AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(2): 1889–1904.
DOI:10.3934/math.2021115
Received: 21 October 2020
Accepted: 27 November 2020
Published: 03 December 2020

Research article

Fractional inequalities of the Hermite–Hadamard type for m-polynomial convex and harmonically convex functions

Eze R. Nwaeze[1], Muhammad Adil Khan[2], Ali Ahmadian[3,*], Mohammad Nazir Ahmad[3] and Ahmad Kamil Mahmood[4]

[1] Department of Mathematics and Computer Science, Alabama State University, Montgomery, AL 36101, USA

[2] Department of Mathematics, University of Peshawar, Peshawar, Pakistan

[3] Institute of IR 4.0, The National University of Malaysia, 43600 Bangi, Selangor, Malaysia

[4] High Performance Computing Centre, CISD, Universiti Teknologi Petronas, Seri Iskandar, Perak, Malaysia

* **Correspondence:** Email: ahmadian.hosseini@gmail.com.

Abstract: In this paper, it is our purpose to establish some new fractional inequalities of the Hermite–Hadamard type for the m-polynomial convex and harmonically convex functions. Our results involve the Caputo–Fabrizio and ζ-Riemann–Liouville fractional integral operators. They generalize, complement and extend existing results in the literature. By taking $m \geq 2$, we deduce loads of new and interesting inequalities. We expect that the thought laid out in this work will provoke advance examinations in this course.

Keywords: Hermite–Hadamard; m-polynomial convex; m-polynomial harmonically convex; Riemann–Liouville; Caputo–Fabrizio
Mathematics Subject Classification: 26D15, 26A51, 26D10

1. Introduction

The sets $\mathbf{T}$ and $\mathbf{S} \subseteq \mathbb{R} \setminus \{0\}$ are called convex and harmonically convex, respectively if

$$\begin{cases} \varsigma q + (1-\varsigma)z \in \mathbf{T} & \text{for all} \quad q, z \in \mathbf{T} \quad \text{and} \quad \varsigma \in [0,1]; \\ \frac{qz}{\varsigma q + (1-\varsigma)z} \in \mathbf{S} & \text{for all} \quad q, z \in \mathbf{S} \quad \text{and} \quad \varsigma \in [0,1]. \end{cases}$$

Whenever used, we shall always consider $\mathbf{T}$ as a convex set and $\mathbf{S}$ as a harmonically convex set. Let $m \in \mathbb{N}$. Recall that a function $\varphi : \mathbf{T} \to \mathbb{R}$ is said to be m-polynomial convex [31] on $\mathbf{T}$ if

$$\varphi(\varsigma q + (1-\varsigma)z) \leq \frac{1}{m}\sum_{\vartheta=1}^{m}\left[1-(1-\varsigma)^{\vartheta}\right]\varphi(q) + \frac{1}{m}\sum_{\vartheta=1}^{m}\left[1-\varsigma^{\vartheta}\right]\varphi(z)$$

for all $q, z \in \mathbf{S}$ and $\varsigma \in [0,1]$. For this class of functions, Toplu *et al*. established the following double inequality of the Hermite–Hadamard type.

Theorem 1 ([31])**.** *Let $\varphi : \mathbf{T} \to \mathbb{R}$ be an m-polynomial convex function. If $\xi, \delta \in \mathbf{T}$ with $\xi < \delta$, and φ is Lebesgue integrable on $[\xi, \delta]$, then the following Hermite–Hadamard type inequality holds:*

$$\frac{2^{-1}m}{m+2^{-m}-1}\varphi\left(\frac{\xi+\delta}{2}\right) \leq \frac{1}{\delta-\xi}\int_{\xi}^{\delta}\varphi(r)\,dr \leq \frac{\varphi(\xi)+\varphi(\delta)}{m}\sum_{\vartheta=1}^{m}\frac{\vartheta}{\vartheta+1}. \tag{1.1}$$

The inequality (1.1) boils down to the classical Hermite–Hadamard inequality for convex functions if we take $m = 1$. Recently, Awan *et al*. [2] introduced the notion of m-polynomial harmonically convex functions as follows: a real valued function $\varphi : \mathbf{S} \to \mathbb{R}^{+} := [0, \infty)$ is m-harmonically convex if

$$\varphi\left(\frac{qz}{\varsigma q+(1-\varsigma)z}\right) \leq \frac{1}{m}\sum_{\vartheta=1}^{m}\left[1-(1-\varsigma)^{\vartheta}\right]\varphi(q) + \frac{1}{m}\sum_{\vartheta=1}^{m}\left[1-\varsigma^{\vartheta}\right]\varphi(z) \tag{1.2}$$

for all $q, z \in \mathbf{S}$ and $\varsigma \in [0,1]$. In the same paper, the authors established the following Hermite–Hadamard type inequality for this class of functions:

Theorem 2 ([2])**.** *Let $\varphi : \mathbf{S} \to \mathbb{R}^{+}$ be an m-polynomial harmonically convex function. If $\xi, \delta \in \mathbf{S}$ with $0 < \xi < \delta$, and φ is Lebesgue integrable on $[\xi, \delta]$, then the following Hermite–Hadamard type inequality holds:*

$$\frac{2^{-1}m}{m+2^{-m}-1}\varphi\left(\frac{2\xi\delta}{\xi+\delta}\right) \leq \frac{\xi\delta}{\delta-\xi}\int_{\xi}^{\delta}\frac{\varphi(r)}{r^2}\,dr \leq \frac{\varphi(\xi)+\varphi(\delta)}{m}\sum_{\vartheta=1}^{m}\frac{\vartheta}{\vartheta+1}.$$

In the sequel, we will denote the sets of all m-polynomial convex and m-polynomial harmonically convex functions from A into B by $\mathbf{XP}_m(A,B)$ and $\mathbf{HXP}_m(A,B)$, respectively. The classical Hermite–Hadamard inequality has generated load of generalizations and extensions to other class of convexity. There are dozens of articles in this direction. We invite the interested reader to see the following articles [3–6, 8, 10–20, 22–30, 32–34] and the references cited therein.

Now, recall that the left- and right-sided ζ-Riemann–Liouville fractional integral operators ${}_{\zeta}\mathcal{J}^{\epsilon}_{\xi^{+}}$ and ${}_{\zeta}\mathcal{J}^{\epsilon}_{\delta^{-}}$ of order $\epsilon > 0$, for a real valued continuous function $\varphi(r)$, are defined as ([21]):

$${}_{\zeta}\mathcal{J}^{\epsilon}_{\xi^{+}}\varphi(r) = \frac{1}{\zeta\Gamma_{\zeta}(\epsilon)}\int_{\xi}^{r}(r-\varsigma)^{\frac{\epsilon}{\zeta}-1}\varphi(\varsigma)\,d\varsigma, \quad r > \xi,$$

and

$${}_{\zeta}\mathcal{J}^{\epsilon}_{\delta^{-}}\varphi(r) = \frac{1}{\zeta\Gamma_{\zeta}(\epsilon)}\int_{r}^{\delta}(\varsigma-r)^{\frac{\epsilon}{\zeta}-1}\varphi(\varsigma)\,d\varsigma, \quad r < \delta,$$

where $\zeta > 0$, and Γ_ζ is the ζ-gamma function given by

$$\Gamma_\zeta(r) := \int_0^\infty \varsigma^{r-1} e^{-\frac{\varsigma^\zeta}{\zeta}}\, d\varsigma, \quad Re(r) > 0,$$

with the properties $\Gamma_\zeta(r+\zeta) = r\Gamma_\zeta(r)$ and $\Gamma_\zeta(\zeta) = 1$. If $\zeta = 1$, we simply write

$${}_1\mathcal{J}^{\epsilon}_{\xi^+}\varphi = \mathcal{J}^{\epsilon}_{\xi^+}\varphi \quad \text{and} \quad {}_1\mathcal{J}^{\epsilon}_{\delta^-}\varphi = \mathcal{J}^{\epsilon}_{\delta^-}\varphi.$$

The beta function $\mathcal{B}$ is defined by

$$\mathcal{B}(u,v) = \int_0^1 \varsigma^{u-1}(1-\varsigma)^{v-1}\, d\varsigma \qquad \text{for} \quad Re(u) > 0, Re(v) > 0. \tag{1.3}$$

Another fractional integral operators of interest is the Caputo–Fabrizio operators [1]: let $L^2(\xi,\delta)$ be the space of square integrable functions on the interval (ξ,δ) and

$$H^1(\xi,\delta) := \Big\{ g \,|\, g \in L^2(\xi,\delta) \quad \text{and} \quad g' \in L^2(\xi,\delta) \Big\}.$$

If $\varphi \in H^1(\xi,\delta)$, $\xi < \delta$ and $\mu \in [0,1]$, then the left- and right-sided Caputo–Fabrizio fractional integral operators ${}^{cf}\mathcal{I}^{\mu}_{\xi}$ and ${}^{cf}\mathcal{I}^{\mu}_{\delta}$ are defined by

$${}^{cf}\mathcal{I}^{\mu}_{\xi}\varphi(s) = \frac{1-\mu}{B(\mu)}\varphi(s) + \frac{\mu}{B(\mu)}\int_\xi^s \varphi(r)\, dr \tag{1.4}$$

and

$${}^{cf}\mathcal{I}^{\mu}_{\delta}\varphi(s) = \frac{1-\mu}{B(\mu)}\varphi(s) + \frac{\mu}{B(\mu)}\int_s^\delta \varphi(r)\, dr, \tag{1.5}$$

where $B : [0,1] \to (0,\infty)$ is a normalization function satisfying $B(0) = B(1) = 1$.

Using these fractional integral operators in (1.4) and (1.5), Gürbüz *et al*. established the following fractional version of the Hermite–Hadamard inequality:

Theorem 3 ([7]). *Let $\varphi : \mathbf{T} \to \mathbb{R}$ be a convex function on $\mathbf{T}$. If $\xi, \delta \in \mathbf{T}$ with $\xi < \delta$, and φ is Lebesgue integrable on $[\xi,\delta]$, then the following double inequality holds:*

$$\varphi\left(\frac{\xi+\delta}{2}\right) \le \frac{B(\mu)}{\mu(\delta-\xi)}\left[{}^{cf}\mathcal{I}^{\mu}_{\xi}\varphi(s) + {}^{cf}\mathcal{I}^{\mu}_{\delta}\varphi(s) - \frac{2(1-\mu)}{B(\mu)}\varphi(s)\right] \le \frac{\varphi(\xi)+\varphi(\delta)}{2},$$

where $\mu \in [0,1]$, $s \in [\xi,\delta]$ and $B(\mu) > 0$ is a normalization function.

Since the classes of convexity introduced here are new, much work have not been done in this sense. This work is geared towards further development around inequalities for these classes. In view of this, we aim to achieve the following objectives:

1. To establish new Hermite–Hadamard type inequalities for the class of m-polynomial convex functions involving the Caputo–Fabrizio integral operators. Our first result in this direction generalizes and extends Theorem 3.
2. To obtain inequalities of the Hermite–Hadamard type for functions that are m-polynomial harmonically convex functions via the ζ-Riemann–Liouville fractional integral operators. This, in turn, also complement and generalize some existing results in the literature.

2. Main results

2.1. Inequalities for m-polynomial convex functions

Inequalities of the Hermite–Hadamard type, for m-polynomial convex functions, are hereby presented. The results, presented herein, involve the Caputo–Fabrizio operators.

Theorem 4. *Let $\varphi : \mathbf{T} \to \mathbb{R}$ be a Lebesgue integrable function on $[\xi, \delta]$ with $\xi < \delta$ and $\xi, \delta \in \mathbf{T}$. If $\varphi \in \mathbf{XP}_m(\mathbf{T}, \mathbb{R})$, then*

$$\frac{2^{-1}m}{m + 2^{-m} - 1}\varphi\left(\frac{\xi+\delta}{2}\right) \leq \frac{B(\mu)}{\mu(\delta-\xi)}\left[{}^{cf}\mathcal{I}^{\mu}_{\xi}\varphi(s) + {}^{cf}\mathcal{I}^{\mu}_{\delta}\varphi(s) - \frac{2(1-\mu)}{B(\mu)}\varphi(s)\right] \leq \frac{\varphi(\xi)+\varphi(\delta)}{m}\sum_{\vartheta=1}^{m}\frac{\vartheta}{\vartheta+1},$$

where $\mu \in (0, 1]$, $s \in [\xi, \delta]$ and $B(\mu) > 0$ is a normalization function.

Proof. Given that $\varphi \in \mathbf{XP}_m(\mathbf{T}, \mathbb{R})$, it follows from (1.1) that

$$\begin{aligned}\frac{m}{m + 2^{-m} - 1}\varphi\left(\frac{\xi+\delta}{2}\right) &\leq \frac{2}{\delta-\xi}\int_{\xi}^{\delta}\varphi(r)\,dr \\ &= \frac{2}{\delta-\xi}\left[\int_{\xi}^{s}\varphi(r)\,dr + \int_{s}^{\delta}\varphi(r)\,dr\right].\end{aligned} \tag{2.1}$$

Multiplying both sides of (2.1) by $\frac{\mu(\delta-\xi)}{2B(\mu)}$ gives:

$$\frac{\mu(\delta-\xi)}{2B(\mu)}\frac{m}{m + 2^{-m} - 1}\varphi\left(\frac{\xi+\delta}{2}\right) \leq \frac{\mu}{B(\mu)}\left[\int_{\xi}^{s}\varphi(r)\,dr + \int_{s}^{\delta}\varphi(r)\,dr\right]. \tag{2.2}$$

By adding $\frac{2(1-\mu)}{B(\mu)}\varphi(s)$ to both sides of (2.2), we get:

$$\begin{aligned}&\frac{2(1-\mu)}{B(\mu)}\varphi(s) + \frac{\mu(\delta-\xi)}{2B(\mu)}\frac{m}{m + 2^{-m} - 1}\varphi\left(\frac{\xi+\delta}{2}\right) \\ &\leq \frac{2(1-\mu)}{B(\mu)}\varphi(s) + \frac{\mu}{B(\mu)}\left[\int_{\xi}^{s}\varphi(r)\,dr + \int_{s}^{\delta}\varphi(r)\,dr\right] \\ &= \left[\frac{(1-\mu)}{B(\mu)}\varphi(s) + \frac{\mu}{B(\mu)}\int_{\xi}^{s}\varphi(r)\,dr\right] \\ &\qquad + \left[\frac{(1-\mu)}{B(\mu)}\varphi(s) + \frac{\mu}{B(\mu)}\int_{s}^{\delta}\varphi(r)\,dr\right] \\ &= {}^{cf}\mathcal{I}^{\mu}_{\xi}\varphi(s) + {}^{cf}\mathcal{I}^{\mu}_{\delta}\varphi(s).\end{aligned}$$

This implies that

$$\begin{aligned}&\frac{2(1-\mu)}{B(\mu)}\varphi(s) + \frac{\mu(\delta-\xi)}{2B(\mu)}\frac{m}{m + 2^{-m} - 1}\varphi\left(\frac{\xi+\delta}{2}\right) \\ &\leq {}^{cf}\mathcal{I}^{\mu}_{\xi}\varphi(s) + {}^{cf}\mathcal{I}^{\mu}_{\delta}\varphi(s).\end{aligned} \tag{2.3}$$

On the other hand, we also get from (1.1) the following inequality:

$$\frac{2}{\delta-\xi}\int_{\xi}^{\delta}\varphi(r)\,dr \leq \frac{\varphi(\xi)+\varphi(\delta)}{m}\sum_{\vartheta=1}^{m}\frac{2\vartheta}{\vartheta+1}. \tag{2.4}$$

If we multiply (2.4) by $\frac{\mu(\delta-\xi)}{2B(\mu)}$ and then add $\frac{2(1-\mu)}{B(\mu)}\varphi(s)$ to the resulting inequality, we obtain:

$${}^{cf}\mathcal{I}^{\mu}_{\xi}\varphi(s)+{}^{cf}\mathcal{I}^{\mu}_{\delta}\varphi(s) \leq \frac{\mu(\delta-\xi)}{B(\mu)}\frac{\varphi(\xi)+\varphi(\delta)}{m}\sum_{\vartheta=1}^{m}\frac{\vartheta}{\vartheta+1}+\frac{2(1-\mu)}{B(\mu)}\varphi(s). \tag{2.5}$$

Hence, the desired result is obtained by combining (2.3) and (2.5). □

Remark 1. By taking $m = 1$, Theorem 4 becomes Theorem 3.

Theorem 5. *Let $\varphi, \upsilon : \mathbf{T} \to \mathbb{R}$ be two functions such that $\varphi\upsilon$ is Lebesgue integrable function on $[\xi, \delta]$ with $\xi < \delta$ and $\xi, \delta \in \mathbf{T}$. If $\varphi \in \mathbf{XP}_{m_1}(\mathbf{S}, \mathbb{R})$, $\upsilon \in \mathbf{XP}_{m_2}(\mathbf{T}, \mathbb{R})$, then*

$$\begin{aligned}&\frac{B(\mu)}{\mu(\delta-\xi)}\left[{}^{cf}\mathcal{I}^{\mu}_{\xi}\varphi(s)\upsilon(s)+{}^{cf}\mathcal{I}^{\mu}_{\delta}\varphi(s)\upsilon(s)-\frac{2(1-\mu)}{B(\mu)}\varphi(s)\upsilon(s)\right]\\ &\leq \int_0^1\Big[\Delta_1(\varsigma)\varphi(\xi)\upsilon(\xi)+\Delta_2(\varsigma)\varphi(\xi)\upsilon(\delta)+\Delta_3(\varsigma)\varphi(\delta)\upsilon(\xi)+\Delta_4(\varsigma)\varphi(\delta)\upsilon(\delta)\Big]\,d\varsigma,\end{aligned}$$

where $\mu \in (0, 1]$, $s \in [\xi, \delta]$ and $B(\mu) > 0$ is a normalization function, and

$$\begin{aligned}\Delta_1(\varsigma) &:= \frac{1}{m_1}\frac{1}{m_2}\sum_{\vartheta=1}^{m_1}\Big[1-(1-\varsigma)^{\vartheta}\Big]\sum_{\vartheta=1}^{m_2}\Big[1-(1-\varsigma)^{\vartheta}\Big];\\ \Delta_2(\varsigma) &:= \frac{1}{m_1}\frac{1}{m_2}\sum_{\vartheta=1}^{m_1}\Big[1-(1-\varsigma)^{\vartheta}\Big]\sum_{\vartheta=1}^{m_2}\Big[1-\varsigma^{\vartheta}\Big];\\ \Delta_3(\varsigma) &:= \frac{1}{m_1}\frac{1}{m_2}\sum_{\vartheta=1}^{m_1}\Big[1-\varsigma^{\vartheta}\Big]\sum_{\vartheta=1}^{m_2}\Big[1-(1-\varsigma)^{\vartheta}\Big];\\ \Delta_4(\varsigma) &:= \frac{1}{m_1}\frac{1}{m_2}\sum_{\vartheta=1}^{m_1}\Big[1-\varsigma^{\vartheta}\Big]\sum_{\vartheta=1}^{m_2}\Big[1-\varsigma^{\vartheta}\Big].\end{aligned}$$

Proof. Let $\varphi \in \mathbf{XP}_{m_1}(\mathbf{T}, \mathbb{R})$ and $\upsilon \in \mathbf{XP}_{m_2}(\mathbf{T}, \mathbb{R})$. Then for $\varsigma \in [0, 1]$, we have:

$$\varphi(\varsigma\xi+(1-\varsigma)\delta) \leq \frac{1}{m_1}\sum_{\vartheta=1}^{m_1}\Big[1-(1-\varsigma)^{\vartheta}\Big]\varphi(\xi)+\frac{1}{m_1}\sum_{\vartheta=1}^{m_1}\Big[1-\varsigma^{\vartheta}\Big]\varphi(\delta) \tag{2.6}$$

and

$$\upsilon(\varsigma\xi+(1-\varsigma)\delta) \leq \frac{1}{m_1}\sum_{\vartheta=1}^{m_1}\Big[1-(1-\varsigma)^{\vartheta}\Big]\upsilon(\xi)+\frac{1}{m_1}\sum_{\vartheta=1}^{m_1}\Big[1-\varsigma^{\vartheta}\Big]\upsilon(\delta). \tag{2.7}$$

Multiplying (2.6) and (2.7) gives:

$$\begin{aligned}
&\varphi(\varsigma\xi + (1-\varsigma)\delta)\,\upsilon(\varsigma\xi + (1-\varsigma)\delta)\\
&\leq \frac{1}{m_1}\frac{1}{m_2}\sum_{\vartheta=1}^{m_1}\left[1-(1-\varsigma)^{\vartheta}\right]\sum_{\vartheta=1}^{m_2}\left[1-(1-\varsigma)^{\vartheta}\right]\varphi(\xi)\upsilon(\xi)\\
&\quad+\frac{1}{m_1}\frac{1}{m_2}\sum_{\vartheta=1}^{m_1}\left[1-(1-\varsigma)^{\vartheta}\right]\sum_{\vartheta=1}^{m_2}\left[1-\varsigma^{\vartheta}\right]\varphi(\xi)\upsilon(\delta)\\
&\quad+\frac{1}{m_1}\frac{1}{m_2}\sum_{\vartheta=1}^{m_1}\left[1-\varsigma^{\vartheta}\right]\sum_{\vartheta=1}^{m_2}\left[1-(1-\varsigma)^{\vartheta}\right]\varphi(\delta)\upsilon(\xi)\\
&\quad+\frac{1}{m_1}\frac{1}{m_2}\sum_{\vartheta=1}^{m_1}\left[1-\varsigma^{\vartheta}\right]\sum_{\vartheta=1}^{m_2}\left[1-\varsigma^{\vartheta}\right]\varphi(\delta)w(\delta)\\
&:= \Delta_1(\varsigma)\varphi(\xi)\upsilon(\xi)+\Delta_2(\varsigma)\varphi(\xi)\upsilon(\delta)+\Delta_3(\varsigma)\varphi(\delta)\upsilon(\xi)+\Delta_4(\varsigma)\varphi(\delta)\upsilon(\delta).
\end{aligned}$$

This implies that

$$\begin{aligned}
&\varphi(\varsigma\xi + (1-\varsigma)\delta)\,\upsilon(\varsigma\xi + (1-\varsigma)\delta)\\
&\leq \Delta_1(\varsigma)\varphi(\xi)\upsilon(\xi)+\Delta_2(\varsigma)\varphi(\xi)\upsilon(\delta)+\Delta_3(\varsigma)\varphi(\delta)\upsilon(\xi)+\Delta_4(\varsigma)\varphi(\delta)\upsilon(\delta).
\end{aligned}\tag{2.8}$$

Integrating both sides of (2.8) with respect to ς over $[0,1]$ results to:

$$\begin{aligned}
&\frac{2}{\delta-\xi}\int_{\xi}^{\delta}\varphi(r)\upsilon(r)\,dr\\
&\leq 2\int_0^1\left[\Delta_1(\varsigma)\varphi(\xi)\upsilon(\xi)+\Delta_2(\varsigma)\varphi(\xi)\upsilon(\delta)+\Delta_3(\varsigma)\varphi(\delta)\upsilon(\xi)+\Delta_4(\varsigma)\varphi(\delta)\upsilon(\delta)\right]d\varsigma\\
&:= \mathcal{N}(\xi,\delta).
\end{aligned}$$

That is,

$$\begin{aligned}
&\frac{2}{\delta-\xi}\left[\int_{\xi}^{s}\varphi(r)\upsilon(r)\,dr+\int_{s}^{\delta}\varphi(r)\upsilon(r)\,dr\right]\\
&\leq \mathcal{N}(\xi,\delta).
\end{aligned}\tag{2.9}$$

Now, multiplying (2.9) by $\frac{\mu(\delta-\xi)}{2B(\mu)}$ and then adding $\frac{2(1-\mu)}{B(\mu)}\varphi(s)\upsilon(s)$ to the result to obtain:

$$\begin{aligned}
&\frac{\mu}{B(\mu)}\left[\int_{\xi}^{s}\varphi(r)\upsilon(r)\,dr+\int_{s}^{\delta}\varphi(r)\upsilon(r)\,dr\right]+\frac{2(1-\mu)}{B(\mu)}\varphi(s)\upsilon(s)\\
&\leq \frac{\mu(\delta-\xi)}{2B(\mu)}\mathcal{N}(\xi,\delta)+\frac{2(1-\mu)}{B(\mu)}\varphi(s)\upsilon(s).
\end{aligned}$$

Hence,

$$\begin{aligned}
&{}^{cf}\mathcal{I}^{\mu}_{\xi}\varphi(s)\upsilon(s)+{}^{cf}\mathcal{I}^{\mu}_{\delta}\varphi(s)\upsilon(s)\\
&\leq \frac{\mu(\delta-\xi)}{2B(\mu)}\mathcal{N}(\xi,\delta)+\frac{2(1-\mu)}{B(\mu)}\varphi(s)\upsilon(s),
\end{aligned}$$

from which we get the intended inequality. □

Remark 2. Set $m_1 = m_2 = 1$ in Theorem 5. Then we recover [7, Theorem 3].

2.2. Inequalities for m-polynomial harmonically convex functions

In this subsection, we present some new Hermite–Hadamard type results involving the ζ-Riemann–Liouville fractional integral operators.

Theorem 6. *Let $\varphi : \mathbf{S} \to \mathbb{R}^+$ be a Lebesgue integrable function on $[\xi, \delta]$ with $0 < \xi < \delta$ and $\xi, \delta \in \mathbf{S}$. If $\varphi \in \mathbf{HXP}_m(\mathbf{S}, \mathbb{R}^+)$ and $\zeta, \epsilon > 0$, then*

$$\begin{aligned}
&\frac{1}{m + 2^{-m} - 1}\, \varphi\left(\frac{2\xi\delta}{\xi + \delta}\right) \\
&\leq \frac{\Gamma_\zeta(\epsilon + \zeta)}{m}\left(\frac{\xi\delta}{\delta - \xi}\right)^{\frac{\epsilon}{\zeta}}\left[{}_\zeta\mathcal{J}^{\epsilon}_{\frac{1}{\delta}^+}(\varphi \circ \tilde{\varphi})\left(\frac{1}{\xi}\right) + {}_\zeta\,\mathcal{J}^{\epsilon}_{\frac{1}{\xi}^-}(\varphi \circ \tilde{\varphi})\left(\frac{1}{\delta}\right)\right] \\
&\leq \frac{\varphi(\xi) + \varphi(\delta)}{m^2}\sum_{\vartheta=1}^{m}\left[2 - \frac{\epsilon}{\epsilon + \zeta\vartheta} - \frac{\epsilon}{\zeta}\mathcal{B}\left(\frac{\epsilon}{\zeta},\, \vartheta + 1\right)\right],
\end{aligned}$$

where $\tilde{\varphi}(r) = \frac{1}{r}$ and $\mathcal{B}$ is the beta function defined by (1.3).

Proof. Given that $\varphi \in \mathbf{HXP}_m(\mathbf{S}, \mathbb{R}^+)$, we get the following relation:

$$\varphi\left(\frac{qz}{\frac{1}{2}q + \frac{1}{2}z}\right) \leq \frac{1}{m}\sum_{\vartheta=1}^{m}\left[1 - \frac{1}{2^\vartheta}\right]\varphi(q) + \frac{1}{m}\sum_{\vartheta=1}^{m}\left[1 - \frac{1}{2^\vartheta}\right]\varphi(z).$$

This implies that for all $q, z \in \mathbf{S}$:

$$\varphi\left(\frac{2qz}{q + z}\right) \leq \frac{1}{m}\sum_{\vartheta=1}^{m}\left[1 - \frac{1}{2^\vartheta}\right]\Big(\varphi(q) + \varphi(z)\Big). \tag{2.10}$$

Now, let $q = \frac{\xi\delta}{\varsigma\xi + (1-\varsigma)\delta}$ and $z = \frac{\xi\delta}{\varsigma\delta + (1-\varsigma)\xi}$. Then (2.10) becomes:

$$\varphi\left(\frac{2\xi\delta}{\xi + \delta}\right) \leq \frac{1}{m}\sum_{\vartheta=1}^{m}\left(1 - \frac{1}{2^\vartheta}\right)\left\{\varphi\left(\frac{\xi\delta}{\varsigma\xi + (1-\varsigma)\delta}\right) + \varphi\left(\frac{\xi\delta}{\varsigma\delta + (1-\varsigma)\xi}\right)\right\}. \tag{2.11}$$

Multiplying both sides of (2.11) by $\varsigma^{\frac{\epsilon}{\zeta}-1}$ and integrating with respect to ς over $[0,1]$, we get:

$$\begin{aligned}
&\int_0^1 \varsigma^{\frac{\epsilon}{\zeta}-1}\varphi\left(\frac{2\xi\delta}{\xi+\delta}\right)d\varsigma\\
&\leq \frac{1}{m}\sum_{\vartheta=1}^{m}\left(1-\frac{1}{2^{\vartheta}}\right)\int_0^1 \varsigma^{\frac{\epsilon}{\zeta}-1}\left\{\varphi\left(\frac{\xi\delta}{\varsigma\xi+(1-\varsigma)\delta}\right)+\varphi\left(\frac{\xi\delta}{\varsigma\delta+(1-\varsigma)\xi}\right)\right\}d\varsigma\\
&= \frac{1}{m}\sum_{\vartheta=1}^{m}\left(1-\frac{1}{2^{\vartheta}}\right)\left[\int_0^1 \varsigma^{\frac{\epsilon}{\zeta}-1}\varphi\left(\frac{\xi\delta}{\varsigma\xi+(1-\varsigma)\delta}\right)d\varsigma\right.\\
&\qquad\qquad\left.+\int_0^1 \varsigma^{\frac{\epsilon}{\zeta}-1}\varphi\left(\frac{\xi\delta}{\varsigma\delta+(1-\varsigma)\xi}\right)d\varsigma\right]\\
&= \frac{1}{m}\sum_{\vartheta=1}^{m}\left(1-\frac{1}{2^{\vartheta}}\right)\left[\left(\frac{\xi\delta}{\delta-\xi}\right)^{\frac{\epsilon}{\zeta}}\int_{\frac{1}{\delta}}^{\frac{1}{\xi}}\left(\frac{1}{\xi}-r\right)^{\frac{\epsilon}{\zeta}-1}\varphi\left(\frac{1}{r}\right)dr\right.\\
&\qquad\qquad\left.+\left(\frac{\xi\delta}{\delta-\xi}\right)^{\frac{\epsilon}{\zeta}}\int_{\frac{1}{\delta}}^{\frac{1}{\xi}}\left(r-\frac{1}{\delta}\right)^{\frac{\epsilon}{\zeta}-1}\varphi\left(\frac{1}{r}\right)dr\right]\\
&= \frac{\zeta\Gamma_{\zeta}(\epsilon)}{m}\sum_{\vartheta=1}^{m}\left(1-\frac{1}{2^{\vartheta}}\right)\left(\frac{\xi\delta}{\delta-\xi}\right)^{\frac{\epsilon}{\zeta}}\left[\frac{1}{\zeta\Gamma_{\zeta}(\epsilon)}\int_{\frac{1}{\delta}}^{\frac{1}{\xi}}\left(\frac{1}{\xi}-r\right)^{\frac{\epsilon}{\zeta}-1}\varphi\left(\frac{1}{r}\right)dr\right.\\
&\qquad\qquad\left.+\frac{1}{\zeta\Gamma_{\zeta}(\epsilon)}\int_{\frac{1}{\delta}}^{\frac{1}{\xi}}\left(r-\frac{1}{\delta}\right)^{\frac{\epsilon}{\zeta}-1}\varphi\left(\frac{1}{r}\right)dr\right]\\
&= \frac{\zeta\Gamma_{\zeta}(\epsilon)}{m}\sum_{\vartheta=1}^{m}\left(1-\frac{1}{2^{\vartheta}}\right)\left(\frac{\xi\delta}{\delta-\xi}\right)^{\frac{\epsilon}{\zeta}}\left[{}_{\zeta}\mathcal{J}^{\epsilon}_{\frac{1}{\delta}^{+}}(\varphi\circ\tilde{\varphi})\left(\frac{1}{\xi}\right)+{}_{\zeta}\,\mathcal{J}^{\epsilon}_{\frac{1}{\xi}^{-}}(\varphi\circ\tilde{\varphi})\left(\frac{1}{\delta}\right)\right],
\end{aligned}$$

where $\tilde{\varphi}(r)=\frac{1}{r}$. This implies that

$$\begin{aligned}
&\frac{1}{m+2^{-m}-1}\,\varphi\left(\frac{2\xi\delta}{\xi+\delta}\right)\\
&\leq \frac{\Gamma_{\zeta}(\epsilon+\zeta)}{m}\left(\frac{\xi\delta}{\delta-\xi}\right)^{\frac{\epsilon}{\zeta}}\left[{}_{\zeta}\mathcal{J}^{\epsilon}_{\frac{1}{\delta}^{+}}(\varphi\circ\tilde{\varphi})\left(\frac{1}{\xi}\right)+{}_{\zeta}\,\mathcal{J}^{\epsilon}_{\frac{1}{\xi}^{-}}(\varphi\circ\tilde{\varphi})\left(\frac{1}{\delta}\right)\right].
\end{aligned}\tag{2.12}$$

Next, substituting $q=\xi$ and $z=\delta$ in (1.2) gives

$$\varphi\left(\frac{\xi\delta}{\varsigma\xi+(1-\varsigma)\delta}\right)\leq\frac{1}{m}\sum_{\vartheta=1}^{m}\left[1-(1-\varsigma)^{\vartheta}\right]\varphi(\xi)+\frac{1}{m}\sum_{\vartheta=1}^{m}\left[1-\varsigma^{\vartheta}\right]\varphi(\delta).\tag{2.13}$$

Reversing the role of ξ and δ in (2.13) produces:

$$\varphi\left(\frac{\xi\delta}{\varsigma\delta+(1-\varsigma)\xi}\right)\leq\frac{1}{m}\sum_{\vartheta=1}^{m}\left[1-(1-\varsigma)^{\vartheta}\right]\varphi(\delta)+\frac{1}{m}\sum_{\vartheta=1}^{m}\left[1-\varsigma^{\vartheta}\right]\varphi(\xi).\tag{2.14}$$

If we now add (2.13) and (2.15), multiply the resulting inequality by $\varsigma^{\frac{\epsilon}{\zeta}-1}$ and integrate with respect to

$\varsigma \in [0,1]$, then we obtain:

$$\begin{aligned}
&\int_0^1 \varsigma^{\frac{\epsilon}{\zeta}-1}\left\{\varphi\left(\frac{\xi\delta}{\varsigma\xi+(1-\varsigma)\delta}\right)+\varphi\left(\frac{\xi\delta}{\varsigma\delta+(1-\varsigma)\xi}\right)\right\}d\varsigma \\
&\leq \frac{\varphi(\xi)+\varphi(\delta)}{m}\sum_{\vartheta=1}^{m}\int_0^1\left[2\varsigma^{\frac{\epsilon}{\zeta}-1}-\varsigma^{\frac{\epsilon}{\zeta}-1}(1-\varsigma)^{\vartheta}-\varsigma^{\frac{\epsilon}{\zeta}+\vartheta-1}\right]d\varsigma \\
&\leq \frac{\varphi(\xi)+\varphi(\delta)}{m}\sum_{\vartheta=1}^{m}\left[\frac{2\zeta}{\epsilon}-\frac{\zeta}{\epsilon+\zeta\vartheta}-\mathcal{B}\left(\frac{\epsilon}{\zeta},\,\vartheta+1\right)\right].
\end{aligned} \tag{2.15}$$

From (2.15), we get:

$$\begin{aligned}
&\frac{\Gamma_\zeta(\epsilon+\zeta)}{m}\left(\frac{\xi\delta}{\delta-\xi}\right)^{\frac{\epsilon}{\zeta}}\left[{}_\zeta\mathcal{J}^{\epsilon}_{\frac{1}{\delta}^+}(\varphi\circ\tilde{\varphi})\left(\frac{1}{\xi}\right)+{}_\zeta\,\mathcal{J}^{\epsilon}_{\frac{1}{\xi}^-}(\varphi\circ\tilde{\varphi})\left(\frac{1}{\delta}\right)\right] \\
&\leq \frac{\varphi(\xi)+\varphi(\delta)}{m^2}\sum_{\vartheta=1}^{m}\left[2-\frac{\epsilon}{\epsilon+\zeta\vartheta}-\frac{\epsilon}{\zeta}\mathcal{B}\left(\frac{\epsilon}{\zeta},\,\vartheta+1\right)\right].
\end{aligned} \tag{2.16}$$

Combining (2.12) and (2.16), we get the desired result. □

Remark 3. If we take $\epsilon = \zeta = 1$, then Theorem 6 reduces to Theorem 2. If, on the other hand, we let $m = 1$, then we get from Theorem 6 the following corollary:

Corollary 1. *Let $\varphi : \mathbf{S} \to \mathbb{R}^+$ be a Lebesgue integrable function on $[\xi,\delta]$ with $\xi < \delta$ and $\xi,\delta \in \mathbf{S}$. If φ is harmonically convex and $\zeta,\epsilon > 0$, then*

$$\begin{aligned}
&\varphi\left(\frac{2\xi\delta}{\xi+\delta}\right) \\
&\leq \frac{\Gamma_\zeta(\epsilon+\zeta)}{2}\left(\frac{\xi\delta}{\delta-\xi}\right)^{\frac{\epsilon}{\zeta}}\left[{}_\zeta\mathcal{J}^{\epsilon}_{\frac{1}{\delta}^+}(\varphi\circ\tilde{\varphi})\left(\frac{1}{\xi}\right)+{}_\zeta\,\mathcal{J}^{\epsilon}_{\frac{1}{\xi}^-}(\varphi\circ\tilde{\varphi})\left(\frac{1}{\delta}\right)\right] \\
&\leq \frac{\varphi(\xi)+\varphi(\delta)}{2}.
\end{aligned}$$

Theorem 7. *Let $\varphi,\upsilon : \mathbf{S} \to \mathbb{R}^+$ be two functions such that $\varphi\upsilon$ is Lebesgue integrable function on $[\xi,\delta]$ with $0 < \xi < \delta$ and $\xi,\delta \in \mathbf{S}$. If $\varphi \in \mathbf{HXP}_{m_1}(\mathbf{S},\mathbb{R}^+)$, $\upsilon \in \mathbf{HXP}_{m_2}(\mathbf{S},\mathbb{R}^+)$ and $\zeta,\epsilon > 0$, then*

$$\begin{aligned}
&\left(\frac{\xi\delta}{\delta-\xi}\right)^{\frac{\epsilon}{\zeta}}\left[{}_\zeta\mathcal{J}^{\epsilon}_{\frac{1}{\delta}^+}(\varphi\upsilon\circ\tilde{\varphi})\left(\frac{1}{\xi}\right)+{}_\zeta\,\mathcal{J}^{\epsilon}_{\frac{1}{\xi}^-}(\varphi\upsilon\circ\tilde{\varphi})\left(\frac{1}{\delta}\right)\right] \\
&\leq \frac{\mathcal{D}(\xi,\delta)}{\zeta\Gamma_\zeta(\epsilon)}\int_0^1\varsigma^{\frac{\epsilon}{\zeta}-1}\Big[\Delta_1(\varsigma)+\Delta_4(\varsigma)\Big]d\varsigma+\frac{\mathcal{F}(\xi,\delta)}{\zeta\Gamma_\zeta(\epsilon)}\int_0^1\varsigma^{\frac{\epsilon}{\zeta}-1}\Big[\Delta_2(\varsigma)+\Delta_3(\varsigma)\Big]d\varsigma,
\end{aligned}$$

where $\mathcal{D}(\xi,\delta) := \varphi(\xi)\upsilon(\xi)+\varphi(\delta)\upsilon(\delta)$, $\mathcal{F}(\xi,\delta) := \varphi(\xi)\upsilon(\delta)+\varphi(\delta)\upsilon(\xi)$, $\tilde{\varphi}$ is as defined in Theorem 6, and $\Delta_j(\varsigma)$, $j = \overline{1,4}$, as defined in Theorem 5.

Proof. Given that $\varphi \in \mathbf{HXP}_{m_1}(\mathbf{S},\mathbb{R}^+)$ and $\upsilon \in \mathbf{HXP}_{m_2}(\mathbf{S},\mathbb{R}^+)$, we get:

$$\varphi\left(\frac{\xi\delta}{\varsigma\xi+(1-\varsigma)\delta}\right) \leq \frac{1}{m_1}\sum_{\vartheta=1}^{m_1}\Big[1-(1-\varsigma)^{\vartheta}\Big]\varphi(\xi)+\frac{1}{m_1}\sum_{\vartheta=1}^{m_1}\Big[1-\varsigma^{\vartheta}\Big]\varphi(\delta)$$

and

$$\upsilon\left(\frac{\xi\delta}{\varsigma\xi+(1-\varsigma)\delta}\right)\leq\frac{1}{m_1}\sum_{\vartheta=1}^{m_1}\left[1-(1-\varsigma)^{\vartheta}\right]\upsilon(\xi)+\frac{1}{m_1}\sum_{\vartheta=1}^{m_1}\left[1-\varsigma^{\vartheta}\right]\upsilon(\delta).$$

This implies:

$$\begin{aligned}
&\varphi\left(\frac{\xi\delta}{\varsigma\xi+(1-\varsigma)\delta}\right)\upsilon\left(\frac{\xi\delta}{\varsigma\xi+(1-\varsigma)\delta}\right)\\
&\leq\frac{1}{m_1}\frac{1}{m_2}\sum_{\vartheta=1}^{m_1}\left[1-(1-\varsigma)^{\vartheta}\right]\sum_{\vartheta=1}^{m_2}\left[1-(1-\varsigma)^{\vartheta}\right]\varphi(\xi)\upsilon(\xi)\\
&\quad+\frac{1}{m_1}\frac{1}{m_2}\sum_{\vartheta=1}^{m_1}\left[1-(1-\varsigma)^{\vartheta}\right]\sum_{\vartheta=1}^{m_2}\left[1-\varsigma^{\vartheta}\right]\varphi(\xi)\upsilon(\delta)\\
&\quad+\frac{1}{m_1}\frac{1}{m_2}\sum_{\vartheta=1}^{m_1}\left[1-\varsigma^{\vartheta}\right]\sum_{\vartheta=1}^{m_2}\left[1-(1-\varsigma)^{\vartheta}\right]\varphi(\delta)\upsilon(\xi)\\
&\quad+\frac{1}{m_1}\frac{1}{m_2}\sum_{\vartheta=1}^{m_1}\left[1-\varsigma^{\vartheta}\right]\sum_{\vartheta=1}^{m_2}\left[1-\varsigma^{\vartheta}\right]\varphi(\delta)\upsilon(\delta)\\
&:=\Delta_1(\varsigma)\varphi(\xi)\upsilon(\xi)+\Delta_2(\varsigma)\varphi(\xi)\upsilon(\delta)+\Delta_3(\varsigma)\varphi(\delta)\upsilon(\xi)+\Delta_4(\varsigma)\varphi(\delta)\upsilon(\delta).
\end{aligned}$$

This gives:

$$\begin{aligned}
&\varphi\left(\frac{\xi\delta}{\varsigma\xi+(1-\varsigma)\delta}\right)\upsilon\left(\frac{\xi\delta}{\varsigma\xi+(1-\varsigma)\delta}\right)\\
&\leq\Delta_1(\varsigma)\varphi(\xi)\upsilon(\xi)+\Delta_2(\varsigma)\varphi(\xi)\upsilon(\delta)+\Delta_3(\varsigma)\varphi(\delta)\upsilon(\xi)+\Delta_4(\varsigma)\varphi(\delta)\upsilon(\delta).
\end{aligned}\tag{2.17}$$

Similarly, we also have

$$\begin{aligned}
&\varphi\left(\frac{\xi\delta}{\varsigma\delta+(1-\varsigma)\xi}\right)\upsilon\left(\frac{\xi\delta}{\varsigma\delta+(1-\varsigma)\xi}\right)\\
&\leq\Delta_4(\varsigma)\varphi(\xi)\upsilon(\xi)+\Delta_3(\varsigma)\varphi(\xi)\upsilon(\delta)+\Delta_2(\varsigma)\varphi(\delta)\upsilon(\xi)+\Delta_1(\varsigma)\varphi(\delta)\upsilon(\delta).
\end{aligned}\tag{2.18}$$

Adding (2.17) and (2.18), we get

$$\begin{aligned}
&\varphi\left(\frac{\xi\delta}{\varsigma\xi+(1-\varsigma)\delta}\right)\upsilon\left(\frac{\xi\delta}{\varsigma\xi+(1-\varsigma)\delta}\right)+\varphi\left(\frac{\xi\delta}{\varsigma\delta+(1-\varsigma)\xi}\right)\upsilon\left(\frac{\xi\delta}{\varsigma\delta+(1-\varsigma)\xi}\right)\\
&\leq\Big(\varphi(\xi)\upsilon(\xi)+\varphi(\delta)\upsilon(\delta)\Big)\Big[\Delta_1(\varsigma)+\Delta_4(\varsigma)\Big]\\
&\qquad+\Big(\varphi(\xi)\upsilon(\delta)+\varphi(\delta)\upsilon(\xi)\Big)\Big[\Delta_2(\varsigma)+\Delta_3(\varsigma)\Big].
\end{aligned}\tag{2.19}$$

Now, multiplying both sides of (2.19) by $\varsigma^{\frac{\epsilon}{\zeta}-1}$ and integrating with respect to ς over $[0,1]$, gives:

$$\begin{aligned}
&\zeta\Gamma_{\zeta}(\epsilon)\left(\frac{\xi\delta}{\delta-\xi}\right)^{\frac{\epsilon}{\zeta}}\left[{}_{\zeta}\mathcal{J}^{\epsilon}_{\frac{1}{\delta}^{+}}(\varphi\upsilon\circ\tilde{\varphi})\left(\frac{1}{\xi}\right)+_{\zeta}\mathcal{J}^{\epsilon}_{\frac{1}{\xi}^{-}}(\varphi\upsilon\circ\tilde{\varphi})\left(\frac{1}{\delta}\right)\right]\\
&=\int_0^1\varsigma^{\frac{\epsilon}{\zeta}-1}\varphi\left(\frac{\xi\delta}{\varsigma\xi+(1-\varsigma)\delta}\right)\upsilon\left(\frac{\xi\delta}{\varsigma\xi+(1-\varsigma)\delta}\right)d\varsigma\\
&+\int_0^1\varsigma^{\frac{\epsilon}{\zeta}-1}\varphi\left(\frac{\xi\delta}{\varsigma\delta+(1-\varsigma)\xi}\right)\upsilon\left(\frac{\xi\delta}{\varsigma\delta+(1-\varsigma)\xi}\right)d\varsigma\\
&\leq\Big(\varphi(\xi)\upsilon(\xi)+\varphi(\delta)\upsilon(\delta)\Big)\int_0^1\varsigma^{\frac{\epsilon}{\zeta}-1}\Big[\Delta_1(\varsigma)+\Delta_4(\varsigma)\Big]d\varsigma\\
&\qquad+\Big(\varphi(\xi)\upsilon(\delta)+\varphi(\delta)\upsilon(\xi)\Big)\int_0^1\varsigma^{\frac{\epsilon}{\zeta}-1}\Big[\Delta_2(\varsigma)+\Delta_3(\varsigma)\Big]d\varsigma\\
&:=\mathcal{D}(\xi,\delta)\int_0^1\varsigma^{\frac{\epsilon}{\zeta}-1}\Big[\Delta_1(\varsigma)+\Delta_4(\varsigma)\Big]d\varsigma+\mathcal{F}(\xi,\delta)\int_0^1\varsigma^{\frac{\epsilon}{\zeta}-1}\Big[\Delta_2(\varsigma)+\Delta_3(\varsigma)\Big].
\end{aligned}$$

Hence, this completes the proof.

□

Corollary 2. *Let $\varphi,\upsilon:\mathbf{S}\to\mathbb{R}^{+}$ be two functions such that $\varphi\upsilon$ is Lebesgue integrable function on $[\xi,\delta]$ with $0<\xi<\delta$ and $\xi,\delta\in\mathbf{S}$. If φ and υ are harmonically convex and $\zeta,\epsilon>0$, then*

$$\begin{aligned}
&\left(\frac{\xi\delta}{\delta-\xi}\right)^{\frac{\epsilon}{\zeta}}\left[{}_{\zeta}\mathcal{J}^{\epsilon}_{\frac{1}{\delta}^{+}}(\varphi\upsilon\circ\tilde{\varphi})\left(\frac{1}{\xi}\right)+_{\zeta}\mathcal{J}^{\epsilon}_{\frac{1}{\xi}^{-}}(\varphi\upsilon\circ\tilde{\varphi})\left(\frac{1}{\delta}\right)\right]\\
&\leq\frac{\mathcal{D}(\xi,\delta)}{\Gamma_{\zeta}(\epsilon)}\left[\frac{1}{\epsilon}+\frac{2}{\epsilon+2\zeta}-\frac{2}{\epsilon+\zeta}\right]+\frac{\mathcal{F}(\xi,\delta)}{\Gamma_{\zeta}(\epsilon)}\left[\frac{2}{\epsilon+\zeta}-\frac{2}{\epsilon+2\zeta}\right].
\end{aligned}$$

Proof. Let $m_1=m_2=1$. Then, $\Delta_1(\varsigma)=\varsigma^2$, $\Delta_2(\varsigma)=\Delta_3(\varsigma)=\varsigma-\varsigma^2$ and $\Delta_4(\varsigma)=1-2\varsigma+\varsigma^2$. The intended result follows by using Theorem 7. □

Theorem 8. *Let $\varphi,\upsilon:\mathbf{S}\to\mathbb{R}^{+}$ be two functions such that $\varphi\upsilon$ is Lebesgue integrable function on $[\xi,\delta]$ with $0<\xi<\delta$ and $\xi,\delta\in\mathbf{S}$. If $\varphi\in\mathbf{HXP}_{m_1}(\mathbf{S},\mathbb{R}^{+})$, $\upsilon\in\mathbf{HXP}_{m_2}(\mathbf{S},\mathbb{R}^{+})$ and $\zeta,\epsilon>0$, then*

$$\begin{aligned}
&\frac{m_1m_2}{(m_1+2^{-m_1}-1)(m_2+2^{-m_2}-1)}\varphi\left(\frac{2\xi\delta}{\xi+\delta}\right)\upsilon\left(\frac{2\xi\delta}{\xi+\delta}\right)\\
&\leq\Gamma_{\zeta}(\epsilon+\zeta)\left(\frac{\xi\delta}{\delta-\xi}\right)^{\frac{\epsilon}{\zeta}}\left[{}_{\zeta}\mathcal{J}^{\epsilon}_{\frac{1}{\delta}^{+}}(\varphi\upsilon\circ\tilde{\varphi})\left(\frac{1}{\xi}\right)+_{\zeta}\mathcal{J}^{\epsilon}_{\frac{1}{\xi}^{-}}(\varphi\upsilon\circ\tilde{\varphi})\left(\frac{1}{\delta}\right)\right]\\
&\qquad+\frac{\epsilon}{\zeta}\int_0^1\varsigma^{\frac{\epsilon}{\zeta}-1}\Big\{\Big[\Lambda_{m_1}(\varsigma)\tilde{\Lambda}_{m_2}(\varsigma)+\tilde{\Lambda}_{m_1}(\varsigma)\Lambda_{m_2}(\varsigma)\Big]\mathcal{D}(\xi,\delta)\\
&\qquad\qquad+\Big[\Lambda_{m_1}(\varsigma)\Lambda_{m_2}(\varsigma)+\tilde{\Lambda}_{m_1}(\varsigma)\tilde{\Lambda}_{m_2}(\varsigma)\Big]\mathcal{F}(\xi,\delta)\Big\}\,d\varsigma,
\end{aligned}$$

where $\tilde{\varphi}$ is defined in Theorem 6, $\Lambda_m(\varsigma)=\frac{1}{m}\sum_{\vartheta=1}^{m}\Big[1-(1-\varsigma)^{\vartheta}\Big]$ and $\tilde{\Lambda}_m(\varsigma)=\frac{1}{m}\sum_{\vartheta=1}^{m}\Big[1-\varsigma^{\vartheta}\Big]$.

Proof. We start by noticing that:

$$\tilde{\Lambda}_m\left(\frac{1}{2}\right) = \Lambda_m\left(\frac{1}{2}\right) := E_m := \frac{m + 2^{-m} - 1}{m}.$$

Now, let $\varsigma \in [0, 1]$. Hence, from (2.11), one gets:

$$\varphi\left(\frac{2\xi\delta}{\xi+\delta}\right) \leq E_{m_1}\left\{\varphi\left(\frac{\xi\delta}{\varsigma\xi + (1-\varsigma)\delta}\right) + \varphi\left(\frac{\xi\delta}{\varsigma\delta + (1-\varsigma)\xi}\right)\right\}$$

and

$$\upsilon\left(\frac{2\xi\delta}{\xi+\delta}\right) \leq E_{m_1}\left\{\upsilon\left(\frac{\xi\delta}{\varsigma\xi + (1-\varsigma)\delta}\right) + \upsilon\left(\frac{\xi\delta}{\varsigma\delta + (1-\varsigma)\xi}\right)\right\}.$$

Now,

$$\begin{aligned}
&\varphi\left(\frac{2\xi\delta}{\xi+\delta}\right)\upsilon\left(\frac{2\xi\delta}{\xi+\delta}\right)\\
&\leq E_{m_1}E_{m_2}\left[\varphi\left(\frac{\xi\delta}{\varsigma\xi + (1-\varsigma)\delta}\right)\upsilon\left(\frac{\xi\delta}{\varsigma\xi + (1-\varsigma)\delta}\right)\right.\\
&\quad\left.+\,\varphi\left(\frac{\xi\delta}{\varsigma\delta + (1-\varsigma)\xi}\right)\upsilon\left(\frac{\xi\delta}{\varsigma\delta + (1-\varsigma)\xi}\right)\right]\\
&\quad+ E_{m_1}E_{m_2}\left[\varphi\left(\frac{\xi\delta}{\varsigma\xi + (1-\varsigma)\delta}\right)\upsilon\left(\frac{\xi\delta}{\varsigma\delta + (1-\varsigma)\xi}\right)\right.\\
&\quad\left.+\,\varphi\left(\frac{\xi\delta}{\varsigma\delta + (1-\varsigma)\xi}\right)\upsilon\left(\frac{\xi\delta}{\varsigma\xi + (1-\varsigma)\delta}\right)\right]\\
&\leq E_{m_1}E_{m_2}\left[\varphi\left(\frac{\xi\delta}{\varsigma\xi + (1-\varsigma)\delta}\right)\upsilon\left(\frac{\xi\delta}{\varsigma\xi + (1-\varsigma)\delta}\right)\right.\\
&\quad\left.+\,\varphi\left(\frac{\xi\delta}{\varsigma\delta + (1-\varsigma)\xi}\right)\upsilon\left(\frac{\xi\delta}{\varsigma\delta + (1-\varsigma)\xi}\right)\right]\\
&\quad+ E_{m_1}E_{m_2}\left\{\left[\Lambda_{m_1}(\varsigma)\varphi(\xi) + \tilde{\Lambda}_{m_1}(\varsigma)\varphi(\delta)\right]\left[\Lambda_{m_2}(\varsigma)\upsilon(\delta) + \tilde{\Lambda}_{m_2}(\varsigma)\upsilon(\xi)\right]\right.\\
&\qquad\left.+\left[\Lambda_{m_1}(\varsigma)\varphi(\delta) + \tilde{\Lambda}_{m_1}(\varsigma)\varphi(\xi)\right]\left[\Lambda_{m_2}(\varsigma)\upsilon(\xi) + \tilde{\Lambda}_{m_2}(\varsigma)\upsilon(\delta)\right]\right\}\\
&= E_{m_1}E_{m_2}\left[\varphi\left(\frac{\xi\delta}{\varsigma\xi + (1-\varsigma)\delta}\right)\upsilon\left(\frac{\xi\delta}{\varsigma\xi + (1-\varsigma)\delta}\right)\right.\\
&\quad\left.+\,\varphi\left(\frac{\xi\delta}{\varsigma\delta + (1-\varsigma)\xi}\right)\upsilon\left(\frac{\xi\delta}{\varsigma\delta + (1-\varsigma)\xi}\right)\right]\\
&\quad+ E_{m_1}E_{m_2}\left\{\left[\Lambda_{m_1}(\varsigma)\tilde{\Lambda}_{m_2}(\varsigma) + \tilde{\Lambda}_{m_1}(\varsigma)\Lambda_{m_2}(\varsigma)\right]\left[\varphi(\xi)\upsilon(\xi) + \varphi(\delta)\upsilon(\delta)\right]\right.\\
&\qquad\left.+\left[\Lambda_{m_1}(\varsigma)\Lambda_{m_2}(\varsigma) + \tilde{\Lambda}_{m_1}(\varsigma)\tilde{\Lambda}_{m_2}(\varsigma)\right]\left[\varphi(\xi)\upsilon(\delta) + \varphi(\delta)\upsilon(\xi)\right]\right\}\\
&= E_{m_1}E_{m_2}\left[\varphi\left(\frac{\xi\delta}{\varsigma\xi + (1-\varsigma)\delta}\right)\upsilon\left(\frac{\xi\delta}{\varsigma\xi + (1-\varsigma)\delta}\right)\right.\\
&\quad\left.+\,\varphi\left(\frac{\xi\delta}{\varsigma\delta + (1-\varsigma)\xi}\right)\upsilon\left(\frac{\xi\delta}{\varsigma\delta + (1-\varsigma)\xi}\right)\right]\\
&\quad+ E_{m_1}E_{m_2}\left\{\left[\Lambda_{m_1}(\varsigma)\tilde{\Lambda}_{m_2}(\varsigma) + \tilde{\Lambda}_{m_1}(\varsigma)\Lambda_{m_2}(\varsigma)\right]\mathcal{D}(\xi,\delta)\right.\\
&\qquad\left.+\left[\Lambda_{m_1}(\varsigma)\Lambda_{m_2}(\varsigma) + \tilde{\Lambda}_{m_1}(\varsigma)\tilde{\Lambda}_{m_2}(\varsigma)\right]\mathcal{F}(\xi,\delta)\right\}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\varphi\left(\frac{2\xi\delta}{\xi+\delta}\right)\upsilon\left(\frac{2\xi\delta}{\xi+\delta}\right)\\
&\leq E_{m_1}E_{m_2}\left[\varphi\left(\frac{\xi\delta}{\varsigma\xi+(1-\varsigma)\delta}\right)\upsilon\left(\frac{\xi\delta}{\varsigma\xi+(1-\varsigma)\delta}\right)\right.\\
&\quad\left.+\,\varphi\left(\frac{\xi\delta}{\varsigma\delta+(1-\varsigma)\xi}\right)\upsilon\left(\frac{\xi\delta}{\varsigma\delta+(1-\varsigma)\xi}\right)\right]\\
&\quad+E_{m_1}E_{m_2}\left\{\left[\Lambda_{m_1}(\varsigma)\tilde{\Lambda}_{m_2}(\varsigma)+\tilde{\Lambda}_{m_1}(\varsigma)\Lambda_{m_2}(\varsigma)\right]\mathcal{D}(\xi,\delta)\right.\\
&\qquad\left.+\left[\Lambda_{m_1}(\varsigma)\Lambda_{m_2}(\varsigma)+\tilde{\Lambda}_{m_1}(\varsigma)\tilde{\Lambda}_{m_2}(\varsigma)\right]\mathcal{F}(\xi,\delta)\right\}.
\end{aligned}\tag{2.20}$$

Multiplying both sides of (2.20) by $\varsigma^{\frac{\epsilon}{\zeta}-1}$ and integrating with respect to ς over $[0,1]$ to get:

$$\begin{aligned}
&\frac{\zeta}{\epsilon}\varphi\left(\frac{2\xi\delta}{\xi+\delta}\right)\upsilon\left(\frac{2\xi\delta}{\xi+\delta}\right)\\
&=\int_0^1\varsigma^{\frac{\epsilon}{\zeta}-1}\varphi\left(\frac{2\xi\delta}{\xi+\delta}\right)\upsilon\left(\frac{2\xi\delta}{\xi+\delta}\right)d\varsigma\\
&\leq E_{m_1}E_{m_2}\int_0^1\varsigma^{\frac{\epsilon}{\zeta}-1}\left[\varphi\left(\frac{\xi\delta}{\varsigma\xi+(1-\varsigma)\delta}\right)\upsilon\left(\frac{\xi\delta}{\varsigma\xi+(1-\varsigma)\delta}\right)\right.\\
&\quad\left.+\,\varphi\left(\frac{\xi\delta}{\varsigma\delta+(1-\varsigma)\xi}\right)\upsilon\left(\frac{\xi\delta}{\varsigma\delta+(1-\varsigma)\xi}\right)\right]d\varsigma\\
&\quad+E_{m_1}E_{m_2}\int_0^1\varsigma^{\frac{\epsilon}{\zeta}-1}\left\{\left[\Lambda_{m_1}(\varsigma)\tilde{\Lambda}_{m_2}(\varsigma)+\tilde{\Lambda}_{m_1}(\varsigma)\Lambda_{m_2}(\varsigma)\right]\mathcal{D}(\xi,\delta)\right.\\
&\qquad\left.+\left[\Lambda_{m_1}(\varsigma)\Lambda_{m_2}(\varsigma)+\tilde{\Lambda}_{m_1}(\varsigma)\tilde{\Lambda}_{m_2}(\varsigma)\right]\mathcal{F}(\xi,\delta)\right\}d\varsigma\\
&=E_{m_1}E_{m_2}\left\{\zeta\Gamma_\zeta(\epsilon)\left(\frac{\xi\delta}{\delta-\xi}\right)^{\frac{\epsilon}{\zeta}}\left[{}_\zeta\mathcal{J}^{\epsilon}_{\frac{1}{\delta}^+}(\varphi\upsilon\circ\tilde{\varphi})\left(\frac{1}{\xi}\right)+{}_\zeta\,\mathcal{J}^{\epsilon}_{\frac{1}{\xi}^-}(\varphi\upsilon\circ\tilde{\varphi})\left(\frac{1}{\delta}\right)\right]\right\}\\
&\qquad+E_{m_1}E_{m_2}\int_0^1\varsigma^{\frac{\epsilon}{\zeta}-1}\left\{\left[\Lambda_{m_1}(\varsigma)\tilde{\Lambda}_{m_2}(\varsigma)+\tilde{\Lambda}_{m_1}(\varsigma)\Lambda_{m_2}(\varsigma)\right]\mathcal{D}(\xi,\delta)\right.\\
&\qquad\left.+\left[\Lambda_{m_1}(\varsigma)\Lambda_{m_2}(\varsigma)+\tilde{\Lambda}_{m_1}(\varsigma)\tilde{\Lambda}_{m_2}(\varsigma)\right]\mathcal{F}(\xi,\delta)\right\}d\varsigma.
\end{aligned}$$

The required result follows. □

Corollary 3. *Let $\varphi,\upsilon:\mathbf{S}\to\mathbb{R}^+$ be two functions such that $\varphi\upsilon$ is Lebesgue integrable function on $[\xi,\delta]$ with $0<\xi<\delta$ and $\xi,\delta\in\mathbf{S}$. If φ and υ are harmonically convex and $\zeta,\epsilon>0$, then*

$$\begin{aligned}
&\varphi\left(\frac{2\xi\delta}{\xi+\delta}\right)\upsilon\left(\frac{2\xi\delta}{\xi+\delta}\right)\\
&\leq\frac{\Gamma_\zeta(\epsilon+\zeta)}{4}\left(\frac{\xi\delta}{\delta-\xi}\right)^{\frac{\epsilon}{\zeta}}\left[{}_\zeta\mathcal{J}^{\epsilon}_{\frac{1}{\delta}^+}(\varphi\upsilon\circ\tilde{\varphi})\left(\frac{1}{\xi}\right)+{}_\zeta\,\mathcal{J}^{\epsilon}_{\frac{1}{\xi}^-}(\varphi\upsilon\circ\tilde{\varphi})\left(\frac{1}{\delta}\right)\right]\\
&\qquad+\frac{1}{2}\left[\frac{\epsilon}{\epsilon+\zeta}-\frac{\epsilon}{\epsilon+2\zeta}\right]\mathcal{D}(\xi,\delta)+\frac{1}{4}\left[1+\frac{2\epsilon}{\epsilon+2\zeta}-\frac{2\epsilon}{\epsilon+\zeta}\right]\mathcal{F}(\xi,\delta).
\end{aligned}$$

Proof. Let $m_1 = m_2 = 1$. Then, $\Lambda_{m_1}(\varsigma) = \Lambda_{m_2}(\varsigma) = \varsigma$ and $\tilde{\Lambda}_{m_1}(\varsigma) = \tilde{\Lambda}_{m_2}(\varsigma) = 1 - \varsigma$. The intended result follows by using Theorem 8. □

3. Conclusion

Utilizing the Caputo–Fabrizio and generalized Riemann–Liouville fractional integral operators, we proved some inequalities of the Hermite–Hadamard kinds for m-polynomial convex and harmonically convex functions. Our results generalize, extend and complement results in [7, 9, 31].

Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

1. T. Abdeljawad, D. Baleanu, On fractional derivatives with exponential kernel and their discrete versions, *Rep. Math. Phys.,* **80** (2017), 11–27.
2. M. U. Awan, N. Akhtar, S. Iftikhar, M. A. Noor, Y. M. Chu, New Hermite–Hadamard type inequalities for n-polynomial harmonically convex functions, *J. Inequal. Appl.,* **2020** (2020), 1–12.
3. Y. M. Chu, M. Adil Khan, T. U. Khan, T. Ali, Generalizations of Hermite–Hadamard type inequalities for MT-convex functions, *J. Nonlinear Sci. Appl.,* **9** (2016), 4305–4316.
4. M. Adil Khan, Y. M. Chu, T. U. Khan, J. Khan, Some new inequalities of Hermite–Hadamard type for s-convex functions with applications, *Open Math.,* **15** (2017), 1414–1430.
5. M. R. Delavar, M. De La Sen, Some generalizations of Hermite–Hadamard type inequalities, *SpringerPlus,* **5** (2016), 1–9.
6. A. Guessab, G. Schmeisser, Sharp integral inequalities of the Hermite–Hadamard type, *J. Approx. Theory,* **115** (2002), 260–288.
7. M. Gürbüz, A. O. Akdemir, S. Rashid, E. Set, Hermite–Hadamard inequality for fractional integrals of Caputo–Fabrizio type and related inequalities, *J. Inequal. Appl.,* **2020** (2020), 1–10.
8. İ. İşcan, Hermite–Hadamard type inequalities for harmonically convex functions, *Hacet. J. Math. Stat.,* **43** (2014), 935–942.
9. İ. İşcan, S. Wu, Hermite–Hadamard type inequalities for harmonically convex functions via fractional integrals, *Appl. Math. Comput.,* **238** (2014), 237–244.
10. A. Iqbal, M. Adil Khan, S. Ullah, Y. M. Chu, Some new Hermite–Hadamard type inequalities associated with conformable fractional integrals and their applications, *J. Funct. Space.,* **2020** (2020), 1–18.
11. A. Iqbal, M. Adil Khan, N. Mohammad, E. R. Nwaeze, Revisiting the Hermite–Hadamard fractional integral inequality via a Green function, *AIMS Math.,* **5** (2020), 6087–6107.
12. A. Iqbal, M. Adil Khan, M. Suleman, Y. M. Chu, The right Riemann-Liouville fractional Hermite-Hadamard type inequalities derived from Green's function, *AIP Adv.,* **10** (2020), 1–10.

13. M. Adil Khan, Y. Khurshid, T. Ali, Hermite–Hadamard Inequality for fractional integrals via η-convex functions, *Acta Math. Univ. Comenianae.,* **86** (2017), 153–164.

14. M. Adil Khan, T. Ali, T. U. Khan, Hermite–Hadamard Type Inequalities with Applications, *Fasciculi Mathematici,* **59** (2017), 57–74.

15. M. Adil Khan, N. Mohammad, E. R. Nwaeze, Y. M. Chu, Quantum Hermite–Hadamard inequality by means of a green function, *Adv. Diff. Equ.,* **2020** (2020), 1–20.

16. T. U. Khan, M. Adil Khan, Hermite-Hadamard inequality for new generalized conformable fractional operators, *AIMS Math.,* **6** (2020), 23–38.

17. P. O. Mohammed, I. Brevik, A new version of the Hermite–Hadamard inequality for Riemann–Liouville fractional integrals, *Symmetry,* **12** (2020), 1–11.

18. P. O. Mohammed, M. Z. Sarikaya, On generalized fractional integral inequalities for twice differentiable convex functions, *J. Comput. Appl. Math.,* **372** (2020), 1–15.

19. P. O. Mohammed, New integral inequalities for preinvex functions via generalized beta function, *J. Interdiscip. Math.,* **22** (2019), 539–549.

20. A. Fernandez, P. O. Mohammed, Hermite-Hadamard inequalities in fractional calculus defined using Mittag-Leffler kernels, *Math. Meth. Appl. Sci.,* (2020), 1–18.

21. S. Mubeen, G. M. Habibullah, k-Fractional Integrals and Aplication, *Int. J. Contemp. Math. Sci.,* **7** (2012), 89–94.

22. E. R. Nwaeze, Inequalities of the Hermite–Hadamard type for Quasi-convex functions via the (k, s)-Riemann–Liouville fractional integrals, *Fractional Differ. Calc.,* **8** (2018), 327–336.

23. E. R. Nwaeze, D. F. M. Torres, Novel results on the Hermite–Hadamard kind inequality for η-convex functions by means of the (k, r)-fractional integral operators. In: S. S. Dragomir, P. Agarwal, M. Jleli, B, Samet (eds.) *Advances in Mathematical Inequalities and Applications (AMIA),* Trends in Mathematics, Birkhäuser, Singapore, 2018, 311–321.

24. E. R. Nwaeze, M. Adil Khan, Y. M. Chu, Fractional inclusions of the Hermite-Hadamard type for m-polynomial convex interval-valued functions, *Adv. Diff. Equ.,* **2020** (2020), 1–17.

25. G. Rahman, K. S. Nisar, S. Rashid, T. Abdeljawad, Certain Gruss-type inequalities via tempered fractional integrals concerning another function, *J. Inequal. Appl.,* **2020** (2020), 1–18.

26. J. Sun, B. Y. Xi, F. Qi, Some new inequalities of the Hermite–Hadamard type for extended s-convex functions, *J. Comput. Anal. Appl.,* **26** (2019), 985–996.

27. S. Salahshour, A. Ahmadian, C. S. Chan, Successive approximation method for Caputo q-fractional IVPs, *Commun. Nonlinear Sci. Numer. Simul.,* **24** (2015), 153–158.

28. M. R. B. Shahriyar, F. Ismail, S. Aghabeigi, A. Ahmadian, S. Salahshour, An eigenvalue-eigenvector method for solving a system of fractional differential equations with uncertainty, *Math. Probl. Eng.,* **2013** (2013), 1–10.

29. A. Ahmadian, S. Salahshour, M. Ali-Akbari, F. Ismail, D. Baleanu, A novel approach to approximate fractional derivative with uncertain conditions, *Chaos, Solitons Fractals,* **104** (2017), 68–76.

30. A. Ahmadian, C. S. Chan, S. Salahshour, V. Vembarasan, FTFBE: A numerical approximation for fuzzy time-fractional Bloch equation, IEEE international conference on fuzzy systems, 2014, 418–423.

31. T. Toplu, M. Kadakal, İ. İşcan, On n-Polynomial convexity and some related inequalities, *AIMS Math.,* **5** (2020), 1304–1318.

32. S. S. Zhou, S. Rashid, M. A. Noor, F. Safdar, New Hermite-Hadamard type inequalities for exponentially convex functions and applications, *AIMS Math.,* **5** (2020), 6874–6901.

33. S. S. Zhou, S. Rashid, F. Jarad, H. Kalsoom, New estimates considering the generalized proportional Hadamard fractional integral operators, *Adv. Diff. Equ.,* **2020** (2020), 1–15.

34. S. S. Zhou, S. Rashid, S. S. Dragomir, M. A. Latif, Some New inequalities involving k-fractional integral for certain classes of functions and their applications, *J. Funct. Space.,* **2020** (2020), 1–14.

AIMS Press