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Research article

Multiplicity of solutions for Schrödinger-Poisson system with critical exponent in $\mathbb{R}^3$

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Abstract: In this paper, we study the following Schrödinger-Poisson system with critical exponent

$$\begin{cases} -\Delta u - k(x)\phi u = \lambda h(x)|u|^{p-2}u + s(x)|u|^4 u, & x \in \mathbb{R}^3, \\ -\triangle\phi = k(x)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $1 < p < 2$ and $\lambda > 0$. Under suitable conditions on k, h and s, we show that there exists $\lambda^* > 0$ such that the above problem possesses infinitely many solutions with negative energy for each $\lambda \in (0, \lambda^*)$. Moreover, we prove the existence of infinitely many solutions with positive energy. The main tools are the concentration compactness principle, Z_2 index theory and Fountain Theorem. These results extend some existing results in the literature.

Keywords: Schrödinger-Poisson system; critical exponent; variational method; genus
Mathematics Subject Classification: 35J62, 35B33

1. Introduction

In this article, we are devoted to the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u - k(x)\phi u = \lambda h(x)|u|^{p-2}u + s(x)|u|^4 u, & x \in \mathbb{R}^3, \\ -\triangle\phi = k(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $1 < p < 2$. To state our results, we impose some conditions on h and k as follows:

(A_1) $h \in L^{\frac{6}{6-p}}(\mathbb{R}^3)$, $h(x) \geq 0$ and $h(x) \not\equiv 0$.

(A_2) $k \in L^2(\mathbb{R}^3)$, $k(x) \geq 0$ and $k(x) \not\equiv 0$.

(A_3) $s \in C(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, $s(x) > 1$.

(A_4) $s \in C(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, $s(0) = 0$, $s(x) > 0$ *a.e.* in $\mathbb{R}^3$ and $\lim_{|x|\to\infty} s(x) = 0$.

As we all know the Schrödinger-Poisson system has a strong physical meaning due to the influence in quantum mechanics models (see e.g. [5, 15]) and in semiconductor theory (see e.g. [18, 19]). The crucial tools to study the existence and multiplicity of solutions about nonlinear differential equations are the variational method and the critical point theory (see e.g. [2, 26]). From an academic point of view, these methods present an interesting competition between local and nonlocal nonlinearities. Problem (1.1) is derived from the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \lambda k(x)\phi u = f(x,u), & x \in \mathbb{R}^3, \\ -\triangle\phi = k(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.2}$$

which is also called the Schrödinger-Maxwell equation, was firstly introduced in [4] while describing the interacting between solitary waves and an electrostatic field in quantum mechanics. While studying the Schrödinger-Poisson system, one has to face many obstacles since the existence of the non-local term, especially in the critical case, the invariance by dilations of $\mathbb{R}^3$ makes the problems much harder to deal with. In the past few years, a number of papers are devoted to the existence of solutions for (1.2) under various assumptions on V, k and f. In [8], D'Aprile and Mugnai firstly proved the existence results in the subcritical case. And the first non-existence result was given in [9] for the critical case. After that, Ruiz in [21] obtained more existence results and properties of the non-local term ϕ. Based on the work of [21], Azzollini and Pomponio [3] obtained the existence of ground state solutions for (1.2) where $f(x,u) = |u|^{p-1}u$ with $2 < p < 5$ when V is a positive constant and $3 < p < 5$ when V is a non-constant potential. After that, Zhang, Ma and Xie [28] studied the following problem with critical exponent

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = |u|^4 u, & x \in \mathbb{R}^3, \\ -\triangle\phi = k(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.3}$$

where $V \in L^{\frac{3}{2}}(\mathbb{R}^3)$, and proved the existence of bound state solutions.

In recent years, some researchers are interested in the existence of solutions involving concave-convex nonlinearities. For example, Zhang [27] obtained ground state and nodal solutions of following problem with critical exponent

$$\begin{cases} -\Delta u + u + k(x)\phi u = a(x)|u|^{p-2}u + |u|^5, & x \in \mathbb{R}^3, \\ -\triangle\phi = k(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.4}$$

where $p \in (4, 6)$. Later, in [14], Li and Tang considered the following Schrödinger-Poisson system with negative coefficient of nonlocal term

$$\begin{cases} -\Delta u - k(x)\phi u = \lambda h(x)|u|^{p-2}u + |u|^4 u, & x \in \mathbb{R}^3, \\ -\triangle\phi = k(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.5}$$

they proved Problem (1.5) possesses at least two solutions by Mountain Pass Theorem and Ekeland's Variational Principle. As for more results treating this problem or similar one, readers can refer to [1, 7, 10, 13, 22–24] and references therein.

All above works are to study existence of solutions of the Schrödinger-Poisson system under different conditions. Here, we have to highlight the fact that one of the main attentions of interest in our present paper is to prove the existence of infinitely many solutions. To the best of our knowledge, it seems that there are no results about infinitely many solutions while concerning negative coefficient nonlocal term. The first purpose in our paper is to establish the multiplicity of solutions possessing negative energy of the problem (1.1). Furthermore, we are also devote to studying the convergent properties of energy corresponding to the solutions. The critical exponential growth makes the problem complicated due to the lack of compactness, thus we use the concentration compactness principle to restore compactness. And we will introduce a cut-off functional which is bounded from below, by analyzing the properties of the cut-off functional, utilizing Z_2 index theory, we can obtain the first result. To demonstrate our second result, we assume some extra conditions on k, h and s, by using Fountain Theorem, we prove the existence of the multiple solutions possessing positive energy.

Next, we will state our main results.

Theorem 1.1. *Suppose* $1 < p < 2$, *the hypotheses* $(A_1), (A_2)$ *and* (A_3) *hold. If* $h(x) > 0$ *is bounded on some open subset* $\Omega \subset \mathbb{R}^3$ *with* $|\Omega| > 0$. *Then there exists* $\lambda^* > 0$ *such that for all* $\lambda \in (0, \lambda^*)$, *Problem (1.1) has infinitely many solutions with negative energy. Moreover, there exists a sequence of the critical values corresponding to the solutions which converges to zero.*

In order to give the second result, we need to introduce some notations. Denote $O(3)$ to be the group of orthogonal linear transformations in $\mathbb{R}^3$ and let $T \subset O(3)$ be a subgroup. Set $|T| := \inf_{x\in\mathbb{R}^3, x\neq 0} |T_x|$, where $T_x := \{\tau x : \tau \in O(3)\}$ for $x \neq 0$. Moreover, a function $f : \mathbb{R}^3 \to \mathbb{R}$ is called T-invariant if $f(\tau x) = f(x)$ for all $\tau \in T$ and $x \in \mathbb{R}^3$.

Theorem 1.2. *Suppose* $1 < p < 2$, *the hypotheses* $(A_1), (A_2)$ *and* (A_4) *hold. Assume* $k(x)$, $h(x)$ *and* $s(x)$ *are* T*-invariant. Moreover, let* $|T| = \infty$. *Then Problem (1.1) has infinitely many solutions with positive energy.*

Remark 1.1. The results obtained in our paper extend the ones in [14]. To be more precise, the authors [14] obtained just two solutions. Here, by the argument of Z_2 index theory, we prove the existence of infinitely many small solutions with negative energy, besides, we also obtain a sequence of high energy solutions by Fountain Theorem.

This paper is organized as follows. In section 2, we give some notations and preliminaries, for the readers' convenience, we also describe the main mathematical tools which we shall use. In section 3, we prove Theorem 1.1 by the truncated technique. Section 4 is devoted to the proof of Theorem 1.2.

2. Preliminaries and variational setting

Hereafter we use the following notations.

$L^s := L^s(\mathbb{R}^3)$ $(1 \leq s < \infty)$ is the usual Lebesgue space with the norm defined by

$$\|u\|_s = \left(\int_{\mathbb{R}^3} |u|^s dx\right)^{\frac{1}{s}},$$

$\|\cdot\|_\infty$ denotes the L^∞-norm and $D^{1,2} := D^{1,2}(\mathbb{R}^3) = \{u \in L^6(\mathbb{R}^3) | \nabla u \in L^2(\mathbb{R}^3)\}$ with the norm defined by

$$\|u\| = \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^{\frac{1}{2}}.$$

For any $\rho > 0$ and $z \in \mathbb{R}^3$, $B_\rho(z)$ denotes the ball of radius ρ centered at z, and $|B_\rho(z)|$ denotes its Lebesgue measure. $C,\ \hat{C},\ C_p,\ C_1,\ C_2, \cdots$ are various positive constants which can change from line to line.

We now recall some known results. For all $u \in D^{1,2}$, the linear functional Lu is defined by

$$Lu(v) = \int_{\mathbb{R}^3} k(x)u^2 v dx.$$

By (A_2), Hölder and Sobolev inequalities, we obtain

$$Lu(v) \le \|k\|_2 \|u^2\|_3 \|v\|_6 \le C_1 \|k\|_2 \|u\|_6^2 \|v\|. \tag{2.1}$$

Thanks to the Lax-Milgram theorem, for every $u \in D^{1,2}$, the Poisson equation

$$-\triangle \phi = k(x)u^2, \quad x \in \mathbb{R}^3$$

exists a unique solution $\phi_u \in D^{1,2}$ and

$$\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{k(x)u^2(y)}{|x-y|} dy.$$

It is easy to see that ϕ_u satisfies

$$\int_{\mathbb{R}^3} \nabla \phi_u \nabla v dx = \int_{\mathbb{R}^3} k(x)u^2 v dx, \tag{2.2}$$

for any $v \in D^{1,2}$. Furthermore, by (2.1), (2.2), Hölder and Sobolev inequalities, the relations

$$\|\phi_u\| \le C_1 S^{-1} \|k\|_2 \|u\|^2, \quad \|\phi_u\|_6 \le C_2 \|\phi_u\|,$$

$$\left|\int_{\mathbb{R}^3} k(x)\phi_u u^2 dx\right| \le \|k\|_2 \|\phi_u\|_6 \|u^2\|_3 \le C_1 C_2 S^{-2} \|k\|_2^2 \|u\|^4 := C_3 \|u\|^4$$

hold, where S is the best Sobolev constant defined by

$$S := \inf_{u \in D^{1,2}\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^3} |u|^6 dx\right)^{\frac{1}{3}}}. \tag{2.3}$$

Substituting ϕ_u into (1.1), we get

$$-\triangle u - k(x)\phi_u u = \lambda h(x)|u|^{p-2}u + s(x)|u|^4 u, \quad x \in \mathbb{R}^3.$$

It is standard to see that the solutions of (1.1) are the critical points of the functional defined by

$$I(u) := \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}^3} k(x)\phi_u u^2 dx - \frac{\lambda}{p}\int_{\mathbb{R}^3} h(x)|u|^p dx - \frac{1}{6}\int_{\mathbb{R}^3} s(x)|u|^6 dx,$$

for $u \in D^{1,2}$. Hence, we just say that $u \in D^{1,2}$, instead of $(u, \phi_u) \in D^{1,2} \times D^{1,2}$, is a weak solution of system (1.1). It is easy to see that $I(u) \in C^1(D^{1,2}, \mathbb{R})$ and

$$\langle I'(u), \varphi \rangle = \int_{\mathbb{R}^3} \nabla u \nabla \varphi dx - \int_{\mathbb{R}^3} k(x)\phi_u(x)u(x)\varphi(x)dx - \lambda \int_{\mathbb{R}^3} h(x)|u|^{p-2}u\varphi dx - \int_{\mathbb{R}^3} s(x)|u|^4 u\varphi dx,$$

for all $\varphi \in D^{1,2}$.

Now we define the operator

$$\Phi : D^{1,2} \to D^{1,2} \; as \; \Phi(u) = \phi_u$$

and set

$$N(u) = \int_{\mathbb{R}^3} k(x)\phi_u u^2 dx.$$

In the following lemma, we conclude some properties of Φ which are useful for studying our problems.

Lemma 2.1. *([21])*

1. *Φ is continuous;*
2. *Φ maps bounded sets into bounded sets;*
3. *$\Phi(tu) = t^2\Phi(u)$ for all $t \in \mathbb{R}$;*
4. *If $u_n \rightharpoonup u \in D^{1,2}$, then $\Phi(u_n) \to \Phi(u)$ in $D^{1,2}$;*
5. *If $u_n \rightharpoonup u \in D^{1,2}$, then $N(u_n) \to N(u)$, as $n \to \infty$.*

Definition 2.2. Let Y be a Banach space and $I : Y \to \mathbb{R}$ be a differentiable functional. A sequence $\{u_k\} \subset Y$ is called a $(PS)_c$ sequence for I if $I(u_k) \to c$ and $I'(u_k) \to 0$ as $k \to \infty$. If every $(PS)_c$ sequence for I has a converging subsequence (*in Y*), we say that I satisfies the $(PS)_c$ condition.

Lemma 2.3. *Assume that (A_1), (A_2) and (A_3) hold. Let $\{u_n\} \subset D^{1,2}$ be a $(PS)_c$ sequence for I, then $\{u_n\}$ is bounded in $D^{1,2}$. Moreover, if $c < 0$, there exists $\lambda^{**} > 0$ such that I satisfies the $(PS)_c$ condition for all $\lambda \in (0, \lambda^{**})$.*

Proof. Since $\{u_n\}$ is a $(PS)_c$ sequence, we have

$$I(u_n) \to c, \quad I'(u_n) \to 0, \;\; as \;\; n \to \infty. \tag{2.4}$$

On one hand, by (2.4), we can easily get

$$I(u_n) - \frac{1}{4}\langle I'(u_n), u_n \rangle = c + o_n(1). \tag{2.5}$$

On the other hand, we have

$$I(u_n) - \frac{1}{4}\langle I'(u_n), u_n \rangle = \frac{1}{4}||u_n||^2 + \frac{1}{12}\int_{\mathbb{R}^3} s(x)|u_n|^6 dx - (\frac{1}{p} - \frac{1}{4})\lambda \int_{\mathbb{R}^3} h(x)|u_n|^p dx. \tag{2.6}$$

By (A_1), Sobolev and Hölder inequalities, we find

$$\int_{\mathbb{R}^3} h(x)|u_n|^p dx \leq C_p||u_n||^p. \tag{2.7}$$

In view of (2.5)–(2.7), we get

$$c + o_n(1) \geq \frac{1}{4}\|u_n\|^2 - (\frac{1}{p} - \frac{1}{4})C_p\lambda\|u_n\|^p. \tag{2.8}$$

Since $1 < p < 2$, we obtain that $\{u_n\}$ is bounded in $D^{1,2}$. Thus there exists a subsequence, still denoted by $\{u_n\}$, and $u \in D^{1,2}$, such that

$$\begin{aligned} u_n &\rightharpoonup u, \quad in \ D^{1,2}, \\ u_n &\to u, \quad a.e. \ x \in \mathbb{R}^3. \end{aligned}$$

Moreover, we get $|u_n|^p \rightharpoonup |u|^p$ in $L^{\frac{6}{p}}$ (see Proposition 4.7.12 in [6]). By (A_1), we can conclude that

$$\int_{\mathbb{R}^3} h(x)|u_n|^p dx \to \int_{\mathbb{R}^3} h(x)|u|^p dx, \ \ as \ \ n \to \infty. \tag{2.9}$$

Next, we want to use the concentration compactness principle to restore the compactness. Using the fact that $\{u_n\}$ is bounded in $D^{1,2}$, by the concentration compactness principle in [16, 17], we may suppose there exists a subsequence, still denoted by $\{u_n\}$, such that

$$|\nabla u_n|^2 \rightharpoonup \mu \geq |\nabla u|^2 + \sum_{i\in\Gamma} \mu_i\delta_{a_i}, \quad |u_n|^6 \rightharpoonup \nu = |u|^6 + \sum_{i\in\Gamma} \nu_i\delta_{a_i}, \quad \sum_{i\in\Gamma} \nu_i^{\frac{1}{3}} < \infty, \tag{2.10}$$

where $\mu,\ \mu_i,\ \nu$ and ν_i are nonnegative measures, Γ is an at most countable index set, $\{a_i\} \subset \mathbb{R}^3$ is a sequence and δ_{a_i} is the Dirac mass at a_i. Moreover, we have

$$\mu_i,\ \nu_i \geq 0,\ S\nu_i^{\frac{1}{3}} \leq \mu_i, \tag{2.11}$$

where S is given in (2.3).

We claim that Γ is empty. Indeed, if Γ is not empty, then there exists $i \in \Gamma$ such that $\mu_i \neq 0$. For $\varepsilon > 0$ small, we introduce a cut-off function centered at a_i as following

$$\begin{aligned} \varphi_\varepsilon^i(x) &= 1, \text{ for } |x - a_i| \leq \tfrac{\varepsilon}{2}, \\ \varphi_\varepsilon^i(x) &= 0, \text{ for } |x - a_i| \geq \varepsilon \end{aligned}$$

and $0 \leq \varphi_\varepsilon^i(x) \leq 1, |\nabla\varphi_\varepsilon^i(x)| \leq \frac{4}{\varepsilon}$. By (2.4) we can obtain

$$\langle I'(u_n), \varphi_\varepsilon^i(x)u_n\rangle \to 0, \quad as \ \ n \to \infty,$$

which implies

$$\begin{aligned} &\int_{\mathbb{R}^3} (\nabla u_n \nabla\varphi_\varepsilon^i(x))u_n dx + \int_{\mathbb{R}^3} \varphi_\varepsilon^i(x)|\nabla u_n|^2 dx - \int_{\mathbb{R}^3} k(x)\phi_{u_n}(x)\varphi_\varepsilon^i(x)|u_n|^2 dx \\ &\quad = \lambda \int_{\mathbb{R}^3} \varphi_\varepsilon^i(x)h(x)|u_n|^p dx + \int_{\mathbb{R}^3} \varphi_\varepsilon^i(x)s(x)|u_n|^6 dx + o_n(1). \end{aligned} \tag{2.12}$$

Step 1. We prove $\nu_i^{\frac{1}{3}} \geq \sqrt{\frac{S}{s(a_i)}}$.

Since $\{u_n\}$ is bounded, using Hölder inequality, we can obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\left|\int_{\mathbb{R}^3}(\nabla u_n\nabla\varphi_\varepsilon^i(x))u_n dx\right| \\
&\le \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\left(\int_{B_\varepsilon(a_i)}|\nabla u_n|^2dx\right)^{\frac{1}{2}}\left(\int_{B_\varepsilon(a_i)}|\nabla\varphi_\varepsilon^i(x)|^2|u_n|^2dx\right)^{\frac{1}{2}} \\
&\le C\lim_{\varepsilon\to 0}\left(\int_{B_\varepsilon(a_i)}|\nabla\varphi_\varepsilon^i(x)|^2|u|^2dx\right)^{\frac{1}{2}} \\
&\le C\lim_{\varepsilon\to 0}\left(\int_{B_\varepsilon(a_i)}|\nabla\varphi_\varepsilon^i(x)|^3dx\right)^{\frac{1}{3}}\left(\int_{B_\varepsilon(a_i)}|u|^6dx\right)^{\frac{1}{6}} \\
&= 0,
\end{aligned} \tag{2.13}$$

where $B_\varepsilon(a_i) = \{x\in\mathbb{R}^3|\ \ |x-a_i|<\varepsilon\}$. By (2.10), we get

$$\begin{aligned}
\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\int_{\mathbb{R}^3}\varphi_\varepsilon^i(x)|\nabla u_n|^2dx &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^3}\varphi_\varepsilon^i(x)d\mu \\
&\ge \lim_{\varepsilon\to 0}\left(\int_{B_\varepsilon(a_i)}\varphi_\varepsilon^i(x)|\nabla u|^2dx+\mu_i\right) \\
&= \mu_i,
\end{aligned} \tag{2.14}$$

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\int_{\mathbb{R}^3}k(x)\phi_{u_n}(x)\varphi_\varepsilon^i(x)|u_n|^2dx = \lim_{\varepsilon\to 0}\int_{B_\varepsilon(a_i)}k(x)\phi_u(x)\varphi_\varepsilon^i(x)|u|^2dx = 0, \tag{2.15}$$

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\int_{\mathbb{R}^3}\varphi_\varepsilon^i(x)h(x)|u_n|^pdx = \lim_{\varepsilon\to 0}\int_{B_\varepsilon(a_i)}\varphi_\varepsilon^i(x)h(x)|u|^pdx = 0 \tag{2.16}$$

and

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\int_{\mathbb{R}^3}\varphi_\varepsilon^i(x)s(x)|u_n|^6dx = \lim_{\varepsilon\to 0}\int_{B_\varepsilon(a_i)}\varphi_\varepsilon^i(x)s(x)d\nu = s(a_i)\nu_i. \tag{2.17}$$

In view of (2.12)-(2.17), we get $\mu_i\le s(a_i)\nu_i$. By (2.11) we obtain

$$\nu_i^{\frac{1}{3}} \ge \sqrt{\frac{S}{s(a_i)}}. \tag{2.18}$$

Step 2. We prove our claim.
Let $\varphi_R(x)$ be a cut-off function which satisfies

$$\varphi_R(x) = 1,\ \ |x|<R;\quad \varphi_R(x) = 0,\ \ |x|>2R$$

and $0 \le \varphi_R(x) \le 1$, $|\nabla \varphi_R(x)| < \frac{2}{R}$. By (2.5) and (2.9), we obtain

$$\begin{aligned}
c &= \lim_{R\to\infty}\limsup_{n\to\infty}\left(I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle\right) \\
&\ge \lim_{R\to\infty}\limsup_{n\to\infty}\frac{1}{4}\int_{\mathbb{R}^3}|\nabla u_n|^2dx + \lim_{R\to\infty}\limsup_{n\to\infty}\frac{1}{12}\int_{\mathbb{R}^3}s(x)|u_n|^6dx \\
&\quad - \lim_{R\to\infty}\limsup_{n\to\infty}(\frac{1}{p}-\frac{1}{4})\lambda\int_{\mathbb{R}^3}h(x)|u_n|^pdx \\
&\ge \lim_{R\to\infty}\limsup_{n\to\infty}\frac{1}{4}\int_{\mathbb{R}^3}\varphi_R(x)|\nabla u_n|^2dx + \lim_{R\to\infty}\limsup_{n\to\infty}\frac{1}{12}\int_{\mathbb{R}^3}\varphi_R(x)s(x)|u_n|^6dx \\
&\quad - \lim_{R\to\infty}\limsup_{n\to\infty}(\frac{1}{p}-\frac{1}{4})\lambda\int_{\mathbb{R}^3}h(x)|u_n|^pdx \\
&= \lim_{R\to\infty}\frac{1}{4}\int_{\mathbb{R}^3}\varphi_R(x)d\mu + \lim_{R\to\infty}\frac{1}{12}\int_{\mathbb{R}^3}\varphi_R(x)s(x)d\nu - (\frac{1}{p}-\frac{1}{4})\lambda\int_{\mathbb{R}^3}h(x)|u|^pdx \\
&\ge \frac{1}{4}\mu_i + \frac{1}{12}s(a_i)\nu_i + \frac{1}{12}\int_{\mathbb{R}^3}s(x)|u|^6dx - (\frac{1}{p}-\frac{1}{4})\lambda\int_{\mathbb{R}^3}h(x)|u|^pdx.
\end{aligned} \tag{2.19}$$

Using Hölder and Young inequalities (ε small enough), we obtain

$$\begin{aligned}
(\frac{1}{p}-\frac{1}{4})\lambda\int_{\mathbb{R}^3}h(x)|u|^pdx &\le (\frac{1}{p}-\frac{1}{4})\lambda\left(\int_{\mathbb{R}^3}|h(x)|^{\frac{6}{6-p}}dx\right)^{\frac{6-p}{6}}\left(\int_{\mathbb{R}^3}|u|^6dx\right)^{\frac{p}{6}} \\
&\le \varepsilon\int_{\mathbb{R}^3}|u|^6dx + C_\varepsilon\lambda^{\frac{6}{6-p}}.
\end{aligned} \tag{2.20}$$

Hence we have

$$c \ge \frac{1}{4}\mu_i + \frac{1}{12}s(a_i)\nu_i - C_\varepsilon\lambda^{\frac{6}{6-p}}. \tag{2.21}$$

Choose λ_1 small enough such that $\frac{1}{4}\mu_i + \frac{1}{12}s(a_i)\nu_i - C_\varepsilon\lambda^{\frac{6}{6-p}} > 0$ for all $\lambda \in (0, \lambda_1)$, which contradicts to $c < 0$. Thus Γ is empty.

By the claim, we get

$$|u_n|^6dx \rightharpoonup |u|^6dx,$$

which implies

$$\int_{\mathbb{R}^3}|u_n|^6vdx \to \int_{\mathbb{R}^3}|u|^6vdx,\ \forall\ v \in C_0(\mathbb{R}^3),\ \ as\ \ n \to \infty. \tag{2.22}$$

We define

$$\nu_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>R}|u_n|^6dx$$

and

$$\mu_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>R}|\nabla u_n|^2dx.$$

From [16], we know that ν_∞ and μ_∞ satisfy

$$
\begin{aligned}
&(i)\ \limsup_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^6dx=\int_{\mathbb{R}^3}d\nu+\nu_\infty,\\
&(ii)\ \limsup_{n\to\infty}\int_{\mathbb{R}^3}|\nabla u_n|^2dx=\int_{\mathbb{R}^3}d\mu+\mu_\infty,\\
&(iii)\ S\nu_\infty^{\frac{1}{3}}\ \le\ \mu_\infty,
\end{aligned}
$$

where μ and ν are the same as above.

In the following discussion, we want to prove $\mu_\infty=\nu_\infty=0$. Let $\eta_R\in C^1(\mathbb{R}^3)$ be such that

$$
\begin{cases}\eta_R(x)=0, & |x|<R,\\ \eta_R(x)=1, & |x|>2R,\end{cases} \tag{2.23}
$$

with $0\le\eta_R(x)\le 1$ and $|\nabla\eta_R(x)|<\frac{2}{R}$. From (2.4), we get

$$
\langle I'(u_n),\eta_R(x)u_n\rangle\to 0,\quad as\ \ n\to\infty,
$$

which gives

$$
\begin{aligned}
&\int_{\mathbb{R}^3}(\nabla u_n\nabla\eta_R(x))u_ndx+\int_{\mathbb{R}^3}\eta_R(x)|\nabla u_n|^2dx-\int_{\mathbb{R}^3}k(x)\phi_{u_n}(x)\eta_R(x)|u_n|^2dx\\
&\qquad=\lambda\int_{\mathbb{R}^3}\eta_R(x)h(x)|u_n|^pdx+\int_{\mathbb{R}^3}\eta_R(x)s(x)|u_n|^6dx+o_n(1).
\end{aligned} \tag{2.24}
$$

Since $\{u_n\}$ is bounded, by Hölder inequality, we have

$$
\begin{aligned}
&\lim_{R\to\infty}\limsup_{n\to\infty}\left|\int_{\mathbb{R}^3}(\nabla u_n\nabla\eta_R(x))u_ndx\right|\\
&\le\lim_{R\to\infty}\limsup_{n\to\infty}\left(\int_{|x|\ge R}|\nabla u_n|^2dx\right)^{\frac{1}{2}}\left(\int_{|x|\ge R}|\nabla\eta_R(x)|^2|u_n|^2dx\right)^{\frac{1}{2}}\\
&\le C\lim_{R\to\infty}\left(\int_{|x|\ge R}|\nabla\eta_R(x)|^2|u|^2dx\right)^{\frac{1}{2}}\\
&\le C\lim_{R\to\infty}\left(\int_{|x|\ge R}|\nabla\eta_R(x)|^3dx\right)^{\frac{1}{3}}\left(\int_{|x|\ge R}|u|^6dx\right)^{\frac{1}{6}}\\
&=0.
\end{aligned} \tag{2.25}
$$

Moreover, by Lemma 2.1, (2.9) and the definitions of μ_∞ and ν_∞, we obtain

$$
\begin{aligned}
\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^3}\eta_R(x)|\nabla u_n|^2dx&\ge\lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>2R}\eta_R(x)|\nabla u_n|^2dx\\
&=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>2R}|\nabla u_n|^2dx\\
&=\mu_\infty,
\end{aligned} \tag{2.26}
$$

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^3}k(x)\phi_{u_n}\eta_R(x)|u_n|^2dx=\lim_{R\to\infty}\int_{|x|\geq R}k(x)\phi_u\eta_R(x)|u|^2dx=0, \tag{2.27}$$

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^3}\eta_R(x)h(x)|u_n|^pdx=\lim_{R\to\infty}\int_{|x|\geq R}\eta_R(x)h(x)|u|^pdx=0 \tag{2.28}$$

and

$$\begin{aligned}\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^3}\eta_R(x)s(x)|u_n|^6dx&=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|\geq R}\eta_R(x)s(x)|u_n|^6dx\\&\leq\lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|\geq R}s(x)|u_n|^6dx=\|s\|_\infty\nu_\infty.\end{aligned} \tag{2.29}$$

Therefore, by (2.24)–(2.29), we obtain

$$\mu_\infty\leq\|s\|_\infty\nu_\infty.$$

Furthermore, from $S\nu_\infty^{\frac{1}{3}}\leq\mu_\infty$, we have

$$(1^*)\ \nu_\infty=0\ \text{or}\ (2^*)\ \nu_\infty^{\frac{1}{3}}\geq\sqrt{\frac{S}{\|s\|_\infty}}. \tag{2.30}$$

We claim (1^*) holds. In fact, if (2^*) holds, it follows from (2.5), (2.9) and $s(x)>1$ that

$$\begin{aligned}c&=\limsup_{n\to\infty}\left(I(u_n)-\frac{1}{4}\langle I'(u_n),u_n\rangle\right)\\&\geq\limsup_{n\to\infty}\frac{1}{4}\int_{\mathbb{R}^3}|\nabla u_n|^2dx+\limsup_{n\to\infty}\frac{1}{12}\int_{\mathbb{R}^3}|u_n|^6dx\\&\quad-\limsup_{n\to\infty}(\frac{1}{p}-\frac{1}{4})\lambda\int_{\mathbb{R}^3}h(x)|u_n|^pdx\\&=\frac{1}{4}(\int_{\mathbb{R}^3}d\mu+\mu_\infty)+\frac{1}{12}(\int_{\mathbb{R}^3}d\nu+\nu_\infty)-(\frac{1}{p}-\frac{1}{4})\lambda\int_{\mathbb{R}^3}h(x)|u|^pdx\\&\geq\frac{1}{4}\mu_\infty+\frac{1}{12}\nu_\infty+\frac{1}{12}\int_{\mathbb{R}^3}|u|^6dx-(\frac{1}{p}-\frac{1}{4})\lambda\int_{\mathbb{R}^3}h(x)|u|^pdx.\end{aligned} \tag{2.31}$$

From (2.20) and (2.31), we have

$$c\geq\frac{1}{4}\mu_\infty+\frac{1}{12}\nu_\infty-C_\varepsilon\lambda^{\frac{6}{6-p}}. \tag{2.32}$$

Choose λ_2 small enough such that $\frac{1}{4}\mu_\infty+\frac{1}{12}\nu_\infty-C_\varepsilon\lambda^{\frac{6}{6-p}}>0$ for all $\lambda\in(0,\lambda_2)$, which is a contradiction. Thus we have $\mu_\infty=\nu_\infty=0$. By the definitions of μ_∞ and ν_∞, we obtain

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>R}|u_n|^6dx=0. \tag{2.33}$$

Hence,

$$\begin{aligned}
\lim_{n\to\infty}\left|\int_{\mathbb{R}^3}|u_n|^6dx-\int_{\mathbb{R}^3}|u|^6dx\right| &\le \limsup_{n\to\infty}\left|\int_{\mathbb{R}^3}|u_n|^6dx-\int_{\mathbb{R}^3}|u|^6dx\right|\\
&\le \limsup_{n\to\infty}\left|\int_{\mathbb{R}^3}(|u_n|^6-|u|^6)\eta_R(x)dx\right|\\
&\quad+\limsup_{n\to\infty}\left|\int_{\mathbb{R}^3}(|u_n|^6-|u|^6)(1-\eta_R(x))dx\right| \\
&\le \limsup_{n\to\infty}\left|\int_{\mathbb{R}^3}(|u_n|^6-|u|^6)\eta_R(x)dx\right|\\
&\quad+\limsup_{n\to\infty}\int_{|x|\ge R}|u_n|^6dx+\limsup_{n\to\infty}\int_{|x|\ge R}|u|^6dx.
\end{aligned} \tag{2.34}$$

Let $R\to\infty$ in (2.34), from (2.22) and (2.33), we get

$$\lim_{n\to\infty}\left|\int_{\mathbb{R}^3}|u_n|^6dx-\int_{\mathbb{R}^3}|u|^6dx\right|=0,$$

it follows that

$$\int_{\mathbb{R}^3}|u_n|^6dx\to\int_{\mathbb{R}^3}|u|^6dx,\ \ as\ \ n\to\infty. \tag{2.35}$$

On one hand, since $\{u_n\}$ is bounded in $D^{1,2}$, we set $U=\lim\limits_{n\to\infty}\|u_n\|$. Since $\langle I'(u_n),u_n\rangle\to 0\ as\ n\to\infty$, utilizing (2.9) and (2.35), we have

$$\lim_{n\to\infty}\left(\|u_n\|^2-\int_{\mathbb{R}^3}k(x)\phi_{u_n}u_n^2dx\right)=\lambda\int_{\mathbb{R}^3}h(x)|u|^pdx+\int_{\mathbb{R}^3}s(x)|u|^6dx. \tag{2.36}$$

Then by Lemma 2.1, we have

$$U^2-\int_{\mathbb{R}^3}k(x)\phi_u u^2dx=\lambda\int_{\mathbb{R}^3}h(x)|u|^pdx+\int_{\mathbb{R}^3}s(x)|u|^6dx. \tag{2.37}$$

On the other hand, since $\{u_n\}$ is a $(PS)_c$ sequence for I, i.e., $\langle I'(u_n),v\rangle\to 0\ as\,n\to\infty$ for all $v\in D^{1,2}$, that implies

$$\int_{\mathbb{R}^3}\nabla u_n\nabla vdx-\int_{\mathbb{R}^3}k(x)\phi_{u_n}u_nvdx-\lambda\int_{\mathbb{R}^3}h(x)|u_n|^{p-2}u_nvdx-\int_{\mathbb{R}^3}s(x)|u_n|^4u_nvdx\to 0,\ as\ n\to\infty.$$

Combining (2.9), Lemma 2.1 with the fact of $u_n\rightharpoonup u$ in $D^{1,2}$, we have

$$\int_{\mathbb{R}^3}\nabla u\nabla vdx-\int_{\mathbb{R}^3}k(x)\phi_u uvdx=\lambda\int_{\mathbb{R}^3}h(x)|u|^{p-2}uvdx-\int_{\mathbb{R}^3}s(x)|u|^4uvdx. \tag{2.38}$$

Taking $v=u$ in (2.38), we get

$$\|u\|^2-\int_{\mathbb{R}^3}k(x)\phi_u u^2dx=\lambda\int_{\mathbb{R}^3}h(x)|u|^pdx+\int_{\mathbb{R}^3}s(x)|u|^6dx. \tag{2.39}$$

Comparing (2.37) with (2.39), we get $\|u\|=U=\lim\limits_{n\to\infty}\|u_n\|$. Noticing that $D^{1,2}$ is a reflexive Banach space, combining above analysis, we can prove that $u_n\to u$ in $D^{1,2}$ $as\ n\to\infty$. Taking $\lambda^{**}=\min\{\lambda_1,\lambda_2\}$, we conclude that $I(u)$ satisfies the $(PS)_c$ condition for all $\lambda\in(0,\lambda^{**})$. □

In order to continue our proof, we will introduce a truncated functional. By (A_1), Sobolev embedding theorem and above analysis, we have

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}^3} k(x)\phi_u u^2 dx - \frac{\lambda}{p}\int_{\mathbb{R}^3} h(x)|u|^p dx - \frac{1}{6}\int_{\mathbb{R}^3} s(x)|u|^6 dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{C_3}{4}\|u\|^4 - \frac{\lambda}{p}\|h(x)\|_{\frac{6}{6-p}}\|u\|_6^p - \frac{\|s\|_\infty}{6}\int_{\mathbb{R}^3}|u|^6 dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{C_3}{4}\|u\|^4 - \frac{\lambda C_4}{p}\|h(x)\|_{\frac{6}{6-p}}\|u\|^p - \frac{C_5}{6}\|u\|^6 \\ &:= C_6\|u\|^2 - C_7\|u\|^4 - \lambda C_8\|u\|^p - C_9\|u\|^6 \end{aligned} \tag{2.40}$$

for all $u \in D^{1,2}$. Let $g(t) = C_6t^2 - C_7t^4 - \lambda C_8t^p - C_9t^6$. Next we will discuss some properties of $g(t)$.

First of all, it is easy to see that there exist positive constants λ_3, T_1 and T_2 $(T_1 < T_2)$ such that for any $\lambda \in (0, \lambda_3)$, $g(t)$ can take positive maximum value for some $t > 0$, and we have

$$\begin{aligned} &g(T_1) = g(T_2) = 0, \\ &g(t) \leq 0, \forall\, t \in [0, T_1], \\ &g(t) > 0, \forall\, t \in (T_1, T_2), \\ &g(t) \leq 0, \forall\, t \in [T_2, +\infty). \end{aligned}$$

Let $\tau : \mathbb{R}^+ \to [0, 1]$ be C^∞ function such that

$$\tau(t) = 1, \text{ if } t \leq T_1; \quad \tau(t) = 0, \text{ if } t \geq T_2.$$

Now, we give the truncated functional as follows:

$$I_\infty(u) = \frac{1}{2}\|u\|^2 - \frac{\tau(\|u\|)}{4}\int_{\mathbb{R}^3} k(x)\phi_u u^2 dx - \frac{\lambda}{p}\int_{\mathbb{R}^3} h(x)|u|^p dx - \frac{\tau(\|u\|)}{6}\int_{\mathbb{R}^3} s(x)|u|^6 dx.$$

Since $\tau \in C^\infty$, we get $I_\infty(u) \in C^1(D^{1,2}, \mathbb{R})$. Similar to above analysis, we obtain

$$\begin{aligned} I_\infty(u) &\geq \frac{1}{2}\|u\|^2 - \frac{C_3\tau(\|u\|)}{4}\|u\|^4 - \frac{\lambda C_4}{p}\|h(x)\|_{\frac{6}{6-p}}\|u\|^p - \frac{C_5\tau(\|u\|)}{6}\|u\|^6 \\ &:= C_6\|u\|^2 - C_7\tau(\|u\|)\|u\|^4 - \lambda C_8\|u\|^p - C_9\tau(\|u\|)\|u\|^6, \end{aligned}$$

where constants $C_6, \cdots, C_9$ are the same as those in (2.40).

Let $g_\infty(t) = C_6t^2 - C_7\tau(t)t^4 - \lambda C_8t^p - C_9\tau(t)t^6$. We say that $g_\infty(t) \geq g(t)$ for all $t > 0$. In fact, if $0 \leq t \leq T_1$, $g_\infty(t) = g(t)$; If $T_1 < t < T_2$, $0 < g(t) < g_\infty(t)$; If $t \geq T_2$, $g_\infty(t) > 0 \geq g(t)$. Moreover, we obtain that $I_\infty(u) = I(u)$ when $0 \leq \|u\| \leq T_1$.

Lemma 2.4. *If u satisfies that $I_\infty(u) < 0$, then $\|u\| \leq T_1$ and there exists $\epsilon > 0$ such that for all $v \in B_\epsilon(u)$, there holds $I_\infty(v) = I(v)$. Furthermore, there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$, $I_\infty(u)$ satisfies the $(PS)_c$ condition for $c < 0$.*

Proof. We prove by contradiction. If $I_\infty(u) < 0$ and $\|u\| > T_1$, from above analysis, we get $I_\infty(u) \geq g_\infty(\|u\|) > 0$, which is a contradiction, thus we obtain $\|u\| \leq T_1$. Since $I(u)$ and $I_\infty(u)$ are both continuous, we have $I(v) = I_\infty(v)$ for all $v \in B_\epsilon(u)$. Setting $\lambda^* = \min\{\lambda^{**}, \lambda_3\}$, by using Lemma 2.3, we get $I_\infty(u)$ satisfies the $(PS)_c$ condition for $c < 0$. □

To prove our main results, we need the following deformation lemma.

Lemma 2.5. *([20]) Let Y be a Banach space, $f \in C^1(Y,\mathbb{R})$, $c \in \mathbb{R}$ and N is any neighborhood of $K_c \triangleq \{u \in Y|\ f(u) = c, f'(u) = 0\}$. If f satisfies the $(PS)_c$ condition, then there exist $\eta_t(u) \equiv \eta(t,u) \in C([0,1]\times Y, Y)$ and constants $\overline{\epsilon} > \epsilon > 0$ such that*
(1) $\eta_0(u) = u$, $\forall\ u \in Y$,
(2) $\eta_t(u) = u$, $\forall\ u \notin f^{-1}[c-\overline{\epsilon}, c+\overline{\epsilon}]$,
(3) $\eta_t(u) = u$ *is a homeomorphism of* Y *onto* Y, $\forall\ t \in [0,1]$,
(4) $f(\eta_t(u)) \le f(u)$, $\forall\ u \in Y$ *and* $\forall\ t \in [0,1]$,
(5) $\eta_1(f^{c+\epsilon} \setminus N) \subset f^{c-\epsilon}$, *where* $f^c = \{u \in Y|\ f(u) \le c\}$, $\forall\ c \in \mathbb{R}$,
(6) *if* $K_c = \emptyset$, $\eta_1(f^{c+\epsilon}) \subset f^{c-\epsilon}$,
(7) *if* f *is even,* η_t *is odd in* u.

At the end of this section, we point out some concepts and results about Z_2 index theory. Let Y be a Banach space and set

$$\Sigma = \{A \subset Y \setminus \{0\}|\ A \text{ is closed}, -A = A\}$$

and

$$\Sigma_k = \{A \in \Sigma,\ \gamma(A) \ge k\}, \tag{2.41}$$

where $\gamma(A)$ is the Z_2 genus of A defined by

$$\gamma(A) = \begin{cases} 0, \text{ if } A = \emptyset, \\ \inf\{n : \text{there exists an odd, continuous } \phi : A \to \mathbb{R}^n\backslash\{0\}\}, \\ +\infty, \text{ if it does not exist odd, continuous } h : A \to \mathbb{R}^n\backslash\{0\}. \end{cases}$$

In the following lemma, we give the main properties of genus.

Lemma 2.6. *([20]) Let $A, B \in \Sigma$.*
(1) *If there exists an odd map* $f \in C(A,B)$, *then* $\gamma(A) \le \gamma(B)$.
(2) *If* $A \subset B$, *then* $\gamma(A) \le \gamma(B)$.
(3) *If there exists an odd homeomorphism between A and B, then* $\gamma(A) = \gamma(B)$.
(4) *If* S^{N-1} *is the sphere in* $\mathbb{R}^N$, *then* $\gamma(S^{N-1}) = N$.
(5) $\gamma(A \cup B) \le \gamma(A) + \gamma(B)$.
(6) *If* $\gamma(A) < \infty$, *then* $\gamma(\overline{A-B}) \ge \gamma(A) - \gamma(B)$.
(7) *If A is compact, then* $\gamma(A) < \infty$, *and there exists* $\delta > 0$ *such that* $\gamma(A) = \gamma(N_\delta(A))$, *where* $N_\delta(A) = \{x \in Y|\ dist(x,A) \le \delta\}$.
(8) *If* Y_0 *is a subspace of Y with codimension k, and* $\gamma(A) > k$, *then* $A \cap Y_0 \ne \emptyset$.

3. Proof of Theorem 1.1

Now, we will use the genus argument to prove Theorem 1.1.
For any $k \in \mathbb{N}$, we define

$$c_k = \inf_{A\in\Sigma_k} \sup_{u\in A} I_\infty(u).$$

Moreover, by the definition, we get $\Sigma_{k+1} \subset \Sigma_k$, so we have $c_k \leq c_{k+1}$.

Firstly, we prove for any $k \in \mathbb{N}$, there exists $\varepsilon = \varepsilon(k) > 0$ such that

$$\gamma(I_\infty^{-\varepsilon}(u)) \geq k,$$

where $I_\infty^{-\varepsilon}(u) = \{u \in D^{1,2} | \, I_\infty(u) \leq -\varepsilon\}$. Let Ω be an open bounded subset of $\mathbb{R}^3$ with smooth boundary and $h(x) > 0$ in Ω. For fixed $k \in \mathbb{N}$, let X_k be a k-dimension subspace of $D^{1,2}(\Omega)$. Choosing $u \in X_k$ with $\|u\| = 1$, for $0 < \rho \leq T_1$ (T_1 is the same as before), we get

$$I(\rho u) = I_\infty(\rho u) = \frac{1}{2}\rho^2 - \frac{1}{4}\rho^4 \int_{\mathbb{R}^3} k(x)\phi_u u^2 dx - \frac{\lambda}{p}\rho^p \int_{\mathbb{R}^3} h(x)|u|^p dx - \frac{1}{6}\rho^6 \int_{\mathbb{R}^3} s(x)|u|^6 dx. \tag{3.1}$$

Since X_k is a finite dimension space, all the norms are equivalent. For each $u \in X_k$ with $\|u\| = 1$, by (A_1), we know that there exists $\alpha_k > 0$ such that

$$\int_\Omega h(x)|u|^p dx \geq \alpha_k. \tag{3.2}$$

Define

$$\beta_k = \inf_{u \in X_k, \|u\|=1} \int_{\mathbb{R}^3} s(x)|u|^6 dx. \tag{3.3}$$

It is easy to check that $\lim\limits_{k\to\infty} \beta_k = 0$. Hence, by (3.1)–(3.3), we get

$$I_\infty(\rho u) \leq \tfrac{1}{2}\rho^2 - \tfrac{\lambda}{p}\rho^p \alpha_k - \tfrac{1}{6}\rho^6 \beta_k.$$

Since $1 < p < 2$, for $\lambda \in (0, \lambda^*)$, $u \in X_k$ with $\|u\| = 1$, there must be $\rho_0 \in (0, T_1)$ small enough such that

$$I_\infty(\rho_0 u) \leq -\varepsilon,$$

and $\varepsilon = -\frac{1}{2}\rho_0^2 + \frac{\lambda}{p}\rho_0^p \alpha_k + \frac{1}{6}\rho_0^6 \beta_k > 0$.

Let $K_c = \{u \in D^{1,2} | \, I_\infty(u) = c, I'_\infty(u) = 0\}$ and $S_{\rho_0} = \{u \in D^{1,2}(\Omega) | \, \|u\| = \rho_0\}$, then $S_{\rho_0} \cap X_k \subset I_\infty^{-\varepsilon}$. From Lemma 2.6, we have that

$$\gamma(I_\infty^{-\varepsilon}(u)) \geq \gamma(S_{\rho_0} \cap X_k) = k. \tag{3.4}$$

Thus we obtain $I_\infty^{-\varepsilon}(u) \subset \Sigma_k$ and $c = c_k \leq -\varepsilon < 0$. Using Lemma 2.4, we know $I_\infty(u)$ satisfies the $(PS)_c$ condition if $c < 0$, which implies K_c is a compact set.

Next, by using the idea in [11, 12], we give two claims, which are crucial to prove Theorem 1.1.

Claim 1. If $k, l \in \mathbb{N}$ are such that $c = c_k = c_{k+1} = \cdots = c_{k+l}$, then $\gamma(K_c) \geq l + 1$.

Arguing by contradiction that $\gamma(K_c) \leq l$, then there exists a closed, symmetric set U with $K_c \subset U$ and $\gamma(U) \leq l$. Since $I_\infty(u)$ is even, by Lemma 2.5, we can assume an odd homeomorphism

$$\eta : [0, 1] \times D^{1,2} \to D^{1,2}$$

such that $\eta(I_\infty^{c+\delta} \backslash U) \subset I_\infty^{c-\delta}$ for some $\delta \in (0, -c)$. By the hypothesis $c = c_{k+l}$, we know there exists an $A \in \Sigma_{k+l}$ such that

$$\sup_{u \in A} I_\infty(u) < c + \delta,$$

that is to say $A \in I_\infty^{c+\delta}$. Furthermore, we get

$$\eta(A\backslash U) \subset \eta(I_\infty^{c+\delta}\backslash U) \subset I_\infty^{c-\delta}. \tag{3.5}$$

By Lemma 2.6, we know

$$\gamma(\overline{\eta(A\backslash U)}) \geq \gamma(\overline{A\backslash U}) \geq \gamma(A) - \gamma(U) \geq k.$$

Therefore, $\overline{\eta(A\backslash U)} \subset \Sigma_k$. Then from (3.5) we can obtain

$$c = c_k \leq \sup_{u \in \overline{\eta(A\backslash U)}} I_\infty(u) \leq c - \delta,$$

which is a contradiction. Thus we complete the proof of Claim 1.

Claim 2. If $c_k < 0$ is a critical value of $I_\infty(u)$, then there exists a subsequence of $\{c_k\}$, still denoted by $\{c_k\}$ $(k \in \mathbb{N})$, which satisfies

$$c_k \to 0, \;\; as \;\; k \to \infty.$$

Indeed, since $I_\infty(u)$ is bounded from below, it holds that $c_k > -\infty$, and we know that $\Sigma_{k+1} \subset \Sigma_k$ and $c_k \leq c_{k+1} < 0$. Therefore $\{c_k\}$ has a limit, denoted by c_∞ and $c_\infty \leq 0$. If $c_\infty < 0$, we set

$$K = \{u \in D^{1,2} |\; I'_\infty(u) = 0, \; I_\infty(u) \leq c_\infty\}.$$

From above analysis, we know that K is compact, symmetric and $0 \notin K$ on account of $c_\infty < 0$. By Lemma 2.6 (7), we choose $\delta > 0$ small enough such that

$$\gamma(N_\delta(K)) = \gamma(K) = m < +\infty,$$

where $N_\delta(K) = \{u \in D^{1,2} |\; dist(u, K) \leq \delta\}$. By Lemma 2.5 (5) with $c = c_\infty$, there exist $\varepsilon > 0$ and η_1 such that

$$\eta_1(I_\infty^{c_\infty+\varepsilon}\backslash N_\delta(K)) \subset I_\infty^{c_\infty-\varepsilon}. \tag{3.6}$$

Fix an integer $q \in \mathbb{N}$ such that

$$c_\infty - \varepsilon < c_q. \tag{3.7}$$

Choose $\hat{A} \in \Sigma_{m+q}$ such that

$$\sup_{u\in\hat{A}} I_\infty(u) < c_{m+q} + \varepsilon. \tag{3.8}$$

Setting $B = \overline{\hat{A}\backslash N_\delta(K)}$, using (3.6) and (3.8), we have

$$I_\infty(\eta_1(B)) \leq c_\infty - \varepsilon. \tag{3.9}$$

It follows from Lemma 2.6 that $\gamma(B) \geq \gamma(\hat{A}) - \gamma(N_\delta(K)) \geq q$, so $B \in \Sigma_q$. Denoting $D = \eta_1(B)$, then we have $D \in \Sigma_q$. Using (3.7) and (3.9), we get

$$c_\infty - \varepsilon < c_q \leq \sup_{u\in D} I_\infty(u) \leq c_\infty - \varepsilon,$$

which is absurd. Therefore, $c_\infty = 0$.

Now, we conclude the proof of Theorem1.1. For all $k \in \mathbb{N}$, we have $\Sigma_{k+1} \subset \Sigma_k$ and $c_k \leq c_{k+1} < 0$. If every c_k is distinct, then $\gamma(K_{c_k}) \geq 1$ and we know $\{c_k\}$ is a sequence of distinct negative critical values of $I_\infty(u)$. If for some $k_0 \in \mathbb{N}$, there exists a $l \geq 1$ such that $c = c_{k_0} = c_{k_0+1} = \cdots = c_{k_0+l}$, then by Claim 1, we obtain $\gamma(K_c) \geq l+1$, which implies that K_c contains infinitely many distinct elements. Moreover, by Claim 2, we know there exists a subsequence of $\{c_k\}$, still denoted by $\{c_k\}$, satisfying $c_k \to 0$ as $k \to \infty$. By Lemma 2.4 we know that $I(u) = I_\infty(u)$ if $I_\infty(u) < 0$. Hence we conclude that there exist infinitely many critical points of $I(u)$ and the sequence of the negative critical values converges to zero. Thus, we complete our proof of Theorem 1.1.

4. Proof of Theorem 1.2

We denote $D_T^{1,2} = \{u \in D^{1,2} : u(\tau x) = u(x), \tau \in O(3)\}$ and $L_T^6 = \{u \in L^6 : u(\tau x) = u(x), \tau \in O(3)\}$, where $T \subset O(3)$ is a subgroup. By the principle of symmetric criticality, we have the following results.

Lemma 4.1. *([25]) If $I'(u) = 0$ in $D_T^{1,2}$, then $I'(u) = 0$ in $D^{1,2}$.*

Lemma 4.2. *If $|T| = \infty$, $s(0) = 0$ and $\lim\limits_{|x|\to\infty} s(x) = 0$, then I satisfies the $(PS)_c$ condition for all $c \in \mathbb{R}$, where $|T| := \inf\limits_{x\in\mathbb{R}^3, x\neq 0} |T_x|$.*

Proof. Since the proof is similar to Lemma 2.3, we just give a sketch of the proof. Let $\{u_n\}$ be a $(PS)_c$ sequence of I. An argument similar to the one used in proving Lemma 2.3 shows that $\{u_n\}$ is bounded and there exists a measure ν such that (2.10) holds. We claim that the concentration of ν cannot occur at any $a \neq 0$ ($a \in \mathbb{R}^3$). Assuming that $a_k \neq 0$ is a singular point of ν, we can obtain $\nu_k = \nu(a_k) > 0$. Since ν is T-invariant, then $\nu(\tau a_k) = \nu_k$ for all $\tau \in T$. And we can know the sum in (2.11) is infinite due to $|T| = \infty$, which is a contradiction. On the other hand, by $\nu_i \leq s(a_i)\nu_i$ and $s(0) = 0$, we get $\nu_0 := \nu(0) = 0$.

Next, we prove that the concentration of ν cannot occur at infinity. Since $\lim\limits_{|x|\to\infty} s(x) = 0$, we deduce that

$$\lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} s(x)|u_n|^6 dx = 0.$$

By (2.29), we have $\mu_\infty = 0$. From $S\nu_\infty^{\frac{1}{3}} \leq \mu_\infty$, we obtain $\nu_\infty = 0$. Thus we get $u_n \to u$ in $D_T^{1,2}$ as $n \to \infty$. □

Since $D_T^{1,2}$ is a separable Banach space, there exists a linearly independent sequence $\{e_j\}$ such that

$$D_T^{1,2} = \overline{\bigoplus_{j\geq 1} D_j^{1,2}},\ D_j^{1,2} := \mathrm{span}\{e_j\}.$$

Denote $Y_k = \bigoplus\limits_{j\leq k} D_j^{1,2}$ and $Z_k = \overline{\bigoplus\limits_{j\geq k} D_j^{1,2}}$.

Lemma 4.3. *([25]) Let $I \in C^1(D_T^{1,2}, \mathbb{R})$ be an even functional satisfying the $(PS)_c$ condition for every $c > 0$. If for every $k \in \mathbb{N}$ there exist $\rho_k > r_k > 0$ such that*

(a) $\alpha_k := \max\limits_{u\in Y_k, \|u\|=\rho_k} I(u) \leq 0$,

(b) $\beta_k := \inf\limits_{u\in Z_k, \|u\|=r_k} I(u) \to \infty$ *as* $k \to \infty$,

then I has a sequence of critical values which converges to ∞.

Proof of Theorem 1.2. It is easy to see that $I(u)$ is even and $I(u) \in C^1(D_T^{1,2}, \mathbb{R})$. By Lemma 4.2, we know $I(u)$ satisfies the $(PS)_c$ condition for every $c > 0$. From the definition of Y_k and $s(x) > 0$ a.e. in $\mathbb{R}^3$, which imply that there exists a constant $\varepsilon_k > 0$ such that for all $w \in Y_k$ with $\|w\| = 1$, we have

$$\int_{\mathbb{R}^3} s(x)|w|^6 dx \geq \varepsilon_k. \tag{4.1}$$

On the one hand,

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx - \frac{1}{4}\int_{\mathbb{R}^3}k(x)\phi_u u^2dx - \frac{\lambda}{p}\int_{\mathbb{R}^3}h(x)|u|^pdx - \frac{1}{6}\int_{\mathbb{R}^3}s(x)|u|^6dx \\ &\le \frac{1}{2}\|u\|^2 - \frac{1}{6}\int_{\mathbb{R}^3}s(x)|u|^6dx. \end{aligned} \tag{4.2}$$

Hence if $u \in Y_k, u \neq 0$ and writing $u = t_k w$ with $\|w\| = 1$, by (4.1) and (4.2), we have

$$I(u) \le \tfrac{1}{2}t_k^2 - \tfrac{\varepsilon_k}{6}t_k^6 \le 0$$

for t_k large enough. Thus we have proved (a) of Lemma 4.3.

In the following part, we want to verify (b) of Lemma 4.3. Define

$$\upsilon_k := \sup_{u\in Z_k,\|u\|=1}\left(\int_{\mathbb{R}^3}s(x)|u|^6dx\right)^{\frac{1}{6}} \tag{4.3}$$

and

$$\gamma_k := \sup_{u\in Z_k,\|u\|=1}\left(\int_{\mathbb{R}^3}k(x)\phi_u u^2dx\right)^{\frac{1}{4}}. \tag{4.4}$$

It is clear that $0 \le \upsilon_{k+1} \le \upsilon_k$ and $\upsilon_k \to \upsilon_0 \ge 0$. And for every $k \ge 1$, there exists a $u_k \in Z_k$ with $\|u_k\| = 1$ such that

$$\left(\int_{\mathbb{R}^3}s(x)|u_k|^6dx\right)^{\frac{1}{6}} \ge \frac{\upsilon_0}{2}. \tag{4.5}$$

By the definition of Z_k, we get $u_k \rightharpoonup 0$ as $k \to \infty$ in $D_T^{1,2}$. Therefore, there exists ν such that (2.11) holds. Combining the arguments used in Lemma 4.2 with the fact that $|T| = \infty$, we see that the concentration of the measure ν can only occur at 0 and ∞, thus we have $u_k \to 0$ in $L^6(\Omega)$, where $\Omega = \{x \in \mathbb{R}^3 : r < |x| < R\}$ for each $0 < r < R$. Since $s(0) = 0$, $\lim_{|x|\to\infty} s(x) = 0$, by (A_3), for each $\varepsilon > 0$, we can choose r small and R large, such that

$$\left(\int_{\{x\in\mathbb{R}^3:|x|<r\}}s(x)|u_k|^6dx\right)^{\frac{1}{6}} < \frac{\varepsilon}{2},\ \left(\int_{\{x\in\mathbb{R}^3:|x|>R\}}s(x)|u_k|^6dx\right)^{\frac{1}{6}} < \frac{\varepsilon}{2}.$$

Hence by Sobolev embedding theorem, we can obtain

$$\left(\int_{\mathbb{R}^3}s(x)|u_k|^6dx\right)^{\frac{1}{6}} \to 0,\ as\ \ k \to \infty.$$

Using (4.3), we get $\upsilon_0 = 0$.

By Lemma 2.1(5), we obtain $\gamma_k \to 0$ *as* $k \to \infty$. Since $h(x) \ge 0$, $s(x) > 0$ a.e. in $\mathbb{R}^3$ and $\lambda > 0$, for $u \in Z_k$, by (4.3), Sobolev and Young inequalities, we have

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx - \frac{1}{4}\int_{\mathbb{R}^3}k(x)\phi_u u^2dx - \frac{\lambda}{p}\int_{\mathbb{R}^3}h(x)|u|^pdx - \frac{1}{6}\int_{\mathbb{R}^3}s(x)|u|^6dx \\ &\ge \frac{1}{2}\|u\|^2 - \frac{1}{4}\gamma_k^4\|u\|^4 - \frac{\lambda}{p}\|h(x)\|_{\frac{6-p}{6}}\|u\|^p - \frac{\upsilon_k^6}{6}\|u\|^6 \\ &\ge \frac{1}{2}\|u\|^2 - \frac{1}{192} - \frac{2}{3}\gamma_k^6\|u\|^6 - \frac{\lambda}{p}\|h(x)\|_{\frac{6-p}{6}}\|u\|^p - \frac{\upsilon_k^6}{6}\|u\|^6 \\ &= \frac{1}{2}\|u\|^2 - \left(\frac{1}{192} + \frac{\lambda}{p}\|h(x)\|_{\frac{6-p}{6}}\|u\|^p\right) - \left(\frac{2}{3}\gamma_k^6 + \frac{\upsilon_k^6}{6}\right)\|u\|^6. \end{aligned} \tag{4.6}$$

On the other hand, since $1 < p < 2$, then there exists $R > 0$ such that $\frac{1}{4}\|u\|^2 \geq \frac{1}{192} + \frac{\lambda}{p}\|h(x)\|_{\frac{6-p}{6}}\|u\|^p$ for any $\|u\| \geq R$. Taking $\|u\| = r_k := \left(\frac{3}{16\gamma_k^6+4\upsilon_k^6}\right)^{\frac{1}{4}}$, by $\upsilon_k \to 0$ and $\gamma_k \to 0$, we get $r_k \to \infty$ as $k \to \infty$. Furthermore, we have

$$I(u) \geq \frac{1}{4}\|u\|^2 - \left(\frac{2}{3}\gamma_k^6 + \frac{\upsilon_k^6}{6}\right)\|u\|^6 = \frac{1}{8}\|u\|^2 = \frac{1}{8}r_k^2 \to \infty, \ as\ k \to \infty. \tag{4.7}$$

This concludes the proof of Theorem 1.2.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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