Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(2): 1880–1888.
DOI: 10.3934/math.2021114
Received: 02 September 2020
Accepted: 15 October 2020
Published: 02 December 2020

Correction

Correction: Quantum Montgomery identity and quantum estimates of Ostrowski type inequalities

Andrea Aglić Aljinović[1,*], Domagoj Kovačević[1], Mehmet Kunt[2] and Mate Puljiz[1]

[1] University of Zagreb, Faculty of Electrical Engineering and Computing, Unska 3, 10 000 Zagreb, Croatia

[2] Department of Mathematics, Faculty of Sciences, Karadeniz Technical University, 61080, Trabzon, Turkey

* **Correspondence:** Email: andrea.aglic@fer.hr.

Abstract: We disprove and correct some recently obtained results regarding Montgomery identity for quantum integral operator and Ostrowski type inequalities involving convex functions.

Keywords: q-derivative; q-integral; Jackson integral; Montgomery identity; Ostrowski inequality
Mathematics Subject Classification: 05A30, 26D10, 26D15

1. Introduction

In [4] the authors obtained the following generalization of Montgomery identity for quantum calculus.

Lemma 1. *[4] (Quantum Montgomery identity) Let $f : [a, b] \to \mathbb{R}$, be an arbitrary function with $d_q^a f$ quantum integrable on $[a, b]$, then the following quantum identity holds:*

$$f(x) - \frac{1}{b-a}\int_a^b f(t)\, d_q^a t = (b-a)\int_0^1 K_{q,x}(t)\, D_q^a f(tb + (1-t)a)\, d_q^0 t \tag{1.1}$$

where $K_{q,x}(t)$ is defined by

$$K_{q,x}(t) = \begin{cases} qt, & 0 \leq t \leq \dfrac{x-a}{b-a}, \\ qt - 1, & \dfrac{x-a}{b-a} < t \leq 1. \end{cases} \tag{1.2}$$

Using this identity, the authors have obtained two Ostrowski type inequalities for quantum integrals and applied it in several special cases.

Unfortunately, in the proof of this lemma an error is made when calculating the integrals involving the kernel $K_{q,x}(t)$ on the interval $\left[\frac{x-a}{b-a}, 1\right]$. Also, in the proofs of Theorem 3 and Theorem 4 a small mistake related to the convexity of $\left|D_q^a f\right|^r$ is made.

In the present paper we prove that the identity (1.1) and, thus, all of the consequent results are incorrect and provide corrections for these results.

2. Main results

The q-derivative of a function $f : [a, b] \to \mathbb{R}$ for $q \in \langle 0, 1\rangle$ (see [5] or [2] for $a = 0$) is given by

$$D_q^a f(x) = \frac{f(x) - f(a + q(x-a))}{(1-q)(x-a)}, \text{ for } x \in \langle a, b]$$
$$D_q^a f(a) = \lim_{x \to a} D_q^a f(x)$$

We say that $f : [a, b] \to \mathbb{R}$ is q-differentiable if $\lim_{x\to a} D_q^a f(x)$ exists. The q-derivative is a discretization of the ordinary derivative and if f is a differentiable function then ([1, 3])

$$\lim_{q \to 1} D_q^a f(x) = f'(x).$$

Further, the q-integral of f is defined by

$$\int_a^x f(t)\, d_q^a t = (1-q)(x-a) \sum_{k=0}^{\infty} q^k f\left(a + q^k (x-a)\right), \quad x \in [a, b].$$

If the series on the right hand-side is convergent, then the q-integral $\int_a^x f(t)\, d_q^a t$ exists and $f : [a, b] \to \mathbb{R}$ is said to be q-integrable on $[a, x]$. If f is continuous on $[a, b]$ the series $(1-q)(x-a) \sum_{k=0}^{\infty} q^k f\left(a + q^k (x-a)\right)$ tends to the Riemann integral of f as $q \to 1$ ([1], [3])

$$\lim_{q \to 1} \int_a^x f(t)\, d_q^a t = \int_a^x f(t)\, dt.$$

If $c \in \langle a, x\rangle$ the q-integral is defined by

$$\int_c^x f(t)\, d_q^a t = \int_a^x f(t)\, d_q^a t - \int_a^c f(t)\, d_q^a t.$$

Obviously, the q-integral depends on the values of f at the points outside the interval of integration and an important difference between the definite q-integral and Riemann integral is that even if we are integrating a function over the interval $[c, x]$, $a < c < x < b$, for q-integral we have to take into account its behavior at $t = a$ as well as its values on $[a, x]$. This is the main reason for mistakes made in [4] since in the proof of Lemma 1 the following error was made:

$$\int_{\frac{x-a}{b-a}}^{1} K_{q,x}(t)\, D_q^a f(tb + (1-t)a)\, d_q^0 t$$
$$= \int_0^1 (qt-1)\, D_q^a f(tb + (1-t)a)\, d_q^0 t - \int_0^{\frac{x-a}{b-a}} (qt-1)\, D_q^a f(tb + (1-t)a)\, d_q^0 t.$$

But $K_{q,x}(t) \neq (qt-1)$ for $t \in [0,1]$ or for $t \in \left[0, \frac{x-a}{b-a}\right]$, so the equality does not hold.

Now, we give a proof that the quantum Montgomery identity (1.1) is not correct, since it does not hold for all $x \in [a, b]$. As we shall see, the identity (1.1) is valid only if $x = a + q^{m+1}(b-a)$ for some $m \in \mathbb{N} \cup \{0\}$. We have

$$(b-a)\int_0^1 K_{q,x}(t)\, D_q^a f(tb + (1-t)a)\, d_q^0 t$$
$$= (b-a)(1-q)\sum_{k=0}^{\infty} q^k K_{q,x}\left(q^k\right) D_q^a f\left(a + q^k(b-a)\right).$$

For $q \in \langle 0, 1\rangle$ let $m \in \mathbb{N} \cup \{0\}$ be such that

$$q^{m+1} \le \frac{x-a}{b-a} < q^m,$$

in other words

$$m = \left\lceil \log_q \frac{x-a}{b-a} \right\rceil - 1.$$

Then

$$K_{q,x}\left(q^k\right) = \begin{cases} q^{k+1} - 1, & k \le m, \\ q^{k+1}, & k \ge m+1, \end{cases}$$

and

$$(b-a)(1-q)\sum_{k=0}^{\infty} q^k K_{q,x}\left(q^k\right) D_q^a f\left(a + q^k(b-a)\right)$$
$$= (b-a)(1-q)\left(\sum_{k=0}^{m} q^k\left(q^{k+1}-1\right)\frac{f\left(a + q^k(b-a)\right) - f\left(a + q^{k+1}(b-a)\right)}{(1-q)q^k(b-a)}\right.$$
$$\left. + \sum_{k=m+1}^{\infty} q^k\left(q^{k+1}\right)\frac{f\left(a + q^k(b-a)\right) - f\left(a + q^{k+1}(b-a)\right)}{(1-q)q^k(b-a)}\right)$$

$$= -\sum_{k=0}^{m} \left(f\left(a + q^k (b-a)\right) - f\left(a + q^{k+1}(b-a)\right) \right)$$
$$+ \sum_{k=0}^{\infty} q^{k+1} \left(f\left(a + q^k (b-a)\right) - f\left(a + q^{k+1}(b-a)\right) \right)$$
$$= f\left(a + q^{m+1}(b-a)\right) - f(b) + \sum_{k=0}^{\infty} q^{k+1} \left(f\left(a + q^k (b-a)\right) - f\left(a + q^{k+1}(b-a)\right) \right).$$

If we put $S = \sum_{k=0}^{\infty} q^k f\left(a + q^k (b-a)\right) = \frac{1}{(1-q)(b-a)} \int_a^b f(t)\, d_q^a t$, we have

$$\sum_{k=0}^{\infty} \left(q^{k+1}\right) \left(f\left(a + q^k (b-a)\right) - f\left(a + q^{k+1}(b-a)\right) \right) = qS - (S - f(b))$$

and

$$\frac{1}{b-a} \int_a^b f(t)\, d_q^a t + (b-a) \int_0^1 K_{q,x}(t)\, D_q^a f(tb + (1-t)a)\, d_q^0 t$$
$$= \frac{1}{b-a} \int_a^b f(t)\, d_q^a t + \left(f\left(a + q^{m+1}(b-a)\right) - f(b) \right) + qS - (S - f(b))$$
$$= (1-q)S + f\left(a + q^{m+1}(b-a)\right) - f(b) + qS - S + f(b)$$
$$= f\left(a + q^{m+1}(b-a)\right)$$

which is obviously not equal to $f(x)$, unless $x = a + q^{m+1}(b-a)$.

This is no surprise since Jackson integral takes into account only $f\left(a + q^k (x-a)\right)$ for $k \in \mathbb{N} \cup \{0\}$. Thus, we have proved the next lemma which is a corrected version of Lemma 1 from [4]**.**

Lemma 2. *(Quantum Montgomery identity) Let $f : [a,b] \to \mathbb{R}$, be an arbitrary function with $D_q^a f$ quantum integrable on $[a,b]$, then for all $x \in \langle a,b \rangle$ the following quantum identity holds:*

$$f\left(a + q^{\left\lceil \log_q \frac{x-a}{b-a} \right\rceil}(b-a)\right) - \frac{1}{b-a} \int_a^b f(t)\, d_q^a t$$
$$= (b-a) \int_0^1 K_{q,x}(t)\, D_q^a f(tb + (1-t)a)\, d_q^0 t$$

where $K_{q,x}(t)$ is defined by

$$K_{q,x}(t) = \begin{cases} qt, & 0 \le t \le \dfrac{x-a}{b-a}, \\ qt - 1, & \dfrac{x-a}{b-a} < t \le 1. \end{cases}$$

In Theorem 3 and Theorem 4 from [4] the authors have used the identity (1.1) to derive Ostrowski type inequalities for functions f for which $D_q^a f$ is quantum integrable on $[a, b]$ and $\left|D_q^a f\right|^r$, $r \geq 1$ is a convex function. Since these inequalities depends on the validity of Lemma 1, our discussion invalidates all the results from [4].

More precisely, in all the inequalities an additional assumption $x = a + q^m(b-a)$ for some $m \in \mathbb{N} \cup \{0\}$ should be added. In Theorems 3 and 4 $\left|D_q^a f(a)\right|^r$ and $\left|D_q^a f(b)\right|^r$ should be swapped, since in the proofs of Theorem 3 and Theorem 4, when applying the convexity of $\left|D_q^a f\right|^r$ the following mistake was made

$$\left|D_q^a f(tb + (1-t)a)\right|^r \leq t\left|D_q^a f(a)\right|^r + (1-t)\left|D_q^a f(b)\right|^r.$$

Lastly, the integral $K_4(a, b, x, q)$ is incorrectly computed and should read:

$$K_4(a, b, x, q) = \frac{1-q}{1+q}\left(\frac{b-x}{b-a}\right) + \frac{q}{1+q}\left(\frac{b-x}{b-a}\right)^2.$$

3. Conclusions

The main goal of this paper was to point out that some results in [4] are not correct. We have concentrated on Lemma 3 (Quantum Montgomery identity). The statement of that Lemma is not correct as we have shown. We also found and analyzed the mistake in the proof of Lemma 3.

However, we went one step further and stated and proved the correct version of Lemma 3 (it is Lemma 2 in our paper). We have also explained how can all inequalities derived from Quantum Montgomery identity be corrected.

Acknowledgments

Domagoj Kovačević was supported by the QuantiXLie Centre of Excellence, a project co financed by the Croatian Government and European Union through the European Regional Development Fund-the Competitiveness and Cohesion Operational Programme (Grant KK.01.1.1.01.0004).

Conflict of interest

The authors declare that they have no competing interests.

References

1. M. H. Annaby, Z. S. Mansour, *q-Fractional Calculus and Equations*, Heidelberg: Springer, 2012.
2. F. H. Jackson, On q-functions and a certain difference operator, *Trans. R. Soc. Edinb.,* **46** (1908), 253–281.
3. V. Kac, P. Cheung, *Quantum Calculus*, New York: Springer, 2002.
4. M. Kunt, A. Kashuri, T. Du, A. W. Baidar, Quantum Montgomery identity and quantum estimates of Ostrowski type inequalities, *AIMS Math.,* **5** (2020), 5439–5457.
5. J. Tariboon, S. K. Ntouyas, Quantum calculus on finite intervals and application to impulsive difference equations, *Adv. Differ. Equ.,* **2013** (2013), 1–19.

Addendum—Alternate point of view

After our Correction was accepted we were contacted by the first author of [4], Professor Kunt, who suggested an alternate way to correct the results of [4].

The incorrect version of Montgomery identity from [4]

$$f(x) - \frac{1}{b-a}\int_a^b f(t)\, d_q^a t = (b-a)\int_0^1 K_{q,x}(t)\, D_q^a f(tb+(1-t)a)\, d_q^0 t$$

can be fixed in two ways: either by changing the left hand side or by changing the right hand side of this equation. In Lemma 2 we showed how to fix the identity by correcting the left hand side. This makes it easier to salvage the rest of results in [4], as all the results remain valid with the added assumption that $x = a + q^m(b-a)$ for some $m \in \mathbb{N} \cup \{0\}$.

Professor Kunt suggested correcting the right hand side of this equation to obtain the identity:

$$f(x) - \frac{1}{b-a}\int_a^b f(t)\, d_q^a t = (b-a)\left[\int_0^{\frac{x-a}{b-a}} qt\, D_q^a f(tb+(1-t)a)\, d_q^0 t + \int_{\frac{x-a}{b-a}}^1 (qt-1)\, D_q^a f(tb+(1-t)a)\, d_q^0 t\right]. \quad (*)$$

By doing so, the proofs of all the remaining results have to be corrected as the bound used

$$\left|\int_{\frac{x-a}{b-a}}^1 (qt-1)\, D_q^a f(tb+(1-t)a)\, d_q^0 t\right| \le \int_{\frac{x-a}{b-a}}^1 \left|(qt-1)\, D_q^a f(tb+(1-t)a)\right|\, d_q^0 t$$

does not hold for q-integrals in general. This is discussed, for example, on page 12 in [1, Section 1.3.1, Remark (ii)].

When $\frac{x-a}{b-a} = q^m$ or equivalently $x = a + q^m(b-a)$ for some $m \in \mathbb{N} \cup \{0\}$ the bound above does hold, which is why there is no need to change the rest of the results in [4] if one takes our approach. Nevertheless, we list below the results that can be obtained using identity (*). The results below are due to Professor Kunt.

Theorem 3 in [4] should be as follows:

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be an arbitrary function with $D_q^a f$ is quantum integrable on $[a,b]$. If $\left|D_q^a f\right|^r$, $r \ge 1$ is a convex function, then the following quantum integral inequality holds:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\, d_q^a t\right| \tag{3.1}$$
$$\le (b-a)\left[\begin{array}{c} \left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}\left[\left|D_q^a f(b)\right|^r \frac{1}{(1+q)(1+q+q^2)} + \left|D_q^a f(a)\right|^r \frac{q}{1+q+q^2}\right]^{\frac{1}{r}} \\ +\left(\frac{x-a}{b-a}\right)\left[\left|D_q^a f(b)\right|^r \left(\frac{x-a}{b-a}\right)\frac{1}{1+q} + \left|D_q^a f(a)\right|^r \left(1-\left(\frac{x-a}{b-a}\right)\frac{1}{1+q}\right)\right]^{\frac{1}{r}} \end{array}\right]$$

for all $x \in [a,b]$.

Proof. Using convexity of $\left|D_q^a f\right|^r$, we have that

$$\left|D_q^a f(tb+(1-t)a)\right|^r \leq t\left|D_q^a f(b)\right|^r + (1-t)\left|D_q^a f(a)\right|^r. \tag{3.2}$$

By using (∗), quantum power mean inequality and (3.2), we have that

$$\begin{aligned}
&\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\ d_q^a t\right| \qquad (3.3)\\
=\ &(b-a)\left|\int_0^{\frac{x-a}{b-a}} qt\ D_q^a f(tb+(1-t)a)\ \ d_q^0 t\ + \int_{\frac{x-a}{b-a}}^1 (qt-1)\ D_q^a f(tb+(1-t)a)\ \ d_q^0 t\right|\\
=\ &(b-a)\left|\int_0^1 (qt-1)\ D_q^a f(tb+(1-t)a)\ \ d_q^0 t\ + \int_0^{\frac{x-a}{b-a}} D_q^a f(tb+(1-t)a)\ \ d_q^0 t\right|\\
\leq\ &(b-a)\left|\int_0^1 (qt-1)\ D_q^a f(tb+(1-t)a)\ \ d_q^0 t\right| + \left|\int_0^{\frac{x-a}{b-a}} D_q^a f(tb+(1-t)a)\ \ d_q^0 t\right|\\
\leq\ &(b-a)\left[\int_0^1 (1-qt)\ \left|D_q^a f(tb+(1-t)a)\right|\ \ d_q^0 t\ + \int_0^{\frac{x-a}{b-a}} \left|D_q^a f(tb+(1-t)a)\right|\ \ d_q^0 t\right]\\
\leq\ &(b-a)\left[\begin{array}{c} \left(\int_0^1 1-qt\ d_q^0 t\right)^{1-\frac{1}{r}}\left(\int_0^1 (1-qt)\ \left|D_q^a f(tb+(1-t)a)\right|^r\ \ d_q^0 t\right)^{\frac{1}{r}}\\ +\left(\int_0^{\frac{x-a}{b-a}}\ d_q^0 t\right)^{1-\frac{1}{r}}\left(\int_0^{\frac{x-a}{b-a}} \left|D_q^a f(tb+(1-t)a)\right|^r\ \ d_q^0 t\right)^{\frac{1}{r}} \end{array}\right]\\
\leq\ &(b-a)\left[\begin{array}{c} \left(\int_0^1 (1-qt)\ d_q^0 t\right)^{1-\frac{1}{r}}\\ \times\left(\left|D_q^a f(b)\right|^r \int_0^1 (1-qt)\,t\ d_q^0 t\ + \left|D_q^a f(a)\right|^r \int_0^1 (1-qt)(1-t)\ \ d_q^0 t\right)^{\frac{1}{r}}\\ +\left(\int_0^{\frac{x-a}{b-a}}\ d_q^0 t\right)^{1-\frac{1}{r}}\\ \times\left(\left|D_q^a f(b)\right|^r \int_0^{\frac{x-a}{b-a}} t\ d_q^0 t\ + \left|D_q^a f(a)\right|^r \int_0^{\frac{x-a}{b-a}} (1-t)\ \ d_q^0 t\right)^{\frac{1}{r}} \end{array}\right].
\end{aligned}$$

On the other hand, calculating the following quantum integrals we have

$$\int_0^1 (1-qt)\ d_q^0 t\ = (1-q)\sum_{n=0}^{\infty} q^n\left(1-q^{n+1}\right) = (1-q)\left[\frac{1}{1-q} - \frac{q}{1-q^2}\right] = \frac{1}{1+q}, \tag{3.4}$$

$$\begin{aligned}
\int_0^1 (1-qt)\,t\ d_q^0 t\ &= (1-q)\sum_{n=0}^{\infty} q^n\left[\left(1-q^{n+1}\right)q^n\right] = (1-q)\left[\frac{1}{1-q^2} - \frac{q}{1-q^3}\right] \qquad (3.5)\\
&= \frac{1}{1+q} - \frac{q}{1+q+q^2} = \frac{1}{(1+q)\left(1+q+q^2\right)},
\end{aligned}$$

$$\begin{aligned}
\int_0^1 (1-qt)(1-t)\ \ d_q^0 t\ &= \int_0^1 1-qt\ d_q^0 t\ - \int_0^1 (1-qt)\,t\ d_q^0 t \qquad (3.6)\\
&= \frac{1}{1+q} - \frac{1}{(1+q)\left(1+q+q^2\right)} = \frac{q}{1+q+q^2},
\end{aligned}$$

$$\int_0^{\frac{x-a}{b-a}} d_q^0 t = (1-q)\left(\frac{x-a}{b-a}\right)\sum_{n=0}^{\infty} q^n = \frac{x-a}{b-a}, \tag{3.7}$$

$$\int_0^{\frac{x-a}{b-a}} t \ d_q^0 t = (1-q)\left(\frac{x-a}{b-a}\right)\sum_{n=0}^{\infty} q^n \left(q^n\left(\frac{x-a}{b-a}\right)\right) = \left(\frac{x-a}{b-a}\right)^2 \frac{1}{1+q}, \tag{3.8}$$

$$\begin{aligned}\int_0^{\frac{x-a}{b-a}} (1-t) \ d_q^0 t &= \int_0^{\frac{x-a}{b-a}} d_q^0 t - \int_0^{\frac{x-a}{b-a}} t \ d_q^0 t = \frac{x-a}{b-a} - \left(\frac{x-a}{b-a}\right)^2 \frac{1}{1+q} \\ &= \left(\frac{x-a}{b-a}\right)\left[1 - \left(\frac{x-a}{b-a}\right)\frac{1}{1+q}\right].\end{aligned} \tag{3.9}$$

Using (3.4)–(3.9) in (3.3), we have (3.1). □

Theorem 4 in [4] should be as follows:

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ be an arbitrary function with $D_q^a f$ is quantum integrable on $[a,b]$. If $\left|D_q^a f\right|^r$, $r > 1$ and $\frac{1}{r} + \frac{1}{p} = 1$ is convex function, then the following quantum integral inequality holds:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t) \ d_q^a t \right| \tag{3.10}$$

$$\leq (b-a)\left[\begin{array}{l} \left(\int_0^1 (1-qt)^p \ d_q^0 t\right)^{\frac{1}{p}} \left(\left|D_q^a f(b)\right|^r \frac{1}{1+q} + \left|D_q^a f(a)\right|^r \frac{q}{1+q}\right)^{\frac{1}{r}} \\ + \left(\frac{x-a}{b-a}\right)\left[\left|D_q^a f(b)\right|^r \left(\frac{x-a}{b-a}\right)\frac{1}{1+q} + \left|D_q^a f(a)\right|^r \left(1 - \left(\frac{x-a}{b-a}\right)\frac{1}{1+q}\right)\right]^{\frac{1}{r}} \end{array}\right]$$

for all $x \in [a,b]$.

Proof. By using (∗) and quantum Hölder inequality, we have

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a}\int_a^b f(t) \ d_q^a t \right| \\ \leq \ & (b-a)\left|\int_0^1 (qt-1) \ D_q^a f(tb+(1-t)a) \ d_q^0 t\right| + \left|\int_0^{\frac{x-a}{b-a}} D_q^a f(tb+(1-t)a) \ d_q^0 t\right| \\ \leq \ & (b-a)\left[\int_0^1 (1-qt) \ \left|D_q^a f(tb+(1-t)a)\right| \ d_q^0 t + \int_0^{\frac{x-a}{b-a}} \left|D_q^a f(tb+(1-t)a)\right| \ d_q^0 t\right] \\ \leq \ & (b-a)\left[\begin{array}{l} \left(\int_0^1 (1-qt)^p \ d_q^0 t\right)^{\frac{1}{p}} \left(\int_0^1 \left|D_q^a f(tb+(1-t)a)\right|^r \ d_q^0 t\right)^{\frac{1}{r}} \\ + \left(\int_0^{\frac{x-a}{b-a}} d_q^0 t\right)^{\frac{1}{p}} \left(\int_0^{\frac{x-a}{b-a}} \left|D_q^a f(tb+(1-t)a)\right|^r \ d_q^0 t\right)^{\frac{1}{r}} \end{array}\right] \\ \leq \ & (b-a)\left[\begin{array}{l} \left(\int_0^1 (1-qt)^p \ d_q^0 t\right)^{\frac{1}{p}} \left(\int_0^1 \left[t\left|D_q^a f(b)\right|^r + (1-t)\left|D_q^a f(a)\right|^r\right] d_q^0 t\right)^{\frac{1}{r}} \\ + \left(\int_0^{\frac{x-a}{b-a}} d_q^0 t\right)^{\frac{1}{p}} \left(\int_0^{\frac{x-a}{b-a}} \left[t\left|D_q^a f(b)\right|^r + (1-t)\left|D_q^a f(a)\right|^r\right] d_q^0 t\right)^{\frac{1}{r}} \end{array}\right] \end{aligned}$$

$$\leq \quad (b-a)\left[\begin{array}{l} \left(\int_0^1 (1-qt)^p \ \ {}_0d_q t\right)^{\frac{1}{p}} \left(\left|{}^aD_q f(b)\right|^r \int_0^1 t \ {}_0d_q t \ + \left|{}^aD_q f(a)\right|^r \int_0^1 (1-t) \ {}_0d_q t\right)^{\frac{1}{r}} \\ +\left(\int_0^{\frac{x-a}{b-a}} \ {}_0d_q t\right)^{\frac{1}{p}} \left(\left|{}^aD_q f(b)\right|^r \int_0^{\frac{x-a}{b-a}} t \ {}_0d_q t \ + \left|{}^aD_q f(a)\right|^r \int_0^{\frac{x-a}{b-a}} (1-t) \ {}_0d_q t\right)^{\frac{1}{r}} \end{array}\right]$$

$$= \quad (b-a)\left[\begin{array}{l} \left(\int_0^1 (1-qt)^p \ \ {}_0d_q t\right)^{\frac{1}{p}} \left(\left|{}^aD_q f(b)\right|^r \frac{1}{1+q} + \left|{}^aD_q f(a)\right|^r \frac{q}{1+q}\right)^{\frac{1}{r}} \\ +\left(\frac{x-a}{b-a}\right)\left[\left|{}^aD_q f(b)\right|^r \left(\frac{x-a}{b-a}\right)\frac{1}{1+q} + \left|{}^aD_q f(a)\right|^r \left(1-\left(\frac{x-a}{b-a}\right)\frac{1}{1+q}\right)\right]^{\frac{1}{r}} \end{array}\right].$$

□

We conclude this section by noting that the bounds obtained in the original paper [4] which, as we have previously shown, do hold with the added assumption $x = a + q^m(b-a)$ for some $m \in \mathbb{N} \cup \{0\}$, are tighter than the bounds obtained above by Professor Kunt. Professor Kunt's bounds, however, hold for all $x \in [a, b]$.