AIMS Mathematics
http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(2): 1624–1633.
DOI: 10.3934/math.2021097
Received: 13 September 2020
Accepted: 16 November 2020
Published: 25 November 2020

Research article

Global injectivity of differentiable maps via W-condition in $\mathbb{R}^2$

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Abstract: In this paper, we study the intrinsic relations between the global injectivity of the differentiable local homeomorphism map F and the rate of the Spec(F) tending to zero, where Spec(F) denotes the set of all (complex) eigenvalues of Jacobian matrix $JF(x)$, for all $x \in \mathbb{R}^2$. They depend deeply on the W-condition which extends the $*$-condition and the B-condition. The W-condition reveals the rate that tends to zero of the real eigenvalues of JF, which can not exceed $O\Big(x \ln x (\ln \frac{\ln x}{\ln \ln x})^2\Big)^{-1}$ by the half-Reeb component method. This improves the theorems of Gutiérrez [16] and Rabanal [27]. The W-condition is optimal for the half-Reeb component method in this paper setting. This work is related to the Jacobian conjecture.

Keywords: global injectivity; W-condition; half-Reeb component; Jacobian conjecture
Mathematics Subject Classification: 14A25, 14R15, 26B10

1. Introduction

In 1939, Keller ([19]) stated the following Conjecture:

Conjecture 1.1. (Jacobian conjecture). *Let $F : k^n \to k^n$ be a polynomial map, where k is a field of characteristic 0. If the determinant for its Jacobian matrix of the polynomial map is a non-zero constant, i.e.,* $\det JF(x) \equiv C \in k^*$, $\forall x \in k^n$, *then $F(x)$ has a polynomial inverse map.*

On the long-standing Jacobian conjecture, it is still open even in the case $n = 2$.

A very important result, for example, if $k = \mathbb{C}^n$, is the following theorem.

Theorem 1.1. *([8]) Let $F : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map. If F is injective, then F is bijective. Furthermore the inverse is also a polynomial map.*

If $k = \mathbb{R}^n$, then one gets

Conjecture 1.2. *(Real Jacobian Conjecture, for short, RJC) If $F : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial map,* $\det JF(x)$ *is not zero in $\mathbb{R}^n$, then F is a injective map.*

It is false and Pinchuk [25] constructs a counterexample to RJC for $n = 2$.

In 2007, Belov-Kanel and Kontsevich [20] proved that Conjecture 1.1 is stably equivant to the Dixmier conjecture. Conjecture 1.1 is also equivalent to the statement: Any ternary Engel algebra in characteristic 0 satisfying a system of Capelli identities is a Yagzhev algebra (see [1], Page 263). Moreover, Conjecture 1.1 is also equivant to some other conjectures, such as the Amazing Image Conjecture [10], a special case of the Vanishing conjecture [31]. There are many results on it, see for example ([2, 3, 9, 15, 17, 25, 28]).

Fernandes et al. [11] study the Conjecture 1.1 by the eigenvalues of the Jacobian matrix $JF(x)$ in $\mathbb{R}^2$ and obtain:

Theorem 1.2. *([11]) Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable map. For some $\varepsilon > 0$, if*

$$Spec(F) \cap [0, \varepsilon) = \emptyset, \tag{1.1}$$

where $Spec(F)$ denotes the set of all (complex) eigenvalues of Jacobian matrix $JF(x)$, for all $x \in \mathbb{R}^2$, then F is injective.

Theorem 1.2 is deep. If the assumption (1.1) is replaced by $0 \notin$ Spec (F), then the conclusion is false, even for polynomial map F, as the counterexample due to Pinchuck [25]. Pmyth and Xavier [30] proved that there exist $n > 2$ and a non-injective polynomial map F such that $\mathrm{Spec}(F) \cap [0, +\infty) = \emptyset$.

Theorem 1.2 adds a new result on Markus-Yamabe conjecture [24]. This Conjecture has been solved by Gutierrez [13] and Fessler [12] independently in dimension $n = 2$ in 1993. It is false for $n \geqslant 3$ even for polynomial vector field, see [7].

Theorem 1.2 also implies that the following conjecture is true in dimension $n = 2$.

Conjecture 1.3. *([5], Conjecture 2.1) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map. Suppose there exists $\varepsilon > 0$ such that $|\lambda| \geqslant \varepsilon$ for all the eigenvalues λ of Jacobian matrix $JF(x)$ and all $x \in \mathbb{R}^n$. Then F is injective.*

The essential technique is to use the concept of the half-Reeb component (see Definition 2.1 below) to prove Theorem 1.2.

Theorem 1.2 leads to study the eigenvalue conditions of some maps for injectivity in dimension $n = 2$. In 2007, Gutiérrez and Chau [16] studied the geometrical behavior of differentiable maps and the following $*$-condition on the real eigenvalues of JF in $\mathbb{R}^2$ by the half-Reeb component method.

For each $\theta \in \mathbb{R}$, we denote the linear rotation R_θ by

$$R_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{1.2}$$

and define the map $F_\theta = R_\theta \circ F \circ R_{-\theta}$.

Definition 1.1. *([16], $*$-condition) A differentiable F satisfies the $*$-condition if for each $\theta \in \mathbb{R}$, there does not exist a sequence $\mathbb{R}^2 \ni z_k \to \infty$ such that, $F_\theta(z_k) \to T \in \mathbb{R}^2$ and $JF_\theta(z_k)$ has a real eigenvalue $\lambda_k \to 0$.*

Theorem 1.3. *([16]) Suppose that $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a differentiable local homeomorphism.*

(i) If F satisfies the $*$-condition, then F is injective and its image is a convex set.

(ii) F is a global homeomorphism of $\mathbb{R}^2$ if and only if F satisfies the $*$-condition and its image $F(\mathbb{R}^2)$ is dense in $\mathbb{R}^2$.

Since the $*$-condition is somewhat weaker than the condition (1.1), we can obtain Theorem 1.2 from Theorem 1.3 (i) by a standard procedure.

For other new cases, the essential difficulty is how to prove that the eigenvalues of JF which may be tending to zero imply F is injective. Rabanal [27] extended the $*$-condition to the following B-condition.

Definition 1.2. *([27], B-condition) The differentiable map $F : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies the B-condition if for each $\theta \in \mathbb{R}$, there does not exist a sequence $(x_k, y_k) \in \mathbb{R}^2$ with $x_k \to +\infty$ such that $F_\theta(x_k, y_k) \to T \in \mathbb{R}^2$ and $JF_\theta(x_k, y_k)$ has a real eigenvalue λ_k satisfying $\lambda_k x_k \to 0$.*

If one replaced the $*$-condition by the B-conditon, then Theorem 1.3 also holds. Moreover, Rabanal obtained the following theorem.

Theorem 1.4. *([27]) Suppose that the differentiable map $F : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies the B-condition and $\det JF(z) \neq 0, \forall z \in \mathbb{R}^2$, then F is a topological embedding.*

In fact, Theorem 1.4 improves the main result of Gutiérrez [16], see also [26, 29].

In 2014, Braun and Venato-Santos [4] considered the relations between the half-Reeb component and the Palais-Smale condition for global injectivity.

Many references on other aspects of the half Reeb component including higher dimensional situations (see [14, 21–23, 28]).

For example, Gutiérrez and Maquera considered the half-Reeb component for the global injectivity in dimension 3.

Theorem 1.5. *([14]) Let $Y = (f, g, h) : \mathbb{R}^3 \to \mathbb{R}^3$ be a polynomial map such that $Spec(Y) \cap [0, \varepsilon) = \emptyset$, for some $\varepsilon > 0$. If $codim(SY) \geqslant 2$, then Y is a bijection.*

Recently, W. Liu prove the following theorem by the Minimax method.

Theorem 1.6. *([22]) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map, $n \geqslant 2$. If for some $\varepsilon > 0$,*

$$0 \notin Spec(F) \quad and \quad Spec(F + F^T) \subseteq (-\infty, -\varepsilon) \; or \; (\varepsilon, +\infty),$$

then F is globally injective.

Let us return to study the eigenvalues of JF approaching to zero by the half-Reeb component method in $\mathbb{R}^2$.

In this paper, we first define the W-condition. For the convenience of our statement, let us denote the set

$\mathcal{P} := \Big\{P \,\Big|\, \mathbb{R}^+ \to \mathbb{R}^+, P$ is nondecreasing and $\forall M > 0,$ there exists a large constant N which depends on M and P, such that $\displaystyle\int_2^N \frac{1}{P(x)}dx > M\Big\}.$

Obviously, $\mathcal{P}$ contains many functions, such as 1, x, $x\ln(x+1)$, $x\ln(1+x)\ln\big(1+\ln(1+x)\big)$ and it doesn't include $x^\alpha, \forall\alpha > 1;\; x\ln^\beta(x+1), \forall\beta > 1$.

Definition 1.3. *(W-condition)*

A differentiable map F satisfies the W-condition if for each $\theta \in \mathbb{R}$ (see (1.2)), there does not exist a sequence $(x_k, y_k) \in \mathbb{R}^2$ with $x_k \to +\infty$ such that $F_\theta(x_k, y_k) \to T \in \mathbb{R}^2$ and $JF_\theta(x_k, y_k)$ has a real eigenvalue λ_k satisfying $\lambda_k P(x_k) \to 0$, where $P \in \mathcal{P}$.

Remark 1.1. *The W-condition obviously contains the* $*$*-condition and the B-condition. Let* $P(x) = x\ln(x+1) \in \mathcal{P}$*, the W-condition with the function P is weaker than the* $*$*-condition and the B-condition. It seems can't be improved in this setting by making use of the half-Reeb component method. The W-condition profoundly reveals the optimal rate that tends to zero of eigenvalues of JF must be in the interval* $\Big(O(x\ln^{\beta} x)^{-1}, \forall\beta > 1,\ O\Big(x\ln x(\ln\frac{\ln x}{\ln\ln x})^2\Big)^{-1}\Big]$ *by the half-Reeb component method.*

Remark 1.2. *If* x_k *exchanges* y_k *in definition 1.3, then the W-condition is also vaild.*

Remark 1.3. *For example, let* $g(x,y)$ *be a* C^1 *function such that* $g(x,y) = \frac{y}{x\ln x}$ *where* $x \geqslant 2$*. The map* $F(x,y) = (e^{-x}, g(x,y))$ *satisfies* $\det JF = -e^{-x}\frac{1}{x\ln x} \neq 0$*. Then, for* $\{x_k\} \subseteq [2,+\infty)$*,* $F(x_k,0) = (e^{-x_k},0) \to P = (0,0)$*, as* $x_k \to +\infty$*.* $JF(x_k,0)$ *has a real eigenvalue*

$$\frac{1}{x_k\ln x_k} = \lambda_k \to 0.$$

However, the limit of the product $x_k\ln x_k$ *is away from zero.*

We use the W-condition and obtain the main result.

Theorem 1.7. *Let* $F : \mathbb{R}^2 \to \mathbb{R}^2$ *be a differentiable local homeomorphism. If F satisfies the W-condition, then F is injective and* $F(\mathbb{R}^2)$ *is convex.*

Obviously, Theorem 1.7 implies Theorems 1.3 and 1.4(i). Moreover, we have:

Theorem 1.8. *Let* $F : \mathbb{R}^2 \to \mathbb{R}^2$ *be a differentiable Jacobian map. If F satisfies the W-condition, then F is a globally injective, measure-preserving map with convex image.*

It improves the main results of Gutiérrez [16], Rabanal [27] and Gutiérrez [18] .

Because of the injectivity of map F in Theorem 1.8, we obtain the following fixed point theorem.

Corollary 1.1. *If F is as in Theorem 1.8 and* $Spec(F) \subseteq \{z \in \mathbb{C} \big| |z| < 1\}$ *, then F has at most one fixed point.*

Another important property on the Keller map as in corollary 1.1 is the theorem B by Cima et al. [6]. They proved that a global attractor for the discrete dynamical system has a unique fixed point.

By the Inverse Function Theorem, the map F in Theorem 1.7 is locally injective at any point in $\mathbb{R}^2$. However, in general, it's not a global injective map. So our goal is to give the sufficient conditions to obtain the global injectivity of F. Here, we also use the W condition as a sufficient condition to obtain the following results.

Theorem 1.9. *Let* $F = (f,g) : \mathbb{R}^2 \to \mathbb{R}^2$ *be a local homeomorphism such that for some* $s > 0$, $F|_{\mathbb{R}^2\setminus D_s}$ *is differentiable. If F satisfies the W-condition, then it is a globally injective and* $F(\mathbb{R}^2)$ *is a convex set.*

Remark 1.4. *If the graph of F is an algebraic set , then the injectivity of F must be the bijectivity of F.*

The W condition can be also devoted to studying the differentiable map $F : \mathbb{R}^2\setminus D_s \to \mathbb{R}^2$ whose $\mathrm{Spec}(F)$ is disjoint with $[0,+\infty)$.

Theorem 1.10. *Let* $F = (f,g) : \mathbb{R}^2\setminus\overline{D_\sigma} \to \mathbb{R}^2$ *be a differential map which satisfies the W-condition. If* $Spec(F) \cap [0,+\infty) = \emptyset$ *or* $Spec(F) \cap (-\infty,0] = \emptyset$*, then there exists* $s \geqslant \sigma$ *such that* $F|_{\mathbb{R}^2\setminus D_s}$ *can be extended to an injective local homeomorphism* $\widetilde{F} = (\widetilde{f},\widetilde{g}) : \mathbb{R}^2 \to \mathbb{R}^2$.

All these works are related to the Jacobian conjecture which can be reduce to that for any dimension $n \geqslant 2$, a polynomial map $F : \mathbb{C}^n \to \mathbb{C}^n$ of the form $F = x + H$, where H is cube-homogeneous and JH is symmetry, is injective if $\mathrm{Spec}(F) = \{1\}$ (see [3]).

In order to prove our theorems, we need to use the definition and some propositions of the half-Reeb component.

2. Half-Reeb component

In this section, we will introduce some preparation work on the eigenvalue conditions of Spec(F).

Let $h_0(x, y) = xy$ and we consider the set

$$B = \{(x, y) \in [0, 2] \times [0, 2] \big| 0 < x + y \leqslant 2\}.$$

Definition 2.1. *(half-Reeb component [13]) Let F be a differentiable map from* $\mathbb{R}^2 \to \mathbb{R}^2$ *and* $\det JF_p \neq 0,\ \forall p \in \mathbb{R}^2$*, Given* $h \in \{f, g\}$*, we will say that* $\mathcal{A} \subseteq \mathbb{R}^2$ *is a half-Reeb component for* $\mathcal{F}(h)$ (*or simply a hRc for* $\mathcal{F}(h)$)*if there exists a homeomorphism* $H : B \to \mathcal{A}$ *which is a topological equivalence between* $\mathcal{F}(h)|_{\mathcal{A}}$ *and* $\mathcal{F}(h_0)|_B$ *and such that:*

(1) The segment$\{(x, y) \in B : x + y = 2\}$ *is sent by H onto a transversal section for the foliation* $\mathcal{F}(h)$ *in the complement of* $H(1, 1)$*; this section is called the compact edge of* $\mathcal{A}$*;*

(2) Both segments $\{(x, y) \in B : x = 0\}$ *and* $\{(x, y) \in B : y = 0\}$ *are sent by H onto full half-trajectories of* $\mathcal{F}(h)$*. These two semi-trajectories of* $\mathcal{F}(h)$ *are called the noncompact edges of* $\mathcal{A}$*.*

The following Propositions connect the half-Reeb components and injectivity of the map F.

Proposition 2.1. *([11]) Suppose that* $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ *is a differentiable map such that* $0 \notin$ *Spec*(F)*. If F is not injective, then both* $\mathcal{F}(f)$ *and* $\mathcal{F}(g)$ *have half-Reeb components.*

Proposition 2.2. *([11]) Let* $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ *be a non-injective, differentiable map such that* $0 \notin$ *Spec*(F)*. Let* $\mathcal{A}$ *be a hRc of* $\mathcal{F}(f)$ *and let* $(f_\theta, g_\theta) = R_\theta \circ F \circ R_{-\theta}$*, where* $\theta \in \mathbb{R}$ *and* R_θ *is in (1.2). If* $\Pi(x, y) = x$ *is bounded, where* $\Pi : \mathbb{R}^2 \to \mathbb{R}$ *is given by* $\Pi(x, y) = x$*, then there is an* $\varepsilon > 0$ *such that, for all* $\theta \in (-\varepsilon, 0) \cup (0, \varepsilon)$*;* $\mathcal{F}(f_\theta)$ *has a hRc* $\mathcal{A}_\theta$ *such that* $\Pi(\mathcal{A}_\theta)$*is an interval of infinite length.*

3. Half-Reeb component and W-condition

In this section, we will establish the essential fact that the W-condition implies non-existence of half-Reeb component.

Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a local homeomorphism of $\mathbb{R}^2$. For each $\theta \in \mathbb{R}$, we denoted by R_θ the linear rotation (see (1.2)):

$$(x, y) \to (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta),$$

and

$$F_\theta := (f_\theta, g_\theta) = R_\theta \circ F \circ R_{-\theta}.$$

In other words, F_θ represents the linear rotation R_θ in the linear coordinates of $\mathbb{R}^2$.

Proposition 3.1. *A differentiable local homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ which satisfies the W-condition has no half-Reeb components.*

Proof. Suppose by contradiction that F has a half-Reeb component. In order to obtain this result, we consider the map $(f_\theta, g_\theta) = F_\theta$. From Proposition 2.2, there exists some $\theta \in \mathbb{R}$, such that $\mathcal{F}(\mathcal{A}_\theta)$ has a half-Reeb component which $\Pi(\mathcal{A})$ is unbounded interval, where $\Pi(\mathcal{A})$ denote orthgonal projection onto the first coordinate in $\mathcal{A}$. Therefore $\exists\, b$ and a half-Reeb component $\mathcal{A}$, such that $[b, +\infty) \subseteq \Pi(\mathcal{A})$. Then, for large enough $a > b$ and any $x \geqslant a$, the vertical line $\Pi^{-1}(x)$ intersects exactly the one trajectory $\alpha_x \cap [x, +\infty) = x$, i.e. x is maximum of the the trajectory Π_{α_x}. If $x \geqslant a$, the intersection $\alpha_x \cap \Pi^{-1}(x)$ is compact subset in $\mathcal{A}$.

Thus, we can define the function $H : (a, +\infty) \to \mathbb{R}$ by

$$H(x) = \sup\{y : (x, y) \in \Pi^{-1}(x) \cap \alpha_x\}.$$

As $\mathcal{F}(f_\theta)$ is a foliation, one gets

$$\Phi : (a, +\infty) \to \mathcal{A} \ \text{ by } \ \Phi(x) = f_\theta(x, H(x)).$$

We can know that Φ is a bounded, monotone strictly function such that, for a full measure subset $M \subseteq (a, +\infty)$.

Since the image of Φ is contained in $f_\theta(\Gamma)$ where Γ is compact edge of hRc $\mathcal{A}$, the function Φ is bounded in $(a, +\infty)$. Furthermore, Φ is continuous because $\mathcal{F}(f_\theta)$ is a C^0 foliation. Since $\mathcal{F}(f_\theta)$ is transversal to Γ, we have Φ is monotone strictly.

For the measure subset $M \subseteq (a, +\infty)$, such that $\Phi(x)$ is differentiable on M and the Jacobian matrix of $F_\theta(x, y)$ at $(x, H(x))$ is

$$JF_\theta(x, H(x)) = \begin{pmatrix} \Phi'(x) & 0 \\ \partial_x g_\theta(x, H(x)) & \partial_y g_\theta(x, H(x)) \end{pmatrix}.$$

Therefore, $\forall x \in M$, $\Phi'(x) = \partial_x f_\theta(x, H(x))$ is a real eigenvalue of $JF_\theta(x, H(x))$ and we denote it by $\lambda(x) := \Phi'(x)$.

Since F is local homeomorphism, without loss of generality, we assume Φ is strictly monotone increasing, i.e. $\Phi'(x) > 0, \forall x \in M$. Let any function $P \in \mathcal{P}$, where

$\mathcal{P} = \Big\{P \,\Big|\, \mathbb{R}^+ \to \mathbb{R}^+, P$ is nondecreasing and $\forall M > 0$, there exists large constant N which depends on M and P, such that $\displaystyle\int_2^N \frac{1}{P(x)} dx > M\Big\}$.

Claim:

$$\liminf_{x_k \to +\infty} \Phi'(x_k) P(x_k) > 0.$$

Because $P(x)$ and $\Phi'(x)$ are both positive, we can suppose by contradition that $\liminf_{x_k \to +\infty} \Phi'(x_k)P(x_k) = 0$. There exists a subsequence denoted still by $\{x_k\}$, such that $\Phi'(x_k)P(x_k) \to 0$, as $x_k \to +\infty$. That is $\underline{\lambda(x_k)P(x_k)} \to 0$. Since $F_\theta(\mathcal{A})$ is bounded, $F_\theta(x_k, H(x_k))$ converges to a finite value T on compact set $\overline{\mathcal{F}_\theta(\mathcal{A})}$. This contradicts the W-condition.

Therefore, there exist a constant a_0 $(a_0 > 2)$ and a small $\varepsilon_0 > 0$, such that

$$\Phi'(x)P(x) > \varepsilon_0, \ \forall x \geqslant a_0.$$

Since $\Phi(x)$ is bounded, there exists $L > 0$, such that

$$\Phi(x) - \Phi(a_0) \leqslant L, \quad \forall x \geqslant a_0.$$

By the definiton of $\mathcal{P}$, we can choose C large enough, such that

$$\int_{a_0}^{C} \frac{1}{P(x)} dx > \frac{L}{\varepsilon_0}.$$

Thus,

$$L \geqslant \Phi(C) - \Phi(a_0) = \int_{a_0}^{C} \Phi'(x) dx \geqslant \int_{a_0}^{C} \frac{\varepsilon_0}{P(x)} dx > L.$$

It's a contradiction.

□

4. The proof of Theorems 1.7 and 1.8

The proof of Theorem 1.7. By contradiction, we suppose that F is not injective. By Proposition 2.1, we have F has a half-Reeb component. This contradicts Proposition 3.1 that implies F has no half-Reeb component if F satisfies the W-condition. Thus, we complete the proof of Theorem 1.7.

The proof of Theorem 1.8. First, we show that the equivalence of the differential Jacobian map and measure-preserving in any dimension n.

For any nonempty measurable set $\Omega \subset \mathbb{R}^n$. Since $F : \mathbb{R}^n \to \mathbb{R}^n$, we can denote $V := \{F(x) \,|x \in \Omega\}$. Let the components of $F(x)$ be v_i $(i = 1, 2...n)$, i.e. $F(x_1, ...x_n) = (v_1(x_1, ...x_n), ...v_n(x_1...x_n))$. So $dv = \det JF(x)dx$. Since $\det JF(x) \equiv 1$, we have $dv = dx$.

Therefore, $\int_V dv = \int_\Omega dx$. It implies F preserves measure.

Inversely, let $v = F(x)$, $\forall x \in \Omega$. We still denote $V = \{F(x) \,|x \in \Omega\}$.

Since F preserves measure, one gets $\int_V dv = \int_\Omega dx$.

Combining it with $dv = \det JF(x)dx$, we obtain $\int_V dv = \int_\Omega \det JF(x)dx$.

Thus, we have $\int_\Omega dx = \int_\Omega \det JF(x)dx$. That is

$$\int_\Omega (1 - \det JF(x))dx = 0, \ \forall \ \Omega \subset \mathbb{R}^n.$$

Claim: $\det JF(x) \equiv 1$, $\forall x \in \mathbb{R}^n$. It's proof by contradiction. Suppose $\exists \ x_0 \in \mathbb{R}^n, \det JF(x_0) \neq 1$. Without loss of generality, we suppose $\det JF(x_0) > 1$, denote $C = \det JF(x_0) - 1 > 0$. Since $F \in C^1$, $\det JF(x) \in C$. $\exists \, \delta > 0$, such that $\det JF(x) - 1 \geqslant \frac{C}{2}$, $\forall x \in U(x_0, \delta)$.

Choosing $\Omega = U(x_0, \delta)$, thus

$$\int_{U(x_0,\delta)} (1 - \det JF(x))dx \leq \int_{U(x_0,\delta)} -\frac{C}{2} dx = -\frac{C}{2} m(U(x_0, \delta)) < 0,$$

it contradicts.

Next, we obtain the global injectiveity of F by the Theorem 1.7. Forthermore, the image of F is convex.

5. The proof of Theorems 1.9 and 1.10

Before we show that Theorem 1.9, the following proposition is necessary.

Proposition 5.1. *Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a local homeomorphism such that for some $s > 0$, $F|_{\mathbb{R}^2 \setminus D_s}$. If F satisfies the W condition, then*
(1) any half Reeb component of $\mathcal{F}(f)$ or $\mathcal{F}(g)$ is a bounded in $\mathbb{R}^2$;
(2) If F extends to a local homemorphism $\overline{\overline{F}} = \left(\overline{f}, \overline{g}\right) : \mathbb{R}^2 \to \mathbb{R}^2$, $\mathcal{F}(\overline{f})$ and $\mathcal{F}(\overline{g})$ have no half-Reeb components.

Proof. By contradiction, without loss of generality, we consider the $\mathcal{F}(f)$ has an unbounded half Reed component. By the process in Proposition 3.1, we assume that $\mathcal{F}(f)$ has a half Reeb component $\mathcal{A}$ such that $\Pi(\mathcal{A})$ is unbounded interval. Furthermore,

$$JF(x, H(x)) = \begin{pmatrix} \Phi'(x) & 0 \\ \partial_x g(x, H(x)) & \partial_y g(x, H(x)) \end{pmatrix}.$$

If $\liminf_{x_k \to +\infty} \Phi'(x_k)P(x_k) = 0$, where $P \in \mathcal{P}$. There exists a subsequence denoted still $\{x_k\}$ with $x_k \to +\infty$ such that $\Phi'(x_k)P(x_k) \to 0$. That is $\lambda(x_k)P(x_k) \to 0$. Since $F(\mathcal{A})$ is bounded, $F(x_k, H(x_k))$ converges to a finite value T on compact set $\overline{\mathcal{F}(\mathcal{A})}$. This contradicts the W-condition.

If $\liminf_{x_k \to +\infty} \Phi'(x_k)P(x_k) \neq 0$, then $\liminf_{x_k \to +\infty} \Phi'(x_k)P(x_k) > 0$. Thus, there exists $C_0 > 0$ and $l > 0$ such that $\Phi'(x)P(x) > l, \forall x > C_0$. For $C > C_0$, there exists $K > 0$, such that

$$\int_{C_0}^{C} \frac{l}{P(x)} dx > K.$$

Since $\Phi(C) - \Phi(C_0) < K$, we have

$$K < \int_{C_0}^{C} \frac{l}{P(x)} dx \leq \int_{C_0}^{C} \Phi'(x) dx < K.$$

It contradicts. We complete the proof of Proposition 5.1. □

The proof of Theorem 1.9. By Proposition 5.1, it's very easy to know that the image of F is convex. This implies that $\mathcal{F}(f)$ has a half Reeb component. It contradicts the Proposition 3.1. Thus, we complete the proof.

The proof of Theorem 1.10. By similar procedure, we can prove the Theorem 1.10 by half Reeb component and Proposition 5.1.

In finally, we prove the Corollary 1.1.

The proof of Corollary 1.1. We consider $G : \mathbb{R}^2 \to \mathbb{R}^2$ and $G(z) = F(z) - 1, \forall z \in \mathbb{R}^2$. Thus, $G(z)$ has no positive eigenvalue because of $\mathrm{Spec}(G) \subset \{z \in \mathbb{R}^2 : \mathrm{Re}(z) < 0\}$. By Theorem 1.7, we have G is injective. Therefore, F has a fixed point. We complete the proof of the Corollary 1.1.

Remark 5.1. *It's very important and meaningful to study the relations between half-Reeb components in higher dimensions and the rate of tending to zero of eigenvalues of JF.*

Conflict of interest

The author declares no conflict of interest in this paper.

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