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Research article

Local interior regularity for the 3D MHD equations in nonendpoint borderline Lorentz space

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Abstract: We prove local regularity condition for a suitable weak solution to 3D MHD equations. Precisely, if a solution satisfies $u, b \in L^\infty(-(\frac{4}{3})^2, 0; L^{3,q}(B_{\frac{3}{4}}))$, $q \in (3, \infty)$ in Lorentz space, then (u, b) is Hölder continuous in the closure of the set $Q_{\frac{1}{2}}$.

Keywords: local regularity condition; suitable weak solution; 3D MHD equations
Mathematics Subject Classification: 35B65, 76W05

1. Introduction

We study the three-dimensional incompressible magnetohydrodynamic (3D MHD) equations (see e.g. [5]):

$$(MHD)\begin{cases} u_t - \triangle u + (u\cdot\nabla)u - (b\cdot\nabla)b + \nabla\pi = 0 \\ b_t - \triangle b + (u\cdot\nabla)b - (b\cdot\nabla)u = 0 \\ \text{div } u = 0 \quad \text{and} \quad \text{div } b = 0, \\ u(x,0) = u_0(x), \quad b(x,0) = b_0(x) \end{cases} \quad \text{in } \; Q_T := \mathbb{R}^3 \times [0,\, T), \tag{1.1}$$

Here u is the flow velocity vector, b is the magnetic vector and $\pi = p + \dfrac{|b|^2}{2}$ is the scalar pressure. By suitable weak solutions we mean solutions that solves MHD in the sense of distribution and satisfy the local energy inequality (see Definition 2.1 in section 2 for details). For a point $z = (0,0) \in \mathbb{R}^3 \times (0,T)$ by translation, we denote $B_r(x) := B_r = \{y \in \mathbb{R}^3 : |y - x| < r\}$,

$$Q_r(z) := Q_r = B_r \times (-r^2, 0), \quad r < \sqrt{T}.$$

We say that solutions u and b are regular at $z \in \mathbb{R}^3 \times (0, T)$ if u and b are bounded for some Q_r, $r > 0$. Otherwise, it is said that u and b are singular at z. The original paper where the weak solvability of the various boundary value problems was proved is Ladyženskaja and Solonnikov [9]. As in the Navier-Stokes equations, regularity problem remains open in dimension three. On the other hand, He and Xin proved in [8] a suitable weak solution to this equations using the construction arguments of a solution in [4]. Furthermore, they show that a suitable weak solution, (u, b) become regular in the presence of a certain type of scaling invariant local integral conditions for velocity and magnetic fields. Recently, in [14], Phuc give a new regularity condition, that is, $u \in L^\infty(-1, 0; L^{3,q}(B_1))$, a weak solution to the 3D Navier-Stokes equations are regular for $q \neq \infty$ (cf [2]). In this paper, we give a criterion of local interior regularity as like Phuc's result for a suitable weak solution to the 3D MHD equations in Lorentz space which is still unknown (see e.g. [20, 12] for the Naiver-Stokes equations). For proofs, we prove the ϵ- regularity criteria for this solution in Lorentz space (below Proposition 2.3)based on the ϵ-regularity criteria in Sobolev space. After that, using the standard blow-up argument(or contraction argument) and the unique continuation for parabolic equation, we show a solution is regular (see e.g. [1, 3, 6, 7, 13]). In summary, overall, our proof is followed the arguments in [14, 2] which is mainly contained the arguments for the Naiver-Stokes equations. Now we are ready to state the first part of our main result.

Theorem 1.1. *Let a pair of functions u, b and π have the following differentiability properties:*

$$u, b \in L^{2,\infty}(Q_2) \cap W_2^{1,0}(Q_2), \quad \pi \in L^{\frac{3}{2}}(Q_2)$$

Suppose that (u, b, π) satisfy the 3D MHD equations in Q_2 in the sense of distributions. Assume, in addition, that there exists $3 < q < \infty$ such that

$$u, b \in L^\infty(-4, 0; L^{3,q}(B_2)).$$

Then (u, b) is Hölder continuous in the closure of the set $Q_{\frac{1}{2}}$.

2. Preliminaries

In this section we introduce some scaling invariant functionals and suitable weak solutions, and recall an estimation of the Stokes system.

We first start with some notations. Let Ω be an open domain in $\mathbb{R}^3$ and I be a finite time interval. We denote by $L^{p,q}(\mathbb{R}^3)$ with $1 \le p, q \le \infty$ the Lorentz space with the norm [21]

$$\|\varphi\|_{L^{p,q}} = \Big(\int_0^\infty t^q (m(\varphi, t))^{q/p} \, \frac{dt}{t} \Big)^{1/q} < \infty \quad \text{for } 1 \le q < \infty,$$

where $m(\varphi, t)$ is the Lebesgue measure of the set $\{x \in \mathbb{R}^3 : |\varphi(x)| > t\}$, *i.e.*

$$m(\varphi, t) := m\{x \in \mathbb{R}^3 : |\varphi(x)| > t\}.$$

In particular, when $q = \infty$,

$$\|\varphi\|_{L^{p,\infty}} = \sup_{t \ge 0} \{t (m(\varphi, t))^{\frac{1}{p}}\} < \infty \,.$$

The Lorentz space $L^{p,\infty}$ is also called weak L^p space. The norm is equivalent to the norm

$$\|f\|_{L^{q,\infty}} = \sup_{0<|E|<\infty} |E|^{1/q-1} \int_E |f(x)|dx.$$

For a function $f(x,t)$, we denote $\|f\|_{L^{p,q}_{x,t}(\Omega\times I)} = \|f\|_{L^q_t(I;L^p_x(\Omega))} = \|\|f\|_{L^p_x(\Omega)}\|_{L^q_t(I)}$ and vector fields u, v we write $(u_i v_j)_{i,j=1,2,3}$ as $u \otimes v$. We denote by $C = C(\alpha, \beta, ...)$ a constant depending on the prescribed quantities $\alpha, \beta, ...$, which may change from line to line. Next we recall suitable weak solutions for the MHD equations (1.1) in three dimensions.

Definition 2.1. *Let $I = (0, T)$. A triple of (u, b, π) is a suitable weak solution to* (1.1) *if the following conditions are satisfied:*

(a) The functions $u, b : Q_T \to \mathbb{R}^3$ and $\pi : Q_T \to \mathbb{R}$ satisfy

$$u, b \in L^\infty(I; L^2(\mathbb{R}^3)) \cap L^2(I; W^{1,2}(\mathbb{R}^3)), \quad \pi \in L^{\frac{3}{2}}(Q_T),$$

(b) (u, b, π) solves the MHD equations in Q_T in the sense of distributions.
(c) u, b and π satisfy the local energy inequality

$$\begin{aligned}
&\int_B (|u(x,t)|^2 + |b(x,t)|^2)\phi(x,t)dx \\
&+2\int_{t_0}^t \int_B (|\nabla u(x,t')|^2 + |\nabla b(x,t')|^2)\phi(x,t')dxdt' \\
\leq &\int_{t_0}^t \int_B (|u|^2 + |b|^2)(\partial_t \phi + \Delta\phi)dxdt' + \int_{t_0}^t \int_B \left(|u|^2 + |b|^2 + 2\pi\right) u \cdot \nabla\phi dxdt' \\
&- 2\int_{t_0}^t \int_B (b\cdot u)(b \cdot \nabla\phi)dxdt'. \qquad (2.1)
\end{aligned}$$

for all nonnegative function $\phi \in C_0^\infty(\mathbb{R}^3 \times R)$.

The crucial regularity result in [8] and [23] ensures that

Lemma 2.1. *There exists $\epsilon > 0$ such that if (u, b, π) is a suitable weak solution of the 3D MHD equations and for $r > 0$,*

$$\frac{1}{r^2}\int_{Q_{z,r}} |u(y,s)|^3 + |b(y,s)|^3 + |\pi(y,s)|^{\frac{3}{2}} dyds < \epsilon,$$

then z is a regular point.

Before a proof, we know some necessary results, which is crucial role for our analysis (see [2] and [14]). After then, using these result, we prove Theorem 1.1.

Proposition 2.1. *Suppose that the pair of functions (u, b, π) satisfies the 3D MHD equations in $Q := Q_1(0,0) = B_1(0) \times (-1, 0)$ in the sense of distributions and has the following properties*

$$u,\ b \in L^\infty(-1, 0; L^2(B_1)) \cap L^2(-1, 0; W^{1,2}(B_1)),$$

$$\pi \in L^2(-1,0;L^1(B_1)).$$

for some $q \in (3,\infty)$. Then (u,b,π) forms a suitable weak solution to the 3D MHD equations in $Q_{\frac{5}{6}}$ with a generalized energy equality, $u \in L^4(Q)$, and $\pi \in L^2(Q_{\frac{5}{6}})$. Suppose further that

$$u \in L^\infty(-1,0;L^{3,q}(B_1)), \quad b \in L^\infty(-1,0;L^{3,q}(B_1)).$$

In addition, the inequalities

$$\|u(\cdot,t)\|_{L^{3,q}(B_{\frac{3}{4}})} \leq \|u\|_{L^\infty(-(\frac{3}{4})^2,0;\ L^{3,q}(B_{\frac{3}{4}}))},$$

and

$$\|b(\cdot,t)\|_{L^{3,q}(B_{\frac{3}{4}})} \leq \|b\|_{L^\infty(-(\frac{3}{4})^2,0;\ L^{3,q}(B_{\frac{3}{4}}))}$$

hold for all $t \in (-(\frac{3}{4})^2,0)$, and the function

$$t \to \int_{B_{\frac{3}{4}}} u(x,t)w(x)dx$$

is continuous on $[-(\frac{3}{4})^2,0]$ for any $w \in L^{\frac{3}{2},\frac{q}{q-1}}(B_{\frac{3}{4}})$. Here, it is clear that $\frac{q}{q-1}=1$ in the case $q=\infty$.

Proof. By Sobolev's inequality, we know $u \in L^2(-1,0;L^6(B_1))$. And also by the assumptions and interpolative inequality, we have

$$\|u\|_{L^4(B_1)} \leq C\|u\|^{\frac{1}{2}}_{L^{3,q}}\|u\|^{\frac{1}{2}}_{L^6(B_1)}, \tag{2.2}$$

which implies $u \in L^4(Q_1)$. Similarly, we get $b \in L^4(Q_1)$. Thus by Hölder's inequality, we obtain

$$u\cdot\nabla u, b\cdot\nabla b, u\cdot\nabla b, b\cdot\nabla u \in L^{\frac{4}{3}}(Q_1). \tag{2.3}$$

Decompose the pressure so that

$$\pi = \pi_1 + \pi_2,$$

where $\pi_1 := R_iR_j(\chi_{B_\rho}(u_iu_j + b_ib_j))$. Here R_i is Riesz operator and we adopt summation convention. It is not difficult to notice that in B_ρ:

$$\Delta\pi_2 = 0.$$

By Calderón-Zygmund estimate we have

$$\|\pi_1\|_{L^2(B_1)} \leq C(\|u\|^2_{L^4(B_1)} + \|b\|^2_{L^4(B_1)}), \tag{2.4}$$

and thus (2.4), it holds

$$\begin{aligned}\|\pi_2\|_{L^2(-1,0;L^\infty(B_{\frac{5}{6}}))} &\leq C\|\pi_2\|_{L^2(-1,0;L^1(B_1))} = C\|\pi-\pi_1\|_{L^2(-1,0;L^1(B_1))}\\ &\leq C\|\pi\|_{L^2(-1,0;L^1(B_1))} + C(\|u\|^2_{L^4(Q)} + \|b\|^2_{L^4(Q)}).\end{aligned} \tag{2.5}$$

Estimates (2.4) and (2.4) imply that the pressure $\pi \in L^2(Q_{\frac{5}{6}})$. With the energy class, estimate (2.2), (2.3) and (2.5), and the local interior regularity of Stokes systems , we have

$$(\|u\|_{L^4(Q_{\frac{3}{4}})} + \|b\|_{L^4(Q_{\frac{3}{4}})})+$$

$$(\|u_t\|_{L^{\frac{4}{3}}(Q_{\frac{3}{4}})} + \|b_t\|_{L^{\frac{4}{3}}(Q_{\frac{3}{4}})} + (\|\nabla^2 u\|_{L^{\frac{4}{3}}(Q_{\frac{3}{4}})} + \|\nabla^2 b\|_{L^{\frac{4}{3}}(Q_{\frac{3}{4}})} + \|\nabla\pi\|_{L^{\frac{4}{3}}(Q_{\frac{3}{4}})} < \infty.$$

It then follows that

$$u, b \in C(-(\frac{3}{4})^2, 0; L^{\frac{4}{3}}(B_{\frac{3}{4}}))$$

and thus the function

$$g_\varphi(t) := \int_{B_{3/4}} u(x,t)\varphi(x)dx$$

is continuous on $[-(\frac{3}{4})^2, 0]$ for any $\varphi \in C_0^\infty(B_{\frac{3}{4}})$. This yields

$$\Big|\int_{B_{3/4}} u(x,t)\varphi(x)dx\Big| \le C\|\varphi\|_{L^{\frac{3}{2},\frac{q}{q-1}}(B_{\frac{3}{4}})}\|u\|_{L^\infty(-(\frac{4}{3})^2,0;L^{3,q}(B_{\frac{3}{4}}))}.$$

Thus by the density of $C_0^\infty(B_{\frac{3}{4}})$ in $L^{\frac{3}{2},\frac{q}{q-1}}(B_{\frac{3}{4}})$ we see that

$$\|u\|_{L^{3,q}(B_{\frac{3}{4}})} \le C\|u\|_{L^\infty(-(\frac{4}{3})^2,0;L^{3,q}(B_{\frac{3}{4}}))}, \quad t \in [-(\frac{3}{4})^2, 0].$$

Then it can be seen, again by density, that the function g_φ above is actually continuous on $[-(\frac{3}{4})^2, 0]$ for any $\varphi \in L^{\frac{3}{2},\frac{q}{q-1}}(B_{\frac{3}{4}})$. Finally, using $u \in L^4(B_1)$ and a standard mollification in R^{3+1} combined with a truncation in time of test functions, we obtain the local generalized energy equality in $Q_{\frac{5}{6}}$. □

2.1. Some estimates

For simplicity, we write

$$\Phi(r) := A_u(r) + A_b(r) + E_u(r) + E_b(r).$$

where

$$A_u(r) := \sup_{t-r^2\le s<t} \frac{1}{r}\int_{B_r} |u(y,s)|^2 dy, \quad E_u(r) := \frac{1}{r}\int_{Q_r} |\nabla u(y,s)|^2 dyds,$$

$$A_b(r) := \sup_{t-r^2\le s<t} \frac{1}{r}\int_{B_r} |b(y,s)|^2 dy, \quad E_b(r) := \frac{1}{r}\int_{Q_r} |\nabla b(y,s)|^2 dyds,$$

Also, we introduce following the scale invariant functional : for $0 < r < 1$,

$$C_\infty^u(r) = \frac{1}{r^2}\int_{-r^2}^0 \|u(y,s)\|^3_{L^{3,\infty}(B_r)} ds, \quad C_\infty^b(r) = \frac{1}{r^2}\int_{-r^2}^0 \|b(y,s)\|^3_{L^{3,\infty}(B_r)} ds.$$

$$D_\infty(r) = \frac{1}{r^2}\int_{-r^2}^0 \|\pi(y,s)\|^{\frac{3}{2}}_{L^{\frac{3}{2},\infty}(B_r)} ds.$$

Now, we begin with stating a well known algebraic Lemma, whose proof is omitted but found in [4].

Lemma 2.2. *Let $I(s)$ be a bounded non negative function in the interval $[R_1, R_2]$. Assume that for every $s, \rho \in [R_1, R_2]$ and $s < \rho$ we have*

$$I(s) \le [A(\rho - s)^{-\alpha} + B(\rho - s)^{-\beta} + C] + \theta I(\rho)$$

with $A, B, C \ge 0$, $\alpha > \beta > 0$ and $\theta \in [0, 1)$. Then there holds

$$I(R_1) \le c(\alpha, \theta)[A(R_2 - R_1)^{-\alpha} + B(R_2 - R_1)^{-\beta} + C].$$

Lemma 2.3. *Let* (u, b, π) *be a suitable weak solution to 3D MHD equations. Then for* $0 < r$ *the following holds*

$$\Phi(\frac{r}{2}) \leq C(C^u_\infty(r)^{\frac{2}{3}} + C^b_\infty(r)^{\frac{2}{3}} + C^u_\infty(r)^{\frac{4}{3}} + C^b_\infty(r)^{\frac{4}{3}} + D_\infty(r)^{\frac{2}{3}}).$$

Proof. Without loss of generally, consider z_0 to be the origin. Let $0 < \frac{r}{2} \leq s < \rho \leq r < 1$. Let $\eta_1 \in C_0^\infty(B(\rho))$ such that $0 \leq \eta_1 \leq 1$ in $\mathbb{R}^3$ and $\eta_1 = 1$ on $B(s)$. Furthermore for $|\alpha| \leq 2$:

$$|\nabla^\alpha \eta_1| \leq \frac{C}{(\rho - s)^\alpha}.$$

Let $\eta_2 \in C_0^\infty(-\rho^2, \rho^2)$ such that $0 \leq \eta_2 \leq 1$ in $\mathbb{R}$ and $\eta_1 = 1$ on $[-s^2, s^2]$.

$$|\eta_1'| \leq \frac{C}{(\rho^2 - s^2)} \leq \frac{C}{r(\rho - s)} \leq \frac{C}{(\rho - s)^2}.$$

Let $\phi(x, t) := \eta_(t)\eta_2(x)$. Hence:

$$|\nabla\phi| \leq \frac{C}{\rho - s}, \quad |\nabla^2\phi| \leq \frac{C}{(\rho - s)^2}, \quad |\phi_t| \leq \frac{C}{(\rho - s)^2}.$$

From the local energy inequality, we are known

$$\int_{B_r} (|u(x,t)|^2 + |b(x,t)|^2)\phi(x,t)dx + 2\int_{-\rho^2}^0 \int_{B_r} (|\nabla u(x,t')|^2 + |\nabla b(x,t')|^2)\phi(x,t')dxdt'$$

$$\leq \int_{-\rho^2}^0 \int_{B_\rho} (|u|^2 + |b|^2)(\partial_t\phi + \Delta\phi)dxdt'$$

$$+ \int_{-\rho^2}^0 \int_{B_\rho} \left(|u|^2 + |b|^2\right) u \cdot \nabla\phi dxdt' + 2\int_{-\rho^2}^0 \int_{B_\rho} \pi u \cdot \nabla\phi dxdt' - 2\int_{-\rho^2}^0 \int_{B_\rho} (b \cdot u)(b \cdot \nabla\phi)dxdt', \quad (2.6)$$

$$:= \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4$$

for all $t \in I = (-1, 0)$ and for all non-negative functions $\phi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})$. Let us treat the term $\mathcal{E}_1$ first. By O'Neil's inequality in space, the property of ϕ, and then Hölder in time, we have

$$\mathcal{E}_1 \leq \int_{-\rho^2}^0 (\|u\|^2_{L^{3,\infty}(B_\rho)} + \|b\|^2_{L^{3,\infty}(B_\rho)})\|\Delta\phi + \partial_t\phi\|_{L^{3,1}(B_\rho)} ds$$

$$\leq \frac{C\rho}{(\rho - s)^2} \int_{-\rho^2}^0 (\|u\|^2_{L^{3,\infty}(B_\rho)} + \|b\|^2_{L^{3,\infty}(B_\rho)})ds$$

$$\leq \frac{C\rho^{\frac{5}{3}}}{(\rho - s)^2}\Big[\Big(\int_{-\rho^2}^0 \|u\|^3_{L^{3,\infty}(B_\rho)} ds\Big)^{\frac{2}{3}} + \Big(\int_{-\rho^2}^0 \|b\|^3_{L^{3,\infty}(B_\rho)} ds\Big)^{\frac{2}{3}}\Big]. \quad (2.7)$$

Lorentz spaces is characterization as interpolation space between L^2 and L^6 as follows:

$$L^{3,1}(\Omega) = (L^2(\Omega), L^6(\Omega))_{\frac{1}{2},1} \quad (2.8)$$

Before the term $\mathcal{E}_3$ is estimated, we note that

$$\|u\cdot\nabla\phi\|_{L^{3,1}(B_\rho)} \le \|u\cdot\nabla\phi\|^{\frac12}_{L^2(B_\rho)}\|u\cdot\nabla\phi\|^{\frac12}_{L^6(B_\rho)} \le \|u\cdot\nabla\phi\|^{\frac12}_{L^2(B_\rho)}\|\nabla(u\cdot\nabla\phi)\|^{\frac12}_{L^2(B_\rho)}$$

$$\le \frac{C\|u\|_{L^2(B_\rho)}}{(\rho-s)^{\frac32}} + \frac{C\|u\|^{\frac12}_{L^2(B_\rho)}\|\nabla u\|^{\frac12}_{L^2(B_\rho)}}{\rho-s}, \tag{2.9}$$

where we use the interpolation (2.8), Sobolev embedding and the property of ϕ. Set $I(\rho) = \rho\Phi(\rho)$. Using O'Neil inequality and the estimate (2.9), the term $\mathcal{E}_3$ is estimated as follows: for $\rho \le r$,

$$\mathcal{E}_3 \le \int_{-\rho^2}^0 \|u\cdot\nabla\phi\|_{L^{3,1}(B_\rho)}\|\pi\|_{L^{\frac32,\infty}(B_\rho)}ds \le \Big[\frac{C}{(\rho-s)^{\frac32}}\Big(\int_{-\rho^2}^0\|u\|^3_{L^2(B_\rho)}ds\Big)^{\frac13}$$

$$+\frac{C}{\rho-s}\Big(\int_{-\rho^2}^0\|u\|^{\frac32}_{L^2(B_\rho)}\|u\|^{\frac32}_{L^2(B_\rho)}ds\Big)^{\frac13}\Big]\times\Big(\int_{-\rho^2}^0\|\pi\|^{\frac32}_{L^{\frac32,\infty}(B_\rho)}ds\Big)^{\frac23}$$

$$\le C\Big(\frac{r^{\frac23}I(\rho)^{\frac12}}{(\rho-s)^{\frac32}} + \frac{r^{\frac16}}{\rho-s}I(\rho)^{\frac12}\Big)\Big(\int_{-\rho^2}^0\|\pi\|^{\frac32}_{L^{\frac32,\infty}(B_\rho)}ds\Big)^{\frac23}. \tag{2.10}$$

Similarly, we are obtained the following estimate as like $\mathcal{E}_3$:

$$\int_{-\rho^2}^0\int_{B_\rho} 2|u|^2u\cdot\nabla\phi dxdt' \le C\Big(\frac{r^{\frac23}I(\rho)^{\frac12}}{(\rho-s)^{\frac32}} + \frac{r^{\frac16}}{\rho-s}I(\rho)^{\frac12}\Big)\Big(\int_{-\rho^2}^0\|u\|^3_{L^{3,\infty}(B_\rho)}ds\Big)^{\frac23}, \tag{2.11}$$

$$\int_{-\rho^2}^0\int_{B_\rho} 2|b|^2u\cdot\nabla\phi dxdt' \le C\Big(\frac{r^{\frac23}I(\rho)^{\frac12}}{(\rho-s)^{\frac32}} + \frac{r^{\frac16}}{\rho-s}I(\rho)^{\frac12}\Big)\Big(\int_{-\rho^2}^0\|b\|^3_{L^{3,\infty}(B_\rho)}ds\Big)^{\frac23}. \tag{2.12}$$

So thus, with the estimates (2.11) and (2.12), the term $\mathcal{E}_2 + \mathcal{E}_4$ is estimated by

$$\mathcal{E}_2+\mathcal{E}_4 \le C\Big(\frac{r^{\frac23}I(\rho)^{\frac12}}{(\rho-s)^{\frac32}} + \frac{r^{\frac16}}{\rho-s}I(\rho)^{\frac12}\Big)\Big[\Big(\int_{-\rho^2}^0\|u\|^3_{L^{3,\infty}(B_\rho)}ds\Big)^{\frac23} + \Big(\int_{-\rho^2}^0\|b\|^3_{L^{3,\infty}(B_\rho)}ds\Big)^{\frac23}\Big]. \tag{2.13}$$

We combine with the estimate (2.7), (2.10) and (2.13) and Young's inequality to get

$$I(\rho) \le \frac{r^{\frac53}}{(\rho-s)^2}\Big[\Big(\int_{-\rho^2}^0\|u\|^3_{L^{3,\infty}(B_\rho)}ds\Big)^{\frac23} + \Big(\int_{-\rho^2}^0\|u\|^3_{L^{3,\infty}(B_\rho)}ds\Big)^{\frac23}\Big] + \frac12 I(\rho)$$

$$+\Big(\frac{r^{\frac43}}{(\rho-s)^3} + \frac{r^{\frac13}}{(\rho-s)^2}\Big)\Big[\Big(\int_{-\rho^2}^0\|u\|^3_{L^{3,\infty}(B_\rho)}ds\Big)^{\frac43} + \Big(\int_{-\rho^2}^0\|b\|^3_{L^{3,\infty}(B_\rho)}ds\Big)^{\frac43} + \Big(\int_{-\rho^2}^0\|\pi\|^{\frac32}_{L^{\frac32,\infty}(B_\rho)}ds\Big)^{\frac43}\Big]$$

Since $\frac{r}{2} \le s < \rho \le r$ and by Lemma 2.2, we obtain

$$\Phi(\frac{r}{2}) \le r^{-\frac13}\Big[\Big(\int_{-\rho^2}^0\|u\|^3_{L^{3,\infty}(B_\rho)}ds\Big)^{\frac23} + \Big(\int_{-\rho^2}^0\|u\|^3_{L^{3,\infty}(B_\rho)}ds\Big)^{\frac23}\Big]$$

$$+Cr^{-\frac53}\Big[\Big(\int_{-\rho^2}^0\|u\|^3_{L^{3,\infty}(B_\rho)}ds\Big)^{\frac43} + \Big(\int_{-\rho^2}^0\|b\|^3_{L^{3,\infty}(B_\rho)}ds\Big)^{\frac43} + \Big(\int_{-\rho^2}^0\|\pi\|^{\frac32}_{L^{\frac32,\infty}(B_\rho)}ds\Big)^{\frac43}\Big].$$

□

2.2. *Proof of main theorem*

Following the notation in [14], we suppose that $z_0 := (x_0, t_0) \in Q_{\frac{1}{2}}(0,0)$ is a singular point. It means that there exists no neighborhood $\mathcal{N}$ of z_0 such that (u, b) has a Hölder continuous representative on $\mathcal{N} \cap [B_1(0) \times (-1, 0])$. By Theorem 3.2 [13], there exist $c_0 > 0$ and a sequence of numbers $\epsilon_k \in (0, 1)$ such that $\epsilon_k \to 0$ as $k \to \infty$ and

$$\sup_{t_0-\epsilon_k \le s \le t_0} \frac{1}{\epsilon_k} \int_{B(x_0,\epsilon_k)} |u(x,s)|^2 dx + |b(x,s)|^2 dx \ge c_0, \tag{2.14}$$

for any $k \in \mathbb{N}$. Moreover, by Proposition 2.1, we have in particular

$$u(\cdot, t_0) \in L^{3,q}(B_{3/4}(0)), \quad b(\cdot, t_0) \in L^{3,q}(B_{3/4}(0))$$

Recall that we can decompose $\pi = \tilde{\pi} + h$, where h is harmonic in B_1, and $\tilde{\pi} = R_i R_j[(u_i u_j + b_i b_j)\chi_{B_1}]$. For each $Q = \omega \times (a, b)$, where $\omega \in \mathbb{R}^3$ and $-\infty < a < b \le 0$, we choose a large $k_0 = k_0(Q) \ge 1$ so that for any $k \ge k_0$ there hold the implications $x \in \omega \Longrightarrow x_0 + \epsilon_k x \in B_{\frac{2}{3}}$, and $t \in (a, b) \Longrightarrow t_0 + \epsilon_k t \in (-(\frac{2}{3})^2, 0)$, where the sequence ϵ_k is as in (4.7). Set $Q = \omega \times (a, b)$, let us set

$$u_k(x,t) = \epsilon_k u(x_0 + \epsilon_k x, t_0 + \epsilon_k^2 t), \quad b_k(x,t) = \epsilon_k b(x_0 + \epsilon_k x, t_0 + \epsilon_k^2 t),$$

and

$$\pi_k(x,t) = \epsilon_k^2 k\pi(x_0 + \epsilon_k x, t_0 + \epsilon_k^2 t),$$
$$\tilde{\pi}_k(x,t) = \epsilon_k^2 \tilde{\pi}(x_0 + \epsilon_k x, t_0 + \epsilon_k^2 t), \quad \text{and} \quad h_k(x,t) = \epsilon_k^2 h(x_0 + \epsilon_k x, t_0 + \epsilon_k^2 t),$$

for any $(x, t) \in Q$ and $k \ge k_0(Q)$.

The following proposition is a key in the proof of Theorem 1.1, which says the properties in the limit.

Proposition 2.2. *Let $0 < q < \infty$ and $Q = \omega \times (a, b)$ with $\omega \subset \mathbb{R}^3$, $-\infty < a < b \le 0$. There exists a subsequence of (u^k, b^k, π^k), still denoted by (u^k, b^k, π^k), and a pair of functions*

$$(u^\infty,\ b^\infty, \pi^\infty) \in L^\infty(-\infty, 0; L^{3,q}(\mathbb{R}^3)) \times L^\infty(-\infty, 0; L^{3,q}(\mathbb{R}^3)) \times L^\infty(-\infty, 0; L^{\frac{3}{2},\frac{q}{2}}(\mathbb{R}^3))$$

with div $u^\infty = 0$ and div $b^\infty = 0$ in $\mathbb{R}^3 \times (-\infty, 0)$, such that for $s \in (1, 3)$,

$$u^k \to u^\infty \text{ in } C(a, b; L^s(\omega)), \tag{2.15}$$

$$b^k \to b^\infty \text{ in } C(a, b; L^s(\omega)), \tag{2.16}$$

$$\pi^k \to \pi^\infty \text{ weakly}^* \text{ in } L^\infty(a, b; L^{\frac{3}{2},\frac{q}{2}}(\omega)), \tag{2.17}$$

Moreover

$$|u^\infty|^2, |b^\infty|^2, \nabla u^\infty, \nabla b^\infty \in L^2(Q), \tag{2.18}$$

$$\partial_t u^\infty, \partial_t b^\infty, \nabla^2 u^\infty, \nabla^2 b^\infty, \nabla \pi^\infty \in L^{\frac{4}{3}}(Q), \tag{2.19}$$

and $(u^\infty, b^\infty, \pi^\infty)$ satisfies a suitable weak solution to the 3D MHD equations in Q. Additionally, u^∞ and b^∞ satisfy the lower bound satisfies the lower bound

$$\int_Q (|u^\infty|^2 + |b^\infty|^2) dz \ge \varepsilon_3. \tag{2.20}$$

Proof. For each $Q=\omega\times(a,b)$, where for $\omega\subset\mathbb{R}^3$ and $t\in[a,b]$ with $-\infty<a<b\le 0$, we have

$$\|u_k(\cdot,t)\|_{L^{3,q}(\omega)}\le\|u_k(\cdot,t_0+\epsilon_k^2t)\|_{L^{3,q}(B_{\frac{3}{4}})}\le\|u\|_{L^\infty(-1,0);L^{3,q}(B_1)},\tag{2.21}$$

and

$$\|b_k(\cdot,t)\|_{L^{3,q}(\omega)}\le\|b\|_{L^\infty(-1,0);L^{3,q}(B_1)},\tag{2.22}$$

By Calderón-Zygmund estimate, for a.e. $t\in(a,b)$ there holds

$$\|\tilde\pi_k(\cdot,t)\|_{L^{\frac{3}{2},\frac{q}{2}}(\omega)}\le\|\tilde\pi_k(\cdot,t_0+\epsilon_k^2t)\|_{L^{\frac{3}{2},\frac{q}{2}}(B_{\frac{3}{4}})}\le C(\|u\|^2_{L^\infty(-1,0);L^{3,q}(B_1)}+\|b\|^2_{L^\infty(-1,0);L^{3,q}(B_1)}).\tag{2.23}$$

On the other hand, by harmonicity we have

$$\int_a^b\sup_{x\in\omega}|h_k(x,t)|^{\frac{3}{2}}dt\le\epsilon_k\int_{-(3/4)^2}\sup_{x\in\omega}|h_k(x_0+\epsilon_kx,s)|^{\frac{3}{2}}ds\le\epsilon_k\|h\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(-1,0);L^\infty(B_{\frac{3}{4}})}\tag{2.24}$$

$$\le C\epsilon_k(\|u\|^3_{L^\infty((-1,0);L^{3,q}(B_1))}+\|b\|^3_{L^\infty((-1,0);L^{3,q}(B_1))}+\|\pi\|_{L^{\frac{3}{2}}(Q_1)})$$

Thus each (u_k,b_k) is a suitable solution in Q. Then, from the energy estimate follows that

$$\|u_k\|_{L^\infty(a,b;L^2(\omega))}+\|b_k\|_{L^\infty(a,b;L^2(\omega))}+\|\nabla b_k\|_{L^2(Q)}+\|\nabla u_k\|_{L^2(Q)}\le C.\tag{2.25}$$

Using (2.25) and Sobolev embedding, we have $\|u_k\|_{L^2(a,b;L^6(\omega))}\le C$, which by (4.12), interpolation, and Hölder's inequality gives for

$$\|u_k\|_{L^4(Q)}+\|b_k\|_{L^4(Q)}+\|(u_k\cdot\nabla)u_k\|_{L^{\frac{4}{3}}(Q)}+\|(b_k\cdot\nabla)u_k\|_{L^{\frac{4}{3}}(Q)}\le C.$$

From the bounds (2.23) and (2.24), we also have

$$\|\pi_k\|_{L^s(Q)}\le C\|\pi_k\|_{L^2(a,b;L^{\frac{3}{2},\frac{q}{2}}(\omega))}\le C,\quad s\in(0,\frac{3}{2}).\tag{2.26}$$

Using the estimate (2.25)–(2.26), it follows from the local interior regularity of solutions to non-stationary Stokes equations we find

$$\|\partial_tu_k\|_{L^{\frac{4}{3}}(Q)}+\|\nabla^2u_k\|_{L^{\frac{4}{3}}(Q)}+\|\nabla\pi_k\|_{L^{\frac{4}{3}}(Q)}\le C.\tag{2.27}$$

Furthermore, we can easily check the as following:

$$\|\partial_tu_k\|_{L^{\frac{4}{3}}(Q)}+\|\partial_tb_k\|_{L^{\frac{4}{3}}(Q)}+\|\nabla^2u_k\|_{L^{\frac{4}{3}}(Q)}+\|\nabla^2b_k\|_{L^{\frac{4}{3}}(Q)}+\|\nabla\pi_k\|_{L^{\frac{4}{3}}(Q)}\le C.\tag{2.28}$$

Using estimates (2.21)–(2.23), we may get that

$$u_k\rightharpoonup^* u^\infty\quad\text{in } L^\infty(-\infty,0;L^{3,q}(\mathbb{R}^3)).$$

$$b_k\rightharpoonup^* b^\infty\quad\text{in } L^\infty(-\infty,0;L^{3,q}(\mathbb{R}^3)).$$

$$\tilde\pi_k\rightharpoonup^*\tilde\pi^\infty\quad\text{in } L^\infty(-\infty,0;L^{\frac{3}{2},\frac{q}{2}}(\mathbb{R}^3)).$$

Estimates (2.25) and (2.27) yield

$$u_k \rightharpoonup^* u^\infty \quad \text{in } C(-\infty, 0; L^{\frac{4}{3}}(Q)), \tag{2.29}$$

$$b_k \rightharpoonup^* b^\infty \quad \text{in } C(-\infty, 0; L^{\frac{4}{3}}(Q)). \tag{2.30}$$

For any $s \in (1, 3)$, the uniform bound (2.21) and the interpolation inequality

$$\|u_k(\cdot, t) - u_k(\cdot, t')\|_{L^s} \le \|u_k(\cdot, t) - u_k(\cdot, t')\|_{L^{\frac{4}{3}}}^{\frac{12}{5}\left(\frac{1}{s} - \frac{1}{3}\right)} \|u_k(\cdot, t) - u_k(\cdot, t')\|_{L^s}^{\frac{12}{5}\left(\frac{3}{4} - \frac{1}{s}\right)}$$

imply that each $u_k \in C([a, b]; L^s(\omega))$. Thus by using (2.29) and interpolating we obtain (2.15) for any $s \in (1, 3)$. On the other hand, by (2.24), we have

$$h_k \to 0 \text{ strongly in } L^2(a, b; L^\infty(\omega)),$$

Now (2.18)–(2.19) follows from (2.29), (2.30), (2.25) and (2.27) via an argument as in the proof of Proposition 2.1. Finally, note that by (2.41) and a change of variables we have

$$\sup_{-1 \le t \le 0} \int_{B(0,1)} |u_k(x, t)|^2 dx = \sup_{t_0 - \epsilon_k^2 \le t \le r_0} \frac{1}{\epsilon_k} \int_{B(0,1)} |u_k(y, s)|^2 dy \ge C_0.$$

Similarly, $\sup_{-1 \le t \le 0} \int_{B(0,1)} |u_k(x, t)|^2 dx \ge C_0$. Thus using the convergences (2.15) and (2.16) with $s = 2$ we obtain the lower bound (2.20). □

Before proving the main statement we introduce some notation

$$C_u(r) := \frac{1}{r^2} \int_{Q_r} |u|^3 dz, \quad C_b(r) := \frac{1}{r^2} \int_{Q_r} |b|^3 dz, \quad D(r) := \frac{1}{r^2} \int_{Q_r} |\pi|^{\frac{3}{2}} dz.$$

Now, we prove the ϵ- regularity criteria for a suitable weak solution to the 3D MHD equations under our circumstance.

Proposition 2.3. *Let (u, b, π) be a suitable weak solution to 3D MHD equations. Then there exists a universal constants c_0 and $c_{0k}(\epsilon_0)$ (with $k = 1, 2, \cdots$) with the following property. Assume*

$$C^u_\infty(1) + C^b_\infty(1) + D_\infty(1) \le \epsilon_0, \tag{2.31}$$

then for any natural number k, $\nabla^{k-1} u$ is Hölder continuous in $\tilde{Q}_{1/8}$ and the following bound is valid:

$$\sup_{\tilde{Q}_{1/8}} \left(|\nabla^{k-1} u(z)| + |\nabla^{k-1} b(z)|\right) < c_{0k}(\epsilon_0).$$

Proof. From Lemma 2.3 and assumptions (2.31), it follows that

$$A_u(\frac{1}{2}) + A_b(\frac{1}{2}) + E_u(\frac{1}{2}) + E_b(\frac{1}{2}) \le C(\epsilon_0 + \epsilon_0^2)^{\frac{2}{3}}. \tag{2.32}$$

By interpolation and Sobolev embedding theorem one can show that

$$C_u(\frac{1}{2}) \le C[A_u(\frac{1}{2})^{\frac{3}{4}} E_u(\frac{1}{2})^{\frac{3}{4}} + A_u(\frac{1}{2})^{\frac{3}{2}}].$$

Thus, by (2.32) we have
$$C_u(\frac{1}{2}) \leq C(\epsilon_0 + \epsilon_0^2). \tag{2.33}$$
Similarly, we have
$$C_b(\frac{1}{2}) \leq C(\epsilon_0 + \epsilon_0^2). \tag{2.34}$$
For similar reasons it is not so difficult to see that
$$\|\nabla \cdot (u \times u)\|_{L^{\frac{9}{8},\frac{3}{2}}(Q_{\frac{1}{2}})} \leq C[A_u(\frac{1}{2}) + A_u(\frac{1}{2})^{\frac{1}{3}} B_u(\frac{1}{2})^{\frac{2}{3}}].$$
Thus,
$$\|\nabla \cdot (u \times u)\|_{L^{\frac{9}{8},\frac{3}{2}}(Q_{\frac{1}{2}})} \leq C(\epsilon_0 + \epsilon_0^2)^{\frac{2}{3}}. \tag{2.35}$$
Similarly, we have
$$\|\nabla \cdot (b \times b)\|_{L^{\frac{9}{8},\frac{3}{2}}(Q_{\frac{1}{2}})} \leq C(\epsilon_0 + \epsilon_0^2)^{\frac{2}{3}}. \tag{2.36}$$
On the other hand, by Hölder's inequality, it is obvious that
$$\|u\|_{W^{1,0}_{\frac{9}{8},\frac{3}{2}}(Q_{\frac{1}{2}})} \leq C(A_u(\frac{1}{2}) + B_u(\frac{1}{2})) \leq C(\epsilon_0 + \epsilon_0^2)^{\frac{1}{3}}. \tag{2.37}$$
Similarly, we have
$$\|b\|_{W^{1,0}_{\frac{9}{8},\frac{3}{2}}(Q_{\frac{1}{2}})} \leq C(\epsilon_0 + \epsilon_0^2)^{\frac{1}{3}}. \tag{2.38}$$
Using O'Neil's inequality, we have
$$\int_{B(\frac{1}{2})} |\pi(x,t)|^{\frac{9}{8}} dx \leq C\|\pi^{\frac{9}{8}}\|_{L^{\frac{8}{3},\infty}} = C\|\pi\|^{\frac{9}{8}}_{L^{3,\infty}}$$
Hence,
$$\|\pi(x,t)\|_{L^{\frac{9}{8},\frac{3}{2}}} \leq C\epsilon_0^{\frac{2}{3}}. \tag{2.39}$$
Using the local interior regularity theory for Stokes equation, we have
$$\begin{aligned}
&\|u_t\|_{L^{\frac{9}{8},\frac{3}{2}}(Q_{\frac{1}{4}})} + \|\nabla^2 u\|_{L^{\frac{9}{8},\frac{3}{2}}(Q_{\frac{1}{4}})} + \|\nabla\pi\|_{L^{\frac{9}{8},\frac{3}{2}}(Q_{\frac{1}{4}})} \\
&\leq C(\|\nabla \cdot (u \times u)\|_{L^{\frac{9}{8},\frac{3}{2}}(Q_{\frac{1}{2}})} + \|\nabla \cdot (b \times b)\|_{L^{\frac{9}{8},\frac{3}{2}}(Q_{\frac{1}{2}})}) \\
&+\|u\|_{L^{\frac{9}{8},\frac{3}{2}}(Q_{\frac{1}{2}})} + \|\nabla u\|_{L^{\frac{9}{8},\frac{3}{2}}Q_{\frac{1}{2}}} + \|\pi\|_{L^{\frac{9}{8},\frac{3}{2}}(Q_{\frac{1}{2}})}.
\end{aligned}$$
Note that a suitable weak solution (u, b, π) implies that
$$u, b \in W^{2,1}_{\frac{9}{8},\frac{3}{2}}(Q_2) \cap W^{1,0}_{\frac{4}{3}}(Q_2), \quad \pi \in W^{1,0}_{\frac{9}{8},\frac{3}{2}}(Q_2) \cap L^{\frac{4}{3}}(Q_2).$$
(see e.g. [18, 19]). Using this together with the estimates (2.35)– (2.39), we obtain that
$$\|\nabla\pi\|_{L^{\frac{9}{8},\frac{3}{2}}(Q_{\frac{1}{4}})} \leq c[((\epsilon_0 + \epsilon_0^2)^{\frac{1}{3}} + (\epsilon_0 + \epsilon_0^2)^{\frac{2}{3}})].$$

Thus, by the Poincaré inequality, we have

$$\|\pi - [\pi]\|_{L^{\frac{3}{2}}(Q_{\frac{1}{4}})} \le c[((\epsilon_0 + \epsilon_0^2)^{\frac{1}{3}} + (\epsilon_0 + \epsilon_0^2)^{\frac{2}{3}})].$$

Therefore, we conclude

$$\|\pi\|_{L^{\frac{3}{2}}(Q_{\frac{1}{4}})} \le c[((\epsilon_0 + \epsilon_0^2)^{\frac{1}{3}} + (\epsilon_0 + \epsilon_0^2)^{\frac{2}{3}})] \tag{2.40}$$

This along with (2.33), (2.34) and (2.40) gives

$$C_u(\frac{1}{2}) + C_b(\frac{1}{2}) + D(\frac{1}{2}) \le C[((\epsilon_0 + \epsilon_0^2)^{\frac{1}{3}} + (\epsilon_0 + \epsilon_0^2)^{\frac{2}{3}})] \tag{2.41}$$

Choosing ϵ_0 sufficiently small, the estimate (2.41) satisfies the conditions of Theorem 3.3 in [13] and so we complete the proof. □

Proof of Theorem 1.1. The proof is similar to the argument in [13, Theorem 1.1] We now fix such numbers M and N and let $z_1 = (x_1, t_1) \in (\mathbb{R}^3 \backslash \bar{B}_{2N}(0)) \times (-\frac{M}{2}, 0]$. Due to $C_\infty^{u^\infty}(1) + C_\infty^{b^\infty}(1) + D_\infty(1) \le \epsilon_0$, we obtain, by Proposition 2.3

$$\max_{z \in \bar{Q}_{\frac{1}{2}}(z_1)} |\nabla^k u^\infty(z)| \le C(k), \qquad \max_{z \in \bar{Q}_{\frac{1}{2}}(z_1)} |\nabla^k b^\infty(z)| \le C(k), \qquad k = 1, 2, \cdots .$$

On the other hand, on the set $(\mathbb{R}^3 \backslash \bar{B}_{2N}(0)) \times (-\frac{M}{2}, 0]$, we have that there exists $M > 0$ such that

$$|\partial_t W - \Delta W| \le M(|W| + |\nabla W|), \quad \text{and} \quad |W| \le C,$$

for the (15-component) vector-valued function $W = (b^\infty, w^\infty, b^\infty{}_{,1}, b^\infty{}_{,2}, b^\infty{}_{,3})$ where $w^\infty = \nabla \times u^\infty$ given in [13, pp.2922-2923]. Then

$$W = 0 \text{ on } (\mathbb{R}^3 \setminus \overline{B_{4N}}(0)) \times (-\frac{M}{4}, 0].$$

Using the theory of unique continuation for parabolic equation (see [6, Theorem 5]), we see $W(\cdot, t) = 0$ in $\mathbb{R}^3$ for a.e. $t \in (-\frac{M}{4}, 0)$. Thus $u^\infty(\cdot, t) = 0$ is globally harmonic, and using Liouville theorem, it follows that $u^\infty(\cdot, t) = 0$ for a.e. $t \in (-\frac{M}{4}, 0)$. This yields to a contradiction to the lower bound (2.20) and hence completes the proof of Theorem 1.1. □

3. Conclusions

In this paper, we investigete some local regularity condition for a suitable weak solution to 3D MHD equations in Lorentz space. However, it remains an open question to obtain the local regularity condition for only velocity vector u.

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Conflicts of interest

The authors declare that they have no conflicts of interest

References

1. R. P. Agarwal, S. Gala, M. A. Ragusa, A regularity criterion in weak spaces to Boussinesq Equations, *Mathematics,* **8** (2020), 920.
2. T. Barker, Local boundary regularity for the Navier-Stokes equations in nonendpoint borderline Lorentz spaces, *J. Math. Sci.(N.Y.),* **224** (2017), 391–413.
3. S. Benbernou, S. Gala, M. A. Ragusa, On the regularity criteria for the 3D magnetohydrodynamic equations via two components in terms of BMO space, *Math. Methods Appl. Sci.,* **37** (2014), 2320–2325.
4. L. Caffarelli, R. Kohn, L. Nirenberg, Partial regularity of suitable weak solutions of the Navier-Stokes equations, *Comm. Pure Appl. Math.,* **35** (1982), 771–831.
5. P. A. Davidson, *An introduction to magnetohydrodynamics*, Cambridge University Press, Cambridge, 2001.
6. L. Escauriaza, G. Seregin, V. Šverák, $L^{3,\infty}$-solutions of Navier-Stokes equations and backward uniqueness, *Uspekhi Mat. Nauk,* **58** (2003), 211–250.
7. N. S. Khan, Mixed convection in MHD second grade nanofluid flow through a porous medium containing nanoparticles and gyrotactic microorganisms with chemical reaction, *Filomat,* **33** (2019), 4627–4653.
8. C. He, Z. Xin, Partial regularity of suitable weak solutions to the incompressible magnetohydrodynamic equations, *J. Funct. Anal.,* **227** (2005), 113–152.
9. O. A. Ladyženskaja, V. A. Solonnikov, Solution of some non-stationary problems of magnetohydrodynamics for a viscous incompressible fluid, *Trudy Mat. Inst. Steklov., Acad. Sci. USSR, Moscow–Leningrad,* **59** (1960), 115–173.
10. O. A. Ladyženskaja, G. A. Seregin, On partial regularity of suitable weak solutions to the three-dimensional Navier-Stokes equations, *J. Math. Fluid Mech.,* **1** (1999), 356–387.
11. F. Lin, A new proof of the Caffarelli-Kohn-Nirenberg theorem, *Comm. Pure Appl. Math.,* **51** (1998), 241–257.
12. Y. Luo, T. P. Tsai, Regularity criteria in weak L^3 for 3D incompressible Navier-Stokes equations, Funccialaj Ekvacioj, *Comm. Pure Appl. Math.,* **58** (2015), 387–404.
13. A. Mahalov, A. Nicolaenko, A. Shilkin, $L^{3,\infty}$-solutions to the MHD equations, *J. Math. Sci. (N. Y.),* **143** (2007), 2911–2923.
14. N. C. Phuc, The Navier-Stokes equations in nonendpoint borderline Lorentz spaces, *J. Math. Fluid Mech.,* **17** (2015), 741–760.
15. M. Sermange, R. Temam, Some mathematical questions related to the MHD equations, *Comm. Pure Appl. Math.,* **36** (1983), 635–664.

16. G. A. Seregin, Local regularity of suitable weak solutions to the Navier-Stokes equations near the boundary, *J. Math. Fluid Mech.,* **4** (2002), 1–29.

17. G. A. Seregin, Some estimates near the boundary for solutions to the non-stationary linearized Navier-Stokes equations, *J. Math. Sci. (N. Y.),* **115** (2003), 2820–2831.

18. G. A. Seregin, On smoothness of $L_{3,\infty}$-solutions to the Navier-Stokes equations up to boundary, *Math. Ann.,* **332** (2005), 219–238.

19. G. A. Seregin, A note on local boundary regularity for the Stokes system, *J. Math. Sci. (N.Y.) ,* **166** (2010), 86–90.

20. S. Takahashi, On interior regularity criteria for weak solutions of the Navier-Stokes equations, *Manuscripta Math.,* **69** (1990), 237–254.

21. H. Triebel, *Theory of Function Spaces*, Birkhäuser Verlag. Basel-Boston, (1983).

22. V. Vyalov, T. Shilkin, Partial regularity of solutions to the magnetohydrodynamic equations, *J. Math. Sci. (N. Y.),* **150** (2008), 1771–1786.

23. W. Wang, Z. Zhang, Limiting case for the regularity criterion to the 3-D magneto-hydrodynamics equations, *J. Differential Equations,* **252** (2012), 5751–5762.