Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(2): 1905–1919.
DOI:10.3934/math.2021116
Received: 22 October 2020
Accepted: 01 December 2020
Published: 03 December 2020

Research article

Additive ρ-functional inequalities in non-Archimedean 2-normed spaces

Zhihua Wang[1], **Choonkil Park**[2,*] **and Dong Yun Shin**[3,*]

[1] School of Science, Hubei University of Technology, Wuhan, Hubei 430068, P. R. China
[2] Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
[3] Department of Mathematics, University of Seoul, Seoul 02504, Korea

* **Correspondence:** Email: baak@hanyang.ac.kr; dyshin@uos.ac.kr; Fax: +82 222810019.

Abstract: In this paper, we solve the additive ρ-functional inequalities:

$$\|f(x+y)-f(x)-f(y)\| \le \|\rho(2f(\frac{x+y}{2})-f(x)-f(y))\|,$$
$$\|2f(\frac{x+y}{2})-f(x)-f(y)\| \le \|\rho(f(x+y)-f(x)-f(y))\|,$$

where ρ is a fixed non-Archimedean number with $|\rho| < 1$. More precisely, we investigate the solutions of these inequalities in non-Archimedean 2-normed spaces, and prove the Hyers-Ulam stability of these inequalities in non-Archimedean 2-normed spaces. Furthermore, we also prove the Hyers-Ulam stability of additive ρ-functional equations associated with these inequalities in non-Archimedean 2-normed spaces.

Keywords: additive ρ-functional equation; additive ρ-functional inequality; Hyers-Ulam stability; non-Archimedean 2-normed spaces
Mathematics Subject Classification: 39B72, 39B62, 12J25

1. Introduction and preliminaries

The study of stability problems for functional equations is related to a question of Ulam [23] in 1940 concerning the stability of group homomorphisms.

The functional equation

$$f(x+y) = f(x) + f(y) \tag{1.1}$$

is called Cauchy functional equation. Every solution of the Cauchy functional equation is said to be an additive mapping. In 1941, Hyers [10] gave the first affirmative answer to the question of Ulam for

Banach spaces. Hyers' result was generalized by Aoki [1] for additive mappings and by Rassias [20] for linear mappings by considering an unbounded Cauchy difference. In 1994, Găvruţă [3] provided a further generalization of the Rassias' theorem in which he replaced the unbounded Cauchy difference by a general control function.

The following functional equation

$$2f(\frac{x+y}{2}) = f(x) + f(y) \tag{1.2}$$

is called Jensen functional equation. See [15, 16, 19, 22] for more information on functional equations.

Gilányi [7] and Rätz [21] showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \le \|f(xy)\|, \tag{1.3}$$

then f satisfies the Jordan-Von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}).$$

Fechner [2] and Gilányi [8] proved the generalized Hyers-Ulam stability of the functional inequality (1.1). Park *et al.* [17] investigated the generalized Hyers-Ulam stability of functional inequalities associated with Jordon-Von Neumann type additive functional equations. Kim *et al.* [11] solved the additive ρ-functional inequalities in complex normed spaces and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in complex Banach spaces. In 2014, Park [14] considered the following two additive ρ-functional inequalities

$$\|f(x+y) - f(x) - f(y)\| \le \|\rho(2f(\frac{x+y}{2}) - f(x) - f(y))\|, \tag{1.4}$$

$$\|2f(\frac{x+y}{2}) - f(x) - f(y)\| \le \|\rho(f(x+y) - f(x) - f(y))\| \tag{1.5}$$

in non-Archimedean Banach spaces and in complex Banach spaces, where ρ is a fixed non-Archimedean number with $|\rho| < 1$ or ρ is a fixed complex number with $|\rho| < 1$.

In this paper, we establish the solution of the additive ρ-functional inequalities (1.4) and (1.5), and prove the Hyers-Ulam stability of the additive ρ-functional inequalities (1.4) and (1.5) in non-Archimedean 2-Banach spaces. Moreover, we prove the Hyers-Ulam stability of additve ρ-functional equations associated with the additive ρ-functional inequalities (1.4) and (1.5) in non-Archimedean 2-Banach spaces.

Gähler [4, 5] has introduced the concept of linear 2-normed spaces in the middle of the 1960s. Then Gähler [6] and White [24, 25] introduced the concept of 2-Banach spaces. Following [9, 12, 13, 18], we recall some basic facts concerning non-Archimedean normed space and non-Archimedean 2-normed space and some preliminary results.

By a non-Archimedean field we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{K}$ into $[0, \infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r||s|$, and $|r + s| \le \max\{|r|, |s|\}$ for $r, s \in \mathbb{K}$. Clearly $|1| = |-1| = 1$ and $|n| \le 1$ for all $n \in \mathbb{N}$. By the trivial valuation we mean the function $|\cdot|$ taking everything but 0 into 1 and $|0| = 0$.

Definition 1.1. (cf. [9, 13]) Let X be a linear space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : X \to \mathbb{R}$ is called a non-Archimedean norm (valuation) if it

satisfies the following conditions:
(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}$, $x \in X$;
(iii) the strong triangle inequality; namely,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}$$

for all $x, y \in X$. Then $(X, \|\cdot\|)$ is called a non-Archimedean normed space.

Definition 1.2. (cf. [12, 18]) Let X be a linear space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$ with $\dim X > 1$. A function $\|\cdot, \cdot\| : X \to \mathbb{R}$ is called a non-Archimedean 2-norm (valuation) if it satisfies the following conditions:
(NA1) $\|x, y\| = 0$ if and only if x, y are linearly dependent;
(NA2) $\|x, y\| = \|y, x\|$;
(NA3) $\|rx, y\| = |r|\|x, y\|$;
(NA4) $\|x, y + z\| \leq \max\{\|x, y\|, \|x, z\|\}$;
for all $r \in \mathbb{K}$ and all $x, y, z \in X$. Then $(X, \|\cdot, \cdot\|)$ is called a non-Archimedean 2-normed space.

According to the conditions in Definition 1.2, we have the following lemma.

Lemma 1.3. *Let $(X, \|\cdot, \cdot\|)$ be a non-Archimedean 2-normed space. If $x \in X$ and $\|x, y\| = 0$ for all $y \in X$, then $x = 0$.*

Definition 1.4. A sequence $\{x_n\}$ in a non-Archimedean 2-normed space $(X, \|\cdot, \cdot\|)$ is called a Cauchy sequence if there are two linearly independent points $y, z \in X$ such that

$$\lim_{m,n\to\infty} \|x_n - x_m, y\| = 0 \quad \text{and} \quad \lim_{m,n\to\infty} \|x_n - x_m, z\| = 0.$$

Definition 1.5. A sequence $\{x_n\}$ in a non-Archimedean 2-normed space $(X, \|\cdot, \cdot\|)$ is called a convergent sequence if there exists an $x \in X$ such that

$$\lim_{n\to\infty} \|x_n - x, y\| = 0$$

for all $y \in X$. In this case, we call that $\{x_n\}$ converges to x or that x is the limit of $\{x_n\}$, write $\{x_n\} \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$.

By (NA4), we have

$$\|x_n - x_m, y\| \leq \max\{\|x_{j+1} - x_j, y\| : m \leq j \leq n - 1\}, \quad (n > m),$$

for all $y \in X$. Hence, a sequence $\{x_n\}$ is Cauchy in $(X, \|\cdot, \cdot\|)$ if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean 2-normed space $(X, \|\cdot, \cdot\|)$.

Remark 1.6. Let $(X, \|\cdot, \cdot\|)$ be a non-Archimedean 2-normed space. One can show that conditions (NA2) and (NA4) in Definition 1.2 imply that

$$\|x + y, z\| \leq \|x, z\| + \|y, z\| \quad \text{and} \quad |\|x - z\| - \|y, z\|| \leq \|x - y, z\|$$

for all $x, y, z \in X$.

We can easily get the following lemma by Remark1.6.

Lemma 1.7. *For a convergent sequence* $\{x_n\}$ *in a non-Archimedean* 2*-normed space* $(X, \|\cdot, \cdot\|)$*,*

$$\lim_{n\to\infty} \|x_n, y\| = \|\lim_{n\to\infty} x_n, y\|$$

for all $y \in X$.

Definition 1.8. A non-Archimedean 2-normed space, in which every Cauchy sequence is a convergent sequence, is called a non-Archimedean 2-Banach space.

Throughout this paper, let X be a non-Archimedean 2-normed space with $\dim X > 1$ and Y be a non-Archimedean 2-Banach space with $\dim Y > 1$. Let $\mathbb{N} = \{0, 1, 2, \ldots, \}$, and ρ be a fixed non-Archimedean number with $|\rho| < 1$.

2. Solutions and stability of the inequality (1.4)

In this section, we solve and investigate the additive ρ-functional inequality (1.4) in non-Archimedean 2-normed spaces.

Lemma 2.1. *A mapping* $f : X \to Y$ *satisfies*

$$\|f(x + y) - f(x) - f(y), \omega\| \leq \|\rho(2f(\frac{x + y}{2}) - f(x) - f(y)), \omega\| \tag{2.1}$$

for all $x, y \in X$ *and all* $\omega \in Y$ *if and only if* $f : X \to Y$ *is additive.*

Proof. Suppose that f satisfies (2.1). Setting $x = y = 0$ in (2.1), we have $\|f(0), \omega\| \leq \|0, \omega\|$=0 for all $\omega \in Y$ and so $\|f(0), \omega\| = 0$ for all $\omega \in Y$. Hence we get

$$f(0) = 0.$$

Putting $y = x$ in (2.1), we get

$$\|f(2x) - 2f(x), \omega\| \leq \|0, \omega\| \tag{2.2}$$

for all $x \in X$ and all $\omega \in Y$. Thus we have

$$f(\frac{x}{2}) = \frac{1}{2}f(x) \tag{2.3}$$

for all $x \in X$. It follows from (2.1) and (2.3) that

$$\begin{aligned}\|f(x + y) - f(x) - f(y), \omega\| &\leq \|\rho(2f(\frac{x + y}{2}) - f(x) - f(y)), \omega\| \\ &= |\rho|\|f(x + y) - f(x) - f(y), \omega\| \end{aligned} \tag{2.4}$$

for all $x, y \in X$ and all $\omega \in Y$. Hence, we obtain

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in X$.

The converse is obviously true. This completes the proof of the lemma. □

The following corollary can be found in [14, Corollary 2.2].

Corollary 2.2. *A mapping $f : X \to Y$ satisfies*

$$f(x+y) - f(x) - f(y) = \rho(2f(\frac{x+y}{2}) - f(x) - f(y)) \tag{2.5}$$

for all $x, y \in X$ if and only if $f : X \to Y$ is additive.

Theorem 2.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\lim_{j\to\infty} \frac{1}{|2|^j}\varphi(2^j x, 2^j y) = 0 \tag{2.6}$$

for all $x, y \in X$. Suppose that $f : X \to Y$ be a mapping satisfying

$$\|f(x+y) - f(x) - f(y), \omega\| \le \|\rho(2f(\frac{x+y}{2}) - f(x) - f(y)), \omega\| + \varphi(x, y) \tag{2.7}$$

for all $x, y \in X$ and all $\omega \in Y$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x), \omega\| \le \sup_{j\in\mathbb{N}}\{\frac{1}{|2|^{j+1}}\varphi(2^j x, 2^j x)\} \tag{2.8}$$

for all $x \in X$ and all $\omega \in Y$.

Proof. Letting $y = x$ in (2.6), we get

$$\|f(2x) - 2f(x), \omega\| \le \varphi(x, x) \tag{2.9}$$

for all $x \in X$ and all $\omega \in Y$. So

$$\|f(x) - \frac{1}{2}f(2x), \omega\| \le \frac{1}{|2|}\varphi(x, x) \tag{2.10}$$

for all $x \in X$ and all $\omega \in Y$. Hence

$$\begin{aligned}
&\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x), \omega\| \\
&\le \max\{\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^{l+1}}f(2^{l+1}x), \omega\|, \cdots, \|\frac{1}{2^{m-1}}f(2^{m-1}x) - \frac{1}{2^m}f(2^m x), \omega\|\} \\
&\le \max\{\frac{1}{|2|^l}\|f(2^l x) - \frac{1}{2}f(2^{l+1}x), \omega\|, \cdots, \frac{1}{|2|^{m-1}}\|f(2^{m-1}x) - \frac{1}{2}f(2^m x), \omega\|\} \\
&\le \sup_{j\in\{l,l+1,\ldots\}}\{\frac{1}{|2|^{j+1}}\varphi(2^j x, 2^j x)\}
\end{aligned} \tag{2.11}$$

for all nonnegative integers m, l with $m > l$ and for all $x \in X$ and all $\omega \in Y$. It follows from (2.11) that

$$\lim_{l,m\to\infty} \|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x), \omega\| = 0$$

for all $x \in X$ and all $\omega \in Y$. Thus the sequence $\{\frac{f(2^n x)}{2^n}\}$ is a Cauchy sequence in Y. Since Y is a non-Archimedean 2-Banach space, the sequence $\{\frac{f(2^n x)}{2^n}\}$ converges for all $x \in X$. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

for all $x \in X$. That is,

$$\lim_{n\to\infty} \|\frac{f(2^n x)}{2^n} - A(x), \omega\| = 0$$

for all $x \in X$ and all $\omega \in Y$.

By Lemma 1.7, (2.6) and (2.7), we get

$$\begin{aligned}
&\|A(x+y) - A(x) - A(y), \omega\| \\
&= \lim_{n\to\infty} \frac{1}{|2|^n} \|f(2^n(x+y)) - f(2^n x) - f(2^n y), \omega\| \\
&\le \lim_{n\to\infty} \frac{1}{|2|^n} \|\rho(2f(\frac{2^n(x+y)}{2}) - f(2^n x) - f(2^n y)), \omega\| + \lim_{n\to\infty} \frac{1}{|2|^n} \varphi(2^n x, 2^n y) \\
&= \|\rho(2A(\frac{x+y}{2}) - A(x) - A(y)), \omega\| \qquad (2.12)
\end{aligned}$$

for all $x, y \in X$ and all $\omega \in Y$. Thus, the mapping $A : X \to Y$ is additive by Lemma 2.1.

By Lemma 1.7 and (2.11), we have

$$\|f(x) - A(x), \omega\| = \lim_{m\to\infty} \|f(x) - \frac{f(2^m x)}{2^m}, \omega\| \le \sup_{j\in\mathbb{N}}\{\frac{1}{|2|^{j+1}} \varphi(2^j x, 2^j x)\}$$

for all $x \in X$ and all $\omega \in Y$. Hence, we obtain (2.8), as desired.

To prove the uniqueness property of A, Let $A' : X \to Y$ be an another additive mapping satisfying (2.8). Then we have

$$\begin{aligned}
\|A(x) - A'(x), \omega\| &= \|\frac{1}{2^n} A(2^n x) - \frac{1}{2^n} A'(2^n x), \omega\| \\
&\le \max\{\|\frac{1}{2^n} A(2^n x) - \frac{1}{2^n} f(2^n x), \omega\|, \|\frac{1}{2^n} f(2^n x) - \frac{1}{2^n} A'(2^n x), \omega\|\} \\
&\le \sup_{j\in\mathbb{N}}\{\frac{1}{|2|^{n+j+1}} \varphi(2^{n+j} x, 2^{n+j} x)\},
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$ and all $\omega \in Y$. By Lemma 1.3, we can conclude that $A(x) = A'(x)$ for all $x \in X$. This proves the uniqueness of A. □

Corollary 2.4. *Let r, θ be positive real numbers with $r > 1$, and let $f : X \to Y$ be a mapping such that*

$$\|f(x+y) - f(x) - f(y), \omega\| \le \|\rho(2f(\frac{x+y}{2}) - f(x) - f(y)), \omega\| + \theta(\|x\|^r + \|y\|^r)\|\omega\| \qquad (2.13)$$

for all $x, y \in X$ and all $\omega \in Y$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x), \omega\| \le \frac{2}{|2|} \theta \|x\|^r \|\omega\| \qquad (2.14)$$

for all $x \in X$ and all $\omega \in Y$.

Proof. The proof follows from Theorem 2.3 by taking $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)\|\omega\|$ for all $x, y \in X$ and all $\omega \in Y$, as desired. □

Theorem 2.5. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that*

$$\lim_{j\to\infty} |2|^j \varphi(\frac{x}{2^j}, \frac{y}{2^j}) = 0 \tag{2.15}$$

for all $x, y \in X$. Suppose that $f : X \to Y$ be a mapping satisfying

$$\|f(x+y) - f(x) - f(y), \omega\| \le \|\rho(2f(\frac{x+y}{2}) - f(x) - f(y)), \omega\| + \varphi(x,y) \tag{2.16}$$

for all $x, y \in X$ and all $\omega \in Y$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x), \omega\| \le \sup_{j\in\mathbb{N}}\{|2|^j \varphi(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}})\} \tag{2.17}$$

for all $x \in X$ and all $\omega \in Y$.

Proof. It follows from (2.9) that

$$\|f(x) - 2f(\frac{x}{2}), \omega\| \le \varphi(\frac{x}{2}, \frac{x}{2}) \tag{2.18}$$

for all $x \in X$ and all $\omega \in Y$. Hence

$$\begin{aligned}
&\|2^l f(\frac{x}{2^l}) - 2^m f(\frac{x}{2^m}), \omega\| \\
&\le \max\{\|2^l f(\frac{x}{2^l}) - 2^{l+1} f(\frac{x}{2^{l+1}}), \omega\|, \cdots, \|2^{m-1} f(\frac{x}{2^{m-1}}) - 2^m f(\frac{x}{2^m}), \omega\|\} \\
&\le \max\{|2|^l \|f(\frac{x}{2^l}) - 2f(\frac{x}{2^{l+1}}), \omega\|, \cdots, |2|^{m-1}\|f(\frac{x}{2^{m-1}}) - 2f(\frac{x}{2^m}), \omega\|\} \\
&\le \sup_{j\in\{l,l+1,\dots\}} \{|2|^j \varphi(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}})\}
\end{aligned} \tag{2.19}$$

for all nonnegative integers m, l with $m > l$ and for all $x \in X$ and all $\omega \in Y$. It follows from (2.19) that

$$\lim_{l,m\to\infty} \|2^l f(\frac{x}{2^l}) - 2^m f(\frac{x}{2^m}), \omega\| = 0$$

for all $x \in X$ and all $\omega \in Y$. Thus the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence in Y. Since Y is a non-Archimedean 2-Banach space, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges for all $x \in X$. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$. That is,

$$\lim_{n\to\infty} \|2^n f(\frac{x}{2^n}) - A(x), \omega\| = 0$$

for all $x \in X$ and all $\omega \in Y$. By Lemma 1.7 and (2.19), we have

$$\|f(x) - A(x), \omega\| = \lim_{m\to\infty} \|f(x) - 2^m f(\frac{x}{2^m}), \omega\| \le \sup_{j\in\mathbb{N}}\{|2|^j \varphi(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}})\}$$

for all $x \in X$ and all $\omega \in Y$. Hence, we obtain (2.17), as desired. The rest of the proof is similar to that of Theorem 2.3 and thus it is omitted. □

Corollary 2.6. *Let r, θ be positive real numbers with $r < 1$, and let $f : X \to Y$ be a mapping satisfying (2.13) for all $x, y \in X$ and all $\omega \in Y$. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x), \omega\| \le \frac{2}{|2|^r}\theta\|x\|^r\|\omega\| \tag{2.20}$$

for all $x \in X$ and all $\omega \in Y$.

Let $A(x, y) := f(x + y) - f(x) - f(y)$ and $B(x, y) := \rho(2f(\frac{x+y}{2}) - f(x) - f(y))$ for all $x, y \in X$. For $x, y \in X$ and $\omega \in Y$ with $\|A(x, y), \omega\| \le \|B(x, y), \omega\|$, we have

$$\|A(x, y), \omega\| - \|B(x, y), \omega\| \le \|A(x, y) - B(x, y), \omega\|.$$

For $x, y \in X$ and $\omega \in Y$ with $\|A(x, y), \omega\| > \|B(x, y), \omega\|$, we have

$$\begin{aligned}\|A(x, y), \omega\| &= \|A(x, y) - B(x, y) + B(x, y), \omega\| \\ &\le \max\{\|A(x, y) - B(x, y), \omega\|, \|B(x, y), \omega\|\} \\ &= \|A(x, y) - B(x, y), \omega\| \\ &\le \|A(x, y) - B(x, y), \omega\| + \|B(x, y), \omega\|.\end{aligned}$$

So we can obtain

$$\begin{aligned}&\|f(x + y) - f(x) - f(y), \omega\| - \|\rho(2f(\frac{x + y}{2}) - f(x) - f(y)), \omega\| \\ &\le \|f(x + y) - f(x) - f(y) - \rho(2f(\frac{x + y}{2}) - f(x) - f(y)), \omega\|.\end{aligned}$$

As corollaries of Theorems 2.3 and 2.5, we obtain the Hyers-Ulam stability results for the additive ρ-functional equation associated with the additive ρ-functional inequality (1.4) in non-Archimedean 2-Banach spaces.

Corollary 2.7. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying (2.6) and*

$$\|f(x + y) - f(x) - f(y) - \rho(2f(\frac{x + y}{2}) - f(x) - f(y)), \omega\| \le \varphi(x, y) \tag{2.21}$$

for all $x, y \in X$ and all $\omega \in Y$. Then there exists a unique additive mapping $A : X \to Y$ satisfying (2.8) for all $x \in X$ and all $\omega \in Y$.

Corollary 2.8. *Let r, θ be positive real numbers with $r > 1$, and let $f : X \to Y$ be a mapping such that*

$$\|f(x + y) - f(x) - f(y) - \rho(2f(\frac{x + y}{2}) - f(x) - f(y)), \omega\| \le \theta(\|x\|^r + \|y\|^r)\|\omega\| \tag{2.22}$$

for all $x, y \in X$ and all $\omega \in Y$. Then there exists a unique additive mapping $A : X \to Y$ satisfying (2.14) for all $x \in X$ and all $\omega \in Y$.

Corollary 2.9. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying (2.15) and (2.21) for all $x, y \in X$ and all $\omega \in Y$. Then there exists a unique additive mapping $A : X \to Y$ satisfying (2.17) for all $x \in X$ and all $\omega \in Y$.*

Corollary 2.10. *Let r, θ be positive real numbers with $r < 1$, and let $f : X \to Y$ be a mapping satisfying (2.22) for all $x, y \in X$ and all $\omega \in Y$. Then there exists a unique additive mapping $A : X \to Y$ satisfying (2.20) for all $x \in X$ and all $\omega \in Y$.*

3. Solutions and stability of the inequality (1.5)

In this section, we solve and investigate the additive ρ-functional inequality (1.5) in non-Archimedean 2-normed spaces.

Lemma 3.1. *A mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\|2f(\frac{x+y}{2}) - f(x) - f(y), \upsilon\| \leq \|\rho(f(x+y) - f(x) - f(y)), \upsilon\| \tag{3.1}$$

for all $x, y \in X$ and all $\upsilon \in Y$ if and only if $f : X \to Y$ is additive.

Proof. Suppose that f satisfies (3.1). Letting $y = 0$ in (3.1), we have

$$\|2f(\frac{x}{2}) - f(x), \upsilon\| \leq \|0, \upsilon\| = 0 \tag{3.2}$$

for all $x \in X$ and all $\upsilon \in Y$. Thus we have

$$f(\frac{x}{2}) = \frac{1}{2}f(x) \tag{3.3}$$

for all $x \in X$. It follows from (3.1) and (3.3) that

$$\begin{aligned}\|f(x+y) - f(x) - f(y), \upsilon\| &= \|2f(\frac{x+y}{2}) - f(x) - f(y), \upsilon\| \\ &\leq |\rho|\|f(x+y) - f(x) - f(y), \upsilon\| \end{aligned} \tag{3.4}$$

for all $x, y \in X$ and all $\upsilon \in Y$. Hence, we obtain

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$.

The converse is obviously true. This completes the proof of the lemma. □

The following corollary can be found in [14, Corollary 3.2].

Corollary 3.2. *A mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$2f(\frac{x+y}{2}) - f(x) - f(y) = \rho(f(x+y) - f(x) - f(y)) \tag{3.5}$$

for all $x, y \in X$ if and only if $f : X \to Y$ is additive.

Theorem 3.3. *Let $\phi : X^2 \to [0, \infty)$ be a function such that*

$$\lim_{j\to\infty} \frac{1}{|2|^j}\phi(2^j x, 2^j y) = 0 \tag{3.6}$$

for all $x, y \in X$. Suppose that $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\|2f(\frac{x+y}{2}) - f(x) - f(y), \upsilon\| \leq \|\rho(f(x+y) - f(x) - f(y)), \upsilon\| + \phi(x, y) \tag{3.7}$$

for all $x, y \in X$ and all $\upsilon \in Y$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x), \upsilon\| \leq \sup_{j\in\mathbb{N}}\{\frac{1}{|2|^{j+1}}\phi(2^{j+1}x, 0)\} \tag{3.8}$$

for all $x \in X$ and all $\upsilon \in Y$.

Proof. Letting $y = 0$ in (3.6), we get

$$\|2f(\frac{x}{2}) - f(x), \upsilon\| \leq \phi(x, 0) \tag{3.9}$$

for all $x \in X$ and all $\upsilon \in Y$. So

$$\|f(x) - \frac{1}{2}f(2x), \upsilon\| \leq \frac{1}{|2|}\phi(2x, 0) \tag{3.10}$$

for all $x \in X$ and all $\upsilon \in Y$. Hence

$$\begin{aligned}
&\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x), \upsilon\| \\
&\leq \max\{\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^{l+1}}f(2^{l+1}x), \upsilon\|, \cdots, \|\frac{1}{2^{m-1}}f(2^{m-1}x) - \frac{1}{2^m}f(2^m x), \upsilon\|\} \\
&\leq \max\{\frac{1}{|2|^l}\|f(2^l x) - \frac{1}{2}f(2^{l+1}x), \upsilon\|, \cdots, \frac{1}{|2|^{m-1}}\|f(2^{m-1}x) - \frac{1}{2}f(2^m x), \upsilon\|\} \\
&\leq \sup_{j\in\{l,l+1,\ldots\}}\{\frac{1}{|2|^{j+1}}\phi(2^{j+1}x, 0)\}
\end{aligned} \tag{3.11}$$

for all nonnegative integers m, l with $m > l$ and for all $x \in X$ and all $\upsilon \in Y$. It follows from (3.11) that

$$\lim_{l,m\to\infty}\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x), \upsilon\| = 0$$

for all $x \in X$ and all $\upsilon \in Y$. Thus the sequence $\{\frac{f(2^n x)}{2^n}\}$ is a Cauchy sequence in Y. Since Y is a non-Archimedean 2-Banach space, the sequence $\{\frac{f(2^n x)}{2^n}\}$ converges for all $x \in X$. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{f(2^n x)}{2^n}$$

for all $x \in X$. That is,

$$\lim_{n\to\infty}\|\frac{f(2^n x)}{2^n} - A(x), \upsilon\| = 0$$

for all $x \in X$ and all $\upsilon \in Y$.

By Lemma 1.7, (3.6) and (3.7), we get

$$\begin{aligned}
&\|2A(\frac{x+y}{2}) - A(x) - A(y), \upsilon\| \\
&= \lim_{n\to\infty}\frac{1}{|2|^n}\|2f(\frac{2^n(x+y)}{2}) - f(2^n x) - f(2^n y), \upsilon\| \\
&\leq \lim_{n\to\infty}\frac{1}{|2|^n}\|\rho(f(2^n(x+y)) - f(2^n x) - f(2^n y)), \upsilon\| + \lim_{n\to\infty}\frac{1}{|2|^n}\phi(2^n x, 2^n y) \\
&= \|\rho(A(x+y) - A(x) - A(y)), \upsilon\|
\end{aligned} \tag{3.12}$$

for all $x, y \in X$ and all $\upsilon \in Y$. Thus, the mapping $A : X \to Y$ is additive by Lemma 3.1.

By Lemma 1.7 and (3.11), we have

$$\|f(x) - A(x), \upsilon\| = \lim_{m\to\infty} \|f(x) - \frac{f(2^m x)}{2^m}, \upsilon\| \leq \sup_{j\in\mathbb{N}}\{\frac{1}{|2|^{j+1}}\phi(2^{j+1}x, 0)\}$$

for all $x \in X$ and all $\upsilon \in Y$. Hence, we obtain (3.8), as desired.

To prove the uniqueness property of A, Let $A' : X \to Y$ be an another additive mapping satisfying (3.8). Then we have

$$\begin{aligned}\|A(x) - A'(x), \upsilon\| &= \|\frac{1}{2^n}A(2^n x) - \frac{1}{2^n}A'(2^n x), \upsilon\| \\ &\leq \max\{\|\frac{1}{2^n}A(2^n x) - \frac{1}{2^n}f(2^n x), \upsilon\|, \|\frac{1}{2^n}f(2^n x) - \frac{1}{2^n}A'(2^n x), \upsilon\|\} \\ &\leq \sup_{j\in\mathbb{N}}\{\frac{1}{|2|^{n+j+1}}\phi(2^{n+j+1}x, 0)\},\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$ and all $\upsilon \in Y$. By Lemma 1.3, we can conclude that $A(x) = A'(x)$ for all $x \in X$. This proves the uniqueness of A. □

Corollary 3.4. *Let s, δ be positive real numbers with $s > 1$, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\|2f(\frac{x+y}{2}) - f(x) - f(y), \upsilon\| \leq \|\rho(f(x+y) - f(x) - (y)), \upsilon\| + \delta(\|x\|^s + \|y\|^s)\|\upsilon\| \tag{3.13}$$

for all $x, y \in X$ and all $\upsilon \in Y$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x), \upsilon\| \leq \frac{|2|^s}{|2|}\delta\|x\|^s\|\upsilon\| \tag{3.14}$$

for all $x \in X$ and all $\upsilon \in Y$.

Proof. The proof follows from Theorem 3.3 by taking $\phi(x, y) = \delta(\|x\|^s + \|y\|^s)\|\upsilon\|$ for all $x, y \in X$ and all $\upsilon \in Y$, as desired. □

Theorem 3.5. *Let $\phi : X^2 \to [0, \infty)$ be a function such that*

$$\lim_{j\to\infty} |2|^j \phi(\frac{x}{2^j}, \frac{y}{2^j}) = 0 \tag{3.15}$$

for all $x, y \in X$. Suppose that $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\|2f(\frac{x+y}{2}) - f(x) - f(y), \upsilon\| \leq \|\rho(f(x+y) - f(x) - f(y)), \upsilon\| + \phi(x, y) \tag{3.16}$$

for all $x, y \in X$ and all $\upsilon \in Y$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x), \upsilon\| \leq \sup_{j\in\mathbb{N}}\{|2|^j \phi(\frac{x}{2^j}, 0)\} \tag{3.17}$$

for all $x \in X$ and all $\upsilon \in Y$.

Proof. It follows from (3.9) that

$$\|f(x) - 2f(\frac{x}{2}), \upsilon\| \leq \phi(x, 0) \tag{3.18}$$

for all $x \in X$ and all $\upsilon \in Y$. Hence

$$\begin{aligned}
&\|2^l f(\frac{x}{2^l}) - 2^m f(\frac{x}{2^m}), \upsilon\| \\
&\leq \max\{\|2^l f(\frac{x}{2^l}) - 2^{l+1} f(\frac{x}{2^{l+1}}), \upsilon\|, \cdots, \|2^{m-1} f(\frac{x}{2^{m-1}}) - 2^m f(\frac{x}{2^m}), \upsilon\|\} \\
&\leq \max\{|2|^l \|f(\frac{x}{2^l}) - 2f(\frac{x}{2^{l+1}}), \upsilon\|, \cdots, |2|^{m-1} \|f(\frac{x}{2^{m-1}}) - 2f(\frac{x}{2^m}), \upsilon\|\} \\
&\leq \sup_{j \in \{l, l+1, \ldots\}} \{|2|^j \phi(\frac{x}{2^j}, 0)\}
\end{aligned} \tag{3.19}$$

for all nonnegative integers m, l with $m > l$ and for all $x \in X$ and all $\upsilon \in Y$. It follows from (3.19) that

$$\lim_{l,m\to\infty} \|2^l f(\frac{x}{2^l}) - 2^m f(\frac{x}{2^m}), \upsilon\| = 0$$

for all $x \in X$ and all $\upsilon \in Y$. Thus the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence in Y. Since Y is a non-Archimedean 2-Banach space, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges for all $x \in X$. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$. That is,

$$\lim_{n\to\infty} \|2^n f(\frac{x}{2^n}) - A(x), \upsilon\| = 0$$

for all $x \in X$ and all $\upsilon \in Y$. By Lemma 1.7 and (3.19), we have

$$\|f(x) - A(x), \upsilon\| = \lim_{m\to\infty} \|f(x) - 2^m f(\frac{x}{2^m}), \upsilon\| \leq \sup_{j\in\mathbb{N}}\{|2|^j \phi(\frac{x}{2^j}, 0)\}$$

for all $x \in X$ and all $\upsilon \in Y$. Hence, we obtain (3.17), as desired. The rest of the proof is similar to that of Theorem 3.3 and thus it is omitted. □

Corollary 3.6. *Let s, δ be positive real numbers with $s < 1$, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.13) for all $x, y \in X$ and all $\upsilon \in Y$. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x), \upsilon\| \leq \delta \|x\|^s \|\upsilon\| \tag{3.20}$$

for all $x \in X$ and all $\upsilon \in Y$.

Let $\tilde{A}(x, y) := 2f(\frac{x+y}{2}) - f(x) - f(y)$ and $\tilde{B}(x, y) := \rho(f(x + y) - f(x) - f(y))$ for all $x, y \in X$. For $x, y \in X$ and $\upsilon \in Y$ with $\|\tilde{A}(x, y), \upsilon\| \leq \|\tilde{B}(x, y), \upsilon\|$, we have

$$\|\tilde{A}(x, y), \upsilon\| - \|\tilde{B}(x, y), \upsilon\| \leq \|\tilde{A}(x, y) - \tilde{B}(x, y), \upsilon\|.$$

For $x, y \in X$ and $\upsilon \in Y$ with $\|\tilde{A}(x,y), \upsilon\| > \|\tilde{B}(x,y), \upsilon\|$, we have

$$\begin{aligned}\|\tilde{A}(x,y), \upsilon\| &= \|\tilde{A}(x,y) - \tilde{B}(x,y) + \tilde{B}(x,y), \upsilon\| \\ &\leq \max\{\|\tilde{A}(x,y) - \tilde{B}(x,y), \upsilon\|, \|\tilde{B}(x,y), \upsilon\|\} \\ &= \|\tilde{A}(x,y) - \tilde{B}(x,y), \upsilon\| \\ &\leq \|\tilde{A}(x,y) - \tilde{B}(x,y), \upsilon\| + \|\tilde{B}(x,y), \upsilon\|.\end{aligned}$$

So we can obtain

$$\begin{aligned}&\|2f(\frac{x+y}{2}) - f(x) - f(y), \upsilon\| - \|\rho(f(x+y) - f(x) - f(y)), \upsilon\| \\ &\qquad \leq \|2f(\frac{x+y}{2}) - f(x) - f(y) - \rho(f(x+y) - f(x) - f(y)), \upsilon\|.\end{aligned}$$

As corollaries of Theorems 3.3 and 3.5, we obtain the Hyers-Ulam stability results for the additive ρ-functional equation associated with the additive ρ-functional inequality (1.5) in non-Archimedean 2-Banach spaces.

Corollary 3.7. *Let $\phi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (3.6) and*

$$\|2f(\frac{x+y}{2}) - f(x) - f(y) - \rho(f(x+y) - f(x) - f(y)), \upsilon\| \leq \phi(x,y) \tag{3.21}$$

for all $x, y \in X$ and all $\upsilon \in Y$. Then there exists a unique additive mapping $A : X \to Y$ satisfying (3.8) for all $x \in X$ and all $\upsilon \in Y$.

Corollary 3.8. *Let s, δ be positive real numbers with $s > 1$, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\|2f(\frac{x+y}{2}) - f(x) - f(y) - \rho(f(x+y) - f(x) - (y)), \upsilon\| \leq \delta(\|x\|^s + \|y\|^s)\|\upsilon\| \tag{3.22}$$

for all $x, y \in X$ and all $\upsilon \in Y$. Then there exists a unique additive mapping $A : X \to Y$ satisfying (3.14) for all $x \in X$ and all $\upsilon \in Y$.

Corollary 3.9. *Let $\phi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (3.15) and (3.21) for all $x, y \in X$ and all $\upsilon \in Y$. Then there exists a unique additive mapping $A : X \to Y$ satisfying (3.17) for all $x \in X$ and all $\upsilon \in Y$.*

Corollary 3.10. *Let s, δ be positive real numbers with $s < 1$, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.22) for all $x, y \in X$ and all $\upsilon \in Y$. Then there exists a unique additive mapping $A : X \to Y$ satisfying (3.20) for all $x \in X$ and all $\upsilon \in Y$.*

4. Conclusion

In this paper, we have solved the additive ρ-functional inequalities:

$$\|f(x+y) - f(x) - f(y)\| \leq \|\rho(2f(\frac{x+y}{2}) - f(x) - f(y))\|,$$

$$\|2f(\frac{x+y}{2}) - f(x) - f(y)\| \leq \|\rho(f(x+y) - f(x) - f(y))\|,$$

where ρ is a fixed non-Archimedean number with $|\rho| < 1$. More precisely, we have investigated the solutions of these inequalities in non-Archimedean 2-normed spaces, and have proved the Hyers-Ulam stability of these inequalities in non-Archimedean 2-normed spaces. Furthermore, we have also proved the Hyers-Ulam stability of additive ρ-functional equations associated with these inequalities in non-Archimedean 2-normed spaces.

Acknowledgments

We would like to express our sincere gratitude to the anonymous referees for their helpful comments that will help to improve the quality of the manuscript.

Authors' contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

Conflict of interest

The authors declare that they have no competing interests.

References

1. T. Aoki, On the stability of the linear transformation in Banach spaces, *J. Math. Soc. Japan,* **2** (1950), 64–66.
2. W. Fechner, Stability of a functional inequality associated with the Jordan-von Neumann functional equation, *Aequationes Math.,* **71** (2006), 149–161.
3. P. Găvruţă, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.,* **184** (1994), 431–436.
4. S. Gähler, 2-Metrische Räume und ihere topologische struktur, *Math. Nachr.,* **26** (1963), 115–148.
5. S. Gähler, Lineare 2-normierte Räume, *Math. Nachr.,* **28** (1964), 1–43.
6. S. Gähler, Über 2-Banach Räume, *Math. Nachr.,* **42** (1969), 335–347.
7. A. Gilányi, Eine zur Parallelogrammgleichung äquivalente Ungleichung, *Aequationes Math.,* **62** (2001), 303–309.
8. A. Gilányi, On a problem by K. Nikodem, *Math. Inequal. Appl.,* **5** (2002), 707–710.
9. M. Eshaghi Gordji, M. B. Savadkouhi, Stability of a mixed type cubic-quartic functional equation in non-Archimedean spaces, *Appl. Math. Lett.,* **23** (2010), 1198–1202.
10. D. H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci. U.S.A.,* **27** (1941), 222–224.

11. M. Kim, M. Kim, Y. Kim, S. Lee, C. Park, 3-Variable additive ρ-functional inequalities and equations, *Res. Math.,* **66** (2014), 159–179.
12. C. Kim, S. Park, The generalized Hyers-Ulam stability of additive functional inequalities in non-Archimedean 2-normed space, *Korean J. Math.,* **22** (2014), 339–348.
13. M. S. Moslehian, Th. M. Rassias, Stability of functional equations in non-Archimedean spaces, *Appl. Anal. Discrete Math.,* **1** (2007), 325–334.
14. C. Park, Additive ρ-functional inequalities, *J. Nonlinear Sci. Appl.,* **7** (2014), 296–310.
15. C. Park, C^*-Ternary biderivations and C^*-ternary bihomomorphisms, *Math.,* **6** (2018), Article No. 30.
16. C. Park, Bi-additive s-functional inequalities and quasi-$*$-multipliers on Banach algebars, *Math.,* **6** (2018), Article No. 31.
17. C. Park, Y. Cho, M. Han, Functional inequalities associated with Jordan-von Neumann-type additive functional equations, *J. Inequal. Appl.,* **2007** (2007), Article ID 41820.
18. C. Park, M. Eshaghi Gordji, M. B. Ghaemi, H. Majani, Fixed points and approximately octic mappings in non-Archimedean 2-normed spaces, *J. Inequal. Appl.,* **2012** (2012), Paper No. 289.
19. W. Prager, J. Schwaiger, A system of two inhomogeneous linear functional equations, *Acta Math. Hungar.,* **140** (2013), 377–406.
20. Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Am. Math. Soc.,* **72** (1978), 297–300.
21. J. Rätz, On inequalities associated with the Jordan-von Neumann functional equation, *Aequationes Math.,* **66** (2003), 191–200.
22. K. Tamilvanan, J. Lee, C. Park, Hyers-Ulam stability of a finite variable mixed type quadratic-additive functional equation in quasi-Banach spaces, *AIMS Math.,* **5** (2020), 5993–6005.
23. S. M. Ulam, *Problems in Modern Mathematics, Chapter VI*, Science Editions, Wiley, New York, 1964.
24. A. White, 2-Banach spaces, *Doctorial Diss.*, St. Louis Univ., 1968.
25. A. White, 2-Banach spaces, *Math. Nachr.,* **42** (1969), 43–60.

AIMS Press