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Research article

Hyers-Ulam stability of an n-variable quartic functional equation

Vediyappan Govindan[1], **Inho Hwang**[2,*] **and Choonkil Park**[3,*]

[1] Department of Mathematics, DMI St John Baptist University, Mangochi, Malawi

[2] Department of Mathematics, Incheon National University, Incheon 22012, Korea

[3] Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea

* **Correspondence:** Email: ho818@inu.ac.kr, baak@hanyang.ac.kr.

Abstract: In this note we investigate the general solution for the quartic functional equation of the form

$$\begin{aligned}(3n+4)\, f\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=1}^{n} f\left(-nx_j + \sum_{i=1, i\neq j}^{n} x_i\right) &= \left(n^2+2n+1\right) \sum_{i=1, i\neq j\neq k}^{n} f\left(x_i+x_j+x_k\right) \\ &\quad - \frac{1}{2}\left(3n^3-2n^2-13n-8\right) \sum_{i=1, i\neq j}^{n} f\left(x_i+x_j\right) \\ &\quad + \frac{1}{2}\left(n^3+2n^2+n\right) \sum_{i=1, i\neq j}^{n} f\left(x_i-x_j\right) \\ &\quad + \frac{1}{2}\left(3n^4-5n^3-7n^2+13n+12\right) \sum_{i=1}^{n} f\left(x_i\right)\end{aligned}$$

($n \in \mathbb{N}$, $n > 4$) and also investigate the Hyers-Ulam stability of the quartic functional equation in random normed spaces using the direct approach and the fixed point approach.

Keywords: quartic functional equation; fixed point method; Hyers-Ulam stability; random normed space; direct method
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1. Introduction

In 1940, Ulam [25] proposed the following question concerning the stability of group homomorphisms: Under what condition does there is an additive mapping near an approximately additive mapping between a group and a metric group? In the next year, Hyers [7, 8] answered the problem of Ulam under the assumption that the groups are Banach spaces. A generalized version of the theorem of Hyers for approximately linear mappings was given by Rassias [22]. Since then, the stability problems of various functional equation have been extensively investigated by a number of authors (see [3, 6, 13, 14, 16, 17, 22, 24, 26–28]). By regarding a large influence of Ulam, Hyers and Rassias on the investigation of stability problems of functional equations the stability phenomenon that was introduced and proved by Rassias [22] in the year 1978 is called the Hyers-Ulam-Rassias stability.

Consider the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.1}$$

The quadratic function $f(x) = cx^2$ is a solution of this functional Equation (1.1), and so one usually is said the above functional equation to be quadratic [5, 10–12]. The Hyers-Ulam stability problem of the quadratic functional equation was first proved by Skof [24] for functions between a normed space and a Banach space. Afterwards, the result was extended by Cholewa [2] and Czerwik [4].

Now, we consider the following functional equation:

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y). \tag{1.2}$$

It is easy to see that the function $f(x) = cx^4$ satisfies the functional equation (1.2). Hence, it is natural that Eq (1.2) is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic mapping (see [15, 19]).

The theory of random normed spaces (briefly, *RN*-spaces) is important as a generalization of deterministic result of normed spaces and also in the study of random operator equations. The notion of an *RN*-space corresponds to the situations when we do not know exactly the norm of the point and we know only probabilities of possible values of this norm. Random theory is a setting in which uncertainty arising from problems in various fields of science, can be modelled. It is a practical tool for handling situations where classical theories fail to explain. Random theory has many application in several fields, for example, population dynamics, computer programming, nonlinear dynamical system, nonlinear operators, statistical convergence and so forth.

In 2008, Mihet and Radu [18] applied fixed point alternative method to prove the stability theorems of the *Cauchy functional equation:*

$$f(x+y) - f(x) - f(y) = 0$$

in random normed spaces. In 2008, Najati and Moghimi [20] obtained a stability of the functional equation deriving from quadratic and additive function:

$$f(2x+y) + f(2x-y) + 2f(x) - f(x+y) - f(x-y) - 2f(2x) = 0 \tag{1.3}$$

by using the direct method. After that, Jin and Lee [9] proved the stability of the above mentioned functional equation in random normed spaces.

In 2011, Saadati *et al.* [21] proved the nonlinear stability of the quartic functional equation of the form

$$\begin{aligned} 16f(x+4y)+f(4x-y) = \quad & 306\left[9f\left(x+\frac{y}{3}\right)+f(x+2y)\right]+136f(x-y) \\ & -1394f(x+y)+425f(y)-1530f(x) \end{aligned}$$

in the setting of random normed spaces. Furthermore, the interdisciplinary relation among the theory of random spaces, the theory of non-Archimedean spaces, the fixed point theory, the theory of intuitionistic spaces and the theory of functional equations were also presented. Azadi Kenary [1] investigated the Ulam stability of the following nonlinear function equation

$$f\left(f(x)-f(y)\right)+f(x)+f(y)=f(x+y)+f(x-y),$$

in random normed spaces.

In this note, we investigate the general solution for the quartic functional equation of the form

$$\begin{aligned} (3n+4)\,f\left(\sum_{i=1}^{n} x_i\right)+\sum_{j=1}^{n} f\left(-nx_j+\sum_{i=1,i\neq j}^{n} x_i\right) &= \left(n^2+2n+1\right)\sum_{i=1,i\neq j\neq k}^{n} f\left(x_i+x_j+x_k\right) \\ &- \frac{1}{2}\left(3n^3-2n^2-13n-8\right)\sum_{i=1,i\neq j}^{n} f\left(x_i+x_j\right) \\ &+ \frac{1}{2}\left(n^3+2n^2+n\right)\sum_{i=1,i\neq j}^{n} f\left(x_i-x_j\right) \\ &+ \frac{1}{2}\left(3n^4-5n^3-7n^2+13n+12\right)\sum_{i=1}^{n} f\left(x_i\right) \end{aligned} \tag{1.4}$$

($n \in \mathbb{N},\ n > 4$) and also investigate the Hyers-Ulam stability of the quartic functional equation in random normed spaces by using the direct approach and the fixed point approach.

2. Preliminaries

In this part, we make some notations and basic definitions used in this article.

Definition 2.1. A function $T : [0,1]\times[0,1] \to [0,1]$ is called a continuous triangular norm, if T satisfies the following condition

(a) T is commutative and associative;
(b) T is continuous;
(c) $T(a,1) = a$ for all $a \in [0,1]$;
(d) $T(a,b) \le T(c,d)$ when $a \le c$ and $b \le d$ for all $a,b,c,d \in [0,1]$.

Typical examples of continuous $t-$norms are $T_p(a,b) = ab$, $T_m(a,b) = \min(a,b)$ and $T_L(a,b) = \max(a+b-1,0)$ (The Lukasiewicz $t-$norm). Recall [23] that if T is a $t-$norm and $\{x_n\}$ is a given sequence of numbers in $[0,1]$, then $T_{i=1}^{n} x_{n+i}$ is defined recurrently by $T_{i=1}^{'} x_i = x_i$ and

$T_{i=1}^{n} x_i = T\left(T_{i=1}^{n-1} x_i, x_n\right)$ for $n \geq 2$, $T_{i=1}^{\infty} x_i$ is defined as $T_{i=1}^{\infty} x_{n+i}$. It is known that, for the Lukasiewicz $t-$norm, the following implication holds:

$$\lim_{n\to\infty} (T_L)_{i=1}^{\infty} x_{n+i} = 1 \Leftrightarrow \sum_{n=1}^{\infty} (1 - x_n) < \infty.$$

Definition 2.2. A random normed space (briefly, $RN-$space) is a triple (X, μ, T), where X is a vector space. T is a continuous $t-$norm and μ is a mapping from X into D^+ satisfying the following conditions:

(RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
(RN2) $\mu_{\alpha x}(t) = \mu_x\left(\frac{t}{|\alpha|}\right)$ for all $x \in X$, and $alpha \in \Re$ with $\alpha \neq 0$;
(RN3) $\mu_{x+y}(t+s) \geq T\left(\mu_x(t), \mu_y(s)\right)$ for all $x, y \in X$ and $t, s \geq 0$.

Definition 2.3. Let (X, μ, T) be an $RN-$space.

1). A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for any $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x}(\varepsilon) > 1 - \lambda$ for all $n > N$.
2). A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for any $\varepsilon > 0$ *and* $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x_m}(\varepsilon) > 1 - \lambda$ for all $n \geq m \geq N$.
3). An $RN-$space (X, μ, T) is said to be complete, if every Cauchy sequence in X is convergent to a point in X.

Throughout this paper, we use the following notation for a given mapping $f : X \to Y$ as

$$\begin{aligned} Df(x_1, \cdots, x_n) := (3n+4) f\left(\sum_{i=1}^{n} x_i\right) &+ \sum_{j=1}^{n} f\left(-nx_j + \sum_{i=1, i\neq j}^{n} x_i\right) \\ &- \left(n^2 + 2n + 1\right) \sum_{i=1, i\neq j\neq k}^{n} f\left(x_i + x_j + x_k\right) \\ &- \frac{1}{2}\left(3n^3 - 2n^2 - 13n - 8\right) \sum_{i=1, i\neq j}^{n} f\left(x_i + x_j\right) \\ &+ \frac{1}{2}\left(n^3 + 2n^2 + n\right) \sum_{i=1, i\neq j}^{n} f\left(x_i - x_j\right) \\ &+ \frac{1}{2}\left(3n^4 - 5n^3 - 7n^2 + 13n + 12\right) \sum_{i=1}^{n} f(x_i) \end{aligned}$$

for all $x_1, x_2, \cdots, x_n \in X$.

3. A solution of the $n-$variable quartic functional equation (1.4)

In this section we investigate the general solution of the n-variable quartic functional equation (1.4).

Theorem 3.1. *Let X and Y be real vector spaces. If a mapping $f : X \to Y$ satisfies the functional equation (1.4) for all $x_1, \cdots, x_n \in X$, then $f : X \to Y$ satisfies the functional equation (1.2) for all $x, y \in X$.*

Proof. Assume that f satisfies the functional equation (1.4). Putting $x_1 = x_2 = \cdots = x_n = 0$ in (1.4), we get

$$\begin{aligned}(3n+4)\,f(0) = &\left(n^2+2n+1\right)\\ &\left\{6n-20+\frac{4\,(n-4)\,(n-5)}{2}+\frac{(n-4)\,(n-5)\,(n-6)}{6}\right\}f(0)\\ &-\frac{1}{2}\left(3n^3-2n^2-13n-8\right)\left\{4n-10+\frac{\left(n^2-9n+20\right)}{2}\right\}f(0)\\ &+\frac{1}{2}\left(n^3+2n^2+n\right)\left\{4n-10+\frac{\left(n^2-9n+20\right)}{2}\right\}f(0)\\ &-\frac{n}{2}\left(3n^4-5n^3-7n^2+13n+12\right)f(0)\end{aligned} \tag{3.1}$$

It follows from (3.1) that

$$f(0) = 0. \tag{3.2}$$

Replacing $(x_1, x_2, \cdots, x_n)$ by $(x, \underbrace{0, \cdots, 0}_{(n-1)-times})$ in (1.4), we have

$$\begin{aligned}(3n+4)\,f(x)+n^4 f(-x)+(n-1)\,f(x) = &\left(\frac{n^4-n^3-3n^2+n+2}{2}\right)f(x)\\ &-\frac{1}{2}\left(3n^4-5n^3-11n^2+5n+8\right)f(x)\\ &+\frac{1}{2}\left(n^4+n^3+n^2-n\right)f(x)\\ &+\frac{1}{2}\left(3n^4-5n^3-7n^2+13n+12\right)f(x)\end{aligned} \tag{3.3}$$

for all $x \in X$. It follows from (3.3) that

$$f(-x) = f(x)$$

for all $x \in X$. Setting $x_1 = x_2 = \cdots = x_n = x$ in (1.4), we obtain

$$\begin{aligned}(3n+4)\,f(x)+nf(x) = &\left(n^2+2n+1\right)\\ &\left\{6n-20+\frac{4(27)\,(n-4)\,(n-5)}{2}+\frac{(n-4)\,(n-5)\,(n-6)}{6}\right\}f(x)\\ &-\frac{1}{2}\left(3n^3-2n^2-13n-8\right)\left\{4n-10+\frac{\left(n^2-9n+20\right)}{2}\right\}f(2x)\\ &+\frac{1}{2}\left(n^3+2n^2+n\right)\left\{4n-10+\frac{\left(n^2-9n+20\right)}{2}\right\}f(0)\\ &+\frac{n}{2}\left(3n^4-5n^3-7n^2+13n+12\right)f(x)\end{aligned} \tag{3.4}$$

for all $x \in X$. It follows from (3.4) and (3.2) that

$$f(2x) = 16f(x)$$

for all $x \in X$. Setting $x_1 = x_2 = \cdots = x_n = x$ in (1.4), we get

$$\begin{aligned}(3n+4) f(x) + nf(x) = &\left(n^2+2n+1\right)\\ &\left\{6n-20+\frac{4\left(n^2-9n+20\right)}{2}+\frac{\left(n^2-9n+20\right)(n-6)}{6}\right\} f(3x)\\ &-\frac{1}{2}\left(3n^3-2n^2-13n-8\right)\left\{4n-10+\frac{16\left(n^2-9n+20\right)}{2}\right\} f(x)\\ &+\frac{1}{2}\left(n^3+2n^2+n\right)\left\{4n-10+\frac{\left(n^2-9n+20\right)}{2}\right\} f(0)\\ &+\frac{n}{2}\left(3n^4-5n^3-7n^2+13n+12\right) f(x) \qquad (3.5)\end{aligned}$$

for all $x \in X$. It follows from (3.5) and (3.2) that

$$f(3x) = 81f(x)$$

for all $x \in X$. In general for any positive m, we get

$$f(mx) = m^4 f(x)$$

for all $x \in X$. Replacing $(x_1, x_2, \cdots, x_n)$ by $\left(\underbrace{x, x, \cdots, x}_{(n-1)-times}, y\right)$ in (1.4), we get

$$\begin{aligned}&(3n+4)\left(f((n-1)x+y)+(n-1)f(-2x+y)\right)+f((n-1)x-ny)=\\ &\left(n^2+2n+1\right)\left(3n-9+\frac{\left(n^2-3n+2\right)}{2}\right) f(2x+y)\\ &\left(n^2+2n+1\right)\left(3n-11+\left(\frac{3n^2-27n+60}{2}\right)+\left(\frac{(n-4)(n-5)(n-6)}{6}\right)\right) f(3x)\\ &-\frac{1}{2}\left(3n^3-2n^2-13n-8\right)\left\{3n-9+\frac{\left(n^2-9n+20\right)}{2}\right\} f(2x)\\ &-\frac{(n-1)}{2}\left(3n^3-2n^2-13n-8\right) f(x+y)\\ &+\frac{1}{2}\left(n^3+2n^2+n\right)(n-1) f(x-y)\\ &+\frac{1}{2}\left(3n^4-5n^3-7n^2+13n+12\right)\left(f(y)+(n-1)f(x)\right) \qquad (3.6)\end{aligned}$$

for all $x, y \in X$. It follows from (3.6) that

$$\begin{aligned}
&(3n+4)\left(f\left((n-1)x+y\right)+(n-1)f\left(-2x+y\right)\right)+f\left((n-1)x-ny\right)= \\
&\left(n^2+2n+1\right)\left(\frac{n^2-3n+2}{2}\right)f\left(2x+y\right) \\
&+81\left(n^2+2n+1\right)\left(\frac{n^3-5n^2+11n-6}{6}\right)f\left(x\right) \\
&-16\left(3n^3-2n^2-13n-8\right)\left(\frac{n^2-3n+2}{4}\right)f\left(x\right) \\
&-\frac{(n-1)}{2}\left(3n^3-2n^2-13n-8\right)f\left(x+y\right) \\
&+\frac{1}{2}\left(n^3+2n^2+n\right)(n-1)f\left(x-y\right) \\
&+\frac{1}{2}\left(3n^4-5n^3-7n^2+13n+12\right)\left(f\left(y\right)+(n-1)f\left(x\right)\right) \qquad (3.7)
\end{aligned}$$

for all $x, y \in X$. Replacing y by $-y$ in (3.7), we get

$$\begin{aligned}
&(3n+4)\left(f\left((n-1)x-y\right)+(n-1)f\left(2x+y\right)\right)+f\left(()x+y\right)= \\
&\left(n^2+2n+1\right)\left(\frac{n^2-3n+2}{2}\right)f\left(2x-y\right) \\
&+81\left(n^2+2n+1\right)\left(\frac{n^3-5n^2+11n-6}{6}\right)f\left(x\right) \\
&-16\left(3n^3-2n^2-13n-8\right)\left(\frac{n^2-3n+2}{4}\right)f\left(x\right) \\
&-\frac{(n-1)}{2}\left(3n^3-2n^2-13n-8\right)f\left(x-y\right)+\frac{1}{2}\left(n^3+2n^2+n\right)(n-1)f\left(x+y\right) \\
&+\frac{1}{2}\left(3n^4-5n^3-7n^2+13n+12\right)\left(f\left(y\right)+(n-1)f\left(x\right)\right) \qquad (3.8)
\end{aligned}$$

for all $x, y \in X$. Adding (3.7) and (3.8), we get

$$\begin{aligned}
&(3n+4) \\
&\left((n-1)^2\left\{f\left(x+y\right)+f\left(x-y\right)\right\}+2\left\{f\left((n-1)x\right)-(n-1)^2f\left(x\right)\right\}+2\left\{f\left(y\right)-(n-1)^2f\left(y\right)\right\}\right) \\
&+(n-1)\left(f\left(-2x+y\right)+f\left(2x+y\right)\right)+\left(f\left((n-1)x-ny\right)+f\left((n-1)x+ny\right)\right) \\
&=\left(n^2+2n+1\right)\left(\frac{n^2-3n+2}{2}\right)\left(f\left(2x-y\right)+f\left(2x+y\right)\right) \\
&+162\left(n^2+2n+1\right)\left(\frac{n^3-5n^2+11n-6}{6}\right)f\left(x\right) \\
&-32\left(3n^3-2n^2-13n-8\right)\left(\frac{n^2-3n+2}{4}\right)f\left(x\right) \\
&-\frac{(n-1)}{2}\left(3n^3-2n^2-13n-8\right)\left(f\left(x-y\right)+f\left(x+y\right)\right)
\end{aligned}$$

$$+\frac{1}{2}\left(n^3+2n^2+n\right)(n-1)(f(x+y)+f(x-y))$$
$$+\left(3n^4-5n^3-7n^2+13n+12\right)(f(y)+(n-1)f(x)) \tag{3.9}$$

for all $x, y \in X$. It follows from (3.9) that

$$(3n+4)$$
$$\left((n-1)^2\{f(x+y)+f(x-y)\}+2\left\{f((n-1)x)-(n-1)^2f(x)\right\}+2\left\{f(y)-(n-1)^2f(y)\right\}\right)$$
$$+(n-1)(f(2x-y)+f(2x+y))+n^2(n-1)^2(f(x+y)+f(x-y))$$
$$+2\left\{f((n-1)x)-n^2(n-1)^2f(x)+n^4f(y)-n^2(n-1)^2f(y)\right\}$$
$$=\left(n^2+2n+1\right)\left(\frac{n^2-3n+2}{2}\right)(f(2x-y)+f(2x+y))$$
$$+162\left(n^2+2n+1\right)\left(\frac{n^3-5n^2+11n-6}{6}\right)f(x)$$
$$-32\left(3n^3-2n^2-13n-8\right)\left(\frac{n^2-3n+2}{4}\right)f(x)$$
$$-\frac{(n-1)}{2}\left(3n^3-2n^2-13n-8\right)(f(x-y)+f(x+y))$$
$$+\frac{1}{2}\left(n^3+2n^2+n\right)(n-1)(f(x+y)+f(x-y))$$
$$+\left(3n^4-5n^3-7n^2+13n+12\right)(f(y)+(n-1)f(x)) \tag{3.10}$$

for all $x, y \in X$. It follows from (3.10) that

$$-2f(2x+y)-2f(2x-y)=-8f(x+y)-8f(x-y)-48f(x)+12f(y) \tag{3.11}$$

for all $x, y \in X$. From (3.11), we get

$$(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y)$$

for all $x, y \in X$. Thus the mapping $f: X \to Y$ is quartic. □

4. Direct method

In this section, the Ulam-Hyers stability of the quartic functional equation (1.4) in $RN-$space is provided. Throughout this part, let X be a linear space and (Y, μ, T) be a complete $RN-$space.

Theorem 4.1. *Let $j = \pm 1$, $f : X \to Y$ be a mapping for which there exists a mapping $\eta : X^n \to D^+$ satisfying*

$$\lim_{k\to\infty} T_{i=0}^{\infty}\left(\eta_{2^{(k+i)}x_1,2^{(k+i)}x_2,\ldots,2^{(k+i)}x_n}\left(2^{4(k+i+1)j}t\right)\right)=1=\lim_{k\to\infty}\eta_{2^{kj}x_1,2^{kj}x_2,..,2^{kj}x_n}\left(2^{4kj}t\right)$$

such that $f(0) = 0$ and

$$\mu_{Df(x_1,x_2,\ldots,x_n)}(t) \geq \eta_{(x_1,x_2,\ldots,x_n)}(t) \tag{4.1}$$

for all $x_1, x_2, ..., x_n \in X$ and all $t > 0$. Then there exists a unique quartic mapping $Q : X \to Y$ satisfying the functional equation (1.4) and

$$\mu_{Q(x)-f(x)}(t) \geq T_{i=0}^{\infty}\left(\eta_{2^{(i+1)j}x, 2^{(i+1)j}x,...,2^{(i+1)j}x}\left(\frac{1}{4}\left(3n^5 - 5n^4 - 11n^3 + 5n^2 + 8n\right)2^{4(i+1)j}t\right)\right)$$

for all $x \in X$ and all $t > 0$. The mapping $Q(x)$ is defined by

$$\mu_{C(x)}(t) = \lim_{k\to\infty} \mu_{\frac{f\left(2^{kj}x\right)}{2^{4kj}}}(t) \tag{4.2}$$

for all $x \in X$ and all $t > 0$.

Proof. Assume $j = 1$. Setting $(x_1, x_2, ..., x_n)$ by $\left(\underbrace{x, x, ..., x}_{n-times}\right)$ in (4.1), we obtain

$$\mu_{\frac{\left(3n^5-5n^4-11n^3+5n^2+8n\right)}{4}f(2x)-4\left(3n^5-5n^4-11n^3+5n^2+8n\right)f(x)}(t) \geq \eta_{\left(\underbrace{x, x, ..., x}_{n-times}\right)}(t) \tag{4.3}$$

for all $x \in X$ and all $t > 0$. It follows from (4.2) and (RN2) that

$$\mu_{\frac{f(2x)}{16}-f(x)}(t) \geq \eta_{\left(\underbrace{x, x, ..., x}_{n-times}\right)}\left(2^4\left(3n^5 - 5n^4 - 11n^3 + 5n^2 + 8n\right)t\right)$$

for all $x \in X$ and all $t > 0$. Replacing x by $2^k x$ in (4.3), we have

$$\begin{aligned}\mu_{\frac{f\left(2^{k+1}x\right)}{2^{4(k+1)}}-\frac{f\left(2^k x\right)}{2^{4k}}}(t) &\geq \eta_{\left(\underbrace{2^k x, 2^k x, ..., 2^k x}_{n-times}\right)}\left(2^{4k}\left(3n^5 - 5n^4 - 11n^3 + 5n^2 + 8n\right)16t\right)\\ &\geq \eta_{\left(\underbrace{x, x, ..., x}_{n-times}\right)}\left(\frac{2^{4k}\left(3n^5 - 5n^4 - 11n^3 + 5n^2 + 8n\right)16}{\alpha^k}t\right)\end{aligned} \tag{4.4}$$

for all $x \in X$ and all $t > 0$. It follows from $\frac{f(2^n x)}{2^{4n}} - f(x) = \sum_{k=0}^{n-1} \frac{f\left(2^{k+1}x\right)}{2^{4(k+1)}} - \frac{f\left(2^k x\right)}{2^{4k}}$ and (4.4) that

$$\begin{aligned}\mu_{\frac{f(2^n x)}{2^{4n}}-f(x)}\left(t\sum_{k=0}^{n-1}\frac{\alpha^k}{2^{4k}\left(3n^5 - 5n^4 - 11n^3 + 5n^2 + 8n\right)16}\right) &\geq T_{k=0}^{n-1}\left(\eta_{x,x,0,...,0}(t)\right)\\ &= \eta_{\left(\underbrace{x, x, ..., x}_{n-times}\right)}(t),\end{aligned}$$

$$\mu_{\frac{f(2^n x)}{2^{4n}}-f(x)}(t) \geq \eta_{\left(\underbrace{x, x, ..., x}_{n-times}\right)}\left(\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{2^{4k}\left(3n^5-5n^4-11n^3+5n^2+8n\right)2^4}}\right) \tag{4.5}$$

for all $x \in X$ and all $t > 0$. Replacing x by $2^m x$ in (4.5), we get

$$\mu_{\frac{f(2^{n+m}x)}{2^{4(n+m)}} - \frac{f(2^m x)}{2^{4m}}}(t) \geq \eta_{\left(\underbrace{x, x, ..., x}_{n-times}\right)}\left(\frac{t}{\sum_{k=m}^{n+m} \frac{\alpha^k}{2^{4k}\left(3n^5-5n^4-11n^3+5n^2+8n\right)16}}\right). \tag{4.6}$$

Since $\eta_{\left(\underbrace{x, x, ..., x}_{n-times}\right)}\left(\frac{t}{\sum_{k=m}^{n+m} \frac{\alpha^k}{2^{4k}\left(3n^5-5n^4-11n^3+5n^2+8n\right)16}}\right) \to 1$ as $m, n \to \infty$, $\left\{\frac{f(2^n x)}{2^{4n}}\right\}$ is a Cauchy sequence in (Y, μ, T). Since (Y, μ, T) is a complete RN-space, this sequence converges to some point $C(x) \in Y$. Fix $x \in X$ and put $m = 0$ in (4.6). Then we have

$$\mu_{\frac{f(2^n x)}{2^{4n}} - f(x)}(t) \geq \eta_{\left(\underbrace{x, x, ..., x}_{n-times}\right)}\left(\frac{t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{2^{4k}\left(3n^5-5n^4-11n^3+5n^2+8n\right)16}}\right)$$

and so, for every $\delta > 0$, we get

$$\begin{aligned} \mu_{C(x)-f(x)}(t+\delta) &\geq T\left(\mu_{Q(x)-\frac{f(2^n x)}{2^{4n}}}(\delta), \mu_{\frac{f(2^n x)}{2^{4n}-f(x)}}(t)\right) \\ &\geq T\left(\mu_{Q(x)-\frac{f(2^n x)}{2^{4n}}}(\delta), \eta_{\left(\underbrace{x, x, ..., x}_{n-times}\right)}\left(\frac{t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{2^{4k}\left(3n^5-5n^4-11n^3+5n^2+8n\right)16}}\right)\right). \end{aligned} \tag{4.7}$$

Taking the limit as $n \to \infty$ and using (4.7), we have

$$\mu_{C(x)-f(x)}(t+\delta) \geq \eta_{\left(\underbrace{x, x, ..., x}_{n-times}\right)}\left(\left(3n^5 - 5n^4 - 11n^3 + 5n^2 + 8n\right)\left(2^4 - \alpha\right)t\right). \tag{4.8}$$

Since δ is arbitrary, by taking $\delta \to 0$ in (4.8), we have

$$\mu_{Q(x)-f(x)}(t) \geq \eta_{\left(\underbrace{x, x, ..., x}_{n-times}\right)}\left(\left(3n^5 - 5n^4 - 11n^3 + 5n^2 + 8n\right)\left(2^4 - \alpha\right)t\right). \tag{4.9}$$

Replacing $(x_1, x_2, ..., x_n)$ by $(2^n x_1, 2^n x_2, ..., 2^n x_n)$ in (4.1), respectively, we obtain

$$\mu_{Df(2^n x_1, 2^n x_2, ..., 2^n x_n)}(t) \geq \eta_{2^n x_1, 2^n x_2, ..., 2^n x_n}\left(2^{4n} t\right)$$

for all $x_1, x_2, ..., x_n \in X$ and for all $t > 0$. Since

$$\lim_{k\to\infty} T_{i=0}^{\infty}\left(\eta_{2^{k+i}_{x_1}, 2^{k+i}_{x_2}, ..., 2^{k+i}_{x_n}}\left(2^{4(k+i+1)j} t\right)\right) = 1,$$

we conclude that Q fulfils (1.1). To prove the uniqueness of the quartic mapping Q, assume that there exists another quartic mapping D from X to Y, which satisfies (4.9). Fix $x \in X$. Clearly, $Q(2^n x) = 2^{4n} Q(x)$ and $D(2^n x) = 2^{4n} D(x)$ for all $x \in X$. It follows from (4.9) that

$$\mu_{Q(x)-D(x)}(t) = \lim_{n\to\infty} \mu_{\frac{Q(2^n x)}{2^{4n}} \frac{D(2^n x)}{2^{4n}}}(t)$$
$$\mu_{\frac{Q(2^n x}{2^{4n}}}(t) \geq \min\{\mu_{\frac{Q(2^n x)}{2^{4n}} \frac{f(2^n x)}{2^{4n}}}\left(\frac{t}{2}\right), \mu_{\frac{D(2^n x)}{2^{4n}} \frac{f(2^n x)}{2^{4n}}}\left(\frac{t}{2}\right)\}$$

$$\geq \eta_{\left(\underbrace{2^n_x, 2^n_x, ..., 2^n_x}_{n-times}\right)}\left(2^{4n}\left(3n^5 - 5n^4 - 11n^3 + 5n^2 + 8n\right)\left(2^4 - \alpha\right)t\right)$$
$$\geq \eta_{\left(\underbrace{x, x, ..., x}_{n-times}\right)}\left(\frac{2^{4n}\left(3n^5 - 5n^4 - 11n^3 + 5n^2 + 8n\right)\left(2^4 - \alpha\right)t}{\alpha^n}\right).$$

Since $\lim_{n\to\infty}\left(\frac{2^{4n}\left(3n^5-5n^4-11n^3+5n^2+8n\right)\left(2^4-\alpha\right)t}{\alpha^n}\right) = \infty$, we get
$\lim_{n\to\infty} \eta_{x,x,0,...,0}\left(\frac{2^{4n}\left(3n^5-5n^4-11n^3+5n^2+8n\right)\left(2^4-\alpha\right)t}{\alpha^n}\right) = 1.$
Therefore, it follows that $\mu_{Q(x)-D(x)}(t) = 1$ for all $t > 0$ and so $Q(x) = D(x)$.

This completes the proof. □

The following corollary is an immediate consequence of Theorem 4.1, concerning the stability of (1.4).

Corollary 4.2. *Let Ω and $\mho$ be nonnegative real numbers. Let $f : X \to Y$ satisfy the inequality*

$$\mu_{Df(x_1,x_2,...,x_n)}(t) \geq \begin{cases} \eta_\Omega(t), \\ \eta_\Omega \sum_{i=1}^{n} ||x_i||^{\mho}(t) \\ \eta_\Omega\left(\prod_{i=1}^{n} ||x_i||^{n\,\mho}\right)(t), \\ \eta_\Omega\left(\prod_{i=1}^{n} ||x_i||^{\mho} + \sum_{i=1}^{n} ||x_i||^{n\mho}\right)(t) \end{cases}$$

for all $x_1, x_2, x_3, ..., x_n \in X$ and all $t > 0$, then there exists a unique quartic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \begin{cases} \eta_{\frac{\Omega}{|60|\left(3n^5-5n^4-11n^3+5n^2+8n\right)}}(t) & \\ \eta_{\frac{\Omega||x||^{\mho}}{|4|\left(3n^5-5n^4-11n^3+5n^2+8n\right)\left|2^4-2^{\mho}\right|}}(t), & \mho \neq 4 \\ \eta_{\frac{\Omega||x||^{n\,\mho}}{|4|\left(3n^5-5n^4-11n^3+5n^2+8n\right)\left|2^4-2^{n\mho}\right|}}(t), & \mho \neq \frac{4}{n} \\ \eta_{\frac{(n+1)\Omega||x||^{n\mho}}{|4|\left(3n^5-5n^4-11n^3+5n^2+8n\right)\left|2^4-2^{n\,\mho}\right|}}(t), & \mho \neq \frac{4}{n} \end{cases}$$

for all $x \in X$ and all $t > 0$.

5. Fixed point method

In this section, we prove the Ulam-Hyers stability of the functional equation (1.4) in random normed spaces by the using fixed point method.

Theorem 5.1. *Let $f : X \to Y$ be a mapping for which there exists a mapping $\eta : X^n \to D^+$ with the condition*

$$\lim_{k\to\infty} \eta_{\delta_i^k x_1, \delta_i^k x_2, \ldots, \delta_i^k x_n}\left(\delta_i^k t\right) = 1$$

for all $x_1, x_2, x_3, ..., x_n \in X$ and all $t > 0$, where

$$\delta_i = \begin{cases} 2 & i = 0; \\ \frac{1}{2} & i = 1; \end{cases}$$

and satisfy the functional inequality

$$\mu_{Df(x_1,x_2,\ldots,x_n)}(t) \geq \eta_{x_1,x_2,\ldots,x_n}(t)$$

for all $x_1, x_2, x_3, ..., x_n \in X$ and all $t > 0$. If there exists $L = L(i)$ such that the function $x \to \beta(x, t) = \eta_{\left(\underbrace{\frac{x}{2}, \frac{x}{2}, \ldots, \frac{x}{2}}_{n-times}\right)}\left(\left(3n^5 - 5n^4 - 11n^3 + 5n^2 + 8n\right)t\right)$ has the property that

$$\beta(x,t) \leq L\frac{1}{\delta_i^4}\beta(\delta_i x, t) \tag{5.1}$$

for all $x \in X$ and $t > 0$, then there exists a unique quartic mapping $Q : X \to Y$ satisfying the functional equation (1.4) and

$$\mu_{Q(x)-f(x)}\left(\frac{L^{1-i}}{1-L}t\right) \geq \beta(x,t)$$

for all $x \in X$ and $t > 0$.

Proof. Let d be a general metric on Ω such that

$$d(p,q) = \inf\left\{k \in (0,\infty) / \mu_{(p(x)-q(x))}(kt) \geq \beta(x,t), x \in X, t > 0\right\}.$$

It is easy to see that (Ω, d) is complete. Define $T : \Omega \to \Omega$ by $Tp(x) = \frac{1}{\delta_i^4}p(\delta_i x)$ for all $x \in X$. Now assume that for $p, q \in \Omega$, we have $d(p,q) \leq K$. Then

$$\begin{aligned} &\mu_{(p(x)-q(x))}(kt) \geq \beta(x,t) \\ &\Rightarrow \mu_{(p(x)-q(x))}\left(\frac{Kt}{\delta_i^4}\right) \geq \beta(x,t) \\ &\Rightarrow d(Tp(x), Tq(x)) \leq KL \\ &\Rightarrow d(Tp, Tq) \leq Ld(p,q) \end{aligned} \tag{5.2}$$

for all $p, q \in \Omega$. Therefore, T is a strictly contractive mapping on Ω with Lipschitz constant L. It follows from (5.2) that

$$\mu_{\frac{\left(3n^5-5n^4-11n^3+5n^2+8n\right)}{4} f(2x)-4\left(n3n^5-5n^4-11n^3+5n^2+8n\right) f(x)}(t) \geq \eta_{\left(\underbrace{x, x, \ldots, x}_{n-times}\right)}(t) \tag{5.3}$$

for all $x \in X$. It follows from (5.3) that

$$\mu_{\frac{f(2x)}{16}-f(x)}(t) \geq \eta_{\left(\underbrace{x, x, \ldots, x}_{n-times}\right)}\left(\left(3n^5 - 5n^4 - 11n^3 + 5n^2 + 8n\right) 16t\right) \tag{5.4}$$

for all $x \in X$. By using (5.1) for the case $i = 0$, it reduce to

$$\mu_{\frac{f(2x)}{16}-f(x)}(t) \geq L\beta(x, t)$$

for all $x \in X$. Hence we obtain

$$d\left(\mu_{Tf, f}\right) \leq L = L^{1-i} < \infty \tag{5.5}$$

for all $x \in X$. Replacing x by $\frac{x}{2}$ in (5.4), we get

$$\mu_{\frac{f(x)}{16}-f\left(\frac{x}{2}\right)}(t) \geq \eta_{\left(\underbrace{\frac{x}{2}, \frac{x}{2}, \ldots, \frac{x}{2}}_{n-times}\right)}\left(\left(3n^5 - 5n^4 - 11n^3 + 5n^2 + 8n\right) 16t\right)$$

for all $x \in X$. By using (5.1) for the case $i = 1$, it reduce to

$$\mu_{16f\left(\frac{x}{2}\right)-f(x)}(t) \geq \beta(x, t) \Rightarrow \mu_{Tf(x)-f(x)}(t) \geq \beta(x, t)$$

for all $x \in X$. Hence we get

$$d\left(\mu_{Tf, f}\right) \leq L = L^{1-i} < \infty \tag{5.6}$$

for all $x \in X$. From (5.5) and (5.6), we can conclude

$$d\left(\mu_{Tf, f}\right) \leq L = L^{1-i} < \infty$$

for all $x \in X$.

The remaining proof is similar to the proof of Theorem 4.1. So Q is a unique fixed point of T in the set such that

$$\mu_{(f(x)-Q(x))}\left(\frac{L^{1-i}}{1-L} t\right) \geq \beta(x, t)$$

for all $x \in X$ and $t > 0$. This completes the proof of the theorem. □

From Theorem 4.1, we obtain the following corollary concerning the stability for the functional equation (1.4).

Corollary 5.2. *Suppose that a mapping $f : X \to Y$ satisfies the inequality*

$$\mu_{Df(x_1,x_2,...,x_n)}(t) \geq \begin{cases} \eta_{\Omega}(t), \\ \eta_{\Omega} \sum\limits_{i=1}^{n} \|x_i\|^{\mho}(t) \\ \eta_{\Omega}\left(\prod\limits_{i=1}^{n} \|x_i\|^{\mho}\right)(t), \\ \eta_{\Omega}\left(\prod\limits_{i=1}^{n} \|x_i\|^{\mho} + \sum\limits_{i=1}^{n} \|x_i\|^{n\mho}\right)(t), \end{cases}$$

for all $x_1, x_2, ..., x_n \in X$ and all $t > 0$, where $\Omega, \mho$ are constants with $\Omega > 0$. Then there exists a unique quartic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \begin{cases} \eta_{\frac{\Omega}{|60|\left(3n^5-5n^4-11n^3+5n^2+8n\right)}}(t) & \\ \eta_{\frac{\Omega\|x\|^{\mho}}{|4|\left(3n^5-5n^4-11n^3+5n^2+8n\right)\left|2^4-2^{\mho}\right|}}(t), & \mho \neq 4 \\ \eta_{\frac{\Omega\|x\|^{n\mho}}{|4|\left(3n^5-5n^4-11n^3+5n^2+8n\right)\left|2^4-2^{n\mho}\right|}}(t), & \mho \neq \frac{4}{n} \\ \eta_{\frac{(n+1)\Omega\|x\|^{n\mho}}{|4|\left(3n^5-5n^4-11n^3+5n^2+8n\right)\left|2^4-2^{n\mho}\right|}}(t), & \mho \neq \frac{4}{n} \end{cases}$$

for all $x \in X$ and all $t > 0$.

Proof. Set

$$\mu_{Df(x_1,x_2,...,x_n)}(t) \geq \begin{cases} \eta_{\Omega}(t), \\ \eta_{\Omega} \sum\limits_{i=1}^{n} \|x_i\|^{\mho}(t), \\ \eta_{\Omega}\left(\prod\limits_{i=1}^{n} \|x_i\|^{\mho}\right)(t), \\ \eta_{\Omega}\left(\prod\limits_{i=1}^{n} \|x_i\|^{\mho} + \sum\limits_{i=1}^{n} \|x_i\|^{n\mho}\right)(t) \end{cases}$$

for all $x_1, x_2, ..., x_n \in X$ and all $t > 0$. Then

$$\eta_{\left(\delta_i^k x_1, \delta_i^k x_2, ..., \delta_i^k x_n\right)}\left(\delta_i^{4k} t\right) = \begin{cases} \eta_{\Omega\delta_i^{4k}}(t), \\ \eta_{\Omega} \sum\limits_{i=1}^{n} \|x_i\|^{\mho} \delta_i^{(4-n\mho)k}(t), \\ \eta_{\Omega}\left(\prod\limits_{i=1}^{n} \|x_i\|^{\mho} \delta_i^{(4-n\mho)k}\right)(t), \\ \eta_{\Omega}\left(\prod\limits_{i=1}^{n} \|x_i\|^{\mho} \delta_i^{(4-n\mho)k} + \sum\limits_{i=1}^{n} \|x_i\|^{n\mho}\right)(t), \end{cases}$$

$$= \begin{cases} \to \ 1 \ \text{ as } \ k \to \infty \\ \to \ 1 \ \text{ as } \ k \to \infty \\ \to \ 1 \ \text{ as } \ k \to \infty \\ \to \ 1 \ \text{ as } \ k \to \infty. \end{cases}$$

But we have $\beta(x,t) = \eta\left(\underbrace{\frac{x}{2},\frac{x}{2},\ldots,\frac{x}{2}}_{n-times}\right)\left(\frac{1}{4}\left(3n^5-5n^4-11n^3+5n^2+8n\right)t\right)$ has the property $L\frac{1}{\delta_i^4}\beta(\delta_i x,t)$ for all $x \in X$ and $t > 0$. Now

$$\beta(x,t) = \begin{cases} \eta_{\frac{4\Omega}{3n^5-5n^4-11n^3+5n^2+8n}}(t) \\ \eta_{\frac{4\Omega\|x\|^{\mho}}{2^{4\mho}\left(3n^5-5n^4-11n^3+5n^2+8n\right)}}(t) \\ \eta_{\frac{4\Omega\|x\|^{n\mho}}{2^{4\mho}\left(3n^5-5n^4-11n^3+5n^2+8n\right)}}(t) \\ \eta_{\frac{4\Omega\|x\|^{ns}}{2^{n\mho}\left(3n^5-5n^4-11n^3+5n^2+8n\right)}}(t), \end{cases}$$

$$L\frac{1}{\delta_i^4}\beta(\delta_i x,t) = \begin{cases} \eta_{\delta_i^{-4}\beta(x)}(t) \\ \eta_{\delta_i^{\mho-4}\beta(x)}(t) \\ \eta_{\delta_i^{\mho-4}\beta(x)}(t) \\ \eta_{\delta_i^{n\,\mho-4}\beta(x)}(t). \end{cases}$$

By (4.1), we prove the following eight cases:

$$L = 2^{-4} \text{ if } i = 0 \text{ and } L = 2^{4} \text{ if } i = 1$$
$$L = 2^{\mho-4} \text{ for } \mho < 4 \text{ if } i = 0 \text{ and } L = 2^{4-\mho} \text{ for } \mho > 4 \text{ if } i = 1$$
$$L = 2^{n\mho-4} \text{ for } \mho < \frac{4}{n} \text{ if } i = 0 \text{ and } L = 2^{4-n\mho} \text{ for } \mho > \frac{4}{n} \text{ if } i = 1$$
$$L = 2^{n\mho-4} \text{ for } \mho < \frac{4}{n} \text{ if } i = 0 \text{ and } L = 2^{4-n\mho} \text{ for } \mho > \frac{4}{n} \text{ if } i = 1$$

Case 1: $L = 2^{-4}$ if $i = 0$

$$\mu_{f(x)-Q(x)}(t) \geq L\frac{1}{\delta_i^4}\beta(\delta_i x,t)(t) \geq \eta_{\left(\frac{\Omega}{60\left(3n^5-5n^4-11n^3+5n^2+8n\right)}\right)}(t).$$

Case 2: $L = 2^{4}$ if $i = 1$

$$\mu_{f(x)-Q(x)}(t) \geq L\frac{1}{\delta_i^4}\beta(\delta_i x,t)(t) \geq \eta_{\left(\frac{\Omega}{-60\left(3n^5-5n^4-11n^3+5n^2+8n\right)}\right)}(t).$$

Case 3: $L = 2^{\mho-4}$ for $\mho < 4$ if $i = 0$

$$\mu_{f(x)-Q(x)}(t) \geq L\frac{1}{\delta_i^4}\beta(\delta_i x,t)(t) \geq \eta_{\left(\frac{\Omega n\|x\|^{\mho}}{4\left(3n^5-5n^4-11n^3+5n^2+8n\right)\left(2^4-2^{\mho}\right)}\right)}(t).$$

Case 4: $L = 2^{4-\mho}$ for $\mho > 4$ if $i = 1$

$$\mu_{f(x)-Q(x)}(t) \geq L\frac{1}{\delta_i^4}\beta(\delta_i x,t)(t) \geq \eta_{\left(\frac{n\Omega\|x\|^{s}}{4\left(3n^5-5n^4-11n^3+5n^2+8n\right)\left(2^{\mho}-2^4\right)}\right)}(t).$$

Case 5: $L = 2^{n\mho-4}$ for $\mho < \frac{4}{n}$ if $i = 0$

$$\mu_{f(x)-Q(x)}(t) \geq L\frac{1}{\delta_i^4}\beta(\delta_i x,t)(t) \geq \eta_{\left(\frac{\Omega\|x\|^{n\mho}}{4\left(3n^5-5n^4-11n^3+5n^2+8n\right)\left(2^4-2^{n\mho}\right)}\right)}(t).$$

Case 6: $L = 2^{4-n\mho}$ for $\mho > \frac{4}{n}$ if $i = 1$

$$\mu_{f(x)-Q(x)}(t) \geq L\frac{1}{\delta_i^4}\beta(\delta_i x, t)(t) \geq \eta_{\left(\frac{\Omega\|x\|^{n\mho}}{-4\left(3n^5-5n^4-11n^3+5n^2+8n\right)\left(2^{n\mho}-2^4\right)}\right)}(t).$$

Case 7: $L = 2^{n\mho-4}$ for $\mho < \frac{4}{n}$ if $i = 0$

$$\mu_{f(x)-Q(x)}(t) \geq L\frac{1}{\delta_i^4}\beta(\delta_i x, t)(t) \geq \eta_{\left(\frac{(n+1)\|x\|^{n\mho}}{4\left(3n^5-5n^4-11n^3+5n^2+8n\right)\left(2^4-2^{n\mho}\right)}\right)}(t).$$

Case 8 : $L = 2^{4-n\mho}$ for $\mho > \frac{4}{n}$ if $i = 1$

$$\mu_{f(x)-Q(x)}(t) \geq L\frac{1}{\delta_i^4}\beta(\delta_i x, t)(t) \geq \eta_{\left(\frac{(n+1)\Omega\|x\|^{n\mho}}{-4\left(3n^5-5n^4-11n^3+5n^2+8n\right)\left(2^{n\mho}-2^4\right)}\right)}(t).$$

Hence the proof is complete. □

6. Conclusion

In this note we investigated the general solution for the quartic functional equation (1.4) and also investigated the Hyers-Ulam stability of the quartic functional equation (1.4) in random normed space using the direct approach and the fixed point approach. This work can be applied to study the stability in various spaces such as intuitionistic random normed spaces, quasi-Banach spaces and fuzzy normed spaces. Moreover, the results can be applied to investigate quartic homomorphisms and quratic derivations in Banach algebras, random normed algebras, fuzzy Banach algebras and C^*-ternary algebras.

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Conflict of interest

The authors declare that they have no competing interests.

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