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Research article

Fixed point theorem combined with variational methods for a class of nonlinear impulsive fractional problems with derivative dependence

Adnan Khaliq*and Mujeeb ur Rehman

School of Natural Sciences, National University of Sciences and Technology, Islamabad, Pakistan

* **Correspondence:** Email: adnankhaliqsns@gmail.com; Tel: +923226623874.

Abstract: In this article, we deal with a class of nonlinear impulsive problems of fractional-order in which nonlinearity is due to the fractional-order derivative term. The investigation involved a fixed point theorem with a combination of variational approach and critical point theory to establish sufficient conditions for the existence of at least one solution. First, a damped problem is discussed by using the critical point theory and variational approach, then the solutions of the damped problem and the main problem are connected with the assistance of a fixed point theorem. Towards the end, to illustrate our outcomes, two examples are given.

Keywords: fixed point theorem; variational methods; fractional impulsive boundary value problem; critical points
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1. Introduction

Fractional calculus gives generalization to the classical calculus from integer order to an arbitrary order which may be complex or real. Fractional calculus has become most notable and important part of mathematics which provides useful mathematical structures for the physical and biological phenomena, engineering mathematical models etc. To know about the developments in the theory of fractional calculus along with its applications, one can refer to [1–7] and the references there in.

The mathematical modeling of a process in which impulsive conditions (sudden discontinuous jumps) appear is done by using impulsive differential equations. Normally such processes are found in the field of biology, engineering and physics. The mathematical model of the population dynamics, drug administration and aircraft control are some examples of impulsive differential equations [8–11]. Because of their more importance, recently differential equations with impulsive effects of fractional order have gain a lot of contemplation of the researchers. For both linear and nonlinear impulsive fractional differential equations, the multiplicity and existence theory of their solutions is broadly discussed by using different tools such as Morse theory, measure of noncompactness, method of upper and lower solutions and fixed point theorems [12–17]. But these useful techniques are not appropriate and difficult to apply for that problems in which the corresponding integral equation can not be found easily e.g. when both right and left derivatives of fractional order are there in the problem. Such problems can easily be investigated by using another useful approach: variational techniques and the critical point theory. A pioneer work in this direction was that of Jiao and Zhou [18], who implemented the approach for a class of fractional differential equations. Whereas for a class of impulsive second order differential equations, considerable contributions are made by Nieto and O'Regan [19]. Later on, many authors used the critical point theory combined with variational approach to deal with the existence of solution to nonlinear and linear fractional impulsive differential equations [21–27]. Also variational methods along with semi-inverse methods for the establishment of variational formulation are widely used [28–30].

Recently Nieto and Uzal [20] discussed a class of impulsive differential equations of 2nd order in which nonlinearity is due to derivative dependence, where the existence of at least one solution of the problem is guaranteed via variational structure and fixed point theorem. But an impulsive fractional boundary value problem in which nonlinearity is because of derivative term still needs to be explored.

Above cited work gave us enough motivation to study the following nonlinear impulsive boundary value problem of fractional order in which nonlinearity is because of fractional order derivative dependence:

$$\begin{cases} {}_tD_T^{\alpha}\left({}_0^cD_t^{\alpha}y(t)\right) + b(t)y(t) = h(t, y(t), {}_0^cD_t^{\alpha}y(t)); & t \neq t_\lambda, \\ \Delta_t D_T^{\alpha-1}\left({}_0^cD_t^{\alpha}y(t_\lambda)\right) = I_\lambda(y(t_\lambda)); & \lambda = 1, 2, \cdots, n, \\ y(0) = 0 = y(T), \end{cases} \tag{1.1}$$

here $0 = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = T$, ${}_tD_T^{\alpha}$ is the α-order right Riemann-Liouville fractional derivative and ${}_0^cD_t^{\alpha}$ is the α-order left Caputo fractional derivative for $\frac{1}{2} < \alpha \leq 1$, $\Delta_t D_T^{\alpha-1}\left({}_0^cD_t^{\alpha}y(t_k)\right) = {}_tD_T^{\alpha-1}\left({}_0^cD_t^{\alpha}y(t_k^+)\right) - {}_tD_T^{\alpha-1}\left({}_0^cD_t^{\alpha}y(t_k^-)\right)$ and $b : [0, T] \to \mathbb{R}^+$, $I_\lambda : \mathbb{R} \to \mathbb{R}$ and $h : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are the functions satisfying some assumptions.

Remark 1.1. For $\alpha = 1$, it ought to be noticed that, one has ${}_0^cD_t^{\alpha}u(t) = u'(t)$ and ${}_tD_T^{\alpha}u(t) = -u'(t)$, and (1.1) reduces to standard impulsive problem of second order [20]. Therefore our concern (1.1) generalize that of [20].

Since the corresponding integral equation for the problem (1.1) can not be found, therefore we can not use formal analysis approach such as fixed point theorem. Also the problem (1.1) doesn't have a variational structure because of fractional derivative presence in the nonlinear term [20]. So the problem (1.1) looks like unsolvable. To overcome all these obstacles, we shall use a very interesting procedure by considering, for $z \in E_0^{\alpha}$ (which is defined in preliminary section), following a class of associated damped problems which involves no nonlinear dependence on the derivative.

$$\begin{cases} {}_tD_T^{\alpha}\left({}_0^cD_t^{\alpha}y(t)\right) + b(t)y(t) = h(t, y(t), {}_0^cD_t^{\alpha}z(t)); & t \neq t_\lambda, \\ \Delta_t D_T^{\alpha-1}\left({}_0^cD_t^{\alpha}y(t_\lambda)\right) = I_\lambda(y(t_\lambda)); & \lambda = 1, 2, \cdots, n, \\ y(0) = 0 = y(T). \end{cases} \tag{1.2}$$

Above problem has a variational structure and can be solved by applying critical point theory. At the end, we shall join the solutions of (1.2) and of the main problem (1.1) by using fixed point theorem.

This approach is same as that of used in [20] but here we shall extend the results from integer order to fractional order. Within the outer boundaries of our knowledge, problem (1.1) is untouched and going to get first treatment through this paper by using a novel and useful technique.

From start to finish of this paper, we suppose the following notations and conditions are satisfied.

(*M*1) For all $t \in [0, T]$, we have $0 < \underline{b} \le b(t) \le \bar{b}$ where $\underline{b}$ and $\bar{b}$ are constants.

(*M*2) For all $\lambda = 1, 2, \cdots, n$, $I_\lambda : \mathbb{R} \to \mathbb{R}$ and $h : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

(*M*3) There exist $V \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $W \in L^2(0, T)$ such that $|h(t, y, \xi)| \le V(|y|)W(t)$ and $|H_\xi(t, y)| \le V(|y|)W(t)$ where $H_\xi(t, y) = \int_0^y h(t, u, \xi)du$.

Rest of the article is composed in such a way that as per the prerequisites of article, some essential definitions and fundamental outcomes are given in Section 2. In Section 3, solution of damped problem (1.2) is discussed by converting it in a variational form. In Section 4, main theorem about the existence criteria of at least one solution of complete problem (1.1) along with proof is given. Toward the end, two examples are given to illustrate our outcomes.

2. Preliminaries

All the basic results and the definitions from the literature are given in this section which will be used as the building blocks in the construction of our main outcomes.

Definition 2.1. [1,2] Suppose y is defined on $[a, b]$ and $\alpha \in \mathbb{R}^+$. Then ${}_aD_s^{-\alpha}y(s)$(α-order left Riemann-Liouville fractional integral of y) and ${}_sD_b^{-\alpha}y(s)$(α-order right Riemann-Liouville fractional integral of y) are given by

$$ {}_aD_s^{-\alpha}y(s) = \frac{1}{\Gamma(\alpha)} \int_a^s (s - w)^{\alpha-1} y(w)dw, \qquad s \in [a, b], $$

and

$$ {}_sD_b^{-\alpha}y(s) = \frac{1}{\Gamma(\alpha)} \int_s^b (w - s)^{\alpha-1} y(w)dw, \qquad s \in [a, b], $$

respectively, provided right hand side is defined pointwise on $[a, b]$.

Definition 2.2. [1,2] Let y be defined on $[a, b]$ and $\alpha \in \mathbb{R}^+$. Then ${}_aD_s^{\alpha}y(s)$(α-order left Riemann-Liouville fractional derivative of y) and ${}_sD_b^{\alpha}y(s)$(α-order right Riemann-Liouville fractional derivative of y) are given by

$$ \begin{aligned} {}_aD_s^{\alpha}y(s) &= \frac{d^\eta}{ds^\eta} {}_aD_s^{-(\eta-\alpha)}y(s) \\ &= \frac{1}{\Gamma(\eta - \alpha)} \frac{d^\eta}{ds^\eta}\Big(\int_a^s (s - w)^{\eta-\alpha-1} y(w)dw \Big), \end{aligned} $$

and

$$ \begin{aligned} {}_sD_b^{\alpha}y(s) &= (-1)^\eta \frac{d^\eta}{ds^\eta} {}_sD_b^{-(\eta-\alpha)}y(s) \\ &= \frac{1}{\Gamma(\eta - \alpha)} (-1)^\eta \frac{d^\eta}{ds^\eta}\Big(\int_a^s (w - s)^{\eta-\alpha-1} y(w)dw \Big), \end{aligned} $$

respectively. Here $s \in [a, b], \eta - 1 < \alpha \le \eta$ and $\eta \in \mathbb{N}$.

Definition 2.3. [1,2] For $\eta - 1 < \alpha \le \eta$, suppose $y \in AC^{\eta}([a,b],\mathbb{R})$, then ${}_s^cD_b^{\alpha}y(s)$(α-order right Caputo fractional derivative of y) and ${}_a^cD_s^{\alpha}y(s)$(α-order left Caputo fractional derivative of y) are given by

$$\begin{aligned}{}_s^cD_b^{\alpha}y(s) &= (-1)^{\eta}{}_sD_b^{-(\eta-\alpha)}\frac{d^{\eta}}{ds^{\eta}}y(s)\\ &= \frac{(-1)^{\eta}}{\Gamma(\eta-\alpha)}\int_a^s (w-s)^{\eta-\alpha-1}y^{(\eta)}(w)dw,\end{aligned}$$

and

$$\begin{aligned}{}_a^cD_s^{\alpha}y(s) &= {}_aD_s^{-(\eta-\alpha)}\frac{d^{\eta}}{ds^{\eta}}y(s)\\ &= \frac{1}{\Gamma(\eta-\alpha)}\int_a^s (s-w)^{\eta-\alpha-1}y^{(\eta)}(w)dw,\end{aligned}$$

respectively, here $\eta \in \mathbb{N}$ and $s \in [a,b]$.

Lemma 2.4. [1,2] (a). *Let $\eta - 1 < \alpha \le \eta$ and $u, v \in L^2(a,b)$ then*

$$\int_a^b [{}_aD_s^{1-\alpha}u(s)]v(s)ds = \int_a^b u(s)[{}_sD_b^{1-\alpha}v(s)]ds. \tag{2.1}$$

(b). *Let $\eta - 1 < \alpha \le \eta, v \in AC([a,b],\mathbb{R}^N), v' \in L^2([a,b],\mathbb{R}^N)$ and ${}_sD_T^{\alpha}({}_0^cD_s^{\alpha}u(s)) \in AC([a,b],\mathbb{R}^N)$ with ${}_a^cD_s^{\alpha}u(s) \in L^2([a,b],\mathbb{R}^N)$ then*

$$\begin{aligned}\int_a^b ({}_a^cD_s^{\alpha}u(s))({}_a^cD_s^{\alpha}v(s))ds &= \int_a^b ({}_a^cD_s^{\alpha}u(s))({}_a^cD_s^{\alpha-1}v'(s))ds \\ &= \int_a^b {}_sD_T^{\alpha-1}({}_a^cD_s^{\alpha}u(s))v'(s)ds\\ &= {}_sD_T^{\alpha-1}({}_a^cD_s^{\alpha}u(s))v(s)|_{s=a}^{s=b} - \int_a^b \frac{d}{ds}({}_sD_T^{\alpha-1}({}_a^cD_s^{\alpha}u(s)))v(s)ds\\ &= {}_sD_T^{\alpha-1}({}_a^cD_s^{\alpha}u(s))v(s)|_{s=a}^{s=b} + \int_a^b {}_sD_T^{\alpha}({}_a^cD_s^{\alpha}u(s))v(s)ds.\end{aligned} \tag{2.2}$$

E_0^{α}(Fractional Derivative Space)

Our main attention is to apply variational methods and critical point theory for a corresponding functional. So there is a strong need of a fractional derivative space. Below we define such fractional derivative space which is coinciding to the space defined in [18].

First we review the norms $\|\cdot\|$ and $\|\cdot\|_{L^p}$ as follows

$$\|y\|_{\infty} = \max_{s\in[0,T]}|y(s)|,\ y \in C([0,T]), \qquad \|y\|_{L^p} = \left(\int_0^T |y(s)|^p ds\right)^{\frac{1}{p}},\ y \in L^p(0,T).$$

Definition 2.5. [18] Let $\alpha \in (0,1]$ and $C_0^{\infty}([0,T],\mathbb{R})$ be the set of all functions $y \in C^{\infty}([0,T],\mathbb{R})$ with $y(0) = y(T) = 0$, then the fractional derivative space E_0^{α} is defined by the closure of $C_0^{\infty}([0,T],\mathbb{R})$ with respect to the norm

$$\|y\|_{\alpha,2} = \left(\int_0^T (|y(t)|^2 + |{}_0D_t^{\alpha}y(t)|^2)dt\right)^{1/2}, \tag{2.3}$$

Remark 2.6. It is clear from the definition(2.5) that E_0^α is a space of functions y such that $y \in L^2[0,T]$ and ${}_0D_t^\alpha y(t) \in L^2[0,T]$ with $y(0) = 0 = y(T)$.

Lemma 2.7. [18] *If* $\alpha \in (\frac{1}{2}, 1]$ *then for* $y \in E_0^\alpha$ *we have*

$$\|y\|_{L^2} \le \frac{T^\alpha}{\Gamma(\alpha+1)} \|{}_0D_t^\alpha y(t)\|_{L^2}, \tag{2.4}$$

$$\|y\|_\infty \le \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\sqrt{2\alpha-1}} \|{}_0D_t^\alpha y(t)\|_{L^2}. \tag{2.5}$$

Remark 2.8. In the space E_0^α, if $\|\cdot\|_\alpha$ and $\|\cdot\|_{b,\alpha}$ are defined as

$$\|y\|_\alpha = \left(\int_0^T |{}_0D_t^\alpha y(t)|^2 dt\right)^{\frac{1}{2}}, \tag{2.6}$$

and

$$\|y\|_{b,\alpha} = \left(\int_0^T (b(t)|y(t)|^2 + |{}_0D_t^\alpha y(t)|^2) dt\right)^{1/2}, \tag{2.7}$$

then from (2.5) and (M1), it can be easily seen that $\|y\|_{\alpha,2}$ defined in (2.3), $\|y\|_\alpha$ in (2.6), and $\|y\|_{b,\alpha}$ in (2.7) are equivalent norms.

Proposition 2.9. *Let* $y \in E_0^\alpha$, *then the following result is satisfied*

$$\left(\frac{\underline{b}T^\alpha}{\Gamma(\alpha+1)} + 1\right)\|y\|_\alpha \le \|y\|_{b,\alpha} \le \left(\frac{\bar{b}T^\alpha}{\Gamma(\alpha+1)} + 1\right)\|y\|_\alpha. \tag{2.8}$$

Lemma 2.10. [18] *If a sequence is weakly convergent in* E_0^α *then in* $C[0,T]$ *space, it is strongly convergent.*

Lemma 2.11. [18] *For* $\alpha \in (0,1]$, *the fractional derivative space* E_0^α *is a Banach space which is reflexive and separable.*

Theorem 2.12. [31, Theorem 1.1] *Suppose* $\psi : Y \to \mathbb{R}$ *be a sequentially weakly lower semi-continuous functional for a reflexive Banach space* Y. *If* ψ *is strictly convex and coercive, then* ψ *has a unique minimum on* Y.

Theorem 2.13. [32, Schauder] *If* $T : Z \to Z$ *is a continuous and compact map for a nonempty closed and convex subset* Z *of a Banach space* Y, *then* T *has a fixed point.*

3. Damped problem solution

This section is devoted to investigate the variational structure and existence of solution of damped problem (1.2). First an equivalent form of the damped problem is given and then existence of solution is established by Theorem 2.12.

Lemma 3.1. *For* $x \in E_0^\alpha$, *any solution* y *of the problem* (1.2) *will also satisfy the following Eq* (3.1).

$$\int_0^T {}_0^cD_t^\alpha y(t) {}_0^cD_t^\alpha x(t) dt + \int_0^T b(t)y(t)x(t)dt + \sum_{\lambda=1}^n I_\lambda(y(t_\lambda))x(t_\lambda) - \int_0^T h(t, y(t), {}_0^cD_t^\alpha z(t))x(t)dt = 0. \tag{3.1}$$

Proof. Integrating from 0 to T after multiply (1.2) with $x(t) \in E_0^\alpha$, we get

$$\int_0^T {}_tD_T^\alpha({}_0^cD_t^\alpha y(t))x(t)dt + \int_0^T b(t)y(t)x(t)dt = \int_0^T h(t, y(t), {}_0^cD_t^\alpha z(t))x(t)dt \tag{3.2}$$

On the left hand side for the value of first term , using (2.1), (2.2) and, the impulsive and boundary conditions of problem (1.2), we have

$$\begin{aligned}
\int_0^T {}_0^cD_t^\alpha y(t){}_0^cD_t^\alpha x(t)dt &= \sum_{\lambda=0}^{n} \int_{t_\lambda}^{t_{\lambda+1}} {}_0^cD_t^\alpha y(t){}_0^cD_t^\alpha x(t)dt,\\
&= \sum_{\lambda=0}^{n} \int_{t_\lambda}^{t_{\lambda+1}} {}_0^cD_t^\alpha y(t){}_0^cD_t^{\alpha-1} x'(t)dt,\\
&= \sum_{\lambda=0}^{n} \int_{t_\lambda}^{t_{\lambda+1}} {}_tD_T^{\alpha-1}({}_0^cD_t^\alpha y(t))x'(t)dt,\\
&= \sum_{\lambda=0}^{n} {}_tD_T^{\alpha-1}({}_0^cD_t^\alpha y(t))x(t)\Big|_{t_\lambda}^{t_{\lambda+1}} + \sum_{\lambda=0}^{n} \int_{t_\lambda}^{t_{\lambda+1}} {}_tD_T^\alpha({}_0^cD_t^\alpha y(t))x(t)dt,\\
&= \sum_{\lambda=0}^{n} \left[{}_tD_T^\alpha({}_0^cD_t^\alpha y(t_{\lambda+1}^-))x(t_{\lambda+1}) - {}_tD_T^\alpha({}_0^cD_t^\alpha y(t_\lambda^+))x(t_\lambda)\right] + \int_0^T {}_tD_T^\alpha({}_0^cD_t^\alpha y(t))x(t)dt,\\
&= -\sum_{\lambda=1}^{n} I_\lambda(y(t_\lambda))x(t_\lambda) + \int_0^T {}_tD_T^\alpha({}_0^cD_t^\alpha y(t))x(t)dt.
\end{aligned}$$

So we can write

$$\int_0^T {}_tD_T^\alpha({}_0^cD_t^\alpha y(t))x(t)dt = \int_0^T {}_0^cD_t^\alpha y(t){}_0^cD_t^\alpha x(t)dt + \sum_{\lambda=1}^{n} I_\lambda(y(t_\lambda))x(t_\lambda). \tag{3.3}$$

Using Eq (3.3) in (3.2), we get the Eq (3.1) and the proof is completed. □

Using Lemma 3.1, we can introduce notion of the weak solution for the problem (1.1) and (1.2).

Definition 3.2. Let $y \in E_0^\alpha$, then y is a weak solution of the problem (1.2) if (3.1) is satisfied for each $x \in E_0^\alpha$. Also y is a weak solution of the problem (1.1) if it satisfies (3.1) for all $x \in E_0^\alpha$ with $z = y$.

Definition 3.3. Let $\psi_z : E_0^\alpha \to \mathbb{R}$ be a functional defined by

$$\psi_z(y) = \frac{1}{2}\int_0^T \left|{}_0^cD_t^\alpha y(t)\right|^2 + b(t)|y(t)|^2 dt + \sum_{\lambda=1}^{n} \int_0^{y(t_\lambda)} I_\lambda(s)ds - \int_0^T H_{{}_0^cD_t^\alpha z(t)}(t, y(t))dt, \tag{3.4}$$

where $H_\xi(t, y(t)) = \int_0^{y(t)} h(t, u, \xi)du$.

Remark 3.4. For functional ψ_z, if $x(t) \in E_0^\alpha$ then

$$\langle \psi_z'(y), x\rangle = \int_0^T {}_0^cD_t^\alpha y(t){}_0^cD_t^\alpha x(t)dt + \int_0^T b(t)y(t)x(t)dt + \sum_{\lambda=1}^{n} I_\lambda(y(t_\lambda))x(t_\lambda) - \int_0^T h(t, y(t), {}_0^cD_t^\alpha z(t))x(t)dt.$$

Thus in the view of Lemma 3.1, one can see that critical points of the functional ψ_z are precisely weak solutions of the damped problem (1.2).

We are stating a set of conditions which will be used later where they needed.

($M4$) There exist $a_{\lambda,\gamma_\lambda}, b_{\lambda,\gamma_\lambda} \in \mathbb{R}$ where $\lambda \in \{0,1,2,\cdots,n\}$ and $\gamma_0,\cdots,\gamma_n > 0$ such that

$$\int_0^y I_\lambda(s)ds \geq a_{\lambda,\gamma_\lambda}|y|^{\gamma_\lambda} + b_{\lambda,\gamma_\lambda}, \qquad H_\xi(t,y) \leq a_{0,\gamma_0}|y|^{\gamma_0} + b_{0,\gamma_0}|y|.$$

If $K = \{\lambda \in \{1,2,\cdots,n\}; a_{\lambda,\gamma_\lambda} \leq 0\}$, and suppose one condition from the following is satisfied.

- **($M4.1$)** Either $K \neq \phi$.
  - **($M4.1.1a$)** $a_{0,\gamma_0} \leq 0$.
  - **($M4.1.1b$)** $a_{0,\gamma_0} > 0,\ \gamma_0 = 2,\ \frac{min\{1,\alpha\}}{2}\left(1 + \frac{\underline{b}T^\alpha}{\Gamma(\alpha+1)}\right) > \frac{T^{2\alpha}}{[\Gamma(\alpha)]^2(2\alpha-1)}a_{0,2}$.
  - **($M4.1.1c$)** $a_{0,\gamma_0} > 0,\ \gamma_0 < 2$.
  - **($M4.1.2$)** $\gamma_\lambda \leq 2$ for all $\lambda \in K$, and let $\gamma_{\lambda_1} = \gamma_{\lambda_2} = \cdots = \gamma_{\lambda_q} = 2$ and $K_0 = \{\lambda_1,\lambda_2,\cdots,\lambda_q\}$.
    - **($M4.1.2a$)** $a_{0,\gamma_0} \leq 0,\ \frac{min\{\alpha,1\}}{2}\left(\frac{\underline{b}T^\alpha}{\Gamma(\alpha+1)} + 1\right) > \frac{T^{2\alpha-1}}{[\Gamma(\alpha)]^2(2\alpha-1)}\sum_{\lambda\in K_0} a_{\lambda,\gamma_0}$.
    - **($M4.1.2b$)** $a_{0,\gamma_0} > 0, \gamma_0 = 2,\ \frac{min\{\alpha,1\}}{2}\left(\frac{\underline{b}T^\alpha}{\Gamma(\alpha+1)} + 1\right) > \frac{T^{2\alpha-1}}{[\Gamma(\alpha)]^2(2\alpha-1)}\left[\sum_{\lambda=0}^n a_{\lambda,2} + Ta_{0,2}\right]$.
    - **($M4.1.2c$)** $a_{0,\gamma_0} > 0, \gamma_0 < 2,\ \frac{min\{\alpha,1\}}{2}\left(\frac{\underline{b}T^\alpha}{\Gamma(\alpha+1)} + 1\right) > \frac{T^{2\alpha-1}}{[\Gamma(\alpha)]^2(2\alpha-1)}\sum_{\lambda\in K_0} a_{\lambda,\gamma_0}$.
- **($M4.2$)** Or $K = \phi$, therefore $a_{\lambda,\gamma_\lambda} > 0$ for all $\lambda \in \{1,2,\cdots,n\}$.
  - **($M4.2.1$)** $a_{0,\gamma_0} \leq 0$.
  - **($M4.2.2$)** $a_{0,\gamma_0} > 0,\ \gamma_0 \leq 2$.
  - **($M4.2.3$)** $a_{0,\gamma_0} > 0,\ \gamma_0 = 2,\ \frac{min\{\alpha,1\}}{2}\left(\frac{\underline{b}T^\alpha}{\Gamma(\alpha+1)} + 1\right) > \frac{T^{2\alpha}}{[\Gamma(\alpha)]^2(2\alpha-1)}a_{0,2}$.

($M5$) $y \mapsto H_\xi(t,y)$ is concave and $y \mapsto \int_0^y I_\lambda(s)ds$ is convex and one of them is strict.

Lemma 3.5. *Suppose condition M4 is satisfied, then there exists $\beta(s)$ which is independent of ${}_0^cD_T^\alpha z(t)$ such that $\psi_z(y) \geq \beta(\|y\|_\alpha)$ with the property $\beta(s) \to +\infty$ as $s \to \infty$.*

Proof. We shall prove by considering only one item from (M4). For all other considerations of (M4), one can establish the proof in similar fashion. Suppose $K \neq \phi$ and $\gamma_\lambda < 2, \forall\lambda \in K$ with $a_{0,\gamma_0} < 0$. For $y \in E_0^\alpha$, using (2.8),

$$\begin{aligned}
\psi_z(y) &= \frac{1}{2}\int_0^T \left|{}_0^cD_t^\alpha y(t)\right|^2 + b(t)|y(t)|^2 dt + \sum_{\lambda=1}^n \int_0^{y(t_\lambda)} I_\lambda(s)ds - \int_0^T H_{{}_0^cD_t^\alpha z(t)}(t,y(t))dt,\\
&\geq \frac{min\{1,\alpha\}}{2}\left(1 + \frac{\underline{b}T^\alpha}{\Gamma(\alpha+1)}\right)\|y\|_\alpha^2 + \sum_{\lambda=1}^n [a_{\lambda,\gamma_\lambda}|y(t_\lambda)|^{\gamma_\lambda} + b_{\lambda,\gamma_\lambda}] - \int_0^T a_{0,\gamma_0}|y(t)|^{\gamma_0} + b_{0,\gamma_0}dt,\\
&\geq \frac{min\{1,\alpha\}}{2}\left(1 + \frac{\underline{b}T^\alpha}{\Gamma(\alpha+1)}\right)\|y\|_\alpha^2 - \sum_{\lambda\in K}|a_{\lambda,\gamma_\lambda}|\left(\frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\sqrt{2\alpha-1}}\right)^{\gamma_\lambda}\|y(t)\|_\alpha^{\gamma_\lambda} + |a_{0,\gamma_0}|\int_0^T |y(t)|^{\gamma_0}dt\\
&\quad + \sum_{\lambda=1}^n b_{\lambda,\gamma_\lambda} - b_{0,\gamma_0}T,\\
&\geq \frac{min\{1,\alpha\}}{2}\left(1 + \frac{\underline{b}T^\alpha}{\Gamma(\alpha+1)}\right)\|y\|_\alpha^2 - \sum_{\lambda\in K}|a_{\lambda,\gamma_\lambda}|\left(\frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\sqrt{2\alpha-1}}\right)^{\gamma_\lambda}\|y(t)\|_\alpha^{\gamma_\lambda} + \sum_{\lambda=1}^n b_{\lambda,\gamma_\lambda} - b_{0,\gamma_0}T,\\
&\geq \beta(\|y\|_\alpha),
\end{aligned}$$

where $\beta : (\mathbb{R}^+,\mathbb{R}^+)$ is given by

$$\beta(s) = \frac{min\{1,\alpha\}}{2}\left(1 + \frac{\underline{b}T^\alpha}{\Gamma(\alpha+1)}\right)s^2 - \sum_{\lambda\in K}|a_{\lambda,\gamma_\lambda}|\left(\frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\sqrt{2\alpha-1}}\right)^{\gamma_\lambda}s^{\gamma_\lambda} + \sum_{\lambda=1}^n b_{\lambda,\gamma_\lambda} - b_{0,\gamma_0}T.$$

Then β is continuous, independent of ${}_0^cD_t^\alpha z(t)$ and $\beta(s) \to +\infty$ as $s \to \infty$. The proof is completed. □

Theorem 3.6. *Suppose M4 and M5 are fulfilled, then the damped problem 1.2 has a unique weak solution y_z for each $z \in E_0^\alpha$. Moreover there exists $R > 0$ for all $z \in E_0^\alpha$ such that $\|y_z\|_\alpha \le R$.*

Proof. Our attention is to apply Theorem 2.12. Since any norm is convex and $u \mapsto u^2$ is convex on $[0, \infty)$, therefore The functional ψ_z is convex by using M5. Further more, ψ_z is sequentially weakly lower semi-continuous being sum of a weakly and of a convex continuous functions [31, Theorem 1.2, Propositon 1.2]. Actually $\int_0^T \left|{}_0^cD_t^\alpha y(t)\right|^2 + b(t)|y(t)|^2 dt$ is convex and continuous on E_0^α. Using the Lemma 2.10 along with M2 and Lebesgue dominated convergence theorem $\int_0^{y(t_\lambda)} I_\lambda(s)ds - \int_0^T H_{{}_0^cD_t^\alpha z(t)}(t, y(t))dt$ is weakly continuous on E_0^α. According to the Lemma 3.5, ψ_z is coercive. So according to Theorem 2.12, ψ_z has a unique global minimum y_z for each $z \in E_0^\alpha$ which is also a weak solution of the damped problem (1.2). The existence of R is proved as a consequence of Theorem 2.12. The property that R does not depend on z is a consequence of the fact that β in independent of z. □

4. Main problem solution and examples

Up to previous section, we have proved that there exists a unique critical point y_z for each $z \in E_0^\alpha$. Here we define a map $T : z \in E_0^\alpha \to y_z \in E_0^\alpha$ as $Tz = y_z$. It is clear that if there is any fixed point of T then that will be solution of the main problem (1.1).

Lemma 4.1. *If M4 and M5 are satisfied, then the mapping $T : z \in E_0^\alpha \to y_z \in E_0^\alpha$ is continuous and compact.*

Proof. First we prove that $T : z \in E_0^\alpha \to y_z \in E_0^\alpha$ is continuous. Let $\{z_n\}$ be a sequence in E_0^α such that $z_n \to z$. We have to show that $Tz_n \to Tz$. Suppose $Tz_n = y_n$. From Theorem 3.6, there is a $R > 0$ so that $\|y_n\| \le R$. So there exists a weakly convergent subsequence $\{y_{n_k}\}$ such that $y_{n_k} \rightharpoonup y$ (say) in E_0^α and by Lemma 2.10, $y_{n_k} \to y$ in $C([0, T])$. Let $\{y_{n_{k_j}}\}$ be an arbitrary subsequence of $\{y_{n_k}\}$.

As we have $z_{n_k} \to z$ in $E_0^{\alpha,2}$, then ${}_0^cD_t^\alpha z_{n_j}(t) \to {}_0^c D_t^\alpha z(t)$ in $L^2(0, T)$ and for a subsequence $\{z_{n_{k_j}}\}$, we have ${}_0^cD_t^\alpha z_{n_{j_l}}(t) \to {}_0^cD^\alpha{}_t z(t)$ for almost every $t \in [0, T]$. For any $x \in E_0^\alpha$, using Lebesgue's dominated convergence theorem and the fact that functions I_λ and h are continuous, we get

$$\int_0^T {}_0^cD_t^\alpha y_{n_{k_j}}(t){}_0^cD_t^\alpha x(t)dt + \int_0^T b(t)y_{n_{k_j}}(t)x(t)dt + \sum_{\lambda=1}^n I_\lambda(y_{n_{k_j}}(t_\lambda))x(t_\lambda) - \int_0^T h(t, y_{n_{k_j}}(t), {}_0^c D_t^\alpha z_{n_{k_j}}(t))x(t)dt = 0$$

$$\to \int_0^T {}_0^cD_t^\alpha y(t){}_0^cD_t^\alpha x(t)dt + \int_0^T b(t)y(t)x(t)dt + \sum_{\lambda=1}^n I_\lambda(y(t_\lambda))x(t_\lambda) - \int_0^T h(t, y(t), {}_0^c D_t^\alpha z(t))x(t)dt = 0,$$

so we have $Tz = y$ using the uniqueness of critical point and the fact that $\{y_{n_{k_j}}\}$ converges weakly to y. Also if $x = y_{n_{k_j}}$, then $\|y_{n_{k_j}}\| \to \|y\|$. Hence arbitrary subsequence $\{y_{n_k}\}$ of $\{y_n\}$ has a subsequence $\{y_{n_{k_j}}\}$ such that $y_{n_{k_j}} \to Tz$, which shows that $y_n \to Tz$. So T is continuous.

Next we show that T is compact. Consider $\{z_n\}$ is a bounded sequence in E_0^α. We have to prove that sequence $\{Tz_n\}$ has a convergent subsequence. Suppose $Tz_n = y_n$, then as in above discussion, a subsequence $\{Tz_{n_{k_j}}\}$ of $\{Tz_n\}$ exists which converges to $Tz = y$ in E_0^α. This completes the proof. □

Theorem 4.2. *Suppose M4 and M5 are hold then the main problem 1.1 has at least one weak solution.*

Proof. According to the Theorem 3.6 that there is a constant $R > 0$ such that $\|Tz\| \leq R$. Let $T : \overline{B(0,R)} \subset E_0^\alpha \to \overline{B(0,R)}$, then from Lemma 4.1, T is continuous and compact map. Schauder's fixed point guarantees that there exists a fixed point $z \in E_0^\alpha$ such that $Tz = z$, which shows that main problem (1.1) has a solution. This completes the proof. □

Example 4.3. Suppose the following nonlinear BVP,

$$\begin{cases} {}_tD_1^{\frac{4}{5}}({}_0^cD_t^{\frac{4}{5}}y(t)) + y(t) = -1 - y^3(t)(t+1)^5\left(3 + \cos({}_0^cD_t^{\frac{4}{5}}y(t))\right), & t \neq t_1 = 0.5, \\ \Delta_t D_1^{-\frac{1}{5}}({}_0^cD_t^{\frac{4}{5}}y(t_1)) = 1000y^3(t_1), \\ y(0) = 0 = y(1), \end{cases} \tag{4.1}$$

here $h(t,y,\xi) = -1 - y^3\,(3 + \cos\xi)\,(t+1)^5$ and $I_1(\eta) = 1000\eta^3$. For these values we have $a_{0,4} = -\frac{1}{2}$, $b_{0,4} = -1$, $a_{1,4} = 250$, and $b_{1,4} = 0$. Because $a_{1,4} > 0$ so $K = \phi$ and $a_{0,4} < 0$, so we are in case $(M4.2.1)$. Also

$$y \mapsto \int_0^y I_1(\eta)d\eta = 250y^4,$$

is strictly convex and

$$y \mapsto H_\xi(t,y) = -y - \frac{1}{4}\,(3 + \cos\xi)\,(t+1)^5 y^4,$$

is strictly concave. This shows that (M_5) is satisfied. Hence above problem (4.1) has a weak solution according to the Theorem 4.2.

Example 4.4. Suppose the following impulsive nonlinear BVP,

$$\begin{cases} {}_tD_1^{\frac{9}{10}}({}_0^cD_t^{\frac{9}{10}}y(t)) + t^2y(t) = 5 - \frac{1}{2}\left(\arctan({}_0^cD_t^{\frac{9}{10}}y(t)) + \pi\right)(y(t) + \sin y(t)\cos y(t)), & t \neq t_1 = 0.5, \\ \Delta_t D_1^{-\frac{1}{10}}({}_0^cD_t^{\frac{9}{10}}y(t_1)) = -1, \\ y(0) = 0 = y(1), \end{cases} \tag{4.2}$$

here $h(t,y,\xi) = 5 - \frac{1}{2}\,(\cos y \sin y + y)\,(\arctan\xi + \pi)$ and $I_1(\eta) = -1$. For these values we have $a_{0,2} = -\frac{\pi}{8}$, $b_{0,2} = 5$, $a_{1,1} = -1$, and $b_{1,1} = 0$. Because $a_{1,1} < 0$ so $K \neq \phi$ and $a_{0,2} < 0$, so we are in case $(M4.1.1)$. Moreover

$$y \mapsto \int_0^y I_1(\eta)d\eta = -y,$$

is convex and

$$y \mapsto H_\xi(t,y) = 5y + \frac{1}{8}\left(\cos 2y - 2y^2\right)(\arctan\xi + \pi),$$

is strictly concave. This shows that (M_5) is satisfied. Hence above problem (4.2) has a weak solution according to the Theorem 4.2.

5. Conclusions

Fixed point theorems and varitaitonal techniques along with critical point theory are very useful and applicable tools to discuss the existence of solution of differential equations of both integer and fractional orders. In this article by using these useful approaches, we have provided sufficient conditions for the existence of at least one weak solution of a nonlinear impulsive problem of fractional order in which nonlinearity is due to derivative term of fractional order. Our results generalize the nonlinear second order impulsive differential problems with dependence on derivative. The present results can be easily extended to the two-scale fractal calculus.

Conflict of interest

The authors declare no conflict of interest.

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