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Research article

The existence of subdigraphs with orthogonal factorizations in digraphs

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Abstract: Let G be a $[0, k_1 + k_2 + \cdots + k_m - n + 1]$-digraph and $H_1, H_2, \cdots, H_r$ be r vertex-disjoint n-subdigraphs of G, where m, n, r and k_i $(1 \le i \le m)$ are positive integers satisfying $1 \le n \le m$ and $k_1 \ge k_2 \ge \cdots \ge k_m \ge r + 1$. In this article, we verify that there exists a subdigraph R of G such that R possesses a $[0, k_i]_1^n$-factorization orthogonal to every H_i for $1 \le i \le r$.

Keywords: network; digraph; subdigraph; factor; orthogonal factorization
Mathematics Subject Classification: 05C70, 05C20, 68M10

1. Introduction

Many real-world networks can conveniently be simulated by graphs or networks. Examples include the World Wide Web with nodes modelling Web pages and links representing hyperlinks between Web pages, or a communication network with nodes simulating cities, and links modelling communication channels, an interconnection network with nodes representing processors and links simulating communication channels. The factors, fractional factors, factorizations and orthogonal factorizations in graphs or networks attracted a great deal of attention [1–17] due to their applications in network design, combinatorial design, the file transfer problems on computer networks, coding design, scheduling problems, and so on. The file transfer problem can be simulated as $(0, f)$-factorizations in a graph [18]. The design of a Room square with order $2n$ is equivalent to the orthogonal 1-factorization of K_{2n} which is firstly posed by Horton [19]. The design of a pair of orthogonal Latin squares with order n is related to two orthogonal 1-factorizations of $K_{n,n}$ which is firstly posed by Euler [20]. It is well-known that a network can be simulated by a graph. Vertices of the graph represent nodes of the network, and edges of the graph represent links between the nodes in the network. Henceforth, we replace *network* by the term *graph*.

In this article, we discuss finite directed graphs (digraphs) without loops or parallel arcs. Let G be a digraph. We denote the vertex set and arc set of G by $V(G)$ and $E(G)$, respectively. For $x \in V(G)$, we denote by $d_G^-(x)$ the indegree of x in G, and by $d_G^+(x)$ denote the outdegree of x in G. We use xy to denote the arc with tail x and head y. Let $g = (g^-, g^+)$ and $f = (f^-, f^+)$ be pairs of nonnegative integer-valued functions defined on $V(G)$ satisfying $g^-(x) \le f^-(x)$ and $g^+(x) \le f^+(x)$ for every $x \in V(G)$. A spanning subdigraph F of a digraph G is called a (g, f)-factor of G if it satisfies $g^-(x) \le d_F^-(x) \le f^-(x)$ and $g^+(x) \le d_F^+(x) \le f^+(x)$ for all $x \in V(G)$. We call G a (g, f)-digraph if G itself is a (g, f)-factor. For convenience, we write $g \le f$ if $g^-(x) \le f^-(x)$ and $g^+(x) \le f^+(x)$ for every $x \in V(G)$, and write $g \ge k$ if $\min\{g^-(x), g^+(x)\} \ge k$ for every $x \in V(G)$, and write $g = a$ if $g^-(x) = a$ and $g^+(x) = a$ for every $x \in V(G)$, where a is a nonnegative integer. Furthermore, we shall write $mf + n$ for $(mf^- + n, mf^+ + n)$. If $g(x) = a$ and $f(x) = b$ for any $x \in V(G)$, then we call a (g, f)-factor as an $[a, b]$-factor and a (g, f)-digraph as an $[a, b]$-digraph. If $E(G)$ can be decomposed into arc-disjoint $[0, k_1]$-factor F_1, $[0, k_2]$-factor $F_2, \cdots$, $[0, k_m]$-factor F_m, then we say $\mathcal{F} = \{F_1, F_2, \cdots, F_m\}$ is a $[0, k_i]_1^m$-factorization of a digraph G, where k_i is a positive integer for $1 \le i \le m$.

A subdigraph H of a digraph G is called an m-subdigraph if H possesses m arcs in total. Let H be an m-subdigraph of G and $\mathcal{F} = \{F_1, F_2, \cdots, F_m\}$ be a $[0, k_i]_1^m$-factorization of G. If $|E(H) \cap E(F_i)| = 1$ for $1 \le i \le m$, then we say that $\mathcal{F}$ is orthogonal to H, namely, $\mathcal{F}$ is an orthogonal $[0, k_i]_1^m$-factorization of G. Similarly, we may define an orthogonal (g, f)-factorization of G.

Zhou and Sun [21, 22] posed some sufficient conditions for graphs to admit $[1, 2]$-factors with given properties. Zhou [23] studied the existence of $[1, 2]$-factors with given properties. Kouider and Lonc [24] derived some results on $[a, b]$-factors in graphs. Yan, Pan, Wong and Tokuda [25] investigated (g, f)-factorizations of graphs. Alspach, Heinrich and Liu [14] put forward the following problem: Given a subgraph H of G, does there exist a factorization $\mathcal{F}$ of G of certain type orthogonal to H? Liu [26] demonstrated that every $(mg + m - 1, mf - m + 1)$-graph admits a (g, f)-factorization orthogonal to an m-matching. Lam, Liu, Li and Shiu [27] justified that every $(mg + m - 1, mf - m + 1)$-graph admits a (g, f)-factorization orthogonal to k given vertex-disjoint m-subgraphs. Feng and Liu [28] claimed that every $[0, k_1 + k_2 + \cdots + k_m - m + 1]$-graph possesses a $[0, k_i]_1^m$-factorization orthogonal to any given m-subgraph. Wang [29] verified the existence of subgraphs with orthogonal $[0, k_i]_1^n$-factorizations in $[0, k_1 + k_2 + \cdots + k_m - n + 1]$-graphs. Liu [30] investigated orthogonal (g, f)-factorizations of $(mg + m - 1, mf - m + 1)$-digraphs. Wang [31] claimed that every $(mg + k - 1, mf - k + 1)$-digraph includes a subdigraph R such that R admits a (g, f)-factorization orthogonal to n arc-disjoint k-subdigraphs. Zhou and Bian [32] verified that every $(mg + (k - 1)r, mf - (k - 1)r)$-digraph includes a sundigraph R such that R admits a (g, f)-factorization orthogonal to r vertex-disjoint k-subdigraphs. Zhou, Sun and Xu [33] demonstrated that every $(0, mf - m + 1)$-digraph possesses a $(0, f)$-factorization orthogonal to k vertex-disjoint m-subdigraphs.

In this article, we study the following problem: For given r vertex-disjoint n-subdigraphs $H_1, H_2, \cdots, H_r$ of a digraph G, does G admit factorization orthogonal to every H_i $(i = 1, 2, \cdots, r)$? Furthermore, we verify the following theorem, which partly solves the above problem.

Theorem 1. Let G be a $[0, k_1 + k_2 + \cdots + k_m - n + 1]$-digraph, and let $H_1, H_2, \cdots, H_r$ be r vertex-disjoint n-subdigraphs of G, where m, n, r and k_i $(1 \le i \le m)$ are positive integers satisfying $1 \le n \le m$ and $k_1 \ge k_2 \ge \cdots \ge k_m \ge r + 1$. Then there exists a subdigraph R of G such that R possesses a $[0, k_i]_1^n$-factorization orthogonal to every H_i for $1 \le i \le r$.

Obviously, we admit the following result by setting $r = 1$ in Theorem 1.

Corollary 1. Let G be a $[0, k_1 + k_2 + \cdots + k_m - n + 1]$-digraph, and let H be an n-subdigraphs of G, where m, n and k_i $(1 \le i \le m)$ are positive integers satisfying $1 \le n \le m$ and $k_1 \ge k_2 \ge \cdots \ge k_m \ge 2$.

Then there exists a subdigraph R of G such that R possesses a $[0, k_i]_1^n$-factorization orthogonal to H.

2. Preliminary lemmas

Let G be a digraph. For any two vertex subsets S and T of G, we write $E_G(S,T) = \{xy : xy \in E(G), x \in S, y \in T\}$, and let $e_G(S,T) = |E_G(S,T)|$. Let φ be a function defined on $V(G)$. We write $\varphi(S) = \sum_{x\in S} \varphi(x)$ and $\varphi(\emptyset) = 0$. Define

$$\gamma_{1G}(S,T;g,f) = f^+(S) + e_G(V(G) \setminus S, T) - g^-(T) \tag{2.1}$$

and

$$\gamma_{2G}(S,T;g,f) = f^-(T) + e_G(S, V(G) \setminus T) - g^+(S). \tag{2.2}$$

Let S and T be two vertex subsets of G, and E_1 and E_2 be two disjoint subsets of $E(G)$. Put

$$E_{iS} = E_i \cap E_G(S, V(G) \setminus T), \quad E_{iT} = E_i \cap E_G(V(G) \setminus S, T)$$

for $i = 1, 2$, and set

$$\alpha_S(S,T;E_1) = |E_{1S}|, \quad \beta_T(S,T;E_2) = |E_{2T}|, \quad \alpha_T(S,T;E_1) = |E_{1T}|, \quad \beta_S(S,T;E_2) = |E_{2S}|.$$

For simplicity, $\alpha_S(S,T;E_1)$, $\beta_T(S,T;E_2)$, $\alpha_T(S,T;E_1)$ and $\beta_S(S,T;E_2)$ are written as α_S, β_T, α_T and β_S under without ambiguity.

Liu [30] derived a necessary and sufficient condition for a digraph to admit a (g, f)-factor containing E_1 and excluding E_2, which plays a key role in the proof of Theorem 1.

Lemma 1 (Liu [30]). Let G be a digraph, and let $g = (g^-, g^+)$ and $f = (f^-, f^+)$ be pairs of integer-valued functions defined on $V(G)$ satisfying $0 \le g(x) \le f(x)$ for every $x \in V(G)$. Let E_1 and E_2 be two disjoint subsets of $E(G)$. Then G admits a (g, f)-factor F with $E_1 \subseteq E(F)$ and $E_2 \cap E(F) = \emptyset$ if and only if $\gamma_{1G}(S,T;g,f) \ge \alpha_S + \beta_T$ and $\gamma_{2G}(S,T;g,f) \ge \alpha_T + \beta_S$ for all vertex subsets S and T of G.

In what follows, we always assume that G is a $[0, k_1 + k_2 + \cdots + k_m - n + 1]$-digraph, where m, n, r and k_i $(1 \le i \le m)$ are nonnegative integers satisfying $1 \le n \le m$ and $k_1 \ge k_2 \ge \cdots \ge k_m \ge r + 1$. For every $[0, k_i]$-factor F_i and every isolated vertex x of G, we admit $d_{F_i}(x) = 0$. We use I to denote the set of all isolated vertices in G. If $G - I$ admits a $[0, k_i]$-factor, then G admits also a $[0, k_i]$-factor. Hence, we may assume that G does not admit isolated vertices. For arbitrary $x \in V(G)$, we define

$$p^-(x) = \max\{0, d_G^-(x) - (k_1 + k_2 + \cdots + k_{m-1} - n + 2)\},$$

$$p^+(x) = \max\{0, d_G^+(x) - (k_1 + k_2 + \cdots + k_{m-1} - n + 2)\},$$

$$q^-(x) = \min\{k_m, d_G^-(x)\}$$

and

$$q^+(x) = \min\{k_m, d_G^+(x)\}.$$

Write $p(x) = (p^-(x), p^+(x))$ and $q(x) = (q^-(x), q^+(x))$. According to the definitions of $p(x)$ and $q(x)$, we derive

$$0 \le p(x) \le q(x) \le k_m$$

for every $x \in V(G)$.

Lemma 2. Let G be a $[0, k_1 + k_2 + \cdots + k_m]$-digraph, and let $H_1, H_2, \cdots, H_r$ be independent arcs of G, where m, r and k_i $(1 \le i \le m)$ are positive integers with $k_i \ge r + 1$. Then G possesses a $[0, k_1]$-factor containing H_i $(1 \le i \le r)$.

Proof. Let $E_1 = \{H_1, H_2, \cdots, H_r\}$ and $E_2 = \emptyset$. For arbitrary two vertex subsets S and T of G, we define $\alpha_S, \beta_T, \alpha_T$ and β_S as before. In light of the definitions of $\alpha_S, \beta_T, \alpha_T$ and β_S, we derive

$$\alpha_S \le \min\{r, |S|\} \quad \textit{and} \quad \beta_T = 0;$$

$$\alpha_T \le \min\{r, |T|\} \quad \textit{and} \quad \beta_S = 0.$$

Thus, we admit

$$\gamma_{1G}(S, T; 0, k_1) = k_1|S| + e_G(V(G) \setminus S, T) - 0 \cdot |T| \ge |S| \ge \alpha_S = \alpha_S + \beta_T$$

and

$$\gamma_{2G}(S, T; 0, k_1) = k_1|T| + e_G(S, V(G) \setminus T) - 0 \cdot |S| \ge |T| \ge \alpha_T = \alpha_T + \beta_S$$

by $k_1 \ge 2$, where $\gamma_{1G}(S, T; 0, k_1)$ is defined by Equation (2.1) by replacing g and f by 0 and k_1, and $\gamma_{2G}(S, T; 0, k_1)$ is defined by Equation (2.2) by replacing g and f by 0 and k_1. Then it follows from Lemma 1 that G has a $[0, k_1]$-factor containing H_i $(1 \le i \le r)$. We finish the proof of Lemma 2. □

3. Proof of Theorem 1

Proof of Theorem 1. We apply induction on m and n. According to Lemma 2, Theorem 1 is true for $n = 1$. Hence, we may assume that $n \ge 2$. For the inductive step, we assume that Theorem 1 is true for any $[0, k_1 + k_2 + \cdots + k_{m'} - n' + 1]$-digraph G' with $m' < m$, $n' < n$ and $1 \le n' \le m'$, and any vertex-disjoint n'-subdigraphs $H'_1, H'_2, \cdots, H'_r$ of G'. Now, we consider a $[0, k_1 + k_2 + \cdots + k_m - n + 1]$-digraph G and any vertex-disjoint n-subdigraphs $H_1, H_2, \cdots, H_r$ of G.

We take $x_i y_i \in E(H_i)$ for $1 \le i \le r$. Write $E_1 = \{x_1y_1, x_2y_2, \cdots, x_ry_r\}$ and $E_2 = \Big(\bigcup_{i=1}^{r} E(H_i)\Big) \setminus E_1$. Thus, we easily see that $|E_1| = r$ and $|E_2| = (n-1)r$. Let $E_{1S}, E_{2S}, E_{1T}, E_{2T}, \alpha_S, \beta_T, \alpha_T, \beta_S, p(x)$ and $q(x)$ be defined as in Section 2. By the definitions of $\alpha_S, \beta_T, \alpha_T$ and β_S, we derive

$$\alpha_S \le \min\{r, |S|\}, \quad \beta_T \le \min\{(n-1)r, (n-1)|T|\},$$

$$\alpha_T \le \min\{r, |T|\}, \quad \beta_S \le \min\{(n-1)r, (n-1)|S|\}.$$

Now, we define $\gamma_{1G}(S, T; p, q)$ in Equation (2.1) by replacing g and f by p and q, and define $\gamma_{2G}(S, T; p, q)$ in Equation (2.2) by replacing g and f by p and q. Then we select two vertex subsets S and T of G satisfying

(a) $\gamma_{1G}(S, T; p, q) - (\alpha_S(S, T; E_1) + \beta_T(S, T; E_2))$ is minimum;

(b) $|S|$ is minimum subject to (a).

Now, we demonstrate the following claim.

Claim 1. If $S \ne \emptyset$, then $q^+(x) \le d_G^+(x) - 1$ for every $x \in S$, and so $q^+(x) = k_m$ for every $x \in S$.
Proof. Set $S_1 = \{x \in S : q^+(x) \ge d_G^+(x)\}$. In what follows, we verify $S_1 = \emptyset$.

Suppose that $S_1 \neq \emptyset$. Let $S_0 = S \setminus S_1$. Thus, we admit

$$\begin{aligned}
\gamma_{1G}(S,T;p,q) &= q^{+}(S) + e_G(V(G)\setminus S,T) - p^{-}(T)\\
&= q^{+}(S_0) + q^{+}(S_1) + e_G(V(G)\setminus S_0,T) - e_G(S_1,T) - p^{-}(T)\\
&= q^{+}(S_0) + e_G(V(G)\setminus S_0,T) - p^{-}(T) + q^{+}(S_1) - e_G(S_1,T)\\
&= \gamma_{1G}(S_0,T;p,q) + q^{+}(S_1) - e_G(S_1,T)\\
&\geq \gamma_{1G}(S_0,T;p,q) + d_G^{+}(S_1) - e_G(S_1,T)\\
&= \gamma_{1G}(S_0,T;p,q) + e_G(S_1,V(G)\setminus T).
\end{aligned}$$

Note that

$$\alpha_S(S,T;E_1) + \beta_T(S,T;E_2) \leq \alpha_{S_0}(S_0,T;E_1) + \beta_T(S_0,T;E_2) + \alpha_{S_1}(S_1,T;E_1)$$

and

$$e_G(S_1,V(G)\setminus T) \geq \alpha_{S_1}(S_1,T;E_1).$$

Thus, we derive

$$\begin{aligned}
&\gamma_{1G}(S,T;p,q) - (\alpha_S(S,T;E_1) + \beta_T(S,T;E_2))\\
\geq\ &\gamma_{1G}(S_0,T;p,q) + e_G(S_1,V(G)\setminus T) - (\alpha_{S_0}(S_0,T;E_1) + \beta_T(S_0,T;E_2) + \alpha_{S_1}(S_1,T;E_1))\\
\geq\ &\gamma_{1G}(S_0,T;p,q) - (\alpha_{S_0}(S_0,T;E_1) + \beta_T(S_0,T;E_2)),
\end{aligned}$$

which conflicts the choice of S. Hence, $S_1 = \emptyset$. And so, if $S \neq \emptyset$, then $q^{+}(x) \leq d_G^{+}(x) - 1$ for every $x \in S$. Furthermore, we admit $q^{+}(x) = k_m$ for every $x \in S$. We finish the proof of Claim 1. □

The remaining of the proof is dedicated to proving that G possesses a (p,q)-factor F_n with $E_1 \subseteq E(F_n)$ and $E_2 \cap E(F_n) = \emptyset$. According to Lemma 1 and the choice of S and T, it suffices to verify that $\gamma_{1G}(S,T;p,q) \geq \alpha_S + \beta_T$ and $\gamma_{2G}(S,T;p,q) \geq \alpha_T + \beta_S$.

Next, we write $\rho = k_1 + k_2 + \cdots + k_{m-1} - n + 2$, $T_1 = \{x : d_G^{-}(x) - \rho > 0, x \in T\}$ and $T_0 = T \setminus T_1$. It is obvious that $p^{-}(x) = 0$ for any $x \in T_0$ and $p^{-}(x) = d_G^{-}(x) - \rho$ for any $x \in T_1$. By the definition of $\beta_T(S,T;E_2)$, we possess

$$\beta_{T_0}(S,T_0;E_2) + \beta_{T_1}(S,T_1;E_2) = \beta_T(S,T;E_2). \tag{3.1}$$

In light of the definitions of α_S and β_T, we have $\alpha_S \leq \min\{r,|S|\} \leq |S|$ and $\beta_T \leq e_G(V(G)\setminus S,T)$. If $T_1 = \emptyset$, then by Claim 1 we admit

$$\begin{aligned}
\gamma_{1G}(S,T;p,q) &= q^{+}(S) + e_G(V(G)\setminus S,T) - p^{-}(T)\\
&= q^{+}(S) + e_G(V(G)\setminus S,T) - p^{-}(T_0) - p^{-}(T_1)\\
&= k_m|S| + e_G(V(G)\setminus S,T)\\
&\geq |S| + e_G(V(G)\setminus S,T)\\
&\geq \alpha_S + \beta_T.
\end{aligned}$$

If $S = \emptyset$, then we have $\alpha_S = 0$. It follows from Equation (3.1), $r \geq 1$, $2 \leq n \leq m$ and $k_1 \geq k_2 \geq \cdots \geq k_m \geq r+1$ that

$$\gamma_{1G}(S,T;p,q) = q^{+}(S) + e_G(V(G)\setminus S,T) - p^{-}(T)$$

$$
\begin{aligned}
&= e_G(V(G),T) - p^-(T_1)\\
&= d_G^-(T) - p^-(T_1)\\
&= d_G^-(T_0) + d_G^-(T_1) - (d_G^-(T_1) - \rho|T_1|)\\
&= d_G^-(T_0) + \rho|T_1|\\
&= d_G^-(T_0) + (k_1 + k_2 + \cdots + k_{m-1} - n + 2)|T_1|\\
&\geq d_G^-(T_0) + ((m-1)(r+1) - n + 2)|T_1|\\
&\geq d_G^-(T_0) + ((n-1)(r+1) - n + 2)|T_1|\\
&= e_G(V(G)\setminus S, T_0) + ((n-1)r + 1)|T_1|\\
&\geq e_G(V(G)\setminus S, T_0) + (n-1)|T_1|\\
&\geq \beta_{T_0}(S,T_0;E_2) + \beta_{T_1}(S,T_1;E_2)\\
&= \beta_T(S,T;E_2) = \beta_T = \alpha_S + \beta_T.
\end{aligned}
$$

In what follows, we always assume that $S \neq \emptyset$ and $T_1 \neq \emptyset$. To demonstrate Theorem 1, we consider two cases.

Case 1. $|S| \geq |T_1|$.

According to Claim 1, the definition of T_1, $k_m \geq r+1$ and $|S| \geq |T_1|$, we derive

$$
\begin{aligned}
\gamma_{1G}(S,T;p,q) &= q^+(S) + e_G(V(G)\setminus S, T) - p^-(T)\\
&= q^+(S) + e_G(V(G)\setminus S, T) - p^-(T_1)\\
&= k_m|S| + e_G(V(G)\setminus S, T) - (d_G^-(T_1) - \rho|T_1|)\\
&= k_m|S| + e_G(V(G)\setminus S, T) - d_G^-(T_1) + \rho|T_1|\\
&= k_m(|S| - |T_1|) + (\rho + k_m)|T_1| + e_G(V(G)\setminus S, T) - d_G^-(T_1)\\
&\geq k_m(|S| - |T_1|) + d_G^-(T_1) + |T_1| + e_G(V(G)\setminus S, T) - d_G^-(T_1)\\
&= k_m(|S| - |T_1|) + |T_1| + e_G(V(G)\setminus S, T)\\
&= (k_m - 1)(|S| - |T_1|) + |S| + e_G(V(G)\setminus S, T)\\
&\geq |S| + e_G(V(G)\setminus S, T)\\
&\geq \alpha_S + \beta_T.
\end{aligned}
$$

Case 2. $|S| \leq |T_1| - 1$.

By Claim 1, the definitions of T_0 and T_1, we admit

$$
\begin{aligned}
\gamma_{1G}(S,T;p,q) &= q^+(S) + e_G(V(G)\setminus S, T) - p^-(T)\\
&= q^+(S) + e_G(V(G)\setminus S, T_0) + e_G(V(G)\setminus S, T_1) - p^-(T_1)\\
&= k_m|S| + e_G(V(G)\setminus S, T_0) + d_G^-(T_1) - e_G(S,T_1) - p^-(T_1)\\
&= k_m|S| + e_G(V(G)\setminus S, T_0) + (d_G^-(T_1) - p^-(T_1)) - e_G(S,T_1)\\
&= k_m|S| + e_G(V(G)\setminus S, T_0) + \rho|T_1| - e_G(S,T_1),
\end{aligned}
$$

namely,

$$
\gamma_{1G}(S,T;p,q) = k_m|S| + e_G(V(G)\setminus S, T_0) + \rho|T_1| - e_G(S,T_1). \tag{3.2}
$$

Subcase 2.1. $|T_1| \leq k_m - 1$.

It follows from Equations (3.1) and (3.2), $r \geq 1$, $2 \leq n \leq m$ and $k_1 \geq k_2 \geq \cdots \geq k_m \geq r+1$ that

$$\begin{aligned}
\gamma_{1G}(S,T;p,q) &= k_m|S| + e_G(V(G)\setminus S, T_0) + \rho|T_1| - e_G(S,T_1) \\
&\geq k_m|S| + e_G(V(G)\setminus S, T_0) + \rho|T_1| - |S||T_1| \\
&\geq k_m|S| + e_G(V(G)\setminus S, T_0) + ((m-1)(r+1)-n+2)|T_1| - (k_m-1)|S| \\
&= |S| + e_G(V(G)\setminus S, T_0) + ((m-1)(r+1)-n+2)|T_1| \\
&\geq |S| + e_G(V(G)\setminus S, T_0) + ((n-1)(r+1)-n+2)|T_1| \\
&= |S| + e_G(V(G)\setminus S, T_0) + ((n-1)r+1)|T_1| \\
&\geq |S| + e_G(V(G)\setminus S, T_0) + (n-1)|T_1| \\
&\geq \alpha_S + \beta_{T_0}(S,T_0;E_2) + \beta_{T_1}(S,T_1;E_2) \\
&= \alpha_S + \beta_T(S,T;E_2) = \alpha_S + \beta_T.
\end{aligned}$$

Subcase 2.2. $|T_1| \geq k_m$.

Subcase 2.2.1. $|S| \leq (n-1)r-2$.

By $2 \leq n \leq m$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq r+1$ and $\rho = k_1 + k_2 + \cdots + k_{m-1} - n + 2$, we admit

$$\begin{aligned}
\rho - |S| &\geq ((m-1)(r+1)-n+2) - |S| \\
&\geq ((n-1)(r+1)-n+2) - ((n-1)r-2) \\
&= 3,
\end{aligned}$$

that is,

$$\rho - |S| = 3 > 0. \tag{3.3}$$

It follows from Equations (3.2) and (3.3), $|T_1| \geq k_m$, $r \geq 1$, $2 \leq n \leq m$ and $k_1 \geq k_2 \geq \cdots \geq k_m \geq r+1$ that

$$\begin{aligned}
\gamma_{1G}(S,T;p,q) &= k_m|S| + e_G(V(G)\setminus S, T_0) + \rho|T_1| - e_G(S,T_1) \\
&\geq k_m|S| + \rho|T_1| - |S||T_1| \\
&= k_m|S| + (\rho - |S|)|T_1| \\
&\geq k_m|S| + (\rho - |S|)k_m \\
&= \rho k_m \\
&\geq ((m-1)(r+1)-n+2)(r+1) \\
&\geq (2(n-1)-n+2)(r+1) \\
&= n(r+1) \\
&> nr \\
&= r + (n-1)r \\
&\geq \alpha_S + \beta_T.
\end{aligned}$$

Subcase 2.2.2. $|S| \geq (n-1)r-1$.

According to $k_1 \geq k_2 \geq \cdots \geq k_m \geq r+1$, $\rho = k_1 + k_2 + \cdots + k_{m-1} - n + 2$ and G being a $[0, k_1 + k_2 + \cdots + k_m - n + 1]$-digraph, we admit $\rho \geq (m-1)(r+1) - n + 2$ and $d_G^+(S) \leq (k_1 + k_2 + \cdots +$

$k_m - n + 1)|S| = (\rho + k_m - 1)|S|$. Combining these with Claim 1, the definition of T_1, $|S| \le |T_1| - 1$ and $2 \le n \le m$, we derive

$$\begin{aligned}
\gamma_{1G}(S,T;p,q) &= q^+(S) + e_G(V(G)\setminus S, T) - p^-(T) \\
&= q^+(S) + d_G^-(T) - e_G(S,T) - p^-(T_1) \\
&= k_m|S| + d_G^-(T) - e_G(S,T) - (d_G^-(T_1) - \rho|T_1|) \\
&= k_m|S| - e_G(S,T) + \rho|T_1| + d_G^-(T) - d_G^-(T_1) \\
&\ge k_m|S| - e_G(S,T) + \rho|T_1| \\
&= \rho(|T_1| - |S|) + (\rho + k_m)|S| - e_G(S,T) \\
&\ge \rho + (\rho + k_m)|S| - e_G(S,T) \\
&\ge \rho + |S| + d_G^+(S) - e_G(S,T) \\
&= \rho + |S| + e_G(S, V(G)\setminus T) \\
&\ge \rho + |S| \\
&\ge (m-1)(r+1) - n + 2 + ((n-1)r - 1) \\
&\ge (n-1)(r+1) - n + 2 + ((n-1)r - 1) \\
&= 2(n-1)r \\
&\ge nr \\
&= r + (n-1)r \\
&\ge \alpha_S + \beta_T.
\end{aligned}$$

In conclusion, $\gamma_{1G}(S,T;p,q) \ge \alpha_S(S,T;E_1) + \beta_T(S,T;E_2)$. Similarly, we may demonstrate

$$\gamma_{2G}(S,T;p,q) \ge \alpha_T(S,T;E_1) + \beta_S(S,T;E_2).$$

It follows from the choice of S and T that $\gamma_{1G}(S',T';p,q) \ge \alpha_{S'}(S',T';E_1) + \beta_{T'}(S',T';E_2)$ and $\gamma_{2G}(S',T';p,q) \ge \alpha_{T'}(S',T';E_1) + \beta_{S'}(S',T';E_2)$ for any two vertex subsets S' and T' of G. In light of Lemma 1, G possesses a (p,q)-factor F_n with $E_1 \subseteq E(F_n)$ and $E_2 \cap E(F_n) = \emptyset$. Note that F_n is also a $[0,k_n]$-factor of G. It follows from the definitions of $p(x)$ and $q(x)$ that

$$\begin{aligned}
0 &\le d_{G-F_n}^-(x) \\
&= d_G^-(x) - d_{F_n}^-(x) \\
&\le d_G^-(x) - p^-(x) \\
&\le d_G^-(x) - (d_G^-(x) - (k_1 + k_2 + \cdots + k_{m-1} - n + 2)) \\
&= k_1 + k_2 + \cdots + k_{m-1} - (n-1) + 1
\end{aligned}$$

and

$$\begin{aligned}
0 &\le d_{G-F_n}^+(x) \\
&= d_G^+(x) - d_{F_n}^+(x) \\
&\le d_G^+(x) - p^+(x) \\
&\le d_G^+(x) - (d_G^+(x) - (k_1 + k_2 + \cdots + k_{m-1} - n + 2))
\end{aligned}$$

$$= \quad k_1 + k_2 + \cdots + k_{m-1} - (n-1) + 1$$

for any $x \in V(G)$. Therefore, $G - F_n$ is a $[0, k_1 + k_2 + \cdots + k_{m-1} - (n-1) + 1]$-digraph. Let $H_i' = H_i - x_i y_i$ for $1 \leq i \leq r$. Obviously, $H_1', H_2', \cdots, H_r'$ are r vertex-disjoint $(n-1)$-subdigraphs of $G - F_n$. By the induction hypothesis, there exists a subdigraph R' of $G - F_n$ such that R' admits a $[0, k_i]_{i=1}^{n-1}$-factorization orthogonal to every H_i', $1 \leq i \leq r$. We denote by R the subdigraph of G induced by $E(R') \cup E(F_n)$. Hence, R is a subdigraph of G such that R possesses a $[0, k_i]_{i=1}^{n}$-factorization orthogonal to every H_i, $1 \leq i \leq r$. We finish the proof of Theorem 1. □

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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