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Research article

Fuzzy congruences on AG-group

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Abstract: In this paper, we establish the idea of fuzzy congruences on Abel-Grassmann's group (AG-group). We investigate different outcomes of fuzzy-congruences on AG-groups in detail and give some examples to illustrate the newly established results. We develop the relation between fuzzy congruence and fuzzy normal subgroup. In the end, we also provide some results of fuzzy homomorphism on AG-groups.

Keywords: fuzzy equivalence relation; compatible; congruence relation; AG-group; fuzzy normal AG-group
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1. Introduction

With the beginning of fuzzy set theory [1], numerous contributions have been done by utilizing the concept of fuzzy sets from simple theoretical to logical and innovative disciplines. The theoretical aspect of fuzzy set theory deals as a tool, which extends the classical structure of algebra into the new form of arithmetical structures such as fuzzy relations, fuzzy equivalence and fuzzy compatible relations, fuzzy-semigroups and fuzzy-groups. Keeping this in mind, Rosenfeld [2] made excellent contributions in generalizing groupoids and groups via fuzzy set theory. Since then, many researchers explored fuzzy relations and fuzzy equivalence relations in general and particular in groups [3–8]. Murali [9] examined fuzzy-relations on sets and lattices properties of fuzzy equivalence relations.

Kuroki [10] studied the fuzzy-compatibility on groupoids and generalized it to fuzzy-congruence on groups employing fuzzy normal subgroups. Fuzzy-congruences on n-ary semigroups, quotient n-ary semigroups, and isomorphism theorems in n-ary semigroups were established in [11]. They also relate fuzzy congruences and fuzzy normal ideals, and provided that there is a one-to-one mapping from the set of all fuzzy normal ideals of the special n-ary semigroups to the set of all fuzzy congruences in an n-ary semigroup with one zero. The concepts of fuzzy normal congruence and fuzzy coset relation on a group were explored by Shoar in [12] and provided that a level subset of fuzzy normal congruence is also a normal congruence.

After the start of AG-groups, a midway structure between an abelian group and quasigroup investigated by numerous analysts. AG-groups up to order 11 are counted by Shah, and give lower bound for order 12 [13], it appears that from ordered 3–12 there exist $1, 2, 1, 1, 1, 7, 3, 1, 1$ and ≥ 5, non-associative AG-groups, respectively. As each commutative group is an AG-group, but the converse isn't true. In specific, there exist non-abelian AG-groups of order 3, 3^2 or holding the squaring property $(ab)^2 = a^2b^2\ \forall\ a, b \in G$. Moreover, from an abelian group $(G, \cdot)$ one can easily obtain an AG-group under "$*$" given by:

$$s * t = t \cdot s^{-1} \quad \text{or} \quad s * t = s^{-1} \cdot t,$$

for all $s, t \in G$ [14]. The authors of this paper also contributed in AG-groups in many ways see [15–19]. Recently, the notions of congruences and decomposition of the non-associative structure have been introduced, and then the notions of fuzzy congruences in the non-associative structure are also been introduced [20,21]. Further, various other notions like fuzzy normal and self-conjugate are investigated by them and show that fuzzy kernel and traces of a congruence provided a congruence pair. Congruence is one of the fundamental concepts in number theory; used in business, computer science, physics, chemistry, biology, music, and to design round-robin tournaments [22–24]. However, congruence arithmetic has many applications in the foundation of modern cryptography in public-key encryption, secret sharing, wireless authentication, and many other applications for data security [25, 26]. Based on this, the notion of fuzzy congruences is extended to AG-groups.

2. Preliminaries

A fuzzy set is defined by:

$$S = \{(x, \beta(x)) : x \in X, \beta(x) \in [0, 1]\},$$

where the set of all fuzzy sets over X is denoted by $FP(X)$. A function $\beta : X \times X \to [0, 1]$ is a fuzzy relation on X [27]. Let β and γ be any fuzzy relations on X. Then their product is represented by:

$$\beta \circ \gamma(p, q) = \max_{r \in X} (\beta(p, r) \wedge \gamma(r, q)).$$

Therefore, a fuzzy relation on X is a fuzzy equivalence relation: if $\forall p, q, r \in X$. i. $\beta(p, p) = 1$ (fuzzy reflexive), ii. $\beta(p, q) = \beta(q, p)$ (fuzzy symmetric) and iii. $\beta \circ \beta \leq \beta$ (fuzzy transitive) [28]. A fuzzy relation β is fuzzy left (fuzzy right) compatible if $\beta(rp, rq) \geq \beta(p, q)$ ($\beta(pr, qr) \geq \beta(p, q)$); and is fuzzy compatible if $\beta(pr, qs) \geq (\beta(p, q) \wedge \beta(r, s))$ for all $p, q, r, s \in S$ where S is a semigroup [29]. Further, if β is fuzzy left (fuzzy right) compatible and fuzzy equivalence relation on S, then β is called a fuzzy left (fuzzy right) congruence relation on S; and is fuzzy congruence if and only if it is both fuzzy left and fuzzy right congruence [29]. Simply, a fuzzy compatible and a fuzzy equivalence relation on S is called a fuzzy congruence, where $FC(S)$ represents the set of all fuzzy congruences on S. In the rest of the paper, G represents an AG-group where e be the left identity of G. An AG-groupoid G containing left identity and the inverse of each element in G is called an AG-group. A fuzzy AG-group is defined as follow. Let $\beta \in FP(G)$. Then β is a fuzzy AG-group on G if, for all $s, t \in G$, $\beta(st) \geq (\beta(s) \wedge \beta(t))$ and $\beta(s^{-1}) \geq \beta(s)$ or $\beta(st^{-1}) \geq (\beta(s) \wedge \beta(t)) \,\forall\, s, t \in G$. From now on $F(G)$ will represent the set of all fuzzy AG-groups on G [16].

Example 1. *Let $G = \{0, 1, 2, 3\}$ be an AG-group, under the multiplication table:*

$\cdot$	0	1	2	3
0	0	1	2	3
1	3	0	1	2
2	2	3	0	1
3	1	2	3	0

Clearly, $\nu \in F(G)$ where ν is defined by: $\nu(0) = 0.7$, $\nu(1) = \nu(2) = \nu(3) = 0.5$.

Example 2. *Let $G = \{a_0, a_1, a_2, a_3, a_4, a_5\}$ be an AG-group, under the multiplication table:*

$\cdot$	a_0	a_1	a_2	a_3	a_4	a_5
a_0	a_0	a_1	a_2	a_3	a_4	a_5
a_1	a_5	a_0	a_1	a_2	a_3	a_4
a_2	a_4	a_5	a_0	a_1	a_2	a_3
a_3	a_3	a_4	a_5	a_0	a_1	a_2
a_4	a_2	a_3	a_4	a_5	a_0	a_1
a_5	a_1	a_2	a_3	a_4	a_5	a_0

Clearly, $\nu \in F(G)$, where $\nu(a_0) = 0.5$, $\nu(a_2) = 0.4 = \nu(a_4)$, $\nu(a_1) = \nu(a_3) = \nu(a_5) = 0.2$.

From fuzzy AG-group the following are obvious [16]:

(1) Let $\beta, \gamma \in FP(G)$, then $(\beta \circ \gamma)(p) = (\gamma \circ \beta)(pe) \,\forall\, p \in G$.
(2) Let $\beta \in F(G)$, then $\beta(pq) = \beta(qp) \,\forall\, p, q \in G$.
(3) Let $\beta \in F(G)$, then, $\beta(e) \geq \beta(p)$, and $\beta(p^{-1}) = \beta(p) \,\forall\, p \in G$.
(4) Let $\beta \in FP(G)$, then $\beta \in F(G) \Leftrightarrow \beta \circ \beta \subseteq \beta$ and $\beta^{-1} = \beta$.

Let $\delta \in F(G)$. If for any $a, x \in G$

$$\delta\left((ax)a^{-1}\right) = \delta_a(x),$$

then δ is called a fuzzy normal AG-subgroup of G [15]. Henceforth, $FN(G)$ will represent the set of all fuzzy normal AG-subgroup of G.

Theorem 1. *[15] Let $\delta \in F(G)$. The following are equivalent $\forall\, a, x \in G$,*

(i) $\delta\left((ax)a^{-1}\right) = \delta(x)$;

(ii) $\delta\Big((ax)a^{-1}\Big) \geq \delta(x)$;

(iii) $\delta\Big((ax)a^{-1}\Big) \leq \delta(x)$.

Definition 1. Let $\phi : S \to T$ be a homomorphism on semigroups S and T. Then, $Ker(\phi) = \{(s_1, s_2) \in S \times S \ : \ \phi(s_1) = \phi(s_2)\}$ is a congruence on S [30].

3. Results and discussions

In this section we provide some new results about fuzzy congruences on AG-groups.

Theorem 2. *Let $\nu_1, \nu_2 \in F(G)$. Then $\nu_1 \circ \nu_2 \in F(G)$.*

Proof. Let $\nu_1 \circ \nu_2 \in F(G)$. Using left invertive law we have

$$\begin{aligned}(\nu_1 \circ \nu_2) \circ (\nu_1 \circ \nu_2) &= ((\nu_1 \circ \nu_2) \circ \nu_2) \circ \nu_1 \\ &= ((\nu_2 \circ \nu_2) \circ \nu_1) \circ \nu_1 \\ &= (\nu_1 \circ \nu_1) \circ (\nu_2 \circ \nu_2) \\ &\leq (\nu_1 \circ \nu_2).\end{aligned}$$

This implies that $(\nu_1 \circ \nu_2) \circ (\nu_1 \circ \nu_2) \leq (\nu_1 \circ \nu_2)$. Also we have

$$\begin{aligned}(\nu_1 \circ \nu_2)^{-1}(p) &= (\nu_1 \circ \nu_2)(p^{-1}) \\ &= \max_{p^{-1}=st} (\nu_1(s) \wedge \nu_2(t)) \\ &= \max_{p=(st)^{-1}} (\nu_1(s) \wedge \nu_2(t)) \\ &= \max_{p=s^{-1}t^{-1}} \Big(\nu_1(s^{-1}) \wedge \nu_2(t^{-1})\Big); \ \ \nu_1, \nu_2 \in F(G) \\ &= (\nu_1 \circ \nu_2)(p).\end{aligned}$$

This implies that $(\nu_1 \circ \nu_2)^{-1} = (\nu_1 \circ \nu_2)$. Hence, by the result stated above in (4), $\nu_1 \circ \nu_2 \in F(G)$. □

Thus unlike group, the composition of two “fuzzy AG-subgroups” is also “fuzzy AG-subgroups” without the condition of commutativity.

Example 3. *Consider an AG-group defined in Example 1. Clearly, $\mu : G \times G \to [0, 1]$ defined by*

μ	0	1	2	3
0	1	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{2}$
1	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{3}{4}$
2	$\frac{3}{4}$	$\frac{1}{2}$	1	$\frac{1}{2}$
3	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{2}$	1

is fuzzy congruence on G.

Example 4. *Consider an AG-group defined in Example 2. Clearly, $\mu : G \times G \to [0, 1]$ defined by*

μ	0	1	2	3	4	5
0	1	$\frac{1}{6}$	$\frac{3}{5}$	$\frac{1}{6}$	$\frac{3}{5}$	$\frac{1}{6}$
1	$\frac{1}{6}$	1	$\frac{1}{6}$	$\frac{3}{5}$	$\frac{1}{6}$	$\frac{3}{5}$
2	$\frac{3}{5}$	$\frac{1}{6}$	1	$\frac{1}{6}$	$\frac{3}{5}$	$\frac{1}{6}$
3	$\frac{1}{6}$	$\frac{3}{5}$	$\frac{1}{6}$	1	$\frac{1}{6}$	$\frac{3}{5}$
4	$\frac{3}{5}$	$\frac{1}{6}$	$\frac{3}{5}$	$\frac{1}{6}$	1	$\frac{1}{6}$
5	$\frac{1}{6}$	$\frac{3}{5}$	$\frac{1}{6}$	$\frac{3}{5}$	$\frac{1}{6}$	1

Then, $\mu \in FC(G)$.

Lemma 1. *Let β and γ be fuzzy compatible on G, then $\beta \circ \gamma$ is also fuzzy compatible on G.*

Proof. For any $p, q, r, s \in G$ we have $\beta(pr, qs) \geq (\beta(p, q) \wedge \beta(r, s))$ and $\gamma(pr, qs) \geq (\gamma(p, q) \wedge \gamma(r, s))$ as β and γ are compatible. Now,

$$\begin{aligned}
& (\beta \circ \gamma)(pr, qs) = \\
= \ & \max_{t \in G} (\beta(pr, t) \wedge \gamma(t, qs)) \\
= \ & \max_{t=uv \in G} (\beta(pr, uv) \wedge \gamma(uv, qs)) \\
\geq \ & \max_{z=uv \in G} [(\beta(p, u) \wedge \beta(r, v)) \wedge (\gamma(u, q) \wedge \gamma(v, s))] \\
= \ & \left(\max_{u \in G} (\beta(p, u) \wedge \gamma(u, q))\right) \wedge \left(\max_{v \in G} (\beta(r, v) \wedge \gamma(v, s))\right) \\
= \ & (\beta \circ \gamma)(p, q) \wedge (\beta \circ \gamma)(r, s).
\end{aligned}$$

This implies that $(\beta \circ \gamma)(pr, qs) \geq ((\beta \circ \gamma)(p, q) \wedge (\beta \circ \gamma)(r, s))$. Hence, $\beta \circ \gamma$ is fuzzy compatible on G. □

Lemma 2. *A fuzzy relation β on G is fuzzy congruence $\Leftrightarrow$ β is fuzzy left and fuzzy right compatible.*

Proof. Consider $\beta \in FC(G)$, then, $\beta(p, q) = \beta(p, q) \wedge \beta(r, r) \leq \beta(pr, qr)$ and $\beta(p, q) = \beta(r, r) \wedge \beta(p, q) \leq \beta(rp, rq)$ $\forall\, p, q, r \in G$. Hence, β is fuzzy left and fuzzy right compatible.

Conversely, consider β is fuzzy left and fuzzy right compatible, then $\forall\, p, q, u, v \in G$,

$$\begin{aligned}
\beta(p, q) \wedge \beta(u, v) &= \beta(p, q) \wedge \beta(u, u) \wedge \beta(q, q) \wedge \beta(u, v) \\
&\leq \beta(pu, qu) \wedge \beta(qu, qv) \\
&\leq \beta(pu, qv).
\end{aligned}$$

Hence, $\beta \in FC(G)$. □

Theorem 3. *If $\beta, \gamma \in FC(G)$ and $\beta \circ \gamma = \gamma \circ \beta$. Then $\beta \circ \gamma \in FC(G)$.*

Proof. Consider $\beta, \gamma \in FC(G)$, such that $\beta \circ \gamma = \gamma \circ \beta$. First we show that $\beta \circ \gamma$ is an equivalence relation. Clearly, $\beta \circ \gamma(s, s) = 1$. For symmetry take any $s, t \in G$,

$$(\beta \circ \gamma)(s, t) \quad = \quad \max_{u \in G} (\beta(s, u) \wedge \gamma(u, t))$$

$$
\begin{aligned}
&= \max_{u\in G}(\gamma(u,t)\wedge\beta(s,u)) \\
&= \max_{u\in G}(\gamma(t,u)\wedge\beta(u,s))\,;\ \ (\beta,\gamma\in FC(G)) \\
&= (\gamma\circ\beta)(t,s) \\
&= (\beta\circ\gamma)(t,s).
\end{aligned}
$$

$\Rightarrow \beta\circ\gamma$ is fuzzy symmetric.

Using medial law, we get $(\beta\circ\gamma)\circ(\beta\circ\gamma) = (\beta\circ\beta)\circ(\gamma\circ\gamma) \leq \beta\circ\gamma$. Therefore, $\beta\circ\gamma$ is an equivalence relation and by Lemma 1, $\beta\circ\gamma$ is compatible. Hence, $\beta\circ\gamma\in FC(G)$. □

Corollary 1. *Let $\beta,\gamma\in FC(G)$. If $\beta\circ\gamma\in FC(G)$, then $\beta\circ\gamma=\beta\vee\gamma$.*

Proof. Consider $\beta\circ\gamma\in FC(G)$ where $\beta,\gamma\in FC(G)$. To show that $\beta\circ\gamma=\beta\vee\gamma$, take any $p,q\in G$

$$
\begin{aligned}
(\beta\circ\gamma)(s,t) &= \max_{u\in G}(\beta(s,u)\wedge\gamma(u,t)) \\
&\geq \beta(s,t)\wedge\gamma(t,t) \\
&= \beta(s,t).
\end{aligned}
$$

This implies that $\beta\circ\gamma\geq\beta$. Similarly, $\beta\circ\gamma\geq\gamma$. Now take $\delta\in FC(G)$ such that $\delta\geq\beta$ and $\delta\geq\gamma$. Then,

$$
\begin{aligned}
(\beta\circ\gamma)(s,t) &= \max_{u\in G}(\beta(s,u)\wedge\gamma(u,t)) \\
&\leq \max_{u\in G}(\delta(s,u)\wedge\delta(u,t)) \\
&= \delta(s,t).
\end{aligned}
$$

This implies that $\beta\circ\gamma\leq\delta$. Thus, $\beta\circ\gamma=\beta\vee\gamma$. □

Theorem 4. *If $\beta,\gamma\in FC(G)$. Then show that the following conditions are equivalent:*

(1) $\beta\circ\gamma$ is a fuzzy congruence.
(2) $\beta\circ\gamma$ is a fuzzy equivalence.
(3) $\beta\circ\gamma$ is a fuzzy symmetric.
(4) $\beta\circ\gamma=\gamma\circ\beta$.

Proof. Obviously, (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

To show that (3) $\Rightarrow$ (4), take any $p,q\in G$,

$$
\begin{aligned}
(\beta\circ\gamma)(p,q) &= \bigvee_{r\in G}(\beta(p,r)\wedge\gamma(r,q)) \\
&= \bigvee_{r\in G}(\gamma(r,q)\wedge\beta(p,r)) \\
&= \bigvee_{r\in G}(\gamma(q,r)\wedge\beta(r,p));\quad (\beta,\gamma\in FC(G)) \\
&= (\gamma\circ\beta)(q,p) \\
&= (\gamma\circ\beta)(p,q).
\end{aligned}
$$

This implies that $\beta\circ\gamma=\gamma\circ\beta$.

Now consider (4) holds. We show that $\beta \circ \gamma \in FC(G)$. As $(\beta \circ \gamma)(p, p) = \bigvee_{q \in G}(\beta(p, q) \wedge \gamma(q, p)) \geq \beta(p, p) \wedge \gamma(p, p) = 1$, so that $(\beta \circ \gamma)(p, p) = 1$. Thus, $\beta \circ \gamma$ is fuzzy reflexive. Now for any $p, q \in G$, we have

$$\begin{aligned}
(\beta \circ \gamma)(p, q) &= (\gamma \circ \beta)(p, q) \\
&= \max_{r \in G}(\gamma(p, r) \wedge \beta(r, q)) \\
&= \max_{r \in G}(\beta(r, q) \wedge \gamma(p, r)) \\
&= \max_{r \in G}(\beta(q, r) \wedge \gamma(r, p)); \quad (\beta, \gamma \in FC(G)) \\
&= (\beta \circ \gamma)(q, p).
\end{aligned}$$

Thus, $\beta \circ \gamma$ is fuzzy symmetric. Using medial law and fuzzy transitivity we have,

$$\begin{aligned}
(\beta \circ \gamma) \circ (\beta \circ \gamma) &= ((\beta \circ \gamma) \circ \gamma)) \circ \beta \\
&= ((\gamma \circ \gamma) \circ \beta) \circ \beta \\
&= (\beta \circ \beta) \circ (\gamma \circ \gamma) \\
&\leq \beta \circ \gamma.
\end{aligned}$$

Therefore, $\beta \circ \gamma$ is a fuzzy equivalence relation on G. Compatibility follows by Lemma 1. Hence, $\beta \circ \gamma \in FC(G)$. □

Theorem 5. *If $\beta, \gamma \in FC(G)$. Then $\beta \circ \gamma = \gamma \circ \beta$.*

Proof. Let $p, q \in G$, then

$$\begin{aligned}
(\beta \circ \gamma)(p, q) &= \max_{r \in G}(\beta(p, r) \wedge \gamma(r, q)) \\
&= \max_{r \in G}(\gamma(r, q) \wedge \beta(p, r)) \\
&= \max_{r=(ps^{-1})q \in G}\left(\gamma((ps^{-1})q, q) \wedge \beta(p, (ps^{-1})q)\right) \\
&= \max_{r=(ps^{-1})q \in G}\left(\gamma((ps^{-1})q, eq) \wedge \beta(ep, (ps^{-1})q)\right) \\
&= \max_{r=(qs^{-1})p \in G}\left(\gamma\left((qs^{-1})p, ((ss^{-1})q)\right) \wedge \beta\left((ss^{-1})p, (ps^{-1})q\right)\right) \\
&= \max_{r=(qs^{-1})p \in G}\left(\gamma\left((qs^{-1})p, (qs^{-1})s)\right) \wedge \beta\left((ps^{-1})s, (ps^{-1})q\right)\right) \\
&= \max_{s \in G}(\gamma(up, us) \wedge \beta(vs, vq)) \\
&\geq \max_{s \in G}(\gamma(p, s) \wedge \beta(s, q)) \\
&= (\gamma \circ \beta)(p, q).
\end{aligned}$$

Similarly, we can show that $\gamma \circ \beta \geq \beta \circ \gamma$. Hence, $\beta \circ \gamma = \gamma \circ \beta$. □

Theorem 6. *Let $\beta \in FC(G)$. Then $\beta\left(u^{-1}, v^{-1}\right) = \beta(u, v)\ \forall\, u, v \in G$.*

Proof. For any $u, v \in G$,

$$\beta\left(u^{-1}, v^{-1}\right) = \beta\left(eu^{-1}, ev^{-1}\right)$$

$$
\begin{aligned}
&= \beta\left((v^{-1}v)u^{-1}, (u^{-1}u)v^{-1}\right) \\
&= \beta\left((u^{-1}v)v^{-1}, (u^{-1}u)v^{-1}\right) \\
&\geq \beta\left(u^{-1}v, u^{-1}u\right) \\
&\geq \beta(v, u) \\
&= \beta(u, v);\ \ \beta \in FC(G).
\end{aligned}
$$

This implies $\beta\left(u^{-1}, v^{-1}\right) \geq \beta(u, v)$. Also,

$$
\begin{aligned}
\beta(u, v) &= \beta(v, u);\ \ \beta \in FC(G). \\
&= \beta\left((uu^{-1})v, (vv^{-1})u\right) \\
&= \beta\left((vu^{-1})u, (vv^{-1})u\right) \\
&\geq \beta\left(vu^{-1}, vv^{-1}\right) \\
&\geq \beta\left(u^{-1}, v^{-1}\right).
\end{aligned}
$$

This implies that $\beta(u, v) \geq \beta\left(u^{-1}, v^{-1}\right)$. Hence, $\beta\left(u^{-1}, v^{-1}\right) = \beta(u, v)\ \forall\, u, v \in G$. □

Theorem 7. *Let $\gamma \in FN(G)$. Define a fuzzy relation by $\beta(p, q) = \gamma(pq^{-1})\ \forall\, p, q \in G$. Then $\beta \in FC(G)$.*

Proof. Consider $\gamma \in FN(G)$, and a fuzzy relation β defined by: $\beta(p, q) = \gamma(pq^{-1})\ \forall\, p, q \in G$. We show that $\beta \in FC(G)$. Let $p \in G$. Since, $\beta(p, p) = \gamma(pp^{-1}) = \gamma(e) = 1$, β is fuzzy reflexive. Let $p, q \in G$, then

$$
\begin{aligned}
\beta(p, q) &= \gamma(pq^{-1}) \\
&= \gamma\left((pq^{-1})^{-1}\right);\ \left(\gamma(u^{-1}) = \gamma(u)\ \forall\, u \in G\right) \\
&= \gamma\left(p^{-1}q\right) \\
&= \gamma\left(qp^{-1}\right);\ (\gamma(ab) = \gamma(ba)\ \forall\, a, b \in G) \\
&= \beta(q, p),
\end{aligned}
$$

β is fuzzy symmetric. Let $p, q, r \in G$, then

$$
\begin{aligned}
(\beta \circ \beta)(p, q) &= \bigvee_{r \in G}\{\beta(p, r) \wedge \beta(r, q)\} \\
&= \bigvee_{r \in G}\{\gamma(pr^{-1}) \wedge \gamma(rq^{-1})\} \\
&\leq \bigvee_{r \in G}\left\{\gamma\left((r^{-1}p)(rq^{-1})\right)\right\} \\
&= \gamma(r^{-1}r)(pq^{-1});\ \text{(by medial law)} \\
&= \gamma\left(e(pq^{-1})\right) \\
&= \gamma(pq^{-1}) \\
&= \beta(p, q).
\end{aligned}
$$

Therefore, β is fuzzy transitive as $\beta \circ \beta \leq \beta$. Hence, β is fuzzy equivalence on G. Since $\gamma \in FN(G)$. Therefore, for fuzzy compatibility, we have

$$\begin{aligned}
\beta(rp, rq) &= \gamma((rp)(rq)^{-1}) \\
&= \gamma((rq)^{-1}(rp)) \\
&= \gamma\left(\left(r^{-1}q^{-1}\right)(rp)\right) \\
&= \gamma\left(\left(r^{-1}r\right)(q^{-1}p)\right) \\
&= \gamma\left(e \cdot (q^{-1}p)\right) \\
&\geq \gamma(e) \wedge \gamma(q^{-1}p) \\
&= \gamma(q^{-1}p); \quad (\gamma(e) = 1) \\
&= \gamma(pq^{-1}) \\
&= \beta(p, q).
\end{aligned}$$

This implies that $\beta(rp, rq) \geq \beta(p, q)$. Similarly, $\beta(pr, qr) \geq \beta(p, q)$. Hence, $\beta \in FC(G)$. □

Theorem 8. *Let $\beta \in FC(G)$, for any fuzzy set δ of G defined by $\delta(p) = \beta(p, e)\ \forall\ p \in G$. Then δ is fuzzy normal.*

Proof. Since, $\beta \in FC(G)$. Therefore, by fuzzy transitivity, for any $p, q \in G$, we have

$$\begin{aligned}
\delta(pq) &= \beta(pq, e) \\
&= \beta(pq, q^{-1}q) \\
&\geq \beta(p, q^{-1}) \\
&\geq (\beta \circ \beta)(p, q^{-1}) \\
&= \bigvee_{r \in G} \left(\beta(p, r) \wedge \beta(r, q^{-1})\right) \\
&\geq \beta(p, e) \wedge \beta(e, q^{-1}) \\
&= \beta(p, e) \wedge \beta(qq^{-1}, eq^{-1}) \\
&\geq \beta(p, e) \wedge \beta(q, e) \\
&= \delta(p) \wedge \delta(q).
\end{aligned}$$

This implies that $\delta(pq) \geq \delta(p) \wedge \delta(q)$. Using fuzzy symmetry,

$$\begin{aligned}
\delta(p^{-1}) &= \beta(p^{-1}, e) \\
&= \beta(ep^{-1}, pp^{-1}) \\
&\geq \beta(e, p) \\
&= \beta(p, e) \\
&= \delta(p).
\end{aligned}$$

This implies that $\delta(p^{-1}) \geq \delta(p)$. Replacing p^{-1} by p we get $\delta(p) \geq \delta(p^{-1})$. This implies that $\delta(p^{-1}) = \delta(p)$. Therefore, $\delta \in F(G)$. For fuzzy normality we have

$$
\begin{aligned}
\delta((pq)p^{-1}) &= \beta((pq)p^{-1}, e) \\
&= \beta((qp)p^{-1}, pp^{-1}) \\
&\geq \beta(qp, p) = \beta(qp, ep) \\
&\geq \beta(q, e) \\
&= \delta(q).
\end{aligned}
$$

This implies that $\delta((pq)p^{-1}) \geq \delta(q)$. Therefore, by Theorem 1 we get $\delta \in FN(G)$. □

Theorem 9. *Show that the set of all fuzzy congruences on G is semilattice.*

Proof. Let $\beta, \gamma \in FC(G)$ and for any $p, q \in G$. Then

$$
\begin{aligned}
(\beta \circ \gamma)(p, q) &= \max_{r \in G}(\beta(p, r) \wedge \gamma(r, q)) \\
&= \max_{r \in G}(\gamma(r, q) \wedge \beta(p, r)) \\
&= \max_{r \in G}\left(\gamma(er, (rr^{-1})q) \wedge \beta((rr^{-1})p, er)\right) \\
&= \max_{r \in G}\left(\gamma(er, (qr^{-1})r) \wedge \beta((pr^{-1})r, er)\right) \\
&\geq \max_{r \in G}\left(\gamma\left(e, qr^{-1}\right) \wedge \beta\left((pr^{-1}), e\right)\right) \\
&= \max_{r \in G}\left(\gamma\left(pp^{-1}, (p^{-1}p)(qr^{-1})\right) \wedge \beta\left((q^{-1}q)(pr^{-1}), qq^{-1}\right)\right) \\
&= \max_{r \in G}\left(\gamma\left(pp^{-1}, ((qr^{-1})p)p^{-1}\right) \wedge \beta\left(((pr^{-1})q)q^{-1}, qq^{-1}\right)\right) \\
&\geq \max_{r \in G}(\gamma(p, (qr^{-1})p) \wedge \beta((pr^{-1})q, q)) \\
&= \max_{r \in G}(\gamma(p, (pr^{-1})q) \wedge \beta((pr^{-1})q, q)) \\
&= (\gamma \circ \beta)(p, q).
\end{aligned}
$$

This implies that $(\beta \circ \gamma) \geq (\gamma \circ \beta)$. Similarly, $(\gamma \circ \beta) \geq (\beta \circ \gamma)$. Thus, $(\beta \circ \gamma) = (\gamma \circ \beta)$, and by Theorem 4, $\beta \circ \gamma \in FC(G)$. On the other hand,

$$
\begin{aligned}
(\beta \circ \beta)(p, q) &= \bigvee_{r \in G}\{\beta(p, r) \wedge \beta(r, q)\} \\
&\geq (\beta(p, p) \wedge \beta(p, q)) \\
&= 1 \wedge \beta(p, q) \\
&= \beta(p, q).
\end{aligned}
$$

This implies that $\beta \circ \beta \geq \beta$. As $\beta \in FC(G)$ therefore, $\beta \circ \beta \leq \beta$. Thus $\beta \circ \beta = \beta$. Hence, $FC(G)$ is semilattice. □

Now in the following section fuzzy factor AG-group are discussed and find out the application of fuzzy factor AG-group and provided fuzzy homomorphism theorem of fuzzy AG-groups.

Theorem 10. *Show that there exists one-to-one correspondence between $FN(G)$ and $FC(G)$.*

Proof. The proof follows from Theorem 5.2.10 [31]. □

4. Fuzzy quotient AG-group

Definition 2. Let ρ be a binary relation on G, then the characteristic function represented by χ_ρ and defined by:

$$\chi_\rho(s,t) = \begin{cases} 1 & \text{if } (s,t) \in \rho, \\ 0 & \text{if } (s,t) \notin \rho. \end{cases}$$

Lemma 3. *A relation ρ on G is an equivalence $\Leftrightarrow$ χ_ρ is a fuzzy equivalence.*

Proof. Assume that χ_ρ is fuzzy equivalence. Therefore, by Definition 2, $\chi_\rho(p,p) = 1 \Rightarrow (p,p) \in \rho \Rightarrow \rho$ is reflexive. Let $(p,q) \in \rho \Rightarrow \chi_\rho(p,q) = 1 \Rightarrow \chi_\rho(q,p) = 1 \Rightarrow (q,p) \in \rho \Rightarrow \rho$ is symmetric. Also let (p,r) and $(r,q) \in \rho \Rightarrow \left(\chi_\rho \circ \chi_\rho\right)(p,q) = \max_{r\in G}\left(\chi_\rho(p,r) \wedge \chi_\rho(r,q)\right) = 1 \Rightarrow (p,q) \in \rho \Rightarrow \rho$ is transitive. Therefore ρ is a equivalence relation.

Conversely, consider ρ is an equivalence, then by Definition 2, $\chi_\rho(p,p) = 1$, as $(p,p) \in \rho$. Also $\chi_\rho(p,q) = 1 = \chi_\rho(q,p)$, as $(p,q) \in \rho \Rightarrow (q,p) \in \rho \; \forall \, p,q \in G$, and

$$\begin{aligned} \left(\chi_\rho \circ \chi_\rho\right)(p,q) &= \max_{r\in G}\left(\chi_\rho(p,r) \wedge \chi_\rho(r,q)\right) \\ &= 1 \wedge 1 = 1 = \chi_\rho(p,q), \end{aligned}$$

as $(p,r) \in \rho$ and $(r,q) \in \rho \Rightarrow (q,p) \in \rho \; \forall \, p,q,r \in G$. This implies that $\chi_\rho \circ \chi_\rho \leq \chi_\rho$. Hence, χ_ρ is a fuzzy equivalence relation. □

Theorem 11. *Any binary relation ρ on G is a congruence if and only if χ_ρ is a fuzzy congruence.*

Proof. Consider $\rho \in FC(G)$. As ρ is an equivalence, therefore, by Lemma 3 χ_ρ is fuzzy equivalence. Now for fuzzy compatibility let $(p,q) \in \rho \Rightarrow (pr,qr) \in \rho$ and $(rp,rq) \in \rho \, \forall p,q,r \in G$. If $(p,q) \notin \rho$, then

$$\chi_\rho(pr,qr) \geq 0 \;\; = \;\; \chi_\rho(p,q),$$

and

$$\chi_\rho(rp,rq) \geq 0 \;\; = \;\; \chi_\rho(p,q).$$

Therefore, $\chi_\rho \in FC(G)$ on G.

Conversely, let $\chi_\rho \in FC(G)$. Therefore, by Lemma 3, ρ is an equivalence as χ_ρ is fuzzy equivalence. For compatibility of ρ, let $(p,q) \in \rho \Rightarrow \chi_\rho(rp,rq) \geq \chi_\rho(p,q) = 1$, and $\chi_\rho(pr,qr) \geq \chi_\rho(p,q) = 1$. Therefore, $\chi_\rho(rp,rq) = 1$ and $\chi_\rho(pr,qr) = 1 \Rightarrow (pr,qr) \in \rho$ and $(rp,rq) \in \rho$. Hence, ρ is a congruence on G. □

Definition 3. Let β be a fuzzy equivalence relation on G. If a fuzzy set β_u on G, is defined by:

$$\beta_u(p) = \beta(u,p) \, \forall \, u,p \in G,$$

is called fuzzy equivalence class of β containing $u \in G$.

Theorem 12. *For any fuzzy equivalence relation β on G, $\beta_u = \beta_v \Leftrightarrow \beta(u,v) = 1 \, \forall \, u,v \in G$.*

Proof. For any fuzzy equivalence relation β on G, assume that $\beta_u = \beta_v$, to show that $\beta(u, v) = 1$. By Definition 3, $\beta(u, v) = \beta_u(v) = \beta_v(v) = \beta(v, v) = 1$; which is required result.

Conversely, consider $\beta(u, v) = 1$. Then, $\forall\, p \in G$,

$$\begin{aligned}\beta_u(p) = \beta(u, p) &\geq (\beta \circ \beta)(u, p) = \bigvee_{r \in G} (\beta(u, r) \wedge \beta(r, p)) \\ &\geq (\beta(u, v) \wedge \beta(v, p)) \\ &= 1 \wedge \beta(v, p) = \beta(v, p) = \beta_v(p) \Rightarrow \beta_u \geq \beta_v,\end{aligned}$$

now $\beta(v, u) = \beta(u, v) = 1$, as β is symmetric. Thus, $\beta_v \geq \beta_u$. Hence, $\beta_u = \beta_v$. □

Theorem 13. *Let $\beta \in FC(G)$. Then, the set $\dfrac{G}{\beta} = \{\beta_a : a \in G\}$ forms an AG-group under "$\star$" defined by $\beta_a \star \beta_b = \beta_{ab}$ for all $\beta_a, \beta_b \in \dfrac{G}{\beta}$.*

Proof. Let $\beta \in FC(G)$. To show that the binary operation "$\star$" is well-defined on $\dfrac{G}{\beta}$. Consider, $\beta_a = \beta_b$ and $\beta_c = \beta_d$. Then by Theorem 3, we have $\beta(a, b) = \beta(c, d) = 1$. Thus, $\beta(ac, bd) \geq (\beta \circ \beta)(ac, bd) = \max_{e \in G} (\beta(ac, e) \wedge \beta(e, bd)) \geq \beta(ac, bc) \wedge \beta(bc, bd) \geq \beta(a, b) \wedge \beta(c, d) = 1 \wedge 1 = 1 \Rightarrow \beta(ac, bd) = 1$. Thus by Theorem 3, we get

$$\beta_a \star \beta_c = \beta_{ac} = \beta_{bd} = \beta_b \star \beta_d.$$

Hence, "$\star$" is well-defined on $\dfrac{G}{\beta}$. To show that $\dfrac{G}{\beta}$ is an AG-group under "$\star$". Clearly, "$\star$" is closed in $\dfrac{G}{\beta}$. Thus, $\dfrac{G}{\beta}$ is a groupoid. Left invertive law under "$\star$" also hold in $\dfrac{G}{\beta}$. That is, for all $a, b, c \in G$, we have $(\beta_a \star \beta_b) \star \beta_c = \beta_{ab} \star \beta_c = \beta_{(ab)c} = \beta_{(cb)a} = \beta_{(cb)} \star \beta_a = (\beta_c \star \beta_b) \star \beta_a$. Hence, $\dfrac{G}{\beta}$ is an AG-groupoid. $\dfrac{G}{\beta}$ under "$\star$" is non-associative as: $(\beta_a \star \beta_b) \star \beta_c = \beta_{ab} \star \beta_c = \beta_{(ab)c} \neq \beta_{a(bc)} = \beta_a \star \beta_{(bc)} = \beta_a \star (\beta_b \star \beta_c)$. For all $a \in G$, $(\beta_e \star \beta_a) = \beta_{ea} = \beta_a$, but $(\beta_a \star \beta_e) = \beta_{ae} \neq \beta_a$. Thus, β_e is the left identity of $\dfrac{G}{\beta}$. Thus $\forall \beta_a \in \dfrac{G}{\beta} \exists \beta_{a^{-1}} \in \dfrac{G}{\beta} \ni (\beta_a \star \beta_{a^{-1}}) = \beta_{aa^{-1}} = \beta_e = \beta_{(a^{-1}a)} = (\beta_{a^{-1}} \star \beta_a)$. Hence, $\dfrac{G}{\beta}$ is an AG-group. □

Thus an AG-group $\dfrac{G}{\beta}$, defined in the above Theorem 13, is known as fuzzy quotient AG-group.

Theorem 14. *Let $\beta \in FC(G)$. Then, $\beta^{-1}(1) = \{(a, b) \in G \times G : \beta(a, b) = 1\}$ is congruence on G.*

Proof. The proof follows from Theorem 5.3.4 [31]. □

Using Definition 1, and Theorem 11, it is clear that $\chi_{ker(\phi)}$ is fuzzy congruence. Keeping in view this, we define the fuzzy kernel of ϕ as follow:

$$\chi_{ker(\phi)}(s, t) = \begin{cases} 1 & \text{if } \phi(s) = \phi(t), \\ 0 & \text{if } \phi(s) \neq \phi(t). \end{cases} \tag{4.1}$$

Theorem 15. *If $\phi : G \to G'$ be a onto homomorphism on G, then* $\dfrac{G}{\chi_{ker(\phi)}} \cong G'$.

Proof. Let $\psi : \dfrac{G}{\chi_{ker(\phi)}} \to G'$, defined by $\psi\left(\chi_{ker(\phi)}(g)\right) = \phi(g)$. First we show that the mapping is well-defined. Let $\chi_{ker(\phi)}(g_1) = \chi_{ker(\phi)}(g_2)$, then by Theorem 12, $\chi_{ker(\phi)}(g_1, g_2) = 1$. Using Equation (4.1), we get $\phi(g_1) = \phi(g_2) \Rightarrow \psi\left(\chi_{ker(\phi)}(g_1)\right) = \psi\left(\chi_{ker(\phi)}(g_2)\right)$. This shows that ψ is well-defined. To see that ψ is one-to-one, let $\phi(g_1) = \phi(g_2)$. Then by Definition 1, $(g_1, g_2) \in Ker(\phi)$. Using Equation (4.1), we get $\chi_{ker(\phi)}(g_1, g_2) = 1$. Hence by Theorem 12, $\chi_{ker(\phi)}(g_1) = \chi_{ker(\phi)}(g_2)$. Thus ψ is one-to-one mapping. At the end we show that ψ is a homomorphism on G. Take $\psi\left[\left(\chi_{ker(\phi)}(g_1)\right) \star \left(\chi_{ker(\phi)}(g_2)\right)\right] = \psi\left(\chi_{ker(\phi)}(g_1 g_2)\right) = \phi(g_1 g_2) = \phi(g_1) \cdot \phi(g_2) = \psi\left(\chi_{ker(\phi)}(g_1)\right) \cdot \psi\left(\chi_{ker(\phi)}(g_2)\right)$. As, ψ is a bijective homomorphism, therefore, $\dfrac{G}{\chi_{ker(\phi)}} \cong G'$. □

5. Conclusion

In this paper, a relation on AG-group particularly congruence relation and fuzzy congruence relation on AG-group are provided with suitable examples. Moreover, various results on and fuzzy congruences on AG-groups are explored in the detailed. Further, we prove in the article that fuzzy-congruences and fuzzy normal subgroups imply each other, and each fuzzy-congruences in AG-group are a semilattice. We introduce fuzzy equivalence classes on AG-groups and fuzzy quotient AG-group. We also show fuzzy equivalence classes on AG-groups form an equivalence relation. In the end, some applications of fuzzy congruences in the form of fuzzy homomorphism theorems are also provided. However, AG-groups still needed further attention. In future, the idea can be further extended to fuzzy congruences in rings, near rings and near LA-rings.

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Conflict of interest

The authors declare that they have no competing interests concerning the publication of this article.

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