AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(2): 1851–1864.
DOI:10.3934/math.2021112
Received: 14 June 2020
Accepted: 05 November 2020
Published: 02 December 2020

Research article

Fixed point results in double controlled quasi metric type spaces

Abdullah Shoaib[1,*], Poom Kumam[2,3,4,*], Shaif Saleh Alshoraify[5] and Muhammad Arshad[5]

[1] Department of Mathematics and Statistics, Riphah International University, Islamabad, Pakistan
[2] Fixed Point Research Laboratory, Fixed Point Theory and Applications Research Group, Center of Excellence in Theoretical and Computational Science (TaCS-CoE), Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha Uthit Rd., Bang Mod, Thung Khru, Bangkok 10140, Thailand
[3] Center of Excellence in Theoretical and Computational Science (TaCS-CoE), Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha Uthit Rd., Bang Mod, Thung Khru, Bangkok 10140, Thailand
[4] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
[5] Department of Mathematics and Statistics, International Islamic University, H-10, Islamabad 44000, Pakistan

* **Correspondence:** Email: abdullahshoaib15@yahoo.com; poom.kum@kmutt.ac.th; Tel: +923218555352.

Abstract: Abdeljawad et.al (Mathematics, 6(12), 320, 2018) introduced a new concept, named double controlled metric type spaces, as a generalization of the notion of extended b-metric spaces. In this paper, we introduce double controlled quasi metric type spaces and obtain common fixed points of multivalued mappings satisfying rational type, Reich type and Kannan type contractions in double controlled quasi metric type spaces. Some concrete examples are also provided to illustrate the superiority of our results over other existing results.

Keywords: common fixed point; left Cauchy sequence; contractive multivalued mappings; double controlled quasi metric type space
Mathematics Subject Classification: 47H10, 54H25

1. Introduction and preliminaries

The theory of fixed points takes an important place in the transition from classical analysis to modern analysis. One of the most remarkable works on fixed point of functions was done by Banach [9].

Various generalizations of Banach fixed point result were made by numerous mathematicians [1–3, 6, 12–14, 17, 19, 20, 25, 26]. One of the generalizations of the metric space is the quasi metric space that was introduced by Wilson [30]. The commutativity condition does not hold in general in quasi metric spaces. Several authors used these concepts to prove some fixed point theorems, see [7, 10, 15, 21]. On the other hand Bakhtin [8] and Czerwik [11] generalized the triangle inequality by multiplying the right hand side of triangle inequality in metric spaces by a parameter $b \geq 1$ and defined b-metric spaces, for more results, see [4, 23, 24]. Fixed point results for multivalued mappings generalizes the results for single-valued mappings. Many interesting results have been proved in the setting of multivalued mappings, for example, see [5, 26, 29]. Kamran et al. [12] introduced a new concept of generalized b metric spaces, named as extended b-metric spaces, see also [22]. They replaced the parameter $b \geq 1$ in triangle inequality by the control function $\theta : X \times X \to [1, \infty)$. Mlaiki et al. [16] replaced the triangle inequality in b-metric spaces by using control function in a different style and introduced controlled metric type spaces. Abdeljawad et al. [1] generalized the idea of controlled metric type spaces and introduced double controlled metric type spaces. They replaced the control function $\xi(x, y)$ in triangle inequality by two functions $\xi(x, y)$ and $\top(x, y)$, see also [27, 28]. In this paper, the concept of double controlled quasi metric type spaces has been discussed. Fixed point results and several examples are established. First of all, we discuss the previous concepts that will be useful to understand the paper.

Now, we define double controlled metric type space.

Definition 1.1. [1] Given non-comparable functions $\xi, \top : X \times X \to [1, \infty)$. If $\triangle : X \times X \to [0, \infty)$ satisfies:

(q1) $\triangle(\omega, v) = 0$ if and only if $\omega = v$,

(q2) $\triangle(\omega, v) = \triangle(v, \omega)$,

(q3) $\triangle(\omega, v) \leq \xi(\omega, e)\triangle(\omega, e) + \top(e, v)\triangle(e, v)$,

for all $\omega, v, e \in X$. Then, $\triangle$ is called a double controlled metric with the functions ξ, $\top$ and the pair $(X, \triangle)$ is called double controlled metric type space with the functions $\xi, \top$.

The classical result to obtain fixed point of a mapping in double controlled metric type space is given below.

Theorem 1.2. [1] Let $(X, \triangle)$ be a complete double controlled metric type space with the functions $\xi, \top : X \times X \to [1, \infty)$ and let $T : X \to X$ be a given mapping. Suppose that there exists $k \in (0, 1)$ such that

$$\triangle(T(x), T(y) \leq k\triangle(x, y), \text{ for all } x, y \in X.$$

For $\omega_0 \in X$, choose $\omega_g = T^g\omega_0$. Assume that

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\xi(\omega_{i+1}, \omega_{i+2})}{\xi(\omega_i, \omega_{i+1})} \top(\omega_{i+1}, \omega_m) < \frac{1}{k}.$$

In the addition, assume that, for every $\omega \in X$, we have

$$\lim_{g\to\infty} \xi\left(\omega, \omega_g\right), \text{ and } \lim_{g\to\infty} \top\left(\omega_g, \omega\right) \text{ exists and are finite.}$$

Then T has a unique fixed point $\omega^* \in X$.

Now, we are introducing the concept of double controlled quasi metric type space and controlled quasi metric type space.

Definition 1.3. Given non-comparable functions $\xi, \top : X \times X \to [1, \infty)$. If $\triangle : X \times X \to [0, \infty)$ satisfies

(q1) $\triangle(\omega, v) = 0$ if and only if $\omega = v$,
(q2) $\triangle(\omega, v) \le \xi(\omega, e)\triangle(\omega, e) + \top(e, v)\triangle(e, v)$,
for all $\omega, v, e \in X$. Then, $\triangle$ is called a double controlled quasi metric type with the functions ξ, $\top$ and $(X, \triangle)$ is called a double controlled quasi metric type space. If $\top(e, v) = \xi(e, v)$ then $(X, \triangle)$ is called a controlled quasi metric type space.

Remark 1.4. Any quasi metric space or any double controlled metric type space is also a double controlled quasi metric type space but, the converse is not true in general, see examples (1.5, 2.4, 2.12 and 2.15).

Example 1.5. Let $X = \{0, 1, 2\}$. Define $\triangle : X \times X \to [0,\infty)$ by $\triangle(0, 1) = 4$, $\triangle(0, 2) = 1$, $\triangle(1, 0) = 3 = \triangle(1, 2)$, $\triangle(2, 0) = 0$, $\triangle(2, 1) = 2$, $\triangle(0, 0) = \triangle(1, 1) = \triangle(2, 2) = 0$.
Define $\xi, \top : G \times G \to [1,\infty)$ as $\xi(0, 1) = \xi(1, 0) = \xi(1, 2) = 1$, $\xi(0, 2) = \frac{5}{4}$, $\xi(2, 0) = \frac{10}{9}$, $\xi(2, 1) = \frac{20}{19}$, $\xi(0, 0) = \xi(1, 1) = \xi(2, 2) = 1$, $\top(0, 1) = \top(1, 0) = \top(0, 2) = \top(1, 2) = 1$, $\top(2, 0) = \frac{3}{2}$, $\top(2, 1) = \frac{11}{8}$, $\top(0, 0) = \top(1, 1) = \top(2, 2) = 1$. It is clear that $\triangle$ is double controlled quasi metric type with the functions ξ, $\top$. Let $w = 0$, $e = 2$, $v = 1$, we have

$$\triangle(0, 1) = 4 > 3 = \triangle(0, 2) + \triangle(2, 1).$$

So $\triangle$ is not a quasi metric. Also, it is not a controlled quasi metric type. Indeed,

$$\triangle(0, 1) = 4 > \frac{13}{4} = \xi(0, 2)\triangle(0, 2) + \xi(2, 1)\triangle(2, 1).$$

Moreover, it is not double controlled metric type space because, we have

$$\triangle(0, 1) = \xi(0, 2)\triangle(0, 2) + \top(2, 1)\triangle(2, 1) = \frac{55}{16} \neq \triangle(1, 0).$$

The convergence of a sequence in double controlled quasi metric type space is defined as:

Definition 1.6. Let $(X, \triangle)$ be a double controlled quasi metric type space with two functions. A sequence $\{u_t\}$ is convergent to some u in X if and only if $\lim_{t\to\infty} \triangle(u_t, u) = \lim_{t\to\infty} \triangle(u, u_t) = 0$.

Now, we discuss left Cauchy, right Cauchy and dual Cauchy sequences in double controlled quasi metric type space.

Definition 1.7. Let $(X, \triangle)$ be a double controlled quasi metric type space with two functions.
(i) The sequence $\{u_t\}$ is a left Cauchy if and only if for every $\varepsilon > 0$ such that $\triangle(u_m, u_t) < \varepsilon$, for all $t > m > t_\varepsilon$, where t_ε is some integer or $\lim_{t,m\to\infty} \triangle(u_m, u_t) = 0$.
(ii) The sequence $\{u_t\}$ is a right Cauchy if and only if for every $\varepsilon > 0$ such that $\triangle(u_m, u_t) < \varepsilon$, for all $m > t > t_\varepsilon$, where t_ε is some integer.
(iii) The sequence $\{u_t\}$ is a dual Cauchy if and only if it is both left as well as right Cauchy.

Now, we define left complete, right complete and dual complete double controlled quasi metric type spaces.

Definition 1.8. Let $(X, \triangle)$ be a double controlled quasi metric type space. Then $(X, \triangle)$ is left complete, right complete and dual complete if and only if each left-Cauchy, right Cauchy and dual Cauchy sequence in X is convergent respectively.

Note that every dual complete double controlled quasi metric type space is left complete but the converse is not true in general, so it is better to prove results in left complete double controlled quasi metric type space instead of dual complete.

Best approximation in a set and proximinal set are defined as:

Definition 1.9. Let $(\Im, \triangle)$ be a double controlled quasi metric type space. Let A be a non-empty set and $l \in \Im$. An element $y_0 \in A$ is called a best approximation in A if

$$\begin{aligned} \triangle(l, A) &= \triangle(l, y_0), \text{ where } \triangle(l, A) = \inf_{y \in A} \triangle(l, y) \\ \text{and } \triangle(A, l) &= \triangle(y_0, l), \text{ where } \triangle(A, l) = \inf_{y \in A} \triangle(y, l). \end{aligned}$$

If each $l \in \Im$ has a best approximation in A, then A is know as proximinal set. $P(\Im)$ is equal to the set of all proximinal subsets of $\Im$.

Double controlled Hausdorff quasi metric type space is defined as:

Definition 1.10. The function $H_{\triangle} : P(E) \times P(E) \to [0, \infty)$, defined by

$$H_{\triangle}(C, F) = \max\left\{\sup_{a \in C} \triangle(a, F), \ \sup_{b \in F} \triangle(C, b)\right\}$$

is called double controlled quasi Hausdorff metric type on $P(E)$. Also $(P(E), H_{\triangle})$ is known as double controlled Hausdorff quasi metric type space.

The following lemma plays an important role in the proof of our main result.

Lemma 1.11. Let $(X, \triangle)$ be a double controlled quasi metric type space. Let $(P(E), H_{\triangle})$ be a double controlled Hausdorff quasi metric type space on $P(E)$. Then, for all $C, F \in P(E)$ and for each $c \in C$, there exists $f_c \in F$, such that $H_{\triangle}(C, F) \geq \triangle(c, f_c)$ and $H_{\triangle}(F, C) \geq \triangle(f_c, c)$.

2. Main results

Let $(X, \triangle)$ be a double controlled quasi metric type space, $u_0 \in X$ and $T : X \to P(X)$ be multifunctions on X. Let $u_1 \in Tu_0$ be an element such that $\triangle(u_0, Tu_0) = \triangle(u_0, u_1)$, $\triangle(Tu_0, u_0) = \triangle(u_1, u_0)$. Let $u_2 \in Tu_1$ be such that $\triangle(u_1, Tu_1) = \triangle(u_1, u_2)$, $\triangle(Tu_1, u_1) = \triangle(u_2, u_1)$. Let $u_3 \in Tu_2$ be such that $\triangle(u_2, Tu_2) = \triangle(u_2, u_3)$ and so on. Thus, we construct a sequence u_t of points in X such that $u_{t+1} \in Tu_t$ with $\triangle(u_t, Tu_t) = \triangle(u_t, u_{t+1})$, $\triangle(Tu_t, u_t) = \triangle(u_{t+1}, u_t)$, where $t = 0, 1, 2, \cdots$. We denote this iterative sequence by $\{XT(u_t)\}$. We say that $\{XT(u_t)\}$ is a sequence in X generated by u_0 under double controlled quasi metric $\triangle$. If $\triangle$ is quasi b-metric then, we say that $\{XT(u_t)\}$ is a sequence in X generated by u_0 under quasi b-metric $\triangle$. We can define $\{XT(u_t)\}$ in other metrics in a similar way.

Now, we define double controlled rational contracion which is a generalization of many other classical contractions.

Definition 2.1. Let $(X, \triangle)$ be a complete double controlled quasi-metric type space with the functions $\alpha, \mu : X \times X \to [1, \infty)$. A multivalued mapping $T : X \to P(X)$ is called a double controlled rational contracion if the following conditions are satisfied:

$$H_{\triangle}(Tx, Ty) \leq k(Q(x, y)), \tag{2.1}$$

for all $x, y \in X$, $0 < k < 1$ and

$$Q(x, y) = \max\left\{\triangle(x, y), \triangle(x, Tx), \frac{\triangle(x, Tx)\triangle(x, Ty) + \triangle(y, Ty)\triangle(y, Tx)}{\triangle(x, Ty) + \triangle(y, Tx)}\right\}.$$

Also, for the terms of the sequence $\{XT(u_t)\}$, we have

$$\sup_{m\geq 1}\lim_{i\to\infty}\frac{\alpha(u_{i+1},u_{i+2})}{\alpha(u_i,u_{i+1})}\mu(u_i,u_m)<\frac{1}{k} \tag{2.2}$$

and for every $u\in X$

$$\lim_{t\to\infty}\alpha(u,u_t) \text{ and } \lim_{t\to\infty}\mu(u_t,u) \text{ are finite.} \tag{2.3}$$

Now, we prove that an operator T satisfying certain rational contraction condition has a fixed point in double controlled quasi metric type space.

Theorem 2.2. Let $(X,\triangle)$ be a left complete double controlled quasi-metric type space with the functions $\alpha,\mu : X\times X\to[1,\infty)$ and $T : X\to P(X)$ be a double controlled rational contracion. Then, T has a fixed point $u^*\in X$.

Proof. By Lemma 1.11 and using inequality (2.1), we have

$$\triangle(u_t,u_{t+1})\leq H_\triangle(Tu_{t-1},Tu_t)\leq k\left(Q(u_{t-1},u_t)\right).$$

$$\begin{aligned} Q(u_{t-1},u_t) &\leq \max\Big\{\triangle(u_{t-1},u_t),\triangle(u_{t-1},u_t),\\ &\quad \left.\frac{\triangle(u_{t-1},u_t)\triangle(u_{t-1},Tu_t)+\triangle(u_t,u_{t+1})\triangle(u_t,Tu_{t-1})}{\triangle(u_{t-1},Tu_t)+\triangle(u_t,Tu_{t-1})}\right\}\\ &= \triangle(u_{t-1},u_t). \end{aligned}$$

Therefore,

$$\triangle(u_t,u_{t+1})\leq k\triangle(u_{t-1},u_t). \tag{2.4}$$

Now,

$$\triangle(u_{t-1},u_t)\leq H_\triangle(Tu_{t-2},Tu_{t-1})\leq k\left(Q(u_{t-2},u_{t-1})\right).$$

$$Q(u_{t-2},u_{t-1})=\max\Big\{\triangle(u_{t-2},u_{t-1}),\triangle(u_{t-2},u_{t-1}),$$
$$\left.\frac{\triangle(u_{t-2},u_{t-1})\triangle(u_{t-2},Tu_{t-1})+\triangle(u_{t-1},u_t)\triangle(u_{t-1},Tu_{t-2})}{\triangle(u_{t-2},Tu_{t-1})+\triangle(u_{t-1},Tu_{t-2})}\right\}.$$

Therefore,

$$\triangle(u_{t-1},u_t)\leq k\triangle(u_{t-2},u_{t-1}). \tag{2.5}$$

Using (2.5) in (2.4), we have

$$\triangle(u_t,u_{t+1})\leq k^2\triangle(u_{t-2},u_{t-1}).$$

Continuing in this way, we obtain

$$\triangle(u_t,u_{t+1})\leq k^t\triangle(u_0,u_1). \tag{2.6}$$

Now, by using (2.6) and by using the technique given in [1], it can easily be proved that $\{u_t\}$ is a left Cauchy sequence. So, for all natural numbers with $t<m$, we have

$$\lim_{t,m\to\infty}\triangle(u_t,u_m)=0. \tag{2.7}$$

Since $(X,\triangle)$ is a left complete double controlled quasi metric type space, there exists some $u^*\in X$ such that

$$\lim_{t\to\infty}\triangle(u_t,u^*)=\lim_{t\to\infty}\triangle(u^*,u_t)=0. \tag{2.8}$$

By using triangle inequality and then (2.1), we have

$$\triangle(u^*, Tu^*) \leq \alpha(u^*, u_{t+1})\triangle(u^*, u_{t+1}) + \mu(u_{t+1}, Tu^*)\triangle(u_{t+1}, Tu^*)$$

$$\leq \alpha(u^*, u_{t+1})\triangle(u^*, u_{t+1}) + \mu(u_{t+1}, Tu^*)\max\left\{\triangle(u_t, u^*), \triangle(u_t, u_{t+1}), \frac{\triangle(u_t, u_{t+1})\triangle(u_t, Tu^*) + \triangle(u^*, Tu^*)\triangle(u^*, u_t)}{\triangle(u_t, Tu^*) + \triangle(u^*, u_t)}\right\}.$$

Using (2.3), (2.7) and (2.8), we get $\triangle(u^*, Tu^*) \leq 0$. That is, $u^* \in Tu^*$. Thus u^* is a fixed point of T. □

If we take single-valued mapping instead of multivalued mapping, then we obtain the following result.

Theorem 2.3. Let $(X, \triangle)$ be a left complete double controlled quasi-metric type space with the functions $\alpha, \mu : X \times X \to [1, \infty)$ and $T : X \to X$ be a mapping such that:

$$\triangle(Tx, Ty) \leq k(Q(x, y)),$$

for all $x, y \in X$, $0 < k < 1$ and

$$Q(x, y) = \max\left\{\triangle(x, y), \triangle(x, Tx), \frac{\triangle(x, Tx)\triangle(x, Ty) + \triangle(y, Ty)\triangle(y, Tx)}{\triangle(x, Ty) + \triangle(y, Tx)}\right\},$$

Suppose that, for every $u \in X$ and for the Picard sequence $\{u_t\}$

$$\lim_{t\to\infty} \alpha(u, u_t), \ \lim_{t\to\infty} \mu(u_t, u) \text{ are finite and}$$

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\alpha(u_{i+1}, u_{i+2})}{\alpha(u_i, u_{i+1})}\mu(u_i, u_m) < \frac{1}{k}.$$

Then, T has a fixed point $u^* \in X$.

We present the following example to illustrate Theorem 2.3.

Example 2.4. Let $X = \{0, 1, 2, 3\}$. Define $\triangle : X \times X \to [0,\infty)$ by: $\triangle(0, 1) = 1$, $\triangle(0, 2) = 4$, $\triangle(0, 3) = 5$, $\triangle(1, 0) = 0$, $\triangle(1, 2) = 10$, $\triangle(1, 3) = 1$, $\triangle(2, 0) = 7$, $\triangle(2, 1) = 3$, $\triangle(2, 3) = 5$, $\triangle(3, 0) = 3$, $\triangle(3, 1) = 6$, $\triangle(3, 2) = 2$, $\triangle(0, 0) = \triangle(1, 1) = \triangle(2, 2) = \triangle(3, 3) = 0$. Define $\alpha, \mu : X \times X \to [1,\infty)$ as: $\alpha(0, 1) = 2$, $\alpha(1, 2) = \alpha(0, 2) = \alpha(1, 0) = \alpha(2, 0) = \alpha(3, 1) = \alpha(2, 3) = \alpha(0, 3) = 1$, $\alpha(1, 3) = 2$, $\alpha(2, 1) = \frac{7}{3}$, $\alpha(3, 0) = \frac{4}{3}$, $\alpha(3, 2) = \frac{3}{2}$, $\alpha(0, 0) = \alpha(1, 1) = \alpha(2, 2) = \alpha(3, 3) = 1$, $\mu(1, 2) = \mu(2, 1) = \mu(2, 0) = \mu(3, 0) = \mu(0, 3) = 1$, $\mu(1, 0) = \frac{3}{2}$, $\mu(0, 1) = 2$, $\mu(1, 3) = 3$, $\mu(3, 1) = 1$, $\mu(3, 2) = 4$, $\mu(2, 3) = 1$, $\mu(0, 2) = \frac{5}{2}$, $\mu(0, 0) = \mu(1, 1) = \mu(2, 2) = \mu(3, 3) = 1$. Clearly $(X, \triangle)$ is a double controlled quasi metric type space, but it is not a controlled quasi metric type space. Indeed,

$$\triangle(1, 2) = 10 > 4 = \alpha(1, 0)\triangle(1, 0) + \alpha(0, 2)\triangle(0, 2).$$

Also, it is not a double controlled metric type space. Take $T0 = T1 = \{0\}$, $T2 = T3 = \{1\}$ and $k = \frac{1}{3}$. We observe that

$$\triangle(Tx, Ty) \leq k(Q(x, y)), \text{ for all } x, y \in X.$$

Let $u_0 = 2$, we have $u_1 = Tu_0 = T2 = 1$, $u_2 = Tu_1 = 0$, $u_3 = Tu_2 = 0, \cdots$.

$$\sup_{m\geq 1}\lim_{i\to\infty}\frac{\alpha(u_{i+1},u_{i+2})}{\alpha(u_i,u_{i+1})}\mu(u_i,u_m) = 2 < 3 = \frac{1}{k}.$$

Also, for every $u \in X$, we have

$$\lim_{t\to\infty}\alpha(u,u_t) < \infty \text{ and } \lim_{t\to\infty}\mu(u_t,u) < \infty.$$

All hypotheses of Theorem 2.3 are satisfied and $u^* = 0$ is a fixed point.

As every quasi b-metric space is double controlled quasi metric type space but the converse is not true in general, so we obtain a new result in quasi b-metric space as a corollary of Theorem 2.3.

Theorem 2.5. Let $(X,\triangle)$ be a left complete quasi b-metric space and $T : X \to X$ be a mapping. Suppose that there exists $k \in (0,1)$ such that

$$\triangle(Tx,Ty) \leq k(Q(x,y))$$

whenever,

$$Q(x,y) = \max\left\{\triangle(x,y), \triangle(x,Tx), \frac{\triangle(x,Tx)\triangle(x,Ty)+\triangle(y,Ty)\triangle(y,Tx)}{\triangle(x,Ty)+\triangle(y,Tx)}\right\},$$

for all $x, y \in X$. Assume that $0 < bk < 1$. Then, T has a fixed point $u^* \in X$.

Remark 2.6. In the Example 2.3, note that $\triangle$ is quasi b-metric with $b = \frac{10}{3}$, but we can not apply Theorem 2.5 for any $b = \frac{10}{3}$ and $k = \frac{1}{3}$, because $bk \nless 1$.

Quasi metric version of Theorem 2.2 is given below:

Theorem 2.7. Let $(X,\triangle)$ be a left complete quasi metric space and $T : X \to P(X)$ be a multivalued mapping. Suppose that there exists $0 < k < 1$ such that

$$\begin{aligned} H_\triangle(Tx,Ty) \leq\ & k(\max\{\triangle(x,y), \triangle(x,Tx), \\ & \left.\left.\frac{\triangle(x,Tx)\triangle(x,Ty)+\triangle(y,Ty)\triangle(y,Tx)}{\triangle(x,Ty)+\triangle(y,Tx)}\right\}\right) \end{aligned}$$

for all $x, y \in X$. Then T has a fixed point $u^* \in X$.

Now, we extend the sequence $\{XT(u_t)\}$ for two mappings. Let $(X,\triangle)$ be a double controlled quasi metric type space, $u_0 \in X$ and $S, T : X \to P(X)$ be the multivalued mappings on X. Let $u_1 \in Su_0$ such that $\triangle(u_0, Su_0) = \triangle(u_0,u_1)$ and $\triangle(Su_0,u_0) = \triangle(u_1,u_0)$. Now, for $u_1 \in X$, there exist $u_2 \in Tu_1$ such that $\triangle(u_1,Tu_1) = \triangle(u_1,u_2)$ and $\triangle(Tu_1,u_1) = \triangle(u_2,u_1)$. Continuing this process, we construct a sequence u_t of points in X such that $u_{2t+1} \in Su_{2t}$, and $u_{2t+2} \in Tu_{2t+1}$ with $\triangle(u_{2t},Su_{2t}) = \triangle(u_{2t},u_{2t+1})$, $\triangle(Su_{2t},u_{2t}) = \triangle(u_{2t+1},u_{2t})$ and $\triangle(u_{2t+1},Tu_{2t+1}) = \triangle(u_{2t+1},u_{2t+2})$, $\triangle(Tu_{2t+1},u_{2t+1}) = \triangle(u_{2t+2},u_{2t+1})$. We denote this iterative sequence by $\{TS(u_t)\}$ and say that $\{TS(u_t)\}$ is a sequence in X generated by u_0.

Now, we introduce double controlled Reich type contraction.

Definition 2.8. Let X be a non empty set, $(X,\triangle)$ be a left complete double controlled quasi-metric type space with the functions $\alpha,\mu : X\times X \to [1,\infty)$ and $S,T : X \to P(X)$ be a multivalued mappings. Suppose that the following conditions are satisfied:

$$H_\triangle(Sx,Ty) \leq c(\triangle(x,y)) + k(\triangle(x,Sx)+\triangle(y,Ty)), \tag{2.9}$$

$$H_{\triangle}(Tx, Sy) \leq c(\triangle(x, y)) + k(\triangle(x, Tx) + \triangle(y, Sy)), \tag{2.10}$$

for each $x, y \in X$, $0 < c + 2k < 1$. For $u_0 \in X$, choose $u_t \in \{TS(u_t)\}$, we have

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\alpha(u_{i+1}, u_{i+2})}{\alpha(u_i, u_{i+1})} \mu(u_{i+1}, u_m) < \frac{1-k}{c+k}. \tag{2.11}$$

Then the pair (S, T) is called a double controlled Reich type contraction.

The following results extend the results of Reich [18].

Theorem 2.9. Let $S, T : X \to P(X)$ be the multivalued mappings, $(X, \triangle)$ be a left complete double controlled quasi metric type space and (S, T) be a pair of double controlled Reich type contraction. Suppose that, for all $u \in X$

$$\lim_{t \to \infty} \alpha(u, u_t) \text{ is finite and } \lim_{t \to \infty} \mu(u_t, u) < \frac{1}{k}. \tag{2.12}$$

Then, S and T have a common fixed point $\dot{z}$ in X.

Proof. Consider the sequence $\{TS(u_t)\}$. Now, by Lemma 1.11, we have

$$\triangle(u_{2t}, u_{2t+1}) \leq H_{\triangle}(Tu_{2t-1}, Su_{2t}) \tag{2.13}$$

By using the condition (2.10), we get

$$\begin{aligned} \triangle(u_{2t}, u_{2t+1}) &\leq c(\triangle(u_{2t-1}, u_{2t}) + k(\triangle(u_{2t-1}, Tu_{2t-1}) + \triangle(u_{2t}, Su_{2t}) \\ &\leq c(\triangle(u_{2t-1}, u_{2t}) + k(\triangle(u_{2t-1}, u_{2t}) + \triangle(u_{2t}, u_{2t+1})) \end{aligned}$$

$$\triangle(u_{2t}, u_{2t+1}) \leq \eta(\triangle(u_{2t-1}, u_{2t})), \tag{2.14}$$

where $\eta = \frac{c+k}{1-k}$. Now, by Lemma 1.11, we have

$$\triangle(u_{2t-1}, u_{2t}) \leq H_{\triangle}(Su_{2t-2}, Tu_{2t-1}).$$

So, by using the condition (2.9), we have

$$\begin{aligned} \triangle(u_{2t-1}, u_{2t}) &\leq c\triangle(u_{2t-2}, u_{2t-1}) + k(\triangle(u_{2t-2}, Su_{2t-2}) + \triangle(u_{2t-1}, Tu_{2t-1})) \\ &\leq c\triangle(u_{2t-2}, u_{2t-1}) + k(\triangle(u_{2t-2}, u_{2t-1}) + \triangle(u_{2t-1}, u_{2t})) \end{aligned}$$

$$\triangle(u_{2t-1}, u_{2t}) \leq \frac{c+k}{1-k}(\triangle(u_{2t-2}, u_{2t-1})) = \eta(\triangle(u_{2t-2}, u_{2t-1})). \tag{2.15}$$

Using (2.14) in (2.15), we have

$$\triangle(u_{2t}, u_{2t+1}) \leq \eta^2 \triangle(u_{2t-2}, u_{2t-1}). \tag{2.16}$$

Now, by Lemma 1.11 we have

$$\triangle(u_{2t-2}, u_{2t-1}) \leq H_{\triangle}(Tu_{2t-3}, Su_{2t-2}).$$

Using the condition (2.10), we have

$$\triangle(u_{2t-2}, u_{2t-1}) \leq c\triangle(u_{2t-3}, u_{2t-2}) + k(\triangle(u_{2t-3}, u_{2t-2}) + \triangle(u_{2t-2}, u_{2t-1}))$$

implies

$$\triangle(u_{2t-2}, u_{2t-1}) \leq \eta\left(\triangle(u_{2t-3}, u_{2t-2})\right). \tag{2.17}$$

From (2.16) and (2.17), we have

$$\eta^2(\triangle(u_{2t-2}, u_{2t-1})) \leq \eta^3(\triangle(u_{2t-3}, u_{2t-2})). \tag{2.18}$$

Using (2.18) in (2.14), we have

$$\triangle(u_{2t}, u_{2t+1}) \leq \eta^3(\triangle(u_{2t-3}, u_{2t-2})).$$

Continuing in this way, we get

$$\triangle(u_{2t}, u_{2t+1}) \leq \eta^{2t}(\triangle(u_0, u_1)). \tag{2.19}$$

Similarly, by Lemma 1.11, we have

$$\triangle(u_{2t-1}, u_{2t}) \leq \eta^{2t-1}(\triangle(u_0, u_1)).$$

Now, we can write inequality (2.19) as

$$\triangle(u_t, u_{t+1}) \leq \eta^t(\triangle(u_0, u_1)). \tag{2.20}$$

Now, by using (2.20) and by using the technique given in [1], it can easily be proved that $\{u_t\}$ is a left Cauchy sequence. So, for all natural numbers with $t < m$, we have

$$\lim_{t,m\to\infty} \triangle(u_t, u_m) = 0. \tag{2.21}$$

Since $(X, \triangle)$ is a left complete double controlled quasi metric type space. So $\{u_t\} \to \dot{z} \in X$, that is

$$\lim_{t\to\infty} \triangle(u_t, \dot{z}) = \lim_{t\to\infty} \triangle(\dot{z}, u_t) = 0. \tag{2.22}$$

Now, we show that $\dot{z}$ is a common fixed point. We claim that $\triangle(\dot{z}, T\dot{z}) = 0$. On contrary suppose $\triangle(\dot{z}, T\dot{z}) > 0$. Now by Lemma 1.11, we have

$$\triangle(u_{2t+1}, T\dot{z}) \leq H_{\triangle}(Su_{2t}, T\dot{z}).$$

$$\triangle(u_{2t+1}, T\dot{z}) \leq c(\triangle(u_{2t}, \dot{z})) + k\left[\triangle(u_{2t}, u_{2t+1}) + \triangle(\dot{z}, T\dot{z})\right] \tag{2.23}$$

Taking $\lim_{t\to\infty}$ on both sides of inequality (2.23), we get

$$\lim_{t\to\infty}\triangle(u_{2t+1}, T\dot{z}) \leq c\lim_{t\to\infty}\triangle(u_{2t}, \dot{z}) + k\lim_{t\to\infty}\left[\triangle(u_{2t}, u_{2t+1}) + \triangle(\dot{z}, T\dot{z})\right]$$

By using inequalities (2.21) and (2.22), we get

$$\lim_{t\to\infty}\triangle(u_{2t+1}, T\dot{z}) \leq k(\triangle(\dot{z}, T\dot{z})) \tag{2.24}$$

Now,

$$\triangle(\dot{z}, T\dot{z}) \leq \alpha(\dot{z}, u_{2t+1})\triangle(\dot{z}, u_{2t+1}) + \mu(u_{2t+1}, T\dot{z})\triangle(u_{2t+1}, T\dot{z})$$

Taking $\lim_{t\to\infty}$ and by using inequalities (2.12), (2.22) and (2.24), we get

$$\triangle(\dot{z}, T\dot{z}) < \triangle(\dot{z}, T\dot{z}).$$

It is a contradiction, therefore

$$\triangle(\dot{z}, T\dot{z}) = 0.$$

Thus, $\dot{z} \in T\dot{z}$. Now, suppose $\triangle(\dot{z}, S\dot{z}) > 0$. By Lemma 1.11, we have

$$\triangle(u_{2t}, S\dot{z}) \leq H_{\triangle}(Tu_{2t-1}, S\dot{z}).$$

By inequality (2.10), we get

$$\triangle(u_{2t}, S\dot{z}) \leq c(\triangle(u_{2t-1}, \dot{z})) + k\left[\triangle(u_{2t-1}, u_{2t}) + \triangle(\dot{z}, S\dot{z})\right].$$

Taking $\lim_{t\to\infty}$ on both sides of above inequality, we get

$$\lim_{t\to\infty}\triangle(u_{2t}, S\dot{z}) \leq k(\triangle(\dot{z}, S\dot{z})). \tag{2.25}$$

Now,

$$\triangle(\dot{z}, S\dot{z}) \leq \alpha(\dot{z}, u_{2t})\triangle(\dot{z}, u_{2t}) + \mu(u_{2t}, S\dot{z})\triangle(u_{2t}, S\dot{z}).$$

Taking $\lim_{t\to\infty}$ and by using inequality (2.12), (2.22) and (2.25), we get

$$\triangle(\dot{z}, S\dot{z}) < \triangle(\dot{z}, S\dot{z}).$$

It is a contradiction. Hence, $\dot{z} \in S\dot{z}$. Thus, $\dot{z}$ is a common fixed point for S and T. □

Theorem 2.10 presents a result for single-valued mapping which is a consequence of the previous result.

Theorem 2.10. Let $(X, \triangle)$ be a left complete double controlled quasi-metric type space with the functions $\alpha, \mu : X \times X \to [1, \infty)$ and $S, T : X \to X$ be the mappings such that:

$$\triangle(Sx, Ty) \leq c(\triangle(x, y)) + k(\triangle(x, Sx) + \triangle(y, Ty))$$

and

$$\triangle(Tx, Sy) \leq c(\triangle(x, y)) + k(\triangle(x, Tx) + \triangle(y, Sy)),$$

for each $x, y \in X$, $0 < c + 2k < 1$. Suppose that, for every $u \in X$ and for the Picard sequence $\{u_t\}$

$$\lim_{t\to\infty}\alpha(u, u_t) \text{ is finite, } \lim_{t\to\infty}\mu(u_t, u) < \frac{1}{k} \text{ and}$$

$$\sup_{m\geq 1}\lim_{i\to\infty}\frac{\alpha(u_{i+1}, u_{i+2})}{\alpha(u_i, u_{i+1})}\mu(u_i, u_m) < \frac{1-k}{c+k}.$$

Then S and T have a common fixed point $u^* \in X$.

Quasi b-metric version of Theorem 2.10 is given below:

Theorem 2.11. Let $(X, \triangle)$ be a left complete quasi b-metric type space with the functions $\alpha, \mu : X \times X \to [1, \infty)$ and $S, T : X \to X$ be the mappings such that:

$$\triangle(Sx, Ty) \leq c(\triangle(x, y)) + k(\triangle(x, Sx) + \triangle(y, Ty))$$

and

$$\triangle(Tx, Sy) \leq c(\triangle(x, y)) + k(\triangle(x, Tx) + \triangle(y, Sy)),$$

for each $x, y \in X$, $0 < c + 2k < 1$ and $b < \frac{1-k}{c+k}$. Then S and T have a common fixed point $u^* \in X$.

The following example shows that how double controlled quasi metric type spaces can be used where the quasi b-metric spaces cannot.

Example 2.12. Let $X = \left\{0, \frac{1}{2}, \frac{1}{4}, 1\right\}$. Define $\triangle : X \times X \to [0,\infty)$ by $\triangle(0, \frac{1}{2}) = 1$, $\triangle(0, \frac{1}{4}) = \frac{1}{3}$, $\triangle(\frac{1}{4}, 0) = \frac{1}{5}$, $\triangle(\frac{1}{2}, 0) = 1$, $\triangle\left(\frac{1}{4}, \frac{1}{2}\right) = 3$, $\triangle\left(\frac{1}{4}, 1\right) = \frac{1}{2}$, $\triangle\left(1, \frac{1}{4}\right) = \frac{1}{3}$ and $\triangle(x, y) = |x - y|$, if otherwise. Define $\alpha, \mu : X \times X \to [1,\infty)$ as follows $\alpha\left(\frac{1}{2}, \frac{1}{4}\right) = \frac{16}{5}$, $\alpha(0, \frac{1}{4}) = \frac{3}{2}$, $\alpha(\frac{1}{4}, 1) = 3$, $\alpha(1, \frac{1}{4}) = \frac{12}{5}$ and $\alpha(x, y) = 1$, if otherwise. $\mu(0, \frac{1}{2}) = \frac{14}{5}$, $\mu(1, \frac{1}{2}) = 3$ and $\mu(x, y) = 1$, if otherwise. Clearly $\triangle$ is double controlled quasi metric type for all $x, y, z \in X$. Let, $T0 = \{0\}$, $T\frac{1}{2} = \left\{\frac{1}{4}\right\}$, $T\frac{1}{4} = \{0\}$, $T1 = \left\{\frac{1}{4}\right\}$, $S0 = S\frac{1}{4} = \{0\}$, $S\frac{1}{2} = \left\{\frac{1}{4}\right\}$, $S1 = \{0\}$ and $c = \frac{2}{5}$, $k = \frac{1}{4}$. Now, if we take the case $x = \frac{1}{2}$, $y = \frac{1}{4}$, we have

$H_\triangle(S\frac{1}{2}, T\frac{1}{4}) = H_\triangle\left(\left\{\frac{1}{4}\right\}, \{0\}\right) = \triangle\left(\frac{1}{4}, 0\right) = \frac{1}{5} \leq \frac{17}{80} = c(\triangle(x, y)) + k(\triangle(x, Sx) + \triangle(y, Ty))$. Also, satisfied for all cases $x, y \in X$. That are inequalities (2.9) and (2.10) hold. Take $u_0 = 1$, then $u_1 = Su_0 = 0$, $u_2 = Tu_1 = 0$, $u_3 = Su_2 = 0 \cdots$.

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\alpha(u_{i+1}, u_{i+2})}{\alpha(u_i, u_{i+1})} \mu(u_{i+1}, u_m) = 1 < \frac{15}{13} = \frac{1-k}{c+k}.$$

which shows that inequality (2.11) holds. Thus the pair (S, T) is double controlled Reich type contraction. Finally, for every $u \in X$, we obtain

$$\lim_{t \to \infty} \alpha(u, u_t) \text{ is finite, } \lim_{t \to \infty} \mu(u_t, u) \leq \frac{1}{k}.$$

All hypotheses of Theorem 2.9 are satisfied and $\dot{z} = 0$ is a common fixed point.

Note that $\triangle$ is quasi b-metric with $b = 3$, but Theorem 2.11 can not be applied because $b \nless \frac{1-k}{c+k}$, for all $b = 3$. Therefore, this example shows that generalization from a quasi b-metric spaces to a double controlled quasi metric type spaces is real.

Taking $c = 0$ in Theorem 2.9, we get the following result of Kannan-type.

Theorem 2.13. Let $(X, \triangle)$ be a left complete double controlled quasi-metric type space with the functions $\alpha, \mu : X \times X \to [1, \infty)$ and $S, T : X \to P(X)$ be the multivalued mappings such that:

$$H_\triangle(Sx, Ty) \leq k(\triangle(x, Sx) + \triangle(y, Ty))$$

and

$$H_\triangle(Tx, Sy) \leq k(\triangle(x, Tx) + \triangle(y, Sy)),$$

for each $x, y \in X$, $0 < c + 2k < 1$. Suppose that, for every $u \in X$ and for the sequence $\{TS(u_t)\}$, we have

$$\lim_{t \to \infty} \alpha(u, u_t) \text{ is finite, } \lim_{t \to \infty} \mu(u_t, u) < \frac{1}{k} \text{ and}$$

$$\sup_{m\geq 1}\lim_{i\to\infty}\frac{\alpha(u_{i+1},u_{i+2})}{\alpha(u_i,u_{i+1})}\mu(u_i,u_m)<\frac{1-k}{k}.$$

Then S and T have a common fixed point $u^* \in X$. Then, S and T have a common fixed point $\dot{z}$ in X.

Taking $c = 0$ and $S = T$ in Theorem 2.9, we get the following result.

Theorem 2.14. Let $(X, \triangle)$ be a left complete double controlled quasi-metric type space with the functions $\alpha, \mu : X \times X \to [1, \infty)$ and $T : X \to P(X)$ be a multivalued mapping such that:

$$H_{\triangle}(Tx, Ty) \leq k(\triangle(x, Tx) + \triangle(y, Ty))$$

for each $x, y \in X$, $0 < c + 2k < 1$. Suppose that, for every $u \in X$ and for the sequence $\{T(u_t)\}$, we have

$$\lim_{t\to\infty}\alpha(u,u_t)\ \text{is finite,}\ \lim_{t\to\infty}\mu(u_t,u)<\frac{1}{k}\ \text{and}$$

$$\sup_{m\geq 1}\lim_{i\to\infty}\frac{\alpha(u_{i+1},u_{i+2})}{\alpha(u_i,u_{i+1})}\mu(u_i,u_m)<\frac{1-k}{k}.$$

Then T has a fixed point.

Now, the following example illustrates Theorem 2.14.

Example 2.15. Let $X = [0, 3)$. Define $\triangle : X \times X \to [0,\infty)$ as

$$\triangle(x,y)=\begin{cases}0, \text{ if } x=y,\\ (x-y)^2+x, \text{ if } x\neq y.\end{cases}$$

with

$$\alpha(x,y)=\begin{cases}2, \text{ if } x,y\geq 1,\\ \frac{x+2}{2}, \text{ otherwise.}\end{cases},\ \mu(x,y)=\begin{cases}1, \text{ if } x,y\geq 1,\\ \frac{y+2}{2}, \text{ otherwise.}\end{cases}$$

Clearly $(X, \triangle)$ is double controlled quasi metric type space. Choose $Tx = \left\{\frac{x}{4}\right\}$ and $k = \frac{2}{5}$. It is clear that T is Kannan type double controlled contraction. Also, for each $u \in X$, we have

$$\lim_{t\to\infty}\alpha(u,u_t)<\infty,\ \lim_{t\to\infty}\mu(u_t,u)<\frac{1}{k}.$$

Thus, all hypotheses of Theorem 2.14 are satisfied and $\dot{z} = 0$ is the fixed point.

Quasi b-metric version of Theorem 2.14 is given below:

Theorem 2.16. Let $(X, \triangle)$ be a left complete quasi b-metric space and $T : X \to P(X)$ be a mapping such that:

$$H_{\triangle}(Tx,Ty)\leq k\left[\triangle(x,Tx)+\triangle(y,Ty)\right],$$

for all $x, y \in X$, $k \in [0, \frac{1}{2})$ and

$$b<\frac{1-k}{k}.$$

Then T has a fixed point $u^* \in X$.

The following remark compare, distinguish and relate the quasi b-metric with the double controlled quasi metric type spaces and illustrate the importance of double controlled quasi metric type spaces.

Remark 2.17. In the Example 2.15, $\triangle(x, y) = (x - y)^2 + x$ is a quasi b-metric with $b \geq 2$, but we can not apply Theorem 2.16 because T is not Kannan type b-contraction. Indeed $b \nless \frac{1-k}{k}$, for all $b \geq 2$.

3. Conclusion and future work

It has been shown that double controlled quasi metric is general and better than other metrics, like controlled quasi metric, controlled metric, extended quasi-b-metric, extended b-metric, quasi-b-metric and quasi metric. Also, left, right and dual completeness has been discussed. Results in dual complete spaces can be obtained as corollaries. These results may be extended to obtain results for other contractions. Double controlled quasi metric like ordered spaces can be introduced to establish new results. These results may be applied to find applications to random impulsive differential equations, dynamical systems, graph theory and integral equations.

Acknowledgments

The authors would also like to thank the editor and the reviewers for their helpful comments and suggestions. This research was funded by the Center of Excellence in Theoretical and Computational Science (TaCS-CoE), KMUTT.

Conflict of interest

The authors declare that they do not have any competing interests.

References

1. T. Abdeljawad, N. Mlaiki, H. Aydi, N. Souayah, Double controlled metric type spaces and some fixed point results, *Mathematics*, **6** (2018), 1–10.
2. M. Arshad, A. Shoaib, P. Vetro, Common fixed point of a pair of Hardy Rogers type mapping on closed ball in order dislocated metric space, *J. Funct. Space. Appl.*, **2013** (2013),1–9.
3. M. Arshad, Z. Kadelburg, S. Radenović, A. Shoaib, S. Shukla, Fixed points of α-dominated mappings on dislocated quasi metric spaces, *Filomat*, **31** (2017), 3041–3056.
4. H. Aydi, R. Banković, I. Mitrović, M. Nazam, Nemytzki-Edelstein-Meir-Keeler type results in b-metric spaces, *Discrete Dyn. Nat. Soc.*, **2018** (2018), 1–7.
5. H . Aydi, M. F. Bota, E. Karapinar, S. Mitrović, Fixed point theorem for set-valued quasi-contractions in b-metric spaces, *Fixed Point Theory Appl.*, **2012** (2012), 1–8.
6. A. Azam, M. Arshad, Kannan fixed point theorems on generalized metric spaces, *J. Nonlinear Sci. Appl.*, **1** (2008), 45–48.
7. A. Azam, M. Waseem, M. Rashid, Fixed point theorems for fuzzy contractive mappings in quasi-pseudo metric spaces, *Fixed Point Theory Appl.*, **2013** (2013), 1–14.
8. I. A. Bakhtin, The contraction mapping principle in almost metric spaces, *Funct. Anal.*, **30** (1989), 26–37.
9. S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integrales, *Fund. Math.*, **3** (1992), 133–181.
10. N. Bilgili, E. Karapınar, B. Samet, Generalized $\alpha - \psi$ contractive mappings in quasi-metric spaces and related fixed point theorems, *Fixed Point Theory Appl.*, **2014** (2014), 1–15.

11. S. Czerwik, Contraction mappings in b-metric spaces, *Acta Mathematica et Informatica Universitatis Ostraviensis*, **1** (1993), 5–11.
12. T. Kamran, M. Samreen, Q. U. Ain, A generalization of b-metric space and some fixed point theorems, *Mathematics*, **5** (2017), 1–7.
13. R. Kannan, Some results on fixed points, *Bull. Calcutta Math. Soc.*, **60** (1968), 71–76.
14. Z. Li, X. Shu, F. Xu, The existence of upper and lower solutions to second order random impulsive differential equation with boundary value problem, *AIMS Mathematics*, **5** (2020), 6189–6210.
15. J. Marín, S. Romaguera, P. Tirado, Weakly contractive multivalued maps and wdistances on complete quasi-metric spaces, *Fixed Point Theory Appl.*, **2011** (2011), 1–9.
16. N. Mlaiki, H. Aydi, N. Souayah, T. Abdeljawad, Controlled metric type spaces and the related contraction principle, *Mathematics*, **6** (2018), 1–7.
17. T. Rasham, A. Shoaib, N. Hussain, B. A. S. Alamri, M. Arshad, Multivalued fixed point results in dislocated b- metric spaces with application to the system of nonlinear integral equations, *Symmetry*, **11** (2019), 1–16.
18. S. Reich, Some remarks concerning contraction mappings, *Can. Math. Bull.*, **14** (1971), 121–124.
19. S. Reich, Fixed points of contractive functions, *B. Unione. Mat. Ital.*, **5** (1972), 26–42.
20. S. Reich, Some fixed point problems, *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti*, **57** (1974), 194–198.
21. I. L. Reilly, P. V. Semirahmanya, M. K. Vamanamurthy, Cauchy sequences in quasi-pseudo-metric spaces, *Monatshefte für Mathematik*, **93** (1982), 127–140.
22. M. Samreen, T. Kamran, M. Postolache, Extended b-metric space, extended b-comparison function and nonlinear contractions, *U. P. B. Sci. Bull., Series A*, **80** (2018), 21–28.
23. W. Shatanawi, Fixed and common fixed point for mapping satisfying some nonlinear contraction in b-metric spaces, *J. Math. Anal.*, **7** (2016), 1–12.
24. A. Shoaib, P. Kumam, A. Shahzad, S. Phiangsungnoen, Q. Mahmood, Fixed point results for fuzzy mappings in a b-metric space, *Fixed Point Theory Appl.*, **2018** (2018), 1–12.
25. A. Shoaib, S. Mustafa, A. Shahzad, Common Fixed Point of Multivalued Mappings in Ordered Dislocated Quasi G-Metric Spaces, *Punjab University Journal of Mathematics*, **52** (2020), 1–23.
26. A. Shoaib, A. Azam, M. Arshad, A. Shahzad, Fixed point results for the multivalued mapping on closed ball in dislocated fuzzy metric space, *J. Math. Anal.*, **8** (2017), 98–106.
27. A. Shoaib, S. Kazi, A. Tassaddiq, S. Alshoraify, T. Rasham, Double Controlled Quasi-Metric Type Spaces and Some Results, *Complexity*, **2020** (2020), 1–8.
28. A. Shoaib, S. Alshoraify, M. Arshad, Double Controlled Dislocated Quasi-Metric Type Spaces and Some Results, *Journal of Mathematics*, **2020** (2020), 1–8.
29. M. Sgroi, C. Vetro, Multi-valued $F-$contractions and the solution of certain functional and integral equations, *Filomat*, **27** (2013), 1259–1268.
30. W. A. Wilson, On quasi metric spaces, *Am. J. Math.*, **53** (1931), 675–684.

AIMS Press