Local interior regularity for the 3D MHD equations in nonendpoint borderline Lorentz space

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Abstract: We prove local regularity condition for a suitable weak solution to 3D MHD equations. Precisely, if a solution satisfies 
\[ u, b \in L^\infty(-\frac{4}{3}^2, 0; L^3_{\infty}(B_{z})) \], 
\[ q \in (3, \infty) \] in Lorentz space, then \((u, b)\) is H"older continuous in the closure of the set \( Q_{\frac{r}{2}} \).

Keywords: local regularity condition; suitable weak solution; 3D MHD equations

Mathematics Subject Classification: 35B65, 76W05

1. Introduction

We study the three-dimensional incompressible magnetohydrodynamic (3D MHD) equations (see e.g. [5]):

\[
\begin{cases}
  u_t - \Delta u + (u \cdot \nabla)u - (b \cdot \nabla)b + \nabla \pi = 0 \\
  b_t - \Delta b + (u \cdot \nabla)b - (b \cdot \nabla)u = 0 \\
  \text{div } u = 0 \quad \text{and} \quad \text{div } b = 0, \\
  u(x, 0) = u_0(x), \quad b(x, 0) = b_0(x)
\end{cases}
\text{ in } Q_T := \mathbb{R}^3 \times [0, T),
\]

(1.1)

Here \( u \) is the flow velocity vector, \( b \) is the magnetic vector and \( \pi = p + \frac{|b|^2}{2} \) is the scalar pressure. By suitable weak solutions we mean solutions that solves MHD in the sense of distribution and satisfy the local energy inequality (see Definition 2.1 in section 2 for details). For a point \( z = (0, 0) \in \mathbb{R}^3 \times (0, T) \) by translation, we denote \( B_r(x) := B_r = \{ y \in \mathbb{R}^3 : |y - x| < r \} \),

\[ Q_r(z) := Q_r = B_r \times (-r^2, 0), \quad r < \sqrt{T}. \]
We say that solutions \( u \) and \( b \) are regular at \( z \in \mathbb{R}^3 \times (0, T) \) if \( u \) and \( b \) are bounded for some \( Q_r, r > 0 \). Otherwise, it is said that \( u \) and \( b \) are singular at \( z \). The original paper where the weak solvability of the various boundary value problems was proved is Ladyženskaja and Solonnikov [9]. As in the Navier-Stokes equations, regularity problem remains open in dimension three. On the other hand, He and Xin proved in [8] a suitable weak solution to this equations using the construction arguments of a solution in [4]. Furthermore, they show that a suitable weak solution, \((u, b)\) become regular in the presence of a certain type of scaling invariant local integral conditions for velocity and magnetic fields. Recently, in [14], Phuc give a new regularity condition, that is, \( u \in L^\infty(-1, 0; L^{3,q}(B_1)) \), a weak solution to the 3D Navier-Stokes equations are regular for \( q \neq \infty \) (cf [2]). In this paper, we give a criterion of local interior regularity as like Phuc’s result for a suitable weak solution to the 3D MHD equations in Lorentz space which is still unknown (see e.g. [20, 12] for the Naiver-Stokes equations). For proofs, we prove the \( \epsilon \)- regularity criteria for this solution in Lorentz space (below Proposition 2.3) based on the \( \epsilon \)-regularity criteria in Sobolev space. After that, using the standard blow-up argument (or contraction argument) and the unique continuation for parabolic equation, we show a solution is regular (see e.g. [1, 3, 6, 7, 13]). In summary, overall, our proof is followed the arguments in [14, 2] which is mainly contained the arguments for the Naiver-Stokes equations. Now we are ready to state the first part of our main result.

**Theorem 1.1.** Let a pair of functions \( u, b \) and \( \pi \) have the following differentiability properties:

\[
u, b \in L^{2,\infty}(Q_2) \cap W^{1,0}_2(Q_2), \quad \pi \in L^3(Q_2)
\]

Suppose that \((u, b, \pi)\) satisfy the 3D MHD equations in \( Q_2 \) in the sense of distributions. Assume, in addition, that there exists \( 3 < q < \infty \) such that

\[
u, b \in L^\infty(-4, 0; L^{3,q}(B_2)).
\]

Then \((u, b)\) is Hölder continuous in the closure of the set \( Q_{\frac{1}{2}} \).

**2. Preliminaries**

In this section we introduce some scaling invariant functionals and suitable weak solutions, and recall an estimation of the Stokes system.

We first start with some notations. Let \( \Omega \) be an open domain in \( \mathbb{R}^3 \) and \( I \) be a finite time interval. We denote by \( L^{p,q}(\mathbb{R}^3) \) with \( 1 \leq p, q \leq \infty \) the Lorentz space with the norm [21]

\[
\|\varphi\|_{L^{p,q}} = \left( \int_0^\infty t^q (m(\varphi, t))^{q/p} \frac{dt}{t} \right)^{1/q} < \infty \quad \text{for} \quad 1 \leq q < \infty,
\]

where \( m(\varphi, t) \) is the Lebesgue measure of the set \( \{x \in \mathbb{R}^3 : |\varphi(x)| > t\} \), i.e.

\[
m(\varphi, t) := m\{x \in \mathbb{R}^3 : |\varphi(x)| > t\}.
\]

In particular, when \( q = \infty \),

\[
\|\varphi\|_{L^{p,\infty}} = \sup_{t \geq 0} \{t(m(\varphi, t))^{1/p}\} < \infty.
\]
The Lorentz space $L^{p,\infty}$ is also called weak $L^p$ space. The norm is equivalent to the norm

$$
\|f\|_{L^{p,\infty}} = \sup_{0<|E|<\infty} |E|^{1/p-1} \int_E |f(x)|\,dx.
$$

For a function $f(x, t)$, we denote $\|f\|_{L^{p,\infty}(\Omega \times I)} = \|f\|_{L^p(I; L^{\infty}(\Omega))} = \|f\|_{L^p(I; L^q(\Omega))}$ and vector fields $u, v$ we write $(u, v)_{i=1,2,3}$ as $u \otimes v$. We denote by $C = C(\alpha, \beta, \ldots)$ a constant depending on the prescribed quantities $\alpha, \beta, \ldots$, which may change from line to line. Next we recall suitable weak solutions for the MHD equations (1.1) in three dimensions.

**Definition 2.1.** Let $I = (0, T)$. A triple of $(u, b, \pi)$ is a suitable weak solution to (1.1) if the following conditions are satisfied:

(a) The functions $u, b : Q_T \rightarrow \mathbb{R}^3$ and $\pi : Q_T \rightarrow \mathbb{R}$ satisfy

$$
u, b \in L^{\infty}(I; L^2(\mathbb{R}^3)) \cap L^2(I; W^{1,2}(\mathbb{R}^3)), \quad \pi \in L^2(Q_T),$$

(b) $(u, b, \pi)$ solves the MHD equations in $Q_T$ in the sense of distributions.

(c) $u, b$ and $\pi$ satisfy the local energy inequality

$$
\int_B (|u(x, t)|^2 + |b(x, t)|^2) \phi(x, t)\,dx 
+ 2 \int_0^t \int_B (|\nabla u(x, t')|^2 + |\nabla b(x, t')|^2) \phi(x, t')\,dx\,dt' 
\leq \int_0^t \int_B (|u|^2 + |b|^2) (\partial_t \phi + \Delta \phi)\,dx\,dt' 
+ \int_0^t \int_B (|u|^2 + |b|^2 + 2\pi) u \cdot \nabla \phi \,dx\,dt' 
- 2 \int_B \int_0^t (b \cdot u)(b \cdot \nabla \phi) \,dx\,dt'.
$$

(2.1)

for all nonnegative function $\phi \in C^0_0(\mathbb{R}^3 \times \mathbb{R})$.

The crucial regularity result in [8] and [23] ensures that

**Lemma 2.1.** There exists $\epsilon > 0$ such that if $(u, b, \pi)$ is a suitable weak solution of the 3D MHD equations and for $r > 0$,

$$
\frac{1}{r^2} \int_{Q_r} |u(y, s)|^3 + |b(y, s)|^3 + |\pi(y, s)|^2\,dy\,ds < \epsilon,
$$

then $z$ is a regular point.

Before a proof, we know some necessary results, which is crucial role for our analysis (see [2] and [14]). After then, using these result, we prove Theorem 1.1.

**Proposition 2.1.** Suppose that the pair of functions $(u, b, \pi)$ satisfies the 3D MHD equations in $Q := Q_1(0, 0) = B_1(0) \times (-1, 0)$ in the sense of distributions and has the following properties

$$
u, b \in L^{\infty}(-1, 0; L^2(B_1)) \cap L^2(-1, 0; W^{1,2}(B_1)),
$$
\[ \pi \in L^2(-1,0; L^1(B_1)). \]

for some \( q \in (3, \infty) \). Then \((u, b, \pi)\) forms a suitable weak solution to the 3D MHD equations in \( Q^\circ \) with a generalized energy equality, \( u \in L^4(Q) \), and \( \pi \in L^2(Q^\circ) \). Suppose further that

\[ u \in L^\infty(-1,0; L^3 q(B_1)), \quad b \in L^\infty(-1,0; L^3 q(B_1)). \]

In addition, the inequalities

\[ \|u(\cdot,t)\|_{L^3(q(B_1))} \leq \|u\|_{L^\infty(-1,0; L^3 q(B_1))}, \]

and

\[ \|b(\cdot,t)\|_{L^3(q(B_1))} \leq \|b\|_{L^\infty(-1,0; L^3 q(B_1))} \]

hold for all \( t \in (-\frac{3}{4}, 0) \), and the function

\[ t \rightarrow \int_{B_{\frac{3}{4}}} u(x,t)w(x)dx \]

is continuous on \( [-\frac{3}{4}, 0] \) for any \( w \in L^{\frac{3}{2}}(B_{\frac{3}{4}}) \). Here, it is clear that \( \frac{q}{q-1} = 1 \) in the case \( q = \infty \).

**Proof.** By Sobolev’s inequality, we know \( u \in L^2(-1,0; L^6(B_1)) \). And also by the assumptions and interpolative inequality, we have

\[ \|u\|_{L^4(B_1)} \leq C\|u\|_{L^3 q(B_1)}^{\frac{1}{2}} \|u\|_{L^1(B_1)}^{\frac{1}{2}}, \]  

(2.2)

which implies \( u \in L^4(Q_1) \). Similarly, we get \( b \in L^4(Q_1) \). Thus by Hölder’s inequality, we obtain

\[ u \cdot \nabla u, b \cdot \nabla b, u \cdot \nabla b, b \cdot \nabla u \in L^4(Q_1). \]

(2.3)

Decompose the pressure so that

\[ \pi = \pi_1 + \pi_2, \]

where \( \pi_1 := R_iR_i(\chi_{B_1}(u.\nu_j + b \cdot b_j)) \). Here \( R_i \) is Riesz operator and we adopt summation convention. It is not difficult to notice that in \( B_{\frac{3}{4}} \):

\[ \Delta \pi_2 = 0. \]

By Calderón-Zygmund estimate we have

\[ \|\pi_1\|_{L^2(B_1)} \leq C(\|u\|_{L^4(B_1)}^2 + \|b\|_{L^4(B_1)}^2), \]

(2.4)

and thus (2.4), it holds

\[ \|\pi_2\|_{L^2(-1,0; L^\infty(B_1))} \leq C\|\pi_2\|_{L^2(-1,0; L^1(B_1))} = C\|\pi - \pi_1\|_{L^2(-1,0; L^1(B_1))} \]

\[ \leq C(\|\pi\|_{L^2(-1,0; L^1(B_1))} + \|u\|_{L^4(Q_1)}^2 + \|b\|_{L^4(Q_1)}^2), \]  

(2.5)

Estimates (2.4) and (2.5) imply that the pressure \( \pi \in L^2(Q_{\frac{3}{4}}) \). With the energy class, estimate (2.2), (2.3) and (2.5), and the local interior regularity of Stokes systems , we have

\[ (\|u\|_{L^4(Q_{\frac{3}{4}})} + \|b\|_{L^4(Q_{\frac{3}{4}})})^+ \]
\[ (\|u\|_{L^4(Q_2^\frac{3}{2})} + \|b\|_{L^4(Q_2^\frac{3}{2})} + (\|\nabla^2 u\|_{L^4(Q_2^\frac{3}{2})} + \|\nabla^2 b\|_{L^4(Q_2^\frac{3}{2})} + \|\nabla \pi\|_{L^4(Q_2^\frac{3}{2})} < \infty. \]

It then follows that
\[ u, b \in C(-(\frac{3}{4})^2, 0; L^3(B_\frac{3}{4})) \]

and thus the function
\[ g_\varphi(t) := \int_{B_{3/4}} u(x, t) \varphi(x) \, dx \]
is continuous on \([-((\frac{3}{4})^2, 0])\) for any \(\varphi \in C_0^\infty(B_\frac{3}{4})\). This yields
\[ \int_{B_{3/4}} u(x, t) \varphi(x) \, dx \leq C \|\varphi\|_{L^3(\frac{3}{4}^2, 0; L^\infty(B_\frac{3}{4}))} \|u\|_{L^\infty(-((\frac{3}{4})^2, 0; L^\infty(B_\frac{3}{4}))}. \]

Thus by the density of \(C_0^\infty(B_\frac{3}{4})\) in \(L^3(\frac{3}{4}^2, \frac{3}{4}(\frac{3}{4}))\) we see that
\[ \|u\|_{L^\infty(B_\frac{3}{4})} \leq C \|u\|_{L^\infty(-((\frac{3}{4})^2, 0; L^\infty(B_\frac{3}{4}))}, \quad t \in [-((\frac{3}{4})^2, 0)]. \]

Then it can be seen, again by density, that the function \(g_\varphi\) above is actually continuous on \([-((\frac{3}{4})^2, 0])\) for any \(\varphi \in L^3(\frac{3}{4}^2, \frac{3}{4}(\frac{3}{4}))\). Finally, using \(u \in L^3(B_1)\) and a standard mollification in \(R^{1+1}\) combined with a truncation in time of test functions, we obtain the local generalized energy equality in \(Q_\frac{3}{2}\). \qed

2.1. Some estimates

For simplicity, we write
\[ \Phi(r) := A_u(r) + A_b(r) + E_u(r) + E_b(r), \]
where
\[ A_u(r) := \sup_{r^{-2} \leq t < r} \frac{1}{r} \int_{B_t} |u(y, s)|^2 \, dy, \quad E_u(r) := \frac{1}{r} \int_{Q_r} |\nabla u(y, s)|^2 \, dy \, ds, \]
\[ A_b(r) := \sup_{r^{-2} \leq t < r} \frac{1}{r} \int_{B_t} |b(y, s)|^2 \, dy, \quad E_b(r) := \frac{1}{r} \int_{Q_r} |\nabla b(y, s)|^2 \, dy \, ds. \]

Also, we introduce following the scale invariant functional: for \(0 < r < 1\),
\[ C^u_\infty(r) = \frac{1}{r^2} \int_{-r^2}^0 |u(y, s)|^3_{L^\infty(B_r)} \, ds, \quad C^b_\infty(r) = \frac{1}{r^2} \int_{-r^2}^0 |b(y, s)|^3_{L^\infty(B_r)} \, ds. \]
\[ D_\infty(r) = \frac{1}{r^2} \int_{-r^2}^0 |\pi(y, s)|^3_{L^\infty(B_r)} \, ds. \]

Now, we begin with stating a well known algebraic Lemma, whose proof is omitted but found in [4].

\textbf{Lemma 2.2.} Let \(I(s)\) be a bounded non negative function in the interval \([R_1, R_2]\). Assume that for every \(s, \rho \in [R_1, R_2]\) and \(s < \rho\) we have
\[ I(s) \leq [A(\rho - s)^{-\alpha} + B(\rho - s)^{-\beta} + C] + \theta I(\rho) \]
with \(A, B, C \geq 0, \alpha > \beta > 0\) and \(\theta \in [0, 1)\). Then there holds
\[ I(R_1) \leq c(\alpha, \theta)[A(R_2 - R_1)^{-\alpha} + B(R_2 - R_1)^{-\beta} + C]. \]
Lemma 2.3. Let \(( u, b, \pi )\) be a suitable weak solution to 3D MHD equations. Then for \( 0 < r \) the following holds

\[ \Phi(\frac{r}{2}) \leq C (C_\alpha^u(r)^\frac{2}{3} + C_\alpha^b(r)^\frac{2}{3} + C_\alpha^\pi(r)^\frac{2}{3} + C_\alpha^b(r)^\frac{2}{3} + D_\alpha(r)^\frac{2}{3}). \]

Proof. Without loss of generally, consider \( z_0 \) to be the origin. Let \( 0 < \frac{1}{2} \leq s < \rho \leq r < 1 \). Let \( \eta_1 \in C_0^\infty(B(\rho)) \) such that \( 0 \leq \eta_1 \leq 1 \) in \( \mathbb{R}^3 \) and \( \eta_1 = 1 \) on \( B(s) \). Furthermore for \( |a| \leq 2 \):

\[ |\nabla^a \eta_1| \leq \frac{C}{(\rho - s)^a}. \]

Let \( \eta_2 \in C_0^\infty(-\rho^2, \rho^2) \) such that \( 0 \leq \eta_2 \leq 1 \) in \( \mathbb{R} \) and \( \eta_1 = 1 \) on \([-s^2, s^2] \).

\[ |\nabla^a \eta_2| \leq \frac{C}{(\rho^2 - s^2)} \leq \frac{C}{r(\rho - s)} \leq \frac{C}{(\rho - s)^2}. \]

Let \( \phi(x, t) := \eta_1 \eta_2(x) \). Hence:

\[ |\nabla \phi| \leq \frac{C}{\rho - s}, \quad |\nabla^2 \phi| \leq \frac{C}{(\rho - s)^2}, \quad |\phi| \leq \frac{C}{(\rho - s)^2}. \]

From the local energy inequality, we are known

\[ \int_{\Omega} (|u(x, t)|^2 + |b(x, t)|^2) \phi(x, t) dx + 2 \int_{0}^{t} \int_{B_r} (|\nabla u(x, t')|^2 + |\nabla b(x, t')|^2) \phi(x, t') dx dt' \]

\[ \leq \int_{0}^{t} \int_{B_r} (|u|^2 + |b|^2)(\partial_t \phi + \Delta \phi) dx dt' \]

\[ + \int_{0}^{t} \int_{B_r} (|u|^2 + |b|^2) u \cdot \nabla \phi dx dt' + 2 \int_{0}^{t} \int_{B_r} \pi u \cdot \nabla \phi dx dt' - 2 \int_{0}^{t} \int_{B_r} (b \cdot u)(b \cdot \nabla \phi) dx dt', \quad (2.6) \]

\[ := \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4 \]

for all \( t \in I = (-1, 0) \) and for all non-negative functions \( \phi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R}) \). Let us treat the term \( \mathcal{E}_1 \) first. By O’Neil’s inequality in space, the property of \( \phi \), and then Hölder in time, we have

\[ \mathcal{E}_1 \leq \int_{-\rho^2}^{0} (||u||^2_{L^{\infty}(B_r)} + ||b||^2_{L^{\infty}(B_r)}) ||\Delta \phi + \partial_t \phi||_{L^{1,1}(B_r)} ds \]

\[ \leq \frac{C \rho}{(\rho - s)^2} \int_{-\rho^2}^{0} (||u||^2_{L^{\infty}(B_r)} + ||b||^2_{L^{\infty}(B_r)}) ds \]

\[ \leq \frac{C \rho^\frac{1}{2}}{(\rho - s)^\frac{3}{2}} \left( \int_{-\rho^2}^{0} ||u||^3_{L^{\infty}(B_r)} ds \right)^{\frac{1}{3}} + \left( \int_{-\rho^2}^{0} ||b||^3_{L^{\infty}(B_r)} ds \right)^{\frac{1}{3}}. \quad (2.7) \]

Lorentz spaces is characterization as interpolation space between \( L^2 \) and \( L^6 \) as follows:

\[ L^{3,1}(\Omega) = (L^2(\Omega), L^6(\Omega))_{\frac{1}{2}, 1} \]
Before the term $E_3$ is estimated, we note that

$$
\|u \cdot \nabla \phi\|_{L^{3,1}(B_\rho)} \leq \|u \cdot \nabla \phi\|_{L^2(B_\rho)}^{\frac{1}{3}} \|u\|_{L^6(B_\rho)} \leq \|u \cdot \nabla \phi\|_{L^2(B_\rho)}^{\frac{1}{3}} \|\nabla (u \cdot \nabla \phi)\|_{L^2(B_\rho)}^{\frac{1}{3}}
$$

$$
\leq \frac{C\|u\|_{L^2(B_\rho)}}{(\rho - s)^{\frac{1}{2}}} + \frac{C\|u\|_{L^2(B_\rho)} \|\nabla u\|_{L^2(B_\rho)}}{\rho - s},
$$

(2.9)

where we use the interpolation (2.8), Sobolev embedding and the property of $\phi$. Set $I(\rho) = \rho \Phi(\rho)$.

Using O’Neil inequality and the estimate (2.9), the term $E_3$ is estimated as follows: for $\rho \leq r$,

$$
E_3 \leq \int_{-\rho^2}^{0} \|u \cdot \nabla \phi\|_{L^{3,1}(B_\rho)} \|\pi\|_{L^2(B_\rho)} ds \leq \left[ \frac{C}{(\rho - s)^{\frac{1}{2}}} \left( \int_{-\rho^2}^{0} \|u\|_{L^2(B_\rho)}^3 ds \right)^{\frac{1}{2}} + \frac{r^\frac{1}{2}}{\rho - s} \right] \times \left( \int_{-\rho^2}^{0} \|\pi\|_{L^2(B_\rho)}^2 ds \right)^{\frac{1}{2}}
$$

$$
\leq C \left( \frac{r^\frac{3}{2} I(\rho)}{(\rho - s)^{\frac{1}{2}}} + \frac{r^\frac{1}{2}}{\rho - s} I(\rho) \right) \left( \int_{-\rho^2}^{0} \|\pi\|_{L^2(B_\rho)}^2 ds \right)^{\frac{1}{2}}.
$$

(2.10)

Similarly, we obtain the following estimate as like $E_3$:

$$
\int_{-\rho^2}^{0} \int_{B_\rho} 2|u|^2 u \cdot \nabla \phi dx dt' \leq \frac{C}{(\rho - s)^{\frac{1}{2}}} \left( \int_{-\rho^2}^{0} \|u\|_{L^3(B_\rho)}^3 ds \right)^{\frac{1}{2}} + \frac{r^\frac{1}{2}}{\rho - s} I(\rho) \left( \int_{-\rho^2}^{0} \|\pi\|_{L^2(B_\rho)}^2 ds \right)^{\frac{1}{2}},
$$

(2.11)

$$
\int_{-\rho^2}^{0} \int_{B_\rho} 2|b|^2 u \cdot \nabla \phi dx dt' \leq \frac{C}{(\rho - s)^{\frac{1}{2}}} \left( \int_{-\rho^2}^{0} \|u\|_{L^3(B_\rho)}^3 ds \right)^{\frac{1}{2}} + \frac{r^\frac{1}{2}}{\rho - s} I(\rho) \left( \int_{-\rho^2}^{0} \|b\|_{L^3(B_\rho)}^3 ds \right)^{\frac{1}{2}}.
$$

(2.12)

So thus, with the estimates (2.11) and (2.12), the term $E_2 + E_4$ is estimated by

$$
E_2 + E_4 \leq C \left( \frac{r^\frac{3}{2} I(\rho)}{(\rho - s)^{\frac{1}{2}}} + \frac{r^\frac{1}{2}}{\rho - s} I(\rho) \right) \left( \int_{-\rho^2}^{0} \|u\|_{L^3(B_\rho)}^3 ds \right)^{\frac{1}{2}} + \left( \int_{-\rho^2}^{0} \|b\|_{L^3(B_\rho)}^3 ds \right)^{\frac{1}{2}}.
$$

(2.13)

We combine with the estimate (2.7), (2.10) and (2.13) and Young’s inequality to get

$$
I(\rho) \leq \frac{r^\frac{3}{2}}{(\rho - s)^{\frac{1}{2}}} \left( \int_{-\rho^2}^{0} \|u\|_{L^3(B_\rho)}^3 ds \right)^{\frac{1}{2}} + \left( \int_{-\rho^2}^{0} \|u\|_{L^3(B_\rho)}^3 ds \right)^{\frac{1}{2}} + \frac{1}{2} I(\rho)
$$

$$
+ \left( \frac{r^\frac{1}{2}}{(\rho - s)^{\frac{1}{2}}} + \frac{r^\frac{1}{2}}{(\rho - s)^{\frac{1}{2}}} \right) \left( \int_{-\rho^2}^{0} \|u\|_{L^3(B_\rho)}^3 ds \right)^{\frac{1}{2}} + \left( \int_{-\rho^2}^{0} \|b\|_{L^3(B_\rho)}^3 ds \right)^{\frac{1}{2}}.
$$

Since $\frac{s}{2} \leq s < \rho \leq r$ and by Lemma 2.2, we obtain

$$
\Phi(\rho) \leq \frac{r^\frac{3}{2}}{2} \left( \int_{-\rho^2}^{0} \|u\|_{L^3(B_\rho)}^3 ds \right)^{\frac{1}{2}} + \left( \int_{-\rho^2}^{0} \|u\|_{L^3(B_\rho)}^3 ds \right)^{\frac{1}{2}} + Cr^\frac{1}{2} \left( \int_{-\rho^2}^{0} \|u\|_{L^3(B_\rho)}^3 ds \right)^{\frac{1}{2}}.
$$

\qed
2.2. Proof of main theorem

Following the notation in [14], we suppose that $z_0 := (x_0, t_0) \in Q_1(0, 0)$ is a singular point. It means that there exists no neighborhood $N$ of $z_0$ such that $(u, b)$ has a Hölder continuous representative on $N \cap [B_1(0) \times (-1, 0)]$. By Theorem 3.2 [13], there exist $c_0 > 0$ and a sequence of numbers $\epsilon_k \in (0, 1)$ such that $\epsilon_k \to 0$ as $k \to \infty$ and

$$\sup_{|x| \leq \epsilon_k} \frac{1}{\epsilon_k^3} \int_{B(x_0, \epsilon_k)} |u(x, s)|^2 dx + |b(x, s)|^2 dx \geq c_0, \quad \epsilon_k \leq \epsilon_0,$$

for any $k \in \mathbb{N}$. Moreover, by Proposition 2.1, we have in particular

$$u(\cdot, t_0) \in L^{3, q}(B_{3/4}(0)), \quad b(\cdot, t_0) \in L^{3, q}(B_{3/4}(0)).$$

Recall that we can decompose $\pi = \tilde{\pi} + h$, where $h$ is harmonic in $B_1$, and $\tilde{\pi} = R_B[(u, u_j + b_i b_j)(x_B)]$. For each $Q = \omega \times (a, b)$, where $\omega \in \mathbb{R}^3$ and $-\infty < a < b \leq 0$, we choose a large $k_0 = k_0(Q) \geq 1$ so that for any $k \geq k_0$ there hold the implications $x \in \omega \Rightarrow x_0 + \epsilon_k x \in B_{1/2}$, and $t \in (a, b) \Rightarrow t_0 + \epsilon_k t \in (-\frac{1}{2}, 0)$, where the sequence $\epsilon_k$ is as in (4.7). Set $Q = \omega \times (a, b)$, let us set

$$u_k(x, t) = u_k(x_0 + \epsilon_k x, t_0 + \epsilon_k^2 t), \quad b_k(x, t) = b_k(x_0 + \epsilon_k x, t_0 + \epsilon_k^2 t),$$

and

$$\pi_k(x, t) = \epsilon_k^2 k \pi(x_0 + \epsilon_k x, t_0 + \epsilon_k^2 t), \quad \tilde{\pi}_k(x, t) = \epsilon_k^2 \tilde{\pi}(x_0 + \epsilon_k x, t_0 + \epsilon_k^2 t), \quad h_k(x, t) = \epsilon_k^2 h(x_0 + \epsilon_k x, t_0 + \epsilon_k^2 t),$$

for any $(x, t) \in Q$ and $k \geq k_0(Q)$.

The following proposition is a key in the proof of Theorem 1.1, which says the properties in the limit.

**Proposition 2.2.** Let $0 < q < \infty$ and $Q = \omega \times (a, b)$ with $\omega \subset \mathbb{R}^3$, $-\infty < a < b \leq 0$. There exists a subsequence of $(u^k, b^k, \pi^k)$, still denoted by $(u^k, b^k, \pi^k)$, and a pair of functions

$$(u^\infty, b^\infty, \pi^\infty) \in L^\infty(-\infty, 0; L^3(a, b; L^3(\mathbb{R}^3))) \times L^\infty(-\infty, 0; L^3(a, b; L^3(\mathbb{R}^3))) \times L^\infty(-\infty, 0; L^{3, q}(\mathbb{R}^3))$$

with $\text{div } u^\infty = 0$ and $\text{div } b^\infty = 0$ in $\mathbb{R}^3 \times (-\infty, 0)$, such that for $s \in (1, 3),$

$$u^k \to u^\infty \text{ in } C(a, b; L^s(\omega)), \quad \text{ (2.15)}$$

$$b^k \to b^\infty \text{ in } C(a, b; L^s(\omega)), \quad \text{ (2.16)}$$

$$\pi^k \to \pi^\infty \text{ weakly* in } L^\infty(a, b; L^{3, q}(\omega)).$$

**Moreover**

$$|u^\infty|^2, |b^\infty|^2, \nabla u^\infty, \nabla b^\infty \in L^2(Q), \quad \text{ (2.18)}$$

$$\partial_t u^\infty, \partial_i b^\infty, \nabla^2 u^\infty, \nabla^2 b^\infty, \nabla \pi^\infty \in L^3(Q), \quad \text{ (2.19)}$$

and $(u^\infty, b^\infty, \pi^\infty)$ satisfies a suitable weak solution to the 3D MHD equations in $Q$. Additionally, $u^\infty$ and $b^\infty$ satisfy the lower bound satisfies the lower bound

$$\int_Q (|u^\infty|^2 + |b^\infty|^2) dz \geq \varepsilon_3. \quad \text{ (2.20)}$$
Proof. For each $Q = \omega \times (a, b)$, where for $\omega \subset \mathbb{R}^3$ and $t \in [a, b]$ with $-\infty < a < b \leq 0$, we have

$$
\|u_k(\cdot, t)\|_{L^{3,q}(\omega)} \leq \|u_k(\cdot, t_0 + \varepsilon_k^2 t)\|_{L^{3,q}(B_\frac{3}{2})} \leq \|u\|_{L^{\infty}(-1,0);L^{3,q}(B_1)},
$$
(2.21)

and

$$
\|b_k(\cdot, t)\|_{L^{3,q}(\omega)} \leq \|b\|_{L^{\infty}(-1,0);L^{3,q}(B_1)},
$$
(2.22)

By Calderón-Zygmund estimate, for a.e. $t \in (a, b)$ there holds

$$
\|\tilde{\pi}(\cdot, t)\|_{L^{\frac{4}{3}}(\omega)} \leq \|\tilde{\pi}(\cdot, t_0 + \varepsilon_k^2 t)\|_{L^{\frac{4}{3}}(B_\frac{3}{4})} \leq C(\|u\|_{L^{\infty}(-1,0);L^{3,q}(B_1)} + \|b\|_{L^{\infty}(-1,0);L^{3,q}(B_1)}).
$$
(2.23)

On the other hand, by harmonicity we have

$$
\int_a^b \sup_{x \in \omega} |h_k(x, t)|^2 \, dt \leq \varepsilon_k \int_{-(3/4)^2} \sup_{x \in \omega} |h_k(x_0 + \varepsilon_k x, s)|^2 \, ds \leq \varepsilon_k \|h^2\|_{L^2((0,1);L^{3,q}(B_1))}^2 \leq C \varepsilon_k(\|u\|_{L^{\infty}(Q)}^2 + \|b\|_{L^{\infty}(Q)}^2 + \||\tilde{\pi}|\|_{L^\infty(Q)}^4).
$$
(2.24)

Thus each $(u_k, b_k)$ is a suitable solution in $Q$. Then, from the energy estimate follows that

$$
\|u_k\|_{L^{\infty}(Q)} + \|b_k\|_{L^{\infty}(Q)} + \|\nabla b_k\|_{L^2(Q)} + \|\nabla u_k\|_{L^2(Q)} \leq C.
$$
(2.25)

Using (2.25) and Sobolev embedding, we have $\|u_k\|_{L^{2}(a,b;L^2(\omega))} \leq C$, which by (4.12), interpolation, and Hölder’s inequality gives for

$$
\|u_k\|_{L^{2}(Q)} + \|b_k\|_{L^{2}(Q)} + \|(u_k \cdot \nabla) u_k\|_{L^\frac{4}{3}(Q)} + \|(b_k \cdot \nabla) u_k\|_{L^\frac{4}{3}(Q)} \leq C.
$$

From the bounds (2.23) and (2.24), we also have

$$
\|\tilde{\pi}_k\|_{L^\infty(Q)} \leq C\|\tilde{\pi}_k\|_{L^\infty(\omega)} \leq C, \quad s \in (0, \frac{3}{2}).
$$
(2.26)

Using the estimate (2.25)–(2.26), it follows from the local interior regularity of solutions to non-stationary Stokes equations we find

$$
\|\partial_t u_k\|_{L^\frac{4}{3}(Q)} + \|\nabla^2 u_k\|_{L^\frac{4}{3}(Q)} + \|\nabla \pi_k\|_{L^\frac{4}{3}(Q)} \leq C.
$$
(2.27)

Furthermore, we can easily check the as following:

$$
\|\partial_t u_k\|_{L^\frac{4}{3}(Q)} + \|\partial_t b_k\|_{L^\frac{4}{3}(Q)} + \|\nabla^2 u_k\|_{L^\frac{4}{3}(Q)} + \|\nabla^2 b_k\|_{L^\frac{4}{3}(Q)} + \|\nabla \pi_k\|_{L^\frac{4}{3}(Q)} \leq C.
$$
(2.28)

Using estimates (2.21)–(2.23), we may get that

$$
\begin{align*}
    u_k & \rightarrow^* u^\infty \; \text{in} \; L^{\infty}(-\infty, 0; L^3;\mathbb{R}^3), \\
    b_k & \rightarrow^* b^\infty \; \text{in} \; L^{\infty}(-\infty, 0; L^3;\mathbb{R}^3), \\
    \tilde{\pi}_k & \rightarrow^* \tilde{\pi}^\infty \; \text{in} \; L^{\infty}(-\infty, 0; L^\frac{3}{2};\mathbb{R}^3).
\end{align*}
$$
Estimates (2.25) and (2.27) yield
\[ u_k \rightharpoonup^* u^\infty \quad \text{in} \quad C(-\infty, 0; L^\frac{4}{3}(Q)), \tag{2.29} \]
\[ b_k \rightharpoonup^* b^\infty \quad \text{in} \quad C(-\infty, 0; L^\frac{4}{3}(Q)). \tag{2.30} \]
For any \( s \in (1, 3) \), the uniform bound (2.21) and the interpolation inequality
\[ ||u_k(\cdot, t) - u_k(\cdot, t')||_{L^s} \leq ||u_k(\cdot, t) - u_k(\cdot, t')||_{L^\frac{4}{3}} \left( \frac{1}{s} - \frac{1}{3} \right) ||u_k(\cdot, t) - u_k(\cdot, t')||_{L^\frac{4}{3}} \left( \frac{1}{s} - \frac{1}{3} \right) \]
imply that each \( u_k \in C([a, b]; L^s(\omega)) \). Thus by using (2.29) and interpolating we obtain (2.15) for any \( s \in (1, 3) \). On the other hand, by (2.24), we have
\[ h_k \rightarrow 0 \quad \text{strongly in} \quad L^2(a, b; L^\infty(\omega)), \]
Now (2.18)–(2.19) follows from (2.29), (2.30), (2.25) and (2.27) via an argument as in the proof of Proposition 2.1. Finally, note that by (2.41) and a change of variables we have
\[ \text{sup}_{-1 \leq t \leq 0} \int_{B(0,1)} |u_k(x, t)|^2 dx = \text{sup}_{t_0 - \epsilon \leq t \leq t_0} \frac{1}{\epsilon^k} \int_{B(0,1)} |u_k(y, s)|^2 dy \geq C_0. \]
Similarly, \( \text{sup}_{-1 \leq t \leq 0} \int_{B(0,1)} |u_k(x, t)|^2 dx \geq C_0 \). Thus using the convergences (2.15) and (2.16) with \( s = 2 \) we obtain the lower bound (2.20).

Before proving the main statement we introduce some notation
\[ C_u(r) := \frac{1}{r^2} \int_{Q_r} |u|^3 dz, \quad C_b(r) := \frac{1}{r^3} \int_{Q_r} |b|^3 dz, \quad D(r) := \frac{1}{r^2} \int_{Q_r} |\pi|^3 dz. \]
Now, we prove the \( \epsilon \)-regularity criteria for a suitable weak solution to the 3D MHD equations under our circumstance.

**Proposition 2.3.** Let \((u, b, \pi)\) be a suitable weak solution to 3D MHD equations. Then there exists a universal constants \( c_0 \) and \( c_{0k}(\epsilon_0) \) (with \( k = 1, 2, \cdots \)) with the following property. Assume
\[ C_u^w(1) + C_b^w(1) + D(1) \leq \epsilon_0, \tag{2.31} \]
then for any natural number \( k \), \( \nabla^{k-1} u \) is Hölder continuous in \( \tilde{Q}_{1/8} \) and the following bound is valid:
\[ \text{sup}_{\tilde{Q}_{1/8}} \left( |\nabla^{k-1} u(z)| + |\nabla^{k-1} b(z)| \right) < c_{0k}(\epsilon_0). \]

**Proof.** From Lemma 2.3 and assumptions (2.31), it follows that
\[ A_u\left( \frac{1}{2} \right) + A_b\left( \frac{1}{2} \right) + E_u\left( \frac{1}{2} \right) + E_b\left( \frac{1}{2} \right) \leq C(\epsilon_0 + \epsilon_0^3)^{\frac{4}{3}}. \tag{2.32} \]
By interpolation and Sobolev embedding theorem one can show that
\[ C_u\left( \frac{1}{2} \right) \leq C[A_u\left( \frac{1}{2} \right)^3 E_u\left( \frac{1}{2} \right)^3 + A_u\left( \frac{1}{2} \right)^3]. \]
Thus, by (2.32) we have
\[ C_u(\frac{1}{2}) \leq C(e_0 + \epsilon_0^2). \] (2.33)

Similarly, we have
\[ C_b(\frac{1}{2}) \leq C(e_0 + \epsilon_0^2). \] (2.34)

For similar reasons it is not so difficult to see that
\[ \| \nabla \cdot (u \times u) \|_{L^{\frac{9}{2}}(Q_\frac{1}{2})} \leq C[A_u(\frac{1}{2}) + A_u(\frac{1}{2})^\frac{1}{2}B_u(\frac{1}{2})^\frac{1}{2}]. \]

Thus,
\[ \| \nabla \cdot (u \times u) \|_{L^{\frac{9}{2}}(Q_\frac{1}{2})} \leq C(e_0 + \epsilon_0^2)^\frac{1}{2}. \] (2.35)

Similarly, we have
\[ \| \nabla \cdot (b \times b) \|_{L^{\frac{9}{2}}(Q_\frac{1}{2})} \leq C(e_0 + \epsilon_0^2)^\frac{1}{2}. \] (2.36)

On the other hand, by Hölder’s inequality, it is obvious that
\[ \| u \|_{W^{1,0}(Q_\frac{1}{2})} \leq C(A_u(\frac{1}{2}) + B_u(\frac{1}{2})) \leq C(e_0 + \epsilon_0^2)^\frac{1}{2}. \] (2.37)

Similarly, we have
\[ \| b \|_{W^{1,0}(Q_\frac{1}{2})} \leq C(e_0 + \epsilon_0^2)^\frac{1}{2}. \] (2.38)

Using O’Neil’s inequality, we have
\[ \int_{B_1} |\pi(x,t)|^\frac{2}{3} dx \leq C\| \pi \|_{L^{\frac{9}{2}}(\infty)} = C\| \pi \|_{L^{\frac{9}{2}}(\infty)}. \]

Hence,
\[ \| \pi(x,t) \|_{L^{\frac{9}{2}}} \leq C\epsilon_0^\frac{2}{3}. \] (2.39)

Using the local interior regularity theory for Stokes equation, we have
\[ \| u \|_{L^{\frac{9}{2}}(Q_\frac{1}{2})} + \| \nabla^2 u \|_{L^{\frac{9}{2}}(Q_\frac{1}{2})} + \| \nabla \pi \|_{L^{\frac{9}{2}}(Q_\frac{1}{2})} \]
\[ \leq C(\| \nabla \cdot (u \times u) \|_{L^{\frac{9}{2}}(Q_\frac{1}{2})} + \| \nabla \cdot (b \times b) \|_{L^{\frac{9}{2}}(Q_\frac{1}{2})}) \]
\[ + \| u \|_{L^{\frac{9}{2}}(Q_\frac{1}{2})} + \| \nabla u \|_{L^{\frac{9}{2}}(Q_\frac{1}{2})} + \| \pi \|_{L^{\frac{9}{2}}(Q_\frac{1}{2})}. \]

Note that a suitable weak solution \((u, b, \pi)\) implies that
\[ u, b \in W^{2,1}_{\frac{9}{2}}(Q_2) \cap W^{1,0}_{\frac{3}{2}}(Q_2), \quad \pi \in W^{1,0}_{\frac{3}{2}}(Q_2) \cap L^2(Q_2). \]
(see e.g. [18, 19]). Using this together with the estimates (2.35)–(2.39), we obtain that
\[ \| \nabla \pi \|_{L^{\frac{9}{2}}(Q_\frac{1}{2})} \leq c[\| (e_0 + \epsilon_0^2)^\frac{1}{2} + (e_0 + \epsilon_0^2)^\frac{1}{2}]. \]
Thus, by the Poincaré inequality, we have
\[ \| \pi - [\pi] \|_{L^2(Q_{1/4}^1)} \leq c[((\epsilon_0 + \epsilon_0^2)^{1/2} + (\epsilon_0 + \epsilon_0^2)^{1/2})]. \]

Therefore, we conclude
\[ \| \pi \|_{L^2(Q_{1/4}^1)} \leq c[((\epsilon_0 + \epsilon_0^2)^{1/2} + (\epsilon_0 + \epsilon_0^2)^{1/2})] \quad (2.40) \]

This along with (2.33), (2.34) and (2.40) gives
\[ C_u(1) + C_b(1) + D(1) \leq C[((\epsilon_0 + \epsilon_0^2)^{1/3} + (\epsilon_0 + \epsilon_0^2)^{2/3})] \quad (2.41) \]

Choosing \( \epsilon_0 \) sufficiently small, the estimate (2.41) satisfies the conditions of Theorem 3.3 in [13] and so we complete the proof.

\[
\text{Proof of Theorem 1.1.} \quad \text{The proof is similar to the argument in [13, Theorem 1.1]. We now fix such numbers} \ M \text{ and } N \text{ and let} \ z_1 = (x_1, t_1) \in (\mathbb{R}^3 \setminus \bar{B}_{2N}(0)) \times (-M, 0]. \text{ Due to} \ C_{\infty}^u(1) + C_{\infty}^b(1) + D_{\infty}(1) \leq \epsilon_0, \text{ we obtain, by Proposition 2.3}
\]

\[
\max_{z \in \bar{Q}_{1/2}(z_1)} |\nabla u^\infty(z)| \leq C(k), \quad \max_{z \in \bar{Q}_{1/2}(z_1)} |\nabla b^\infty(z)| \leq C(k), \quad k = 1, 2, \cdots
\]

On the other hand, on the set \((\mathbb{R}^3 \setminus \bar{B}_{2N}(0)) \times (-M, 0]\), we have that there exists \( M > 0 \) such that

\[ |\partial_t W - \Delta W| \leq M(|W| + |\nabla W|), \quad \text{and} \quad |W| \leq C, \]

for the (15-component) vector-valued function \( W = (b^\infty, w^\infty, b^\infty_1, b^\infty_2, b^\infty_3) \) where \( w^\infty = \nabla \times u^\infty \) given in [13, pp.2922-2923]. Then

\[ W = 0 \text{ on } (\mathbb{R}^3 \setminus \bar{B}_{4N}(0)) \times (-M/4, 0]. \]

Using the theory of unique continuation for parabolic equation (see [6, Theorem 5]), we see \( W(\cdot, t) = 0 \) in \( \mathbb{R}^3 \) for a.e. \( t \in (-M/4, 0]. \) Thus \( u^\infty(\cdot, t) = 0 \) is globally harmonic, and using Liouville theorem, it follows that \( u^\infty(\cdot, t) = 0 \) for a.e. \( t \in (-M/4, 0]. \) This yields to a contradiction to the lower bound (2.20) and hence completes the proof of Theorem 1.1.

\[
3. \text{Conclusions}
\]

In this paper, we investigate some local regularity condition for a suitable weak solution to 3D MHD equations in Lorentz space. However, it remains an open question to obtain the local regularity condition for only velocity vector \( u. \)

\[
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\]

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Conflicts of interest

The authors declare that they have no conflicts of interest

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