end{cases}$$

$$\delta_0(l) := \int_l^\infty \frac{1}{r^{1/\alpha}(\nu)}\mathrm{d}\nu, \tag{2.1}$$

$$g(l) := \min\{g_i(l) : i = 1, 2, ..., k\}$$

and

$$Q(l) := \frac{\lambda_1 g^{n-2}(l)}{(n-2)!},$$

where c_1, c_2, c_3 and c_4 are any positive constants. It is recognized that the identification of the signs of the solution derivatives is required and, before studying the oscillation of the delay differential equations, causes a significant effect.

Lemma 2.1. *Assume that* $u \in C([l_0, \infty), (0, \infty))$ *is a solution of (1.1). Then,* $\left(r(l)\left(v^{(n-1)}(l)\right)^{\alpha}\right)' \leq 0$, *and one of the next cases holds, for l large enough*

$$\begin{array}{ll} (A) & v(l) > 0, v'(l) > 0, v^{(n-1)}(l) > 0 \text{ and } v^{(n)}(l) < 0; \\ (B) & v(l) > 0, v'(l) > 0, v^{(n-2)}(l) > 0 \text{ and } v^{(n-1)}(l) < 0; \\ (C) & v(l) > 0, (-1)^k v^{(k)}(l) > 0 \text{ for } k = 1, 2, ..., n-1. \end{array}$$

Proof. Assume that u is an eventually positive solution of (1.1). Then, there exists $l_1 \geq l_0$ such that $u(l)$, $u(\tau(l))$ and $u(g(l))$ are positive for all $l \geq l_1$. Hence, by the definition of v, we have that $v(l) > 0$ for $l \geq l_1$. It follows from (1.1) that $\left(r(l)\left(v^{(n-1)}(l)\right)^{\alpha}\right)' \leq 0$. Next, using Lemma 1.3 and considering that n is even, we directly get the cases (**A**) – (**C**). □

Lemma 2.2. *Assume that* $u \in C([l_0, \infty), (0, \infty))$ *is a solution of (1.1) whose* v *satisfies* (**B**). *Then* $\left(v^{(n-2)}(l)\right)^{\beta-\alpha} \geq \eta(l)$, *eventually.*

Proof. The proof for this lemma is analogous to the proof of Lemma 2.1 in [18]. Hence, we omit it here. □

Lemma 2.3. *Assume that* $u \in C([l_0, \infty), (0, \infty))$ *is a solution of (1.1) whose* v *satisfies* (**C**). *Then* $v^{\beta-\alpha}(l) \geq \mu(l)$, *eventually.*

Proof. The proof for this lemma is similar to the proof of Lemma 2.2 in [18]. Hence, we omit it here. □

Lemma 2.4. *Assume that* $u \in C([l_0, \infty), (0, \infty))$ *is a solution of (1.1) whose* v *satisfies* (**C**). *If*

$$\int_{l_0}^\infty \left(\int_l^\infty (\xi - l)^{n-3}\left(\frac{1}{r(\xi)}\int_{l_1}^\xi \sum_{i=1}^k q_i(v)\,\mathrm{d}v\right)^{1/\alpha}\mathrm{d}\xi\right)\mathrm{d}l = \infty, \tag{2.2}$$

then $\lim_{l\to\infty} u(l) = 0$.

Proof. Let $u \in C([l_0,\infty),(0,\infty))$ be a solution of (1.1) and case (**C**) holds. Then, $\lim_{l\to\infty} v(l) = D$. We claim that $D = 0$. Indeed, for the sake of a contradiction, suppose that $D > 0$, there exists a $l_1 \geq l_0$ such that $u(g(l)) \geq D$ for $l \geq l_1$. Integrating (1.1) on $[l_1, l]$, we have

$$\begin{aligned} r(l)\left(v^{(n-1)}(l)\right)^{\alpha} &= r(l_1)\left(v^{(n-1)}(l_1)\right)^{\alpha} - \int_{l_1}^{l} \sum_{i=1}^{k} q_i(v) f(u(g_i(v)))\,\mathrm{d}v \\ &\leq -\varrho D^{\beta} \int_{l_1}^{l} \sum_{i=1}^{k} q_i(v)\,\mathrm{d}v, \end{aligned}$$

that is,

$$v^{(n-1)}(l) \leq -\varrho^{1/\alpha} D^{\beta/\alpha} \left(\frac{1}{r(l)} \int_{l_1}^{l} \sum_{i=1}^{k} q_i(v)\,\mathrm{d}v\right)^{1/\alpha}. \tag{2.3}$$

Integrating (2.3) twice on $[l,\infty)$, we have

$$-v^{(n-2)}(l) \leq -\varrho^{1/\alpha} D^{\beta/\alpha} \int_{l}^{\infty} \left(\frac{1}{r(\xi)} \int_{l_1}^{\xi} \sum_{i=1}^{k} q_i(v)\,\mathrm{d}v\right)^{1/\alpha} \mathrm{d}\xi$$

and

$$\begin{aligned} v^{(n-3)}(l) &\leq -\varrho^{1/\alpha} D^{\beta/\alpha} \int_{l}^{\infty} \int_{\zeta}^{\infty} \left(\frac{1}{r(\xi)} \int_{l_1}^{\xi} \sum_{i=1}^{k} q_i(v)\,\mathrm{d}v\right)^{1/\alpha} \mathrm{d}\xi\mathrm{d}\zeta \\ &\leq -\varrho^{1/\alpha} D^{\beta/\alpha} \int_{l}^{\infty} (\xi - l)\left(\frac{1}{r(\xi)} \int_{l_1}^{\xi} \sum_{i=1}^{k} q_i(v)\,\mathrm{d}v\right)^{1/\alpha} \mathrm{d}\xi. \end{aligned}$$

Similarly, by integrating the above inequality $(n-4)$ times on $[l,\infty)$, we get

$$v'(l) \leq -\varrho^{1/\alpha} D^{\beta/\alpha} \int_{l}^{\infty} (\xi - l)^{n-3}\left(\frac{1}{r(\xi)} \int_{l_1}^{\xi} \sum_{i=1}^{k} q_i(v)\,\mathrm{d}v\right)^{1/\alpha} \mathrm{d}\xi.$$

Integrating this inequality on $[l_1,\infty)$, we find

$$v(l_1) \geq \varrho^{1/\alpha} D^{\beta/\alpha} \int_{l_1}^{\infty} \left(\int_{l}^{\infty} (\xi - l)^{n-3}\left(\frac{1}{r(\xi)} \int_{l_1}^{\xi} \sum_{i=1}^{k} q_i(v)\,\mathrm{d}v\right)^{1/\alpha} \mathrm{d}\xi\right) \mathrm{d}l,$$

which is a contradiction with (2.2). Thus, $D = 0$; moreover the inequality $u \leq v$ implies $\lim_{l\to\infty} u(l) = 0$. The proof of this lemma is complete. □

Theorem 2.1. *Assume that (2.2) holds and $p_0 < 1$. If the first-order delay differential equation*

$$y'(l) + \varrho\left(\frac{\lambda_0 g^{n-1}(l)}{(n-1)!\,r^{1/\alpha}(g(l))}\right)^{\beta} \left(\sum_{i=1}^{k} q_i(l)(1 - p(g_i(l)))^{\beta}\right) y^{\beta/\alpha}(g(l)) = 0 \tag{2.4}$$

is oscillatory for some and

$$\limsup_{l\to\infty}\int_{l_0}^{l}\left(\varrho\eta(v)\,\delta_0^{\alpha}(v)\,Q^{\beta}(v)\sum_{i=1}^{k}q_i(v)\,(1-p(g_i(v)))^{\beta}-\frac{\alpha^{\alpha+1}r^{-1/\alpha}(v)}{(\alpha+1)^{\alpha+1}\,\delta_0(v)}\right)\mathrm{d}v=\infty \tag{2.5}$$

holds for some $\lambda,\lambda_0,\lambda_1\in(0,1)$*, then every solution of (1.1) is either oscillatory or converges to zero as* $l\to\infty$*.*

Proof. Assume the contrary that there is a nonoscillatory solution u of (1.1). Then, we can assume $u(l)$, $u(\tau(l))$ and $u(g(l))$ are positive for $l\geq l_1\geq l_0$. It follows from Lemma 2.1 that the behavior of v and its derivatives is possible in three cases. First, suppose that case (**A**) holds. Based on the definition of v, we see that

$$u(l)=v(l)-p(l)\,u(\tau(l))\geq(1-p(l))\,v(l), \tag{2.6}$$

and so

$$u^{\beta}(g_i(l))\geq(1-p(g_i(l)))^{\beta}\,v^{\beta}(g_i(l)), \tag{2.7}$$

from (iv) and (2.7), we have

$$f(u(g_i(l)))\geq\varrho\,(1-p(g_i(l)))^{\beta}\,v^{\beta}(g_i(l)),$$

which with (1.1) gives

$$\begin{aligned}\left(r(l)\left(v^{(n-1)}(l)\right)^{\alpha}\right)'&\leq-\varrho\sum_{i=1}^{k}q_i(l)\,(1-p(g_i(l)))^{\beta}\,v^{\beta}(g_i(l))\\&\leq-\varrho v^{\beta}(g(l))\sum_{i=1}^{k}q_i(l)\,(1-p(g_i(l)))^{\beta}.\end{aligned} \tag{2.8}$$

From Lemma 1.1, we have

$$v(g(l))\geq\frac{\lambda g^{n-1}(l)}{(n-1)!}v^{(n-1)}(g(l)), \tag{2.9}$$

for every $\lambda\in(0,1)$. From (2.9) and (2.8), we obtain

$$\left(r(l)\left(v^{(n-1)}(l)\right)^{\alpha}\right)'\leq-\varrho\left(\frac{\lambda g^{n-1}(l)}{(n-1)!}\right)^{\beta}\left(v^{(n-1)}(g(l))\right)^{\beta}\sum_{i=1}^{k}q_i(l)\,(1-p(g_i(l)))^{\beta}.$$

Let $y(l)=r(l)\left(v^{(n-1)}(l)\right)^{\alpha}$. Clearly, y is a positive solution of the first-order delay differential inequality

$$y'(l)+\varrho\left(\frac{\lambda g^{n-1}(l)}{(n-1)!r^{1/\alpha}(g(l))}\right)^{\beta}\left(\sum_{i=1}^{k}q_i(l)\,(1-p(g_i(l)))^{\beta}\right)y^{\beta/\alpha}(g(l))\leq 0. \tag{2.10}$$

It follows from [22, Theorem 1] that the corresponding differential equation (2.4) also has a positive solution for all $\lambda_0\in(0,1)$, which is a contradiction.
Next, we assume that the case (**B**) holds. We define the function Φ by

$$\Phi(l)=\frac{r(l)\left(v^{(n-1)}(l)\right)^{\alpha}}{\left(v^{(n-2)}(l)\right)^{\alpha}}. \tag{2.11}$$

Then $\Phi(l) < 0$ for $l \geq l_1$. Since $r(l)\left(v^{(n-1)}(l)\right)^{\alpha}$ is decreasing, we get

$$r^{1/\alpha}(s)\, v^{(n-1)}(s) \leq r^{1/\alpha}(l)\, v^{(n-1)}(l), \; s \geq l \geq l_1. \tag{2.12}$$

Multiplying (2.12) by $r^{-1/\alpha}(s)$ and integrating it on $[l, \infty)$, we obtain

$$0 \leq v^{(n-2)}(l) + r^{1/\alpha}(l)\, v^{(n-1)}(l)\, \delta_0(l),$$

that is,

$$-\frac{r^{1/\alpha}(l)\, v^{(n-1)}(l)\, \delta_0(l)}{v^{(n-2)}(l)} \leq 1.$$

From (2.11), we see that

$$-\Phi(l)\, \delta_0^{\alpha}(l) \leq 1. \tag{2.13}$$

Differentiating (2.11), we have

$$\Phi'(l) = \frac{\left(r(l)\left(v^{(n-1)}(l)\right)^{\alpha}\right)'}{\left(v^{(n-2)}(l)\right)^{\alpha}} - \frac{\alpha r(l)\left(v^{(n-1)}(l)\right)^{\alpha+1}}{\left(v^{(n-2)}(l)\right)^{\alpha+1}},$$

which, in view of (1.1) and (2.11), becomes

$$\Phi'(l) = -\frac{\sum_{i=1}^{k} q_i(l)\, f(u(g_i(l)))}{\left(v^{(n-2)}(l)\right)^{\alpha}} - \frac{\alpha \Phi^{(\alpha+1)/\alpha}(l)}{r^{1/\alpha}(l)}. \tag{2.14}$$

Since $v'(l) > 0$, we get that (2.8) holds. Hence, (2.14) becomes

$$\Phi'(l) \leq -\frac{-\varrho v^{\beta}(g(l)) \sum_{i=1}^{k} q_i(l)\,(1 - p(g_i(l)))^{\beta}}{\left(v^{(n-2)}(l)\right)^{\alpha}} - \frac{\alpha \Phi^{(\alpha+1)/\alpha}(l)}{r^{1/\alpha}(l)}. \tag{2.15}$$

From Lemma 1.1, we find

$$v(g(l)) \geq \frac{\lambda g^{n-2}(l)}{(n-2)!} v^{(n-2)}(g(l)),$$

for all sufficiently large l and for every $\lambda \in (0, 1)$. Then, (2.15) become

$$\begin{aligned} \Phi'(l) \;\; &\leq \;\; -\varrho Q^{\beta}(l)\left(\sum_{i=1}^{k} q_i(l)\,(1 - p(g_i(l)))^{\beta}\right)\left(v^{(n-2)}(g(l))\right)^{\beta-\alpha} \frac{\left(v^{(n-2)}(g(l))\right)^{\alpha}}{\left(v^{(n-2)}(l)\right)^{\alpha}} \\ &\quad - \frac{\alpha \Phi^{(\alpha+1)/\alpha}(l)}{r^{1/\alpha}(l)}. \end{aligned}$$

Since $l \geq g(l)$ and $v^{(n-2)}(l)$ is decreasing, we have

$$\Phi'(l) \leq -\varrho \eta(l)\, Q^{\beta}(l)\left(\sum_{i=1}^{k} q_i(l)\,(1 - p(g_i(l)))^{\beta}\right) - \frac{\alpha \Phi^{(\alpha+1)/\alpha}(l)}{r^{1/\alpha}(l)}. \tag{2.16}$$

Multiplying (2.16) by $\delta_0^{\alpha}(l)$ and integrating it on $[l_1, l]$, we get

$$0 \;\; \geq \;\; \Phi(l)\, \delta_0^{\alpha}(l) - \Phi(l_1)\, \delta_0^{\alpha}(l_1) + \int_{l_1}^{l} \frac{\alpha \delta_0^{\alpha-1}(v)}{r^{1/\alpha}(v)} \Phi(v)\, \mathrm{d}v + \int_{l_1}^{l} \frac{\alpha \delta_0^{\alpha}(v)}{r^{1/\alpha}(v)} \Phi^{(\alpha+1)/\alpha}(v)\, \mathrm{d}v$$

$$+\int_{l_1}^{l}\left(\varrho\eta(v)\,\delta_0^{\alpha}(v)\,Q^{\beta}(v)\sum_{i=1}^{k}q_i(v)\,(1-p(g_i(v)))^{\beta}\right)\mathrm{d}v.$$

Setting $A=\delta_0^{\alpha}(s)/r^{1/\alpha}(s)$, $B=\delta_0^{\alpha-1}(s)/r^{1/\alpha}(s)$ and $\vartheta=-\Phi(s)$, and using Lemma 1.2, we get

$$\int_{l_1}^{l}\left(\varrho\eta(v)\,\delta_0^{\alpha}(v)\,Q^{\beta}(v)\sum_{i=1}^{k}q_i(v)\,(1-p(g_i(v)))^{\beta}-\frac{\alpha^{\alpha+1}r^{-1/\alpha}(v)}{(\alpha+1)^{\alpha+1}\,\delta_0(v)}\right)\mathrm{d}v\leq\frac{\Phi(l_1)}{\delta_0^{-\alpha}(l_1)}+1,$$

due to (2.13), that contradicts (2.5).

Finally, suppose that (**C**) holds. From Lemma 2.4, one can see that $\lim_{l\to\infty}u(l)=0$, which is a contradiction.

This complete the proof. □

Theorem 2.2. *Suppose that the first-order delay differential equation (2.4) is oscillatory for some* $\lambda_0\in(0,1)$ *and (2.5) holds for some* $\lambda_1\in(0,1)$. *If*

$$\tau(g_i(l))=g_i(\tau(l)),\ \tau'(l)\geq\tau_0>0,\ g(l)\leq\tau(l),$$

and

$$\limsup_{l\to\infty}\varrho\delta_{n-2}^{\alpha}(l)\sum_{i=1}^{k}\mu(g_i(l))\int_{l_1}^{l}\Omega_i(v)\,\mathrm{d}v>\kappa\left(1+\frac{p_0^{\beta}}{\tau_0}\right),\tag{2.17}$$

then every solution of (1.1) is oscillatory, where $\Omega_i(l)=\min\{q_i(l),q_i(\tau(l))\}$,

$$\delta_{k+1}(l):=\int_{l}^{\infty}\delta_k(\varrho)\,\mathrm{d}\varrho\ \textit{for}\ k=0,1,...,n-3,$$

and $\kappa=1$ *if* $\beta\in(0,1]$*; otherwise,* $\kappa=2^{\beta-1}$.

Proof. Assume that there is a nonoscillatory solution u of (1.1). Then, we can assume $u(l)$, $u(\tau(l))$ and $u(g(l))$ are positive for $l\geq l_1\geq l_0$. It follows from Lemma 2.1 that the behavior of v and its derivatives is possible in three cases.

The proof of the case where (**A**) or (**B**) holds is the same as that of Theorem 2.1.

Suppose that (**C**) holds. Since $\left(r(l)\left(v^{(n-1)}(l)\right)^{\alpha}\right)'\leq0$, we have that

$$r(\zeta)\left(v^{(n-1)}(\zeta)\right)^{\alpha}-r(l)\left(v^{(n-1)}(l)\right)^{\alpha}\leq0\text{ for all }\zeta\geq l,$$

or

$$v^{(n-1)}(\zeta)\leq r^{1/\alpha}(l)\,v^{(n-1)}(l)\,\frac{1}{r^{1/\alpha}(\zeta)}.$$

Integrating this inequality from l to ∞ and making use of the fact that $v^{(n-2)}$ is a positive decreasing function, we arrive at

$$-v^{(n-2)}(l)\leq r^{1/\alpha}(l)\,v^{(n-1)}(l)\int_{l}^{\infty}\frac{1}{r^{1/\alpha}(\varrho)}\mathrm{d}\varrho=r^{1/\alpha}(l)\,v^{(n-1)}(l)\,\delta_0(l).\tag{2.18}$$

Taking into account the behavior of derivatives of ν and integrating (2.18) $(n-2)$ times from l to ∞, we see that

$$(-1)^{k+1}\nu^{(k)}(l) \le r^{1/\alpha}(l)\,\nu^{(n-1)}(l)\,\delta_{n-k-2}(l), \tag{2.19}$$

for $k = 0, 1, ..., n-3$. From (1.1), we get

$$\left(r(l)\left(\nu^{(n-1)}(l)\right)^{\alpha}\right)' + \varrho\sum_{i=1}^{k} q_i(l)\,u^{\beta}(g_i(l)) \le 0 \tag{2.20}$$

and

$$\frac{1}{\tau'(l)}\left(r(\tau(l))\left(\nu^{(n-1)}(\tau(l))\right)^{\alpha}\right)' + \varrho\sum_{i=1}^{k} q_i(\tau(l))\,u^{\beta}(g_i(\tau(l))) \le 0,$$

that is,

$$\frac{p_0^{\beta}}{\tau_0}\left(r(\tau(l))\left(\nu^{(n-1)}(\tau(l))\right)^{\alpha}\right)' + \varrho\sum_{i=1}^{k} q_i(\tau(l))\,p_0^{\beta}u^{\beta}(g_i(\tau(l))) \le 0. \tag{2.21}$$

From (2.20) and (2.21), we find

$$\begin{aligned}
0 &\ge \left(r(l)\left(\nu^{(n-1)}(l)\right)^{\alpha}\right)' + \frac{p_0^{\beta}}{\tau_0}\left(r(\tau(l))\left(\nu^{(n-1)}(\tau(l))\right)^{\alpha}\right)' \\
&\quad +\varrho\sum_{i=1}^{k} q_i(l)\,u^{\beta}(g_i(l)) + \varrho\sum_{i=1}^{k} q_i(\tau(l))\,p_0^{\beta}u^{\beta}(g_i(\tau(l))) \\
&\ge \left(r(l)\left(\nu^{(n-1)}(l)\right)^{\alpha}\right)' + \frac{p_0^{\beta}}{\tau_0}\left(r(\tau(l))\left(\nu^{(n-1)}(\tau(l))\right)^{\alpha}\right)' \\
&\quad +\varrho\sum_{i=1}^{k} \Omega_i(l)\left(u^{\beta}(g_i(l)) + p_0^{\beta}u^{\beta}(g_i(\tau(l)))\right) \\
&\ge \left(r(l)\left(\nu^{(n-1)}(l)\right)^{\alpha}\right)' + \frac{p_0^{\beta}}{\tau_0}\left(r(\tau(l))\left(\nu^{(n-1)}(\tau(l))\right)^{\alpha}\right)' \\
&\quad +\varrho\sum_{i=1}^{k} \Omega_i(l)\,\frac{1}{\kappa}\left(u(g_i(l)) + p(g_i(l))\,u(\tau(g_i(l)))\right)^{\beta} \\
&= \left(r(l)\left(\nu^{(n-1)}(l)\right)^{\alpha} + \frac{p_0^{\beta}}{\tau_0}r(\tau(l))\left(\nu^{(n-1)}(\tau(l))\right)^{\alpha}\right)' \\
&\quad +\frac{\varrho}{\kappa}\sum_{i=1}^{k} \Omega_i(l)\,\nu^{\beta}(g_i(l)).
\end{aligned}$$

By integrating this inequality from l_1 to l, we get

$$\begin{aligned}
r(l)\left(\nu^{(n-1)}(l)\right)^{\alpha} + \frac{p_0^{\beta}}{\tau_0}r(\tau(l))\left(\nu^{(n-1)}(\tau(l))\right)^{\alpha} &\le r(l_1)\left(\nu^{(n-1)}(l_1)\right)^{\alpha} + \frac{p_0^{\beta}}{\tau_0}r(\tau(l_1))\left(\nu^{(n-1)}(\tau(l_1))\right)^{\alpha} \\
&\quad -\frac{\varrho}{\kappa}\int_{l_1}^{l}\sum_{i=1}^{k}\Omega_i(v)\,\nu^{\beta}(g_i(v))\,\mathrm{d}v
\end{aligned}$$

$$\leq \quad -\frac{\varrho}{\kappa}\sum_{i=1}^{k}v^{\beta}\left(g_i\left(l\right)\right)\int_{l_1}^{l}\Omega_i\left(v\right)\mathrm{d}v.$$

Since $\left(r\left(l\right)v^{(n-1)}\left(l\right)\right)' \leq 0$, we arrive at

$$\begin{aligned}\left(1+\frac{p_0^{\beta}}{\tau_0}\right)r\left(l\right)\left(v^{(n-1)}\left(l\right)\right)^{\alpha} &\leq -\frac{\varrho}{\kappa}\sum_{i=1}^{k}v^{\alpha}\left(g_i\left(l\right)\right)v^{\beta-\alpha}\left(g_i\left(l\right)\right)\int_{l_1}^{l}\Omega_i\left(v\right)\mathrm{d}v \\ &\leq -\frac{\varrho}{\kappa}v^{\alpha}\left(l\right)\sum_{i=1}^{k}\mu\left(g_i\left(l\right)\right)\int_{l_1}^{l}\Omega_i\left(v\right)\mathrm{d}v,\end{aligned}$$

which, with Lemma 2.3, gives

$$\left(1+\frac{p_0^{\beta}}{\tau_0}\right)r\left(l\right)\left(v^{(n-1)}\left(l\right)\right)^{\alpha} \leq -\frac{\varrho}{\kappa}v^{\alpha}\left(l\right)\sum_{i=1}^{k}\mu\left(g_i\left(l\right)\right)\int_{l_1}^{l}\Omega_i\left(v\right)\mathrm{d}v. \tag{2.22}$$

Combining [(2.19), $k=0$] and (2.22), we have that

$$\left(1+\frac{p_0^{\beta}}{\tau_0}\right) \geq \frac{\varrho}{\kappa}\delta_{n-2}^{\alpha}\left(l\right)\sum_{i=1}^{k}\mu\left(g_i\left(l\right)\right)\int_{l_1}^{l}\Omega_i\left(v\right)\mathrm{d}v$$

which is a contradicts with (2.17). This completes the proof. □

In the following theorem, we set new conditions for testing the oscillation of (1.1) when $n=4$, which apply in the ordinary case.

Theorem 2.3. *Assume that $n=4$, $\alpha=\beta=1$, $p_0<1$ and (2.2) hold. Suppose also that*

$$\limsup_{l\to\infty}\int_{l_0}^{l}\left(\frac{\varrho\lambda_1 g^2\left(s\right)\sum_{i=1}^{k}q_i\left(s\right)\left(1-p\left(g_i\left(s\right)\right)\right)}{2!}\delta\left(s\right)-\frac{1}{4r\left(s\right)\delta\left(s\right)}\right)\mathrm{d}s=\infty, \tag{2.23}$$

for some constant $\lambda_1\in(0,1)$. Assume further that there exist two positive functions $\rho\left(l\right),\vartheta\left(l\right)\in C^1\left[l_0,\infty\right)$, such that

$$\int_{l_0}^{\infty}\left(\varrho\rho\left(s\right)\left(\frac{g\left(s\right)}{s}\right)^{3/\lambda}\sum_{i=1}^{k}q_i\left(s\right)\left(1-p\left(g_i\left(s\right)\right)\right)-\frac{1}{2}\frac{\left(\rho'\left(s\right)\right)^2}{\rho\left(s\right)}\frac{r\left(s\right)}{\lambda_2 s^2}\right)\mathrm{d}s=\infty \tag{2.24}$$

and

$$\int_{l_0}^{\infty}\left(\varrho\vartheta\left(\zeta\right)\int_{\zeta}^{\infty}\left(\frac{1}{r\left(v\right)}\int_{v}^{\infty}\sum_{i=1}^{k}q_i\left(s\right)\left(1-p\left(g_i\left(s\right)\right)\right)\left(\frac{g\left(s\right)}{s}\right)^{1/\lambda}\mathrm{d}s\right)\mathrm{d}v-\frac{\left(\vartheta'\left(\zeta\right)\right)^2}{4\vartheta\left(\zeta\right)}\right)\mathrm{d}\zeta=\infty \tag{2.25}$$

where $\lambda_2\in(0,1)$. Then every solution of (1.1) is oscillatory or tends to zero as $l\to\infty$.

Proof. Assume that Eq (1.1) has a positive solution $u(l)$. It follows from (1.1) and Lemma 2.1 that there exist four possible cases for the behavior of v and its derivatives:

$$\begin{aligned}
(i) &: v'(l) > 0,\ v''(l) > 0,\ v'''(l) > 0 \text{ and } v^{(4)}(l) \leq 0;\\
(ii) &: v'(l) > 0,\ v''(l) < 0,\ v'''(l) > 0 \text{ and } v^{(4)}(l) \leq 0;\\
(iii) &: v'(l) < 0,\ v''(l) > 0 \text{ and } v'''(l) < 0\ ;\\
(iv) &: v'(l) > 0,\ v''(l) > 0 \text{ and } v'''(l) < 0.
\end{aligned}$$

Let (i) hold. Now, we define

$$\phi(l) = \rho(l)\frac{r(l)v'''(l)}{v(l)}.$$

Then clearly $\phi(l)$ is positive for $l \geq l_1$ and satisfies

$$\phi'(l) = \frac{\rho'(l)}{\rho(l)}\phi(l) + \rho(l)\left(\frac{(r(l)v'''(l))'}{v(l)} - \frac{r(l)v'''(l)v'(l)}{v^2(l)}\right). \tag{2.26}$$

From (1.1) and (2.26), we have

$$\phi'(l) = \frac{\rho'(l)}{\rho(l)}\phi(l) - \rho(l)\frac{\sum_{i=1}^{k} q_i(l) f(u(g_i(l)))}{v(l)} - \rho(l)\frac{r(l)v'''(l)v'(l)}{v^2(l)}, \tag{2.27}$$

by using (2.8) and (2.27), we get

$$\phi'(l) \leq \frac{\rho'(l)}{\rho(l)}\phi(l) - \rho(l)\frac{\varrho v(g(l))\sum_{i=1}^{k} q_i(l)(1-p(g_i(l)))}{v(l)} - \rho(l)\frac{r(l)v'''(l)v'(l)}{v^2(l)}. \tag{2.28}$$

Now, it follows from Lemmas 1.1 and 1.4 that

$$v'(l) \geq \frac{\lambda_2 l^2}{2}v'''(l) \tag{2.29}$$

and

$$\frac{v(g(l))}{v(l)} \geq \left(\frac{g(l)}{l}\right)^{3/\lambda}, \tag{2.30}$$

respectively. Substituting (2.29) and (2.30) into (2.28), we get

$$\phi'(l) \leq \frac{\rho'(l)}{\rho(l)}\phi(l) - \varrho\rho(l)\left(\frac{g(l)}{l}\right)^{3/\lambda}\sum_{i=1}^{k} q_i(l)(1-p(g_i(l))) - \frac{\lambda_2 l^2}{2}\frac{\rho(l)r(l)(v'''(l))^2}{v^2(l)},$$

from the definition of $\phi(l)$, we obtain

$$\phi'(l) \leq \frac{\rho'(l)}{\rho(l)}\phi(l) - \varrho\rho(l)\left(\frac{g(l)}{l}\right)^{3/\lambda}\sum_{i=1}^{k} q_i(l)(1-p(g_i(l))) - \frac{\lambda_2 l^2}{2\rho(l)r(l)}\phi^2(l).$$

Set $A = \lambda_2 l^2/2\rho(l)r(l)$, $B = \rho'(l)/\rho(l)$ and $s = \phi(s)$. Using Lemma 1.2, we have

$$\phi'(l) \leq -\varrho\rho(l)\left(\frac{g(l)}{l}\right)^{3/\lambda}\sum_{i=1}^{k} q_i(l)(1-p(g_i(l))) + \frac{1}{2}\frac{(\rho'(l))^2}{\rho(l)}\frac{r(l)}{\lambda_2 l^2}, \tag{2.31}$$

integrating (2.31) from l_1 to l, we have

$$\int_{l_1}^{l}\left(\varrho\rho(s)\left(\frac{g(s)}{s}\right)^{3/\lambda}\sum_{i=1}^{k}q_i(s)(1-p(g_i(s)))-\frac{1}{2}\frac{(\rho'(s))^2}{\rho(s)}\frac{r(s)}{\lambda_2 s^2}\right)\mathrm{d}s\leq\phi(l_1),$$

which contradicts (2.24).
Assume that case (*ii*) holds. Define the function $\varphi(l)$ by

$$\varphi(l)=\vartheta(l)\frac{\nu'(l)}{\nu(l)}.$$

Then clearly $\varphi(l)$ is positive for $l\geq l_1$ and satisfies

$$\varphi'(l)=\frac{\vartheta'(l)}{\vartheta(l)}\varphi(l)+\vartheta(l)\left(\frac{\nu''(l)}{\nu(l)}-\frac{(\nu'(l))^2}{\nu^2(l)}\right),$$

from the definition of $\varphi(l)$, we obtain

$$\varphi'(l)=\frac{\vartheta'(l)}{\vartheta(l)}\varphi(l)+\vartheta(l)\frac{\nu''(l)}{\nu(l)}-\frac{\varphi^2(l)}{\vartheta(l)}. \tag{2.32}$$

Now integrating (1.1) from l to ∞, we have

$$-r(l)\nu'''(l)=-\int_{l}^{\infty}\sum_{i=1}^{k}q_i(s)f(u(g_i(s)))\,\mathrm{d}s, \tag{2.33}$$

by using (2.8) and (2.33), we get

$$-r(l)\nu'''(l)\leq-\varrho\int_{l}^{\infty}\sum_{i=1}^{k}q_i(s)(1-p(g_i(s)))\,\nu(g(l))\,\mathrm{d}s. \tag{2.34}$$

From Lemma 1.4, we get

$$\nu(l)\geq l\lambda\nu'(l),$$

that is,

$$\frac{\nu(g(l))}{\nu(l)}\geq\left(\frac{g(l)}{l}\right)^{1/\lambda}. \tag{2.35}$$

Combining (2.35) and (2.34), we get

$$-r(l)\nu'''(l)\leq-\varrho\nu(l)\int_{l}^{\infty}\sum_{i=1}^{k}q_i(s)(1-p(g_i(s)))\left(\frac{g(s)}{s}\right)^{1/\lambda}\mathrm{d}s,$$

that is

$$-\nu'''(l)\leq-\varrho\frac{\nu(l)}{r(l)}\int_{l}^{\infty}\sum_{i=1}^{k}q_i(s)(1-p(g_i(s)))\left(\frac{g(s)}{s}\right)^{1/\lambda}\mathrm{d}s,$$

integrating the above inequality from l to ∞, we have

$$v''(l) \leq -\varrho v(l) \int_l^\infty \left(\frac{1}{r(v)} \int_v^\infty \sum_{i=1}^k q_i(s)(1 - p(g_i(s))) \left(\frac{g(s)}{s} \right)^{1/\lambda} \mathrm{d}s \right) \mathrm{d}v.$$

Combining above inequality with (2.32), we obtain

$$\varphi'(l) \leq -\varrho\vartheta(l) \int_l^\infty \left(\frac{1}{r(v)} \int_v^\infty \sum_{i=1}^k q_i(s)(1 - p(g_i(s))) \left(\frac{g(s)}{s} \right)^{1/\lambda} \mathrm{d}s \right) \mathrm{d}v + \frac{\vartheta'(l)}{\vartheta(l)}\varphi(l) - \frac{\varphi^2(l)}{\vartheta(l)}.$$

Thus, we have

$$\varphi'(l) \leq -\varrho\vartheta(l) \int_l^\infty \left(\frac{1}{r(v)} \int_v^\infty \sum_{i=1}^k q_i(s)(1 - p(g_i(s))) \left(\frac{g(s)}{s} \right)^{1/\lambda} \mathrm{d}s \right) \mathrm{d}v + \frac{(\vartheta'(l))^2}{4\vartheta(l)}, \tag{2.36}$$

integrating (2.36) from l_1 to l, we have

$$\int_{l_1}^l \left(\varrho\vartheta(\zeta) \int_\zeta^\infty \left(\frac{1}{r(v)} \int_v^\infty \sum_{i=1}^k q_i(s)(1 - p(g_i(s))) \left(\frac{g(s)}{s} \right)^{1/\lambda} \mathrm{d}s \right) \mathrm{d}v - \frac{(\vartheta'(\zeta))^2}{4\vartheta(\zeta)} \right) \mathrm{d}\zeta \leq \varphi(l_1),$$

which contradicts (2.25).
The proof of the case where (*iii*) or (*iv*) holds is the same as that of Theorem 2.2 and Theorem 2.1 respectively.
This completes the proof. □

Example 2.1. *Consider the NDDE*

$$\left(l^4 (u(l) + p_0 u(al))''' \right)' + q_1 u(bl) + q_2 u(cl) = 0, \tag{2.37}$$

where $a, b \in (0, 1)$ *and* $g_1 > g_2$. *Then, we note that*

$$n = 4, r(l) = l^4, p(l) = p_0, \tau(l) = al, q(l) = q_1 + q_2 \text{ and } \sigma(l) = bl.$$

Therefore, it is easy to verify that

$$\delta_0(l) = \frac{1}{3l^3}, \delta_1(l) = \frac{1}{6l^2} \text{ and } \delta_2(l) = \frac{1}{6l}.$$

Next, to apply Theorem 2.1. We first check the condition (2.2), (2.4) and (2.5). By substitution and a simple computation, (2.4) becomes

$$y'(l) + \varrho(q_1 + q_2) \frac{\lambda_0(1 - p_0)}{6c} \frac{1}{l} y(cl) = 0. \tag{2.38}$$

By applying a well known criterion [12, Theorem 2.1.1] for first-order delay differential equation (2.38) to be oscillatory, the criterion is immediately obtained.

$$\liminf_{l\to\infty} \int_{cl}^l \varrho(q_1 + q_2) \frac{\lambda_0(1 - p_0)}{6c} \frac{1}{s} \mathrm{d}s > \frac{1}{e},$$

that is,

$$(q_1 + q_2)\ln\left(\frac{1}{c}\right) > \frac{6c}{\varrho\lambda_0 (1 - p_0) e}. \tag{2.39}$$

Now, we note that (2.5) reduces to

$$\limsup_{l\to\infty} \int_{l_0}^{l} \left(\varrho (q_1 + q_2)(1 - p_0)\frac{\lambda_1 c^2}{6} - \frac{3}{4}\right)\frac{1}{v}\mathrm{d}v = \infty,$$

which satisfies if

$$(q_1 + q_2) > \frac{18}{4\varrho c^2 (1 - p_0)}, \tag{2.40}$$

thus, if condition (2.39) and (2.40) hold, then every solution of (2.37) is oscillatory or tends to zero. On the other hand, by Theorem 2.2, we see that (2.17) becomes

$$\limsup_{l\to\infty} \frac{\varrho}{6l}\int_{l_0}^{l} (q_1 + q_2)\,\mathrm{d}v > \left(1 + \frac{p_0}{a}\right)$$

and so

$$(q_1 + q_2) > \frac{6}{\varrho}\left(1 + \frac{p_0}{a}\right), \tag{2.41}$$

thus, if (2.39), (2.40) and (2.41) hold, then every solution of (2.37) is oscillatory.

Example 2.2. *Consider the NDDE*

$$\left(e^{3l}\left(\left(u(l) + \left(1 - \frac{1}{l^2}\right)u(l-a)\right)'''\right)^3\right)' + q_1 e^{3l} u^3(l-b) + q_2 e^{3l} u^3(l-c) = 0, \tag{2.42}$$

where $l \geq 1, 0 < a < b$ and $b > c$. Then, we can clearly note that $\alpha = \beta = 3, n = 4$.

$$r(l) = e^{3l}, p(l) = 1 - 1/l^2, \tau(l) = l - a, q(l) = (q_1 + q_2)e^{3l} \text{ and } \sigma(l) = l - b.$$

Therefore, it is easy to verify that

$$\delta_i(l) = e^{-l} \text{ for } i = 0, 1, 2.$$

By substitution and a simple computation, (2.4) becomes

$$y'(l) + \varrho(q_1 + q_2)e^{3l}\left(\lambda_0\frac{(l-b)}{3!e^{l-b}}\right)^3 y(l-b) = 0. \tag{2.43}$$

Applying a well-known criterion [12, Theorem 2.1.1], we see that (2.38) is oscillatory. Moreover, (2.5) reduces to

$$\limsup_{l\to\infty}\int_{l_0}^{\infty} \varrho\left((q_1 + q_2)\frac{\lambda_1^3}{2^3} - \left(\frac{3}{4}\right)^4\right)\mathrm{d}s = \infty,$$

which satisfies if $(q_1 + q_2) > 81/32$. Thus, every solution of (2.42) is oscillatory or tends to zero if $(q_1 + q_2) > 81/32$.
To apply Theorem 2.2, we see that (2.17) becomes

$$\limsup_{l\to\infty} \varrho e^{-3l}\int_{l_0}^{l} (q_1 + q_2)e^{3(v-a)}\mathrm{d}v > 2^3.$$

that is, $(q_1 + q_2) > 24e^{3a}$. Then, every solution of (2.42) is oscillatory if $(q_1 + q_2) > \max\left\{24e^{3a}, 81/32\right\}$.

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Conflict of interest

There are no competing interests

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