



Research article

Weighted pseudo almost periodicity on neutral type CNNs involving multi-proportional delays and D operator

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Abstract: In this paper, we aim to explore the dynamic behaviors of cellular neural networks with D operator and multi-proportional delays. By using the fixed point theorem and some differential inequality techniques, we derive some new sufficient criteria on the existence and exponential stability of weighted pseudo almost periodic solutions for the proposed neural networks, which generalize and improve all known consequences in Yang et al. [22] and Yu [Neural Process Lett., 45 (2017), 141–151]. Finally, the effectiveness of the obtained results is illustrated by a numerical simulation example.

Keywords: cellular neural networks; D operator; proportional delay; weighted pseudo almost periodic solution; exponential convergence

Mathematics Subject Classification: 34K13, 34C25

1. Introduction

In many areas of the objective world, such as target tracking, machine learning system identification, associative memories, pattern recognition, solving optimization problems, image processing, signal processing, and so on [1–5], a lot of practical problems can be described by delay differential equations (DDEs). Therefore, the research of delay differential equations has been the subject of significant attention [6, 7]. As we all know, time delays are inevitable in population dynamics models. For example, the maturation period should be considered in the study of simulated biological species [8, 9], incubation periods should be considered in epidemiology area [7], and the synaptic transmission time among neurons should be considered in neuroscience field [10]. In particular, the dynamic behavior of most cellular neural network models is significantly affected by time delay, so the investigation on delayed cellular neural networks has been the world-wide focus.

It should be mentioned that proportional delay is one of important time-varying delays, which is

unbounded and monotonically increasing, and is more predictable and controllable than constant delay and bounded time-varying delay. Over past decade, by introducing proportional time delay, investigations of the following neutral type proportional delayed cellular neural networks (CNNs) with D operators:

$$[x_i(t) - p_i(t)x_i(r_i t)]' = -a_i(t)x_i(t) + \sum_{j=1}^n e_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^n b_{ij}(t)g_j(x_j(q_{ij}t)) + I_i(t),$$

$$t \geq t_0 > 0, i \in N = \{1, 2, \dots, n\}, \quad (1.1)$$

with initial value conditions:

$$x_i(s) = \varphi_i(s), s \in [\rho_i t_0, t_0], \varphi_i \in C([\rho_i t_0, t_0], \mathbb{R}), \rho_i = \min\{r_i, \min_{1 \leq j \leq n} \{q_{ij}\}\}, i \in N, \quad (1.2)$$

have attracted great attention of some researchers. The main reason is that its successful applications in variety of areas such as optimization, associative memories, signal processing, automatic control engineering and so on (see [11–15] and the references therein). Here n is the number of units in a neural network, $(x_1(t), x_2(t), \dots, x_n(t))^T$ corresponds to the state vector, the decay rate at time t is designated by $a_i(t)$, coefficients $p_i(t)$, $e_{ij}(t)$ and $b_{ij}(t)$ are the connection weights at the time t , f_j and g_j are the activation functions of signal transmission, $r_i(t) \geq 0$ denotes the transmission delay, r_i and q_{ij} are proportional delay factors and satisfy $0 < r_i, q_{ij} < 1$, $I_i(t)$ is outside input.

As pointed out by the authors of reference [16], the weighted pseudo almost periodic function consists of an almost periodic process plus a weighted ergodic component. It is well known that the weighted pseudo-almost periodic phenomenon is more common in the environment than the periodic, almost periodic and pseudo-almost periodic phenomenon, so the dynamic analysis of the weighted pseudo-almost periodic is more realistic [17–20]. Furthermore, when $p_i(t) \equiv 0$, the existence and exponential stability of weighted pseudo almost periodic solutions (WPAPS) of proportional delayed cellular neural networks (CNNs)

$$x_i'(t) = -a_i(t)x_i(t) + \sum_{j=1}^n e_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^n b_{ij}(t)g_j(x_j(q_{ij}t)) + I_i(t), t \geq t_0 > 0, i \in N, \quad (1.3)$$

have been established in [22] under the following conditions

$$\sup_{t \in \mathbb{R}} \left\{ -\tilde{a}_i(t) + K_i \left[\xi_i^{-1} \sum_{j=1}^n (|e_{ij}(t)|L_j^f + |b_{ij}(t)|L_j^g)\xi_j \right] \right\} < -\gamma_i. \quad (1.4)$$

Here, for $i \in N$, $\tilde{a}_i \in C(\mathbb{R}, (0, +\infty))$ is a bounded function, and $K_i > 0$ is a constant with

$$e^{-\int_s^t a_i(u)du} \leq K_i e^{-\int_s^t \tilde{a}_i(u)du} \quad \text{for all } t, s \in \mathbb{R} \text{ and } t - s \geq 0.$$

In addition, f_j and g_j are the activation functions with Lipschitz constants L_j^f and L_j^g obeying

$$|f_j(u) - f_j(v)| \leq L_j^f |u - v|, |g_j(u) - g_j(v)| \leq L_j^g |u - v| \quad \text{for all } u, v \in \mathbb{R}, i \in N.$$

It should be mentioned that the authors in [22] use (1.4) to show that there exists a constant $\lambda \in (0, \min_{i \in N} \tilde{a}_i^-)$ such that

$$\Pi_i(\lambda) = \sup_{t \in \mathbb{R}} \left\{ \lambda - \tilde{a}_i(t) + K_i \left[\xi_i^{-1} \sum_{j=1}^n (|e_{ij}(t)| L_j^f + |b_{ij}(t)| L_j^g e^{\lambda(1-q_{ij})t}) \xi_j \right] \right\} < 0, \quad i \in N. \quad (1.5)$$

With the aid of the fact that $\lim_{t \rightarrow +\infty} e^{\lambda(1-q_{ij})t} = +\infty$, it is easy to see that (1.4) can not lead to (1.5). Meanwhile, Examples 4.1 and 4.2 in [22] also have the same error, where

$$b_{ij}(t) = \frac{1}{10(i+j)} \sin 2t, \quad i, j = 1, 2,$$

and

$$\left. \begin{aligned} b_{1j}(t) &= \frac{1}{100}(\cos(1+j)t), \quad b_{2j}(t) = \frac{1}{100}(\cos(1+j)t + \cos \sqrt{2}t), \\ b_{3j}(t) &= \frac{1}{100}(\cos(1+j)t + \sin \sqrt{2}t), \end{aligned} \right\} j = 1, 2, 3,$$

can not also meet (1.5). For detail, the biological explanations on equations (1.4) and (1.5) can be found in [22]. Now, in order to improve [22], we will further study the existence and exponential stability of weighted pseudo almost periodic solutions for (1.1) which includes (1.3) as a special case. Moreover, this class of models has not been touched in the existing literature.

On account of the above considerations, in this article, we are to handle the existence and generalized exponential stability of weighted pseudo almost periodic solutions for system (1.1). Readers can find the following Remark 2.1 for extensive information. In a nutshell, the contributions of this paper can be summarized as follows. 1) A class of weighted pseudo almost periodic cellular neural network model with neutral proportional delay is proposed; 2) Our findings not only correct the errors in [22], but also improve and complement the existing conclusions in the recent publications [22, 23]; 3) Numerical simulations including comparison analyses are presented to verify the obtained theoretical results.

The remainder of the paper is organized as follows. We present the basic notations and assumptions in Section 2. The existence and exponential stability of weighted pseudo almost periodic solutions for the addressed neural networks models are proposed in Section 3. The validity of the proposed method is demonstrated in Section 4, and conclusions are drawn in Section 5.

2. Notations and assumptions

Notations. \mathbb{R} and \mathbb{R}^n denote the set of real numbers and the n -dimensional real spaces. For any $x = \{x_{ij}\} \in \mathbb{R}^{mn}$, let $|x|$ denote the absolute value vector given by $|x| = \{|x_{ij}|\}$, and define $\|x\| = \max_{ij \in J} |x_{ij}(t)|$. Given a bounded continuous function h defined on \mathbb{R} , let $h^+ = \sup_{t \in \mathbb{R}} |h(t)|$, $h^- = \inf_{t \in \mathbb{R}} |h(t)|$. We define \mathbb{U} be the collection of functions (weights) $\mu : \mathbb{R} \rightarrow (0, +\infty)$ satisfying

$$\mathbb{U}_\infty := \left\{ \mu \mid \mu \in \mathbb{U}, \inf_{x \in \mathbb{R}} \mu(x) = \mu_0 > 0 \right\},$$

and

$$\mathbb{U}_\infty^+ := \left\{ \mu \mid \mu \in \mathbb{U}_\infty, \limsup_{|x| \rightarrow +\infty} \frac{\mu(\alpha x)}{\mu(x)} < +\infty, \limsup_{r \rightarrow +\infty} \frac{\mu([- \alpha r, \alpha r])}{\mu([-r, r])} < +\infty, \forall \alpha \in (0, +\infty) \right\}.$$

Let $BC(\mathbb{R}, \mathbb{R}^n)$ denote the collection of bounded and continuous functions from \mathbb{R} to \mathbb{R}^n . Then $(BC(\mathbb{R}, \mathbb{R}^n), \|\cdot\|_\infty)$ is a Banach space, where $\|f\|_\infty := \sup_{t \in \mathbb{R}} \|f(t)\|$. Also, this set of the almost periodic functions from \mathbb{R} to \mathbb{R}^n will be designated by $AP(\mathbb{R}, \mathbb{R}^n)$. Furthermore, the class of functions $PAP_0^\mu(\mathbb{R}, \mathbb{R}^n)$ be defined as

$$PAP_0^\mu(\mathbb{R}, \mathbb{R}^n) = \left\{ \varphi \in BC(\mathbb{R}, \mathbb{R}^n) \mid \lim_{r \rightarrow +\infty} \frac{1}{\mu([-r, r])} \int_{-r}^r \mu(t) |\varphi(t)| dt = 0 \right\}.$$

A function $f \in BC(\mathbb{R}, \mathbb{R}^n)$ is said to be weighted pseudo almost periodic if there exist $h \in AP(\mathbb{R}, \mathbb{R}^n)$ and $\varphi \in PAP_0^\mu(\mathbb{R}, \mathbb{R}^n)$ satisfying

$$f = h + \varphi,$$

where h and φ are called the almost periodic component and the weighted ergodic perturbation of weighted pseudo almost periodic function f , respectively. We designate the collection of such functions by $PAP^\mu(\mathbb{R}, \mathbb{R}^n)$. In addition, fixed $\mu \in \mathbb{U}_\infty^+$, $(PAP^\mu(\mathbb{R}, \mathbb{R}^n), \|\cdot\|_\infty)$ is a Banach space and $AP(\mathbb{R}, \mathbb{R}^n)$ is a proper subspace of $PAP^\mu(\mathbb{R}, \mathbb{R}^n)$. For more details about the above definitions can be available from [17, 18] and the references cited therein.

In what follows, for $i, j \in N$, we shall always assume that $e_{ij}, b_{ij}, p_i, I_i \in PAP^\mu(\mathbb{R}, \mathbb{R})$, and

$$a_i \in AP(\mathbb{R}, \mathbb{R}), \quad M[a_i] = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^{t+T} a_i(s) ds > 0. \quad (2.1)$$

For $i, j \in N$, we also make the following technical assumptions:

(H_1) there are a positive function $\tilde{a}_i \in BC(\mathbb{R}, \mathbb{R})$ and a constant $K_i > 0$ satisfying

$$e^{-\int_s^t a_i(u) du} \leq K_i e^{-\int_s^t \tilde{a}_i(u) du} \quad \text{for all } t, s \in \mathbb{R} \text{ and } t - s \geq 0.$$

(H_2) there exist nonnegative constants L_j^f and L_j^g such that

$$|f_j(u) - f_j(v)| \leq L_j^f |u - v|, |g_j(u) - g_j(v)| \leq L_j^g |u - v| \quad \text{for all } u, v \in \mathbb{R}.$$

(H_3) $\mu \in \mathbb{U}_\infty^+$, we can find constants $\xi_i > 0$ and $\Lambda_i > 0$ such that

$$\sup_{t \in \mathbb{R}} \frac{1}{\tilde{a}_i(t)} K_i [|a_i(t) p_i(t)| + \xi_i^{-1} \sum_{j=1}^n (|e_{ij}(t)| L_j^f + |b_{ij}(t)| L_j^g \xi_j)] < \Lambda_i,$$

$$\begin{aligned} \sup_{t \geq t_0} \{ & -\tilde{a}_i(t) + K_i [|a_i(t) p_i(t)| \frac{1}{1-p_i^+} + \xi_i^{-1} \sum_{j=1}^n |e_{ij}(t)| L_j^f \xi_j \frac{1}{1-p_j^+} \\ & + \xi_i^{-1} \sum_{j=1}^n |b_{ij}(t)| L_j^g \xi_j \frac{1}{1-p_j^+}] \} < 0, \end{aligned}$$

and

$$p_i^+ + \Lambda_i < 1, \quad i \in N.$$

Remark 2.1. From (H_1) and (H_2), one can use an argument similar to that applied in Lemma 2.1 of [24] to demonstrate that every solution of initial value problem (1.1) and (1.2) is unique and exists on $[t_0, +\infty)$.

3. Results

In this section, we will establish some results about the global generalized exponential stability of the weighted pseudo almost periodic solutions of (1.1). To do this end, we first show the following Lemma.

Lemma 3.1. (see [22], Lemma 2.1). Assume that $f \in PAP^\mu(\mathbb{R}, \mathbb{R})$ and $\beta \in \mathbb{R} \setminus \{0\}$. Then, $f(\beta t) \in PAP^\mu(\mathbb{R}, \mathbb{R})$.

Using a similar way to that in lemma 2.3 of [22], we can show the following lemma:

Lemma 3.2. Assume that (H_1) and (H_2) hold. Then, the nonlinear operator G :

$$(G\varphi)_i(t) = \int_{-\infty}^t e^{-\int_s^t a_i(u)du} [-a_i(s)p_i(s)\varphi_i(r_i s) + \xi_i^{-1} \sum_{j=1}^n e_{ij}(s)f_j(\xi_j \varphi_j(s)) \\ + \xi_i^{-1} \sum_{j=1}^n b_{ij}(s)g_j(\xi_j \varphi_j(q_{ij}s)) + \xi_i^{-1} I_i(s)] ds, \quad i \in N, \quad \varphi \in PAP^\mu(\mathbb{R}, \mathbb{R}^n),$$

maps $PAP^\mu(\mathbb{R}, \mathbb{R}^n)$ into itself.

Theorem 3.1. Suppose that (H_1) , (H_2) and (H_3) are satisfied. Then, system (1.1) has exactly one WPAPS $x^*(t) \in PAP^\mu(\mathbb{R}, \mathbb{R}^n)$, which is globally generalized exponentially stable, that is, for every solution $x(t)$ agreeing with (1.1) – (1.2), there exists a constant $\sigma \in (0, \min_{i \in N} \tilde{a}_i^-)$ such that

$$x_i(t) - x_i^*(t) = O\left(\frac{1}{1+t}\right)^\sigma \text{ as } t \rightarrow +\infty \text{ for all } i \in N.$$

Proof. With the help of (H_3) , it is easy to see that there are constants $\sigma, \lambda \in (0, \min_{i \in N} \tilde{a}_i^-)$ such that

$$p_i^+ e^{\sigma \ln \frac{1}{r_i}} < 1, \quad \sup_{t \in \mathbb{R}} \frac{e^\lambda}{\tilde{a}_i(t)} K_i[|a_i(t)p_i(t)| + \xi_i^{-1} \sum_{j=1}^n (|e_{ij}(t)|L_j^f + |b_{ij}(t)|L_j^g)\xi_j] < \Lambda_i, \quad i \in N, \quad (3.1)$$

and

$$\sup_{t \geq t_0} \left\{ \sigma - \tilde{a}_i(t) + K_i[|a_i(t)p_i(t)| \frac{1}{1 - p_i^+ e^{\sigma \ln \frac{1}{r_i}}} + \xi_i^{-1} \sum_{j=1}^n |e_{ij}(t)|L_j^f \xi_j \frac{1}{1 - p_j^+ e^{\sigma \ln \frac{1}{r_j}}} \right. \\ \left. + \xi_i^{-1} \sum_{j=1}^n |b_{ij}(t)|L_j^g \xi_j \frac{1}{1 - p_j^+ e^{\sigma \ln \frac{1}{r_j}}} e^{\sigma \ln(\frac{1}{q_{ij}})} \right\} < 0, \quad i \in N, \quad (3.2)$$

which, along with the inequalities

$$\frac{\sigma}{1+t} \leq \sigma, \quad \ln\left(\frac{1+t}{1+r_i t}\right) \leq \ln \frac{1}{r_i}, \quad \ln\left(\frac{1+t}{1+q_{ij} t}\right) \leq \ln \frac{1}{q_{ij}} \text{ for all } t \geq 0, \quad i, j \in N,$$

yield

$$\sup_{t \geq t_0} \left\{ \frac{\sigma}{1+t} - \tilde{a}_i(t) + K_i[|a_i(t)p_i(t)| \frac{1}{1 - p_i^+ e^{\sigma \ln \frac{1}{r_i}}} e^{\sigma \ln \frac{1+s}{1+r_i t}} \right.$$

$$\begin{aligned}
& + \xi_i^{-1} \sum_{j=1}^n |e_{ij}(t)| L_j^f \xi_j \frac{1}{1 - p_j^+ e^{\sigma \ln \frac{1}{r_j}}} \\
& + \xi_i^{-1} \sum_{j=1}^n |b_{ij}(t)| L_j^g \xi_j \frac{1}{1 - p_j^+ e^{\sigma \ln \frac{1}{r_j}}} e^{\sigma \ln(\frac{1+t}{1+q_{ij}t})} \Big] \Big\} \\
& \leq \sup_{t \geq t_0} \left\{ \sigma - \tilde{a}_i(t) + K_i [|a_i(t) p_i(t)| \frac{1}{1 - p_i^+ e^{\sigma \ln \frac{1}{r_i}}} e^{\sigma \ln \frac{1}{r_i}} \right. \\
& + \xi_i^{-1} \sum_{j=1}^n |e_{ij}(t)| L_j^f \xi_j \frac{1}{1 - p_j^+ e^{\sigma \ln \frac{1}{r_j}}} \\
& + \xi_i^{-1} \sum_{j=1}^n |b_{ij}(t)| L_j^g \xi_j \frac{1}{1 - p_j^+ e^{\sigma \ln \frac{1}{r_j}}} e^{\sigma \ln(\frac{1}{q_{ij}})} \Big] \Big\} < 0, \quad i \in N. \tag{3.3}
\end{aligned}$$

Consequently, applying a transformation:

$$y_i(t) = \xi_i^{-1} x_i(t), \quad Y_i(t) = y_i(t) - p_i(t) y_i(r_i t), \quad i \in N,$$

leads to

$$\begin{aligned}
Y_i'(t) &= -a_i(t) Y_i(t) - a_i(t) p_i(t) y_i(r_i t) + \xi_i^{-1} \sum_{j=1}^n e_{ij}(t) f_j(\xi_j y_j(t)) \\
&+ \xi_i^{-1} \sum_{j=1}^n b_{ij}(t) g_j(\xi_j y_j(q_{ij} t)) + \xi_i^{-1} I_i(t), \quad i \in N. \tag{3.4}
\end{aligned}$$

Now, define a mapping $P : PAP^\mu(\mathbb{R}, \mathbb{R}^n) \rightarrow PAP^\mu(\mathbb{R}, \mathbb{R}^n)$ by setting

$$(P\varphi)_i(t) = p_i(t) \varphi_i(r_i t) + (G\varphi)_i(t) \quad \text{for all } i \in N, \quad \varphi \in PAP^\mu(\mathbb{R}, \mathbb{R}^n), \tag{3.5}$$

it follows from Lemma 3.1 and Lemma 3.2 that $P\varphi \in PAP^\mu(\mathbb{R}, \mathbb{R}^n)$.

Moreover, by means of (H_1) , (H_2) and (H_3) , for $\varphi, \psi \in PAP^\mu(\mathbb{R}, \mathbb{R}^n)$, we have

$$\begin{aligned}
& |(P\varphi)_i(t) - (P\psi)_i(t)| \\
&= |p_i(t) [\varphi_i(r_i t) - \psi_i(r_i t)] + \int_{-\infty}^t e^{-\int_s^t a_i(u) du} [\xi_i^{-1} \sum_{j=1}^n e_{ij}(s) (f_j(\xi_j \varphi_j(s)) - f_j(\xi_j \psi_j(s))) \\
&\quad + \xi_i^{-1} \sum_{j=1}^n b_{ij}(s) (g_j(\xi_j \varphi_j(q_{ij} s)) - g_j(\xi_j \psi_j(q_{ij} s)))] ds| \\
&\leq \{p_i^+ + \int_{-\infty}^t e^{-\int_s^t \tilde{a}_i(u) du} K_i [\xi_i^{-1} \sum_{j=1}^n (|e_{ij}(s)| L_j^f + |b_{ij}(s)| L_j^g) \xi_j] ds\} \|\varphi(t) - \psi(t)\|_\infty \\
&\leq \{p_i + \Lambda_i \int_{-\infty}^t e^{-\int_s^t \tilde{a}_i(u) du} \frac{1}{e^\lambda} \tilde{a}_i(s) ds\} \|\varphi(t) - \psi(t)\|_\infty \\
&\leq \{p_i + \Lambda_i \frac{1}{e^\lambda}\} \|\varphi(t) - \psi(t)\|_\infty,
\end{aligned}$$

which and the fact that $0 < \max_{i \in N} \{p_i^+ + \Lambda_i\} < 1$ suggest that the contraction mapping P possesses a unique fixed point

$$y^* = \{y_i^*(t)\} \in PAP^\mu(\mathbb{R}, \mathbb{R}^n), \quad Py^* = y^*.$$

Thus, (1.5) and (3.5) entail that $x^* = \{x_i^*(t)\} = \{\xi_i y_i^*(t)\} \in PAP^\mu(\mathbb{R}, \mathbb{R}^n)$ is a weighted pseudo almost periodic solution of (1.1).

Finally, we demonstrate that x^* is exponentially stable.

Designate $x(t) = \{x_i(t)\}$ be an arbitrary solution of (1.1) with initial value $\varphi(t) = \{\varphi_i(t)\}$ satisfying (1.2).

Label

$$x_i(t) = \varphi_i(t) = \varphi_i(\sigma_i t_0), \text{ for all } t \in [r_i \sigma_i t_0, \sigma_i t_0], \quad (3.6)$$

$$y_i(t) = \xi_i^{-1} x_i(t), \quad y_i^*(t) = \xi_i^{-1} x_i^*(t), \quad z_i(t) = y_i(t) - y_i^*(t), \quad Z_i(t) = z_i(t) - p_i(t) z_i(r_i t), \quad i \in N.$$

Then

$$\begin{aligned} Z_i'(t) = & -a_i(t)Z_i(t) - a_i(t)p_i(t)z_i(r_i t) + \xi_i^{-1} \sum_{j=1}^n e_{ij}(t)(f_j(\xi_j y_j(t)) - f_j(\xi_j y_j^*(t))) \\ & + \xi_i^{-1} \sum_{j=1}^n b_{ij}(t)(g_j(\xi_j y_j(q_{ij} t)) - g_j(\xi_j y_j^*(q_{ij} t))), \quad i \in N. \end{aligned} \quad (3.7)$$

Without loss of generality, let

$$\|\varphi - x^*\|_\xi = \max_{i \in N} \left\{ \sup_{t \in [\rho_i t_0, t_0]} \xi_i^{-1} |[\varphi_i(t) - p_i(t)\varphi_i(r_i t)] - [x_i^*(t) - p_i(t)x_i^*(r_i t)]| \right\} > 0, \quad (3.8)$$

and M be a constant such that

$$M > \sum_{i=1}^N K_i + 1. \quad (3.9)$$

Consequently, for any $\varepsilon > 0$, it is obvious that

$$|Z_i(t)| < M(\|\varphi - x^*\|_\xi + \varepsilon)e^{-\sigma \ln \frac{1+t}{1+t_0}} \text{ for all } t \in (\rho_i t_0, t_0], \quad i \in N. \quad (3.10)$$

Now, we validate that

$$\|Z(t)\| < M(\|\varphi - x^*\|_\xi + \varepsilon)e^{-\sigma \ln \frac{1+t}{1+t_0}} \text{ for all } t > t_0. \quad (3.11)$$

Otherwise, there must exist $i \in N$ and $\theta > t_0$ such that

$$\begin{cases} |Z_i(\theta)| = M(\|\varphi - x^*\|_\xi + \varepsilon)e^{-\sigma \ln \frac{1+\theta}{1+t_0}}, \\ \|Z(t)\| < M(\|\varphi - x^*\|_\xi + \varepsilon)e^{-\sigma \ln \frac{1+t}{1+t_0}} \text{ for all } t \in (\rho_i t_0, \theta). \end{cases} \quad (3.12)$$

Furthermore, from (3.6), we obtain

$$e^{\sigma \ln \frac{1+v}{1+t_0}} |z_j(v)| \leq e^{\sigma \ln \frac{1+v}{1+t_0}} |z_j(v) - p_j(v)z_j(r_j v)| + e^{\sigma \ln \frac{1+v}{1+t_0}} |p_j(v)z_j(r_j v)|$$

$$\begin{aligned}
&\leq e^{\sigma \ln \frac{1+\nu}{1+t_0}} |Z_j(\nu)| + p_j^+ e^{\sigma \ln \frac{1+\nu}{1+r_j \nu}} e^{\sigma \ln \frac{1+r_j \nu}{1+t_0}} |z_j(r_j \nu)| \\
&\leq M(\|\varphi - x^*\|_\xi + \varepsilon) + p_j^+ e^{\sigma \ln \frac{1}{r_j}} \sup_{s \in [\rho_j t_0, r_j t]} e^{\sigma \ln \frac{1+s}{1+t_0}} |z_j(s)| \\
&\leq M(\|\varphi - x^*\|_\xi + \varepsilon) + p_j^+ e^{\sigma \ln \frac{1}{r_j}} \sup_{s \in [\rho_j t_0, t]} e^{\sigma \ln \frac{1+s}{1+t_0}} |z_j(s)|,
\end{aligned} \tag{3.13}$$

for all $\nu \in [\rho_j t_0, t]$, $t \in [t_0, \theta]$, $j \in J$, which entails that

$$e^{\sigma \ln \frac{1+t}{1+t_0}} |z_j(t)| \leq \sup_{s \in [\rho_j t_0, t]} e^{\sigma \ln \frac{1+s}{1+t_0}} |z_j(s)| \leq \frac{M(\|\varphi - x^*\|_\xi + \varepsilon)}{1 - p_j^+ e^{\sigma \ln \frac{1}{r_j}}}, \tag{3.14}$$

for all $t \in [\rho_j t_0, \theta]$, $j \in N$.

Note that

$$\begin{aligned}
&Z_i'(s) + a_i(s)Z_i(s) \\
&= -a_i(s)p_i(s)z_i(r_i s) + \xi_i^{-1} \sum_{j=1}^n e_{ij}(s)(f_j(\xi_j y_j(s)) - f_j(\xi_j y_j^*(s))) \\
&\quad + \xi_i^{-1} \sum_{j=1}^n b_{ij}(s)(g_j(\xi_j y_j(q_{ij}s)) - g_j(\xi_j y_j^*(q_{ij}s))), \quad s \in [t_0, t], \quad t \in [t_0, \theta].
\end{aligned} \tag{3.15}$$

Multiplying both sides of (3.15) by $e^{\int_{t_0}^s a_i(u)du}$, and integrating it on $[t_0, t]$, we get

$$\begin{aligned}
Z_i(t) &= Z_i(t_0)e^{-\int_{t_0}^t a_i(u)du} + \int_{t_0}^t e^{-\int_s^t a_i(u)du} [-a_i(s)p_i(s)z_i(r_i s) + \\
&\quad \xi_i^{-1} \sum_{j=1}^n e_{ij}(s)(f_j(\xi_j y_j(s)) - f_j(\xi_j y_j^*(s))) \\
&\quad + \xi_i^{-1} \sum_{j=1}^n b_{ij}(s)(g_j(\xi_j y_j(q_{ij}s)) - g_j(\xi_j y_j^*(q_{ij}s)))] ds, \quad t \in [t_0, \theta].
\end{aligned}$$

Thus, with the help of (3.3), (3.9), (3.12) and (3.14), we have

$$\begin{aligned}
|Z_i(\theta)| &= \left| Z_i(t_0)e^{-\int_{t_0}^\theta a_i(u)du} + \int_{t_0}^\theta e^{-\int_s^\theta a_i(u)du} [-a_i(s)p_i(s)z_i(r_i s) + \right. \\
&\quad \xi_i^{-1} \sum_{j=1}^n e_{ij}(s)(f_j(\xi_j y_j(s)) - f_j(\xi_j y_j^*(s))) \\
&\quad \left. + \xi_i^{-1} \sum_{j=1}^n b_{ij}(s)(g_j(\xi_j y_j(q_{ij}s)) - g_j(\xi_j y_j^*(q_{ij}s)))] ds \right| \\
&\leq (\|\varphi - x^*\|_\xi + \varepsilon) K_i e^{-\int_{t_0}^\theta \tilde{a}_i(u)du} \\
&\quad + \int_{t_0}^\theta e^{-\int_s^\theta \tilde{a}_i(u)du} K_i [-a_i(s)p_i(s)z_i(r_i s)| + \xi_i^{-1} \sum_{j=1}^n |e_{ij}(s)| L_j^f \xi_j |z_j(s)|] ds
\end{aligned}$$

$$\begin{aligned}
& + \xi_i^{-1} \sum_{j=1}^n |b_{ij}(s)| L_j^g \xi_j |z_j(q_{ij}s)|] ds \\
& \leq (\|\varphi - x^*\|_\xi + \varepsilon) K_i e^{-\int_{t_0}^\theta \tilde{a}_i(u) du} \\
& \quad + \int_{t_0}^\theta e^{-\int_s^\theta \tilde{a}_i(u) du} K_i [|a_i(s)p_i(s)| \frac{M(\|\varphi - x^*\|_\xi + \varepsilon)}{1 - p_i^+ e^{\sigma \ln \frac{1}{r_i}}} e^{-\sigma \ln \frac{1+r_i s}{1+t_0}} \\
& \quad + \xi_i^{-1} \sum_{j=1}^n |e_{ij}(s)| L_j^f \xi_j \frac{M(\|\varphi - x^*\|_\xi + \varepsilon)}{1 - p_j^+ e^{\sigma \ln \frac{1}{r_j}}} e^{-\sigma \ln \frac{1+s}{1+t_0}} \\
& \quad + \xi_i^{-1} \sum_{j=1}^n |b_{ij}(s)| L_j^g \xi_j \frac{M(\|\varphi - x^*\|_\xi + \varepsilon)}{1 - p_j^+ e^{\sigma \ln \frac{1}{r_j}}} e^{-\sigma \ln \frac{1+q_{ij}s}{1+t_0}}] ds \\
& = M(\|\varphi - x^*\|_\xi + \varepsilon) e^{-\sigma \ln \frac{1+\theta}{1+t_0}} \left\{ \frac{K_i}{M} e^{-\int_{t_0}^\theta [\tilde{a}_i(u) - \frac{\sigma}{1+u}] du} \right. \\
& \quad + \int_{t_0}^\theta e^{-\int_s^\theta [\tilde{a}_i(u) - \frac{\sigma}{1+u}] du} K_i [|a_i(s)p_i(s)| \frac{1}{1 - p_i^+ e^{\sigma \ln \frac{1}{r_i}}} e^{\sigma \ln \frac{1+s}{1+r_i s}} \\
& \quad + \xi_i^{-1} \sum_{j=1}^n |e_{ij}(s)| L_j^f \xi_j \frac{1}{1 - p_j^+ e^{\sigma \ln \frac{1}{r_j}}} \\
& \quad \left. + \xi_i^{-1} \sum_{j=1}^n |b_{ij}(s)| L_j^g \xi_j \frac{1}{1 - p_j^+ e^{\sigma \ln \frac{1}{r_j}}} e^{\sigma \ln (\frac{1+s}{1+q_{ij}s})} \right] ds \Big\} \\
& \leq M(\|\varphi - x^*\|_\xi + \varepsilon) e^{-\sigma \ln \frac{1+\theta}{1+t_0}} \left\{ \frac{K_i}{M} e^{-\int_{t_0}^\theta [\tilde{a}_i(u) - \frac{\sigma}{1+u}] du} \right. \\
& \quad + \int_{t_0}^\theta e^{-\int_s^\theta [\tilde{a}_i(u) - \frac{\sigma}{1+u}] du} K_i [|a_i(s)p_i(s)| \frac{1}{1 - p_i^+ e^{\sigma \ln \frac{1}{r_i}}} e^{\sigma \ln \frac{1}{r_i}} \\
& \quad + \xi_i^{-1} \sum_{j=1}^n |e_{ij}(s)| L_j^f \xi_j \frac{1}{1 - p_j^+ e^{\sigma \ln \frac{1}{r_j}}} \\
& \quad \left. + \xi_i^{-1} \sum_{j=1}^n |b_{ij}(s)| L_j^g \xi_j \frac{1}{1 - p_j^+ e^{\sigma \ln \frac{1}{r_j}}} e^{\sigma \ln (\frac{1}{r_j})} \right] ds \Big\} \\
& \leq M(\|\varphi - x^*\|_\xi + \varepsilon) e^{-\sigma \ln \frac{1+\theta}{1+t_0}} \left[1 - \left(1 - \frac{K_i}{M} \right) e^{-\int_{t_0}^\theta (\tilde{a}_i(u) - \frac{\sigma}{1+u}) du} \right] \\
& < M(\|\varphi - x^*\|_\xi + \varepsilon) e^{-\sigma \ln \frac{1+\theta}{1+t_0}}.
\end{aligned}$$

This is a clear contradiction of (3.12). Hence, (3.11) holds. When $\varepsilon \rightarrow 0^+$, we obtained

$$\|Z(t)\| \leq M\|\varphi - x^*\|_\xi e^{-\sigma \ln \frac{1+\theta}{1+t_0}} \quad \text{for all } t > t_0. \quad (3.17)$$

Then, using a similar derivation in the proof of (3.13) and (3.14), with the help of (3.17), we can know that

$$e^{\sigma \ln \frac{1+t}{1+t_0}} |z_j(t)| \leq \sup_{s \in [\rho_j t_0, t]} e^{\sigma \ln \frac{1+s}{1+t_0}} |z_j(s)| \leq \frac{M\|\varphi - x^*\|_\xi}{1 - p_j^+ e^{\sigma \ln \frac{1}{r_j}}},$$

and

$$|z_j(t)| \leq \frac{M\|\varphi - x^*\|_\xi}{1 - p_j^+ e^{\sigma \ln \frac{1}{r_j}}} \left(\frac{1+t_0}{1+t}\right)^\sigma \quad \text{for all } t > t_0, j \in N.$$

The proof of the Theorem 3.1 is now finished.

Theorem 3.2. Let $\mu \in \mathbb{U}_\infty^+$. Assume that (H_1) and (H_2) hold, and there exist constants $\gamma_i, \xi_i > 0$ such that

$$\sup_{t \in \mathbb{R}} \{-\tilde{a}_i(t) + K_i[\xi_i^{-1} \sum_{j=1}^n (|e_{ij}(t)|L_j^f + |b_{ij}(t)|L_j^g)\xi_j]\} < -\gamma_i \quad \text{for all } i \in N, \quad (3.18)$$

holds. Then, system (1.3) has a unique WPAPS $x^*(t) \in PAP^\mu(\mathbb{R}, \mathbb{R}^n)$, and there is a constant $\sigma \in (0, \min_{i \in N} \tilde{a}_i^-)$ such that

$$x_i(t) - x_i^*(t) = O\left(\left(\frac{1}{1+t}\right)^\sigma\right) \quad \text{as } t \rightarrow +\infty,$$

here $i \in N$, $x(t)$ is an arbitrary solution of system (1.3) with initial conditions:

$$x_i(s) = \varphi_i(s), \quad s \in [\rho_i^* t_0, t_0], \quad \varphi_i \in C([\rho_i^* t_0, t_0], \mathbb{R}), \quad \rho_i^* = \min_{1 \leq j \leq n} \{q_{ij}\}, \quad i \in N.$$

Proof. From (3.18) we can pick a positive constant Λ_i^* such that

$$\sup_{t \in \mathbb{R}} \frac{1}{\tilde{a}_i(t)} K_i \left[\xi_i^{-1} \sum_{j=1}^n (|e_{ij}(t)|L_j^f + |b_{ij}(t)|L_j^g)\xi_j \right] < \Lambda_i^* < 1, \quad i \in N. \quad (3.19)$$

According to fact that (1.3) is a special case of (1.1) with $p_i^+ = 0$ ($i \in N$), the proof proceeds in the same way as in Theorem 3.1.

Remark 3.1. Obviously, it is easy to see that all results in [22] are the special case of Theorem 2.2 in this manuscript. In particular, the wrong in (1.5) has been successfully corrected. This indicates that our results supplement and improve the previous references [22, 23]

4. A numerical example

In order to reveal the correctness and feasibility of the obtained results, an example with the simulation is introduced in this section.

Example 4.1. Consider the following CNNs with D operator and multi-proportional delays:

$$\left\{ \begin{array}{l} [x_1(t) - \frac{\sin t}{100} x_1(\frac{1}{3}t)]' \\ = -(\frac{1}{5} + \frac{3}{10} \sin 20t)x_1(t) + \frac{1}{20}(\sin 2t + e^{-t^2}(\sin t)^2)\frac{1}{20} \arctan(x_1(t)) \\ + \frac{1}{20}(\sin 3t + e^{-t^4}(\sin t)^4)\frac{1}{20} \arctan(x_2(t)) \\ + \frac{1}{20}(\cos 2t + e^{-t^2}(\cos t)^2)\frac{1}{20} x_1(\frac{1}{2}t) \\ + \frac{1}{20}(\cos 3t + e^{-t^4}(\cos t)^4)\frac{1}{20} x_2(\frac{1}{3}t) + e^{-t^2} + \sin(\sqrt{3}t), \\ [x_2(t) - \frac{\cos t}{100} x_2(\frac{1}{3}t)]' \\ = -(\frac{1}{5} + \frac{3}{10} \cos 20t)x_2(t) + \frac{1}{20}(\cos 2t + e^{-t^2}(\cos t)^2)\frac{1}{20} \arctan(x_1(t)) \\ + \frac{1}{20}(\cos 3t + e^{-t^4}(\cos t)^4)\frac{1}{20} \arctan(x_2(t)) \\ + \frac{1}{20}(\cos 3t + e^{-t^6}(\cos t)^6)\frac{1}{20} x_1(\frac{1}{3}t) \\ + \frac{1}{20}(\cos 5t + e^{-t^8}(\cos t)^8)\frac{1}{20} x_2(\frac{1}{4}t) + e^{-t^4} + \sin(\sqrt{5}t). \end{array} \right. \quad (4.1)$$

Clearly,

$$n = 2, q_{ij} = \frac{1}{i+j}, t_0 = 1, f_i(x) = \frac{1}{20} \arctan x, g_i(x) = \frac{1}{20}x, i, j = 1, 2.$$

Then, we can take

$$\tilde{a}_i(t) = \frac{1}{5}, \xi_i = 1, L_i^f = L_i^g = \frac{1}{20}, K_i = e^{\frac{3}{10}}, \mu(t) = t^2 + 1, i, j = 1, 2,$$

such that CNNs (1.1) with (4.1) satisfies all the conditions $(H_1) - (H_3)$. By Theorem 2.1, we can conclude that CNNs (4.1) has a unique weighted pseudo almost periodic solution $x^*(t) \in PAP^\mu(\mathbb{R}, \mathbb{R}^2)$, and every solutions of (4.1) is exponentially convergent to $x^*(t)$ as $t \rightarrow +\infty$. Here, the exponential convergence rate $\sigma \approx 0.01$. Simulations in Figure 1 reflect that the theoretical convergence is in sympathy with the numerically observed behaviors.

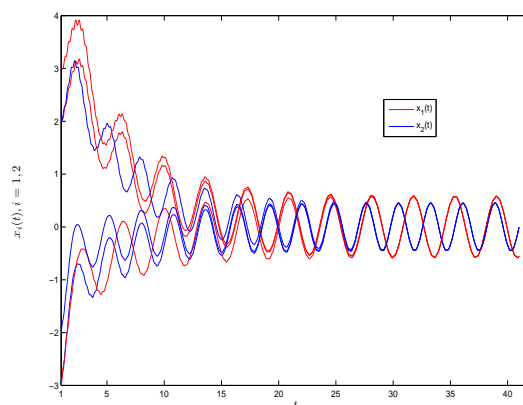


Figure 1. Numerical solutions $x(t)$ to system (3.1) with initial values: $(\varphi_1(s), \varphi_2(s)) = (2, -2), (-3, 2), (3, -2), t_0 = 1$.

As far as we know, the weighted pseudo almost periodic dynamics of cellular neural networks with D operator and multi-proportional delays has never been studied in the previous literature [29–52]. It is easy to see that all results in [16–28] cannot be directly applied to show the case that all solutions of (4.1) converge globally to the weighted pseudo almost periodic solution. In particular, all parameters in system (4.1) are chosen by applying Matlab software. It should be mentioned that the nonlinear activation function $f_i(x) = \frac{1}{20} \arctan x$ has been usually used as the sigmoid functions to agree with the experimental data of signal transmission in the real cellular networks networks.

5. Conclusions

In this paper, we investigate the global dynamic behaviors on a class of neutral type CNNs with D operator and multi-proportional delays. Some new criteria have been gained to guarantee that the existence and exponential stability of weighted pseudo almost periodic solutions for the addressed system by combining the fixed point theorem and some differential inequality techniques. The obtained results are new and complement some corresponding ones of the existing literature. It should be mentioned that the technical assumptions can be easily checked by simple algebra methods and

convenient for application in practice. In addition, this method affords a possible approach to study the weighted pseudo dynamics of other cellular neural networks with D operator and delays. In the future, we will make this further research.

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Conflict of interest

We confirm that we have no conflict of interest.

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