Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(2): 1209–1222.
DOI:10.3934/math.2021074
Received: 04 September 2020
Accepted: 26 October 2020
Published: 12 November 2020

Research article

Periodic wave solutions of a non-Newtonian filtration equation with an indefinite singularity

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Abstract: This paper is concerned with the existence of periodic wave solutions for a type of non-Newtonian filtration equations with an indefinite singularity. A sufficient criterion for the existence of periodic wave solutions for non-Newtonian filtration equation is provided via an innovative method of combining a new continuation theorem with coincidence degree theory as well as mathematical analysis skills. The novelty of the present paper is that it is the first time to discuss the existence of periodic wave solutions for the indefinite singular non-Newtonian filtration equations. Finally, two numerical examples are presented to illustrate the effectiveness and feasibility of the proposed criterion in the present paper.

Keywords: periodic wave solution; singularity; continuation theorem; existence
Mathematics Subject Classification: 34C37, 35C07

1. Introduction

In this paper, we consider the periodic wave solutions problem for a type of non-Newtonian filtration equation with an indefinite singularity as follows:

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial x}\left(\left|\frac{\partial y}{\partial x}\right|^{p-2}\frac{\partial y}{\partial x}\right) + f(y) + \frac{h(t,x)}{y^m}, \tag{1.1}$$

where $p > 1$, $m > 0$, $f \in C(\mathbb{R},\mathbb{R})$, $h \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$. In this equation, the function $\frac{1}{y^m}$ may have a singularity at $y = 0$. Besides this, the signs of $h(t,x)$ are all allowed to change.

Equation (1.1) is known as the evolutionary $p-$Laplacian. Many fluid dynamics models can be described by Eq (1.1), see [1, 2]. For the last forty years, there exist many results about non-Newtonian filtration equation. In 1967, Ladyzhenskaja [1] studied the following non-Newtonian filtration equation:

$$\frac{\partial y}{\partial t}=\frac{\partial}{\partial x}\left(\left|\frac{\partial y}{\partial x}\right|^{p-2}\frac{\partial y}{\partial x}\right)+y^q(1-y)(z-a),\ t\geq 0,\ p>1,\ x\in\mathbb{R}$$

which is the description of incompressible fluids and solvability in the large boundary value. Jin and Yin [3] investigated the traveling wavefronts for a non-Newtonian filtration equation with Hodgkin-Huxley source:

$$\frac{\partial y}{\partial t}=\frac{\partial}{\partial x}\left(\left|\frac{\partial y}{\partial x}\right|^{p-2}\frac{\partial y}{\partial x}\right)+f(y,y_\tau),\ t\geq 0,\ x\in\mathbb{R},$$

where $p>1,\ f(y,z)=y^q(1-y)(z-a),\ q>0,\ a\in(0,1)$ is a constant, $y_\tau=y(x,t-\tau),\ \tau>0$. The more related papers for non-Newtonian filtration equation, see e.g., [4–7].

In recent years, the solitary wave and periodic wave solutions for the non-Newtonian filtration equation have been received great attention. In 2014, Lian etc. [8] studied the following non-Newtonian filtration equation:

$$\frac{\partial q}{\partial t}=\frac{\partial}{\partial x}\left(\left|\frac{\partial q}{\partial x}\right|^{p-2}\frac{\partial q}{\partial x}\right)+f(q)+g(t,x). \tag{1.2}$$

By using an extension of Mawhin's continuation theorem, the authors obtained some existence results of solitary wave and periodic wave solutions for Eq (1.2). Kong etc. [9] considered a non-Newtonian filtration equations with nonlinear sources and the variable delay. In 2017, when $f(q)$ has a singularity (including the attractive singular case or the repulsive singular case) in (1.2), Lian etc. [10] studied the existence and multiplicity of positive periodic wave solutions for Eq (1.2). For more results about periodic solutions and periodic wave solutions, see [11–13].

In Eq (1.1), the signs of function h are allowed to change which means that the singularity of $\frac{1}{y^m}$ has a singularity at $y=0$ can be classified neither as repulsive type nor as attractive type. In this paper, we will use the theorem belonging to [14] to obtain the existence of periodic wave solutions for Eq (1.1). To the best of our knowledge, there is no paper to use the theorem in [14] for studying the non-Newtonian filtration equations with an indefinite singularity, the main purpose is to recommend a new method for the research of non-Newtonian filtration equations with an indefinite singularity. Recent years, second-order indefinite singular equations have been studied by some researchers. Hakl and Zamora [15] studied a second-order indefinite singular equations by using Leray-Schauder degree theory. Fonda and Sfecci [16] investigated the periodic problem of Ambrosetti-Prodi type having a nonlinearity with possibly one or two singularities. In the present paper, we will generalize second-order indefinite singular equations to Eq (1.1). Hence, our research can enrich and develop the study of second-order singular equations. The topics of solitary wave solutions, periodic wave, and traveling wave solutions are interesting. Recently, there are many superior works on these topics, see them in [17–27].

For Eq (1.1), assume that there is a continuous function $h(s)$ such that $h(t,x)=-h(x+ct)=-h(s)$, where $c\in\mathbb{R}$. Let $y(t,x)=u(s)$ with $s=x+ct$ be the solution of Eq (1.1), then Eq (1.1) is changed into the following equation:

$$cu'(s)=(\phi_p(u'(s)))'+f(u)-\frac{h(s)}{u^m}, \tag{1.3}$$

where $\phi_p(u)=|u|^{p-2}u,\ p>1, m>0,\ f,h\in C(\mathbb{R},\mathbb{R})$.

Definition 1.1 Let $T>0$ be a constant. Suppose that $u(s+T)=u(s)$ and $u(s)$ is a solution of Eq

(1.3) for $s \in \mathbb{R}$. In generally, the periodic solution of Eq (1.3) is regarded as periodic wave solution of Eq (1.1).

The highlights of this paper are threefold:
(1) In this paper, we studied a new non-Newtonian filtration equation with an indefinite singularity which is different from the existing non-Newtonian filtration equations, see e.g., [3, 8–12].
(2) We creatively use a new continuation theorem to study a class of strongly nonlinear equations. For estimating the *prior* bounds of periodic wave solutions, we develop some inequality methods and mathematical analysis skills.
(3) Different from the previous results, we introduce a new unified framework to deal with the existence of periodic wave solutions for indefinite singular equations by using Topological degree theory and some mathematical analysis skills, which may be of special interest. It is noted that our main methods can be studied other types of indefinite singular equations.

The following sections are organized as follows: In Section 2, we give some useful lemmas and definitions. In Section 3, main results are obtained for the existence of periodic wave solutions to the non-Newtonian filtration equation (1.1). In Section 4, two examples are given to show the feasibility of our results. Finally, some conclusions and discussions are given about this paper.

2. Preliminaries

Definition 2.1. [14] Let $\mathcal{X}$ and $\mathcal{Z}$ be two Banach spaces with norms $\|\cdot\|_{\mathcal{X}}, \|\cdot\|_{\mathcal{Z}}$, respectively. A continuous operator

$$\mathcal{M} : \mathcal{X} \cap dom\mathcal{M} \to \mathcal{Z}$$

is called to be quasi-linear if

(i) $Im\mathcal{M} := \mathcal{M}(\mathcal{X} \cap dom\mathcal{M})$ is a closed subset of $\mathcal{Z}$;

(ii) $Ker\mathcal{M} := \{x \in \mathcal{X} \cap dom\mathcal{M} : \mathcal{M}x = 0\}$ is linearly homeomorphic to $\mathbb{R}^n$, $n < \infty$,
where $dom\mathcal{M}$ is the domain of $\mathcal{M}$.

Definition 2.2. [14] Let $\Omega \subset \mathcal{X}$ be an open and bounded set with the origin $\theta \in \Omega$. $N_\lambda : \bar{\Omega} \to \mathcal{Z}$, $\lambda \in [0, 1]$ is said to be $\mathcal{M}$-compact in $\bar{\Omega}$ if there exists subset $\mathcal{Z}_1$ of $\mathcal{Z}$ satisfying $dim\mathcal{Z}_1 = dimKer\mathcal{M}$ and an operator $R : \bar{\Omega} \times [0, 1] \to \mathcal{X}_2$ being continuous and compact such that for $\lambda \in [0, 1]$,

(a) $(I - Q)N_\lambda(\bar{\Omega}) \subset Im\mathcal{M} \subset (I - Q)\mathcal{Z}$,

(b) $QN_\lambda x = 0, \ \lambda \in (0, 1) \Leftrightarrow QNx = 0, \forall x \in \Omega$,

(c) $R(\cdot, 0) \equiv 0$ and $R(\cdot, \lambda)|_{\Sigma_\lambda} = (I - P)|_{\Sigma_\lambda}$,

(d) $\mathcal{M}[Pu + R(\cdot, \lambda)] = (I - Q)N_\lambda, \ \lambda \in [0, 1]$,

where $\mathcal{X}_2$ is a the complement space of Ker$\mathcal{M}$ in $\mathcal{X}$, i.e., $\mathcal{X} = Ker\mathcal{M} \oplus \mathcal{X}_2$; $P, \ Q$ are two projectors satisfying $ImP = Ker\mathcal{M}, \ ImQ = \mathcal{Z}_1, N = N_1, \ \Sigma_\lambda = \{x \in \bar{\Omega} : \mathcal{M}x = N_\lambda x\}$.

Lemma 2.1. [14] Let $\mathcal{X}$ and $\mathcal{Z}$ be two Banach spaces with norms $\|\cdot\|_{\mathcal{X}}, \|\cdot\|_{\mathcal{Z}}$, respectively. Let $\Omega \subset X$ be an open and bounded nonempty set. Suppose

$$\mathcal{M} : \mathcal{X} \cap dom\mathcal{M} \to \mathcal{Z}$$

is quasi-linear and $N_\lambda : \bar{\Omega} \to \mathcal{Z}, \ \lambda \in [0, 1]$ is $\mathcal{M}$−compact in $\bar{\Omega}$. In addition, if the following conditions hold:

(A$_1$) $\mathcal{M}x \neq N_\lambda x, \ \ \forall (x,\lambda) \in \partial\Omega \times (0,1)$;
(A$_2$) $QNx \neq 0, \ \ \forall x \in Ker\mathcal{M} \cap \partial\Omega$;
(A$_3$) $deg\{JQN, \Omega \cap Ker\mathcal{M}, 0\} \neq 0, \ J : ImQ \to \ Ker\mathcal{M}$ *is a homeomorphism.*

Then the abstract equation $\mathcal{M}x = Nx$ has at least one solution in $dom\mathcal{M} \cap \bar{\Omega}$.

From Lemma 2.1, [28] and [29], we have the following lemma:

Lemma 2.2. Consider the following $p-$Laplacian equation

$$(\phi_p(u'(s)))' = f(t,u,u'), \tag{2.1}$$

where $p > 1, \ f \in C(\mathbb{R}^3, \mathbb{R})$ with $f(t+T,\cdot,\cdot) = f(t,\cdot,\cdot)$. Assume that Ω is an open bounded set in C^1_T such that the following conditions hold.

(1) For each $\lambda \in (0,1)$, the problem

$$(\phi_p(u'))' = \lambda f(t,u,u'), \ \ u(0) = u(T), \ u'(0) = u'(T),$$

has no solution on $\partial\Omega$.

(2) The equation

$$\mathcal{F}(a) = \frac{1}{T}\int_0^T f(t,a,0)dt = 0,$$

has no solution on $\partial\Omega \cap \mathbb{R}$.

(3)The Brouwer degree

$$d_B(\mathcal{F}, \Omega \cap \mathbb{R}, 0) \neq 0.$$

Then Eq (2.1) has at least one $T-$periodic solution in $\overline{\Omega}$.

Remark 2.1. Lemma 2.2 is derived from the Lemma 2.1 which is convenient for studying the existence of periodic wave solutions to the non-Newtonian filtration equation (1.1).

3. Main results

Denote

$$C_T = \{x | x \in C(\mathbb{R},\mathbb{R}), \ x(t+T) \equiv x(t), \ \forall t \in \mathbb{R}\}$$

with the norm

$$|\varphi|_0 = \max_{t\in[0,T]} |\varphi(t)|, \ \ \forall \varphi \in C_T$$

and

$$C^1_T = \{x | x \in C^1(\mathbb{R},\mathbb{R}), \ x(t+T) \equiv x(t), \ \forall t \in \mathbb{R}\}$$

with the norm

$$|\varphi|_\infty = \max_{t\in[0,T]} \{|\varphi|_0, \ |\varphi'|_0\}, \ \ \forall \varphi \in C^1_T.$$

Clearly, C_T and C^1_T are Banach spaces. For each $\phi \in C_T$, let

$$\phi_+(t) = \max\{\phi(t), 0\}, \ \phi_-(t) = \max\{-\phi(t), 0\},$$

$$\overline{\phi} = \frac{1}{T}\int_0^T \phi(s)ds, \ ||\phi||_p = \left(\int_0^T |\phi(s)|^p ds\right)^{\frac{1}{p}}, \ p > 1.$$

Clearly, for $t \in \mathbb{R}$, $\phi(t) = \phi_+(t) - \phi_-(t)$, $\overline{\phi} = \overline{\phi_+} - \overline{\phi_-}$. Consider the following equations family:

$$(\phi_p(u'(s)))' = c\lambda u'(s) - \lambda f(u) + \lambda \frac{h(s)}{u^m}, \quad \lambda \in (0, 1]. \tag{3.1}$$

Let

$$\Omega = \left\{ u \in C_T^1 : (\phi_p(u'(s)))' = c\lambda u'(s) - \lambda f(u) + \lambda \frac{h(s)}{u^m}, \ \lambda \in (0, 1], \ u > 0 \right\}.$$

Lemma 3.1. Assume that the function f such that

$$f_L < f(u) < f_M, \ f'(u) > 0, \quad \forall u > 0,$$

where f_L and f_M are positive constants. Furthermore, assume $\overline{h} > 0$. Then for each $u \in \Omega$, there are constants $\xi_1, \xi_2 \in [0, T]$ such that

$$u(\xi_1) \le \left(\frac{\overline{h^+}}{f_L} \right)^{\frac{1}{m}} := A_1$$

and

$$u(\xi_2) \ge \left(\frac{\overline{h}}{f_M} \right)^{\frac{1}{m}} := A_2.$$

Proof. Let $u \in \Omega$, we have (3.1) holds. Dividing both sides of (3.1) by $f(u)$ and integrating them on $[0, T]$, we have

$$\int_0^T \frac{(\phi_p(u'(s)))'}{f(u)} ds = -\lambda T + \lambda \int_0^T \frac{h(s)}{f(u)u^m} ds, \quad \lambda \in (0, 1]. \tag{3.2}$$

Note that

$$\begin{aligned} \int_0^T \frac{(\phi_p(u'(s)))'}{f(u)} ds &= \int_0^T \frac{1}{f(u)} d\phi_p(u'(s)) \\ &= \int_0^T f^{-2}(u) f'(u) |u'|^p ds \quad \ge 0, \end{aligned} \tag{3.3}$$

where we use $f'(u) > 0$. By (3.2) and (3.3) we have

$$T \le \int_0^T \frac{h(s)}{f(u)u^m} ds \le \int_0^T \frac{h^+(s)}{f_L u^m} ds.$$

By mean value theorem of integrals, there exists a point $\xi_1 \in [0, T]$ such that

$$u^m(\xi_1) \le \frac{\overline{h^+}}{f_L},$$

i.e.,

$$u(\xi_1) \le \left(\frac{\overline{h^+}}{f_L} \right)^{\frac{1}{m}} := A_1.$$

Multiplying both sides of (3.1) by u^m and integrating them on $[0, T]$, we have

$$\int_0^T (\phi_p(u'(s)))' u^m ds = -\lambda \int_0^T f(u) u^m ds + \lambda \int_0^T h(s) ds, \quad \lambda \in (0, 1]. \tag{3.4}$$

Note that

$$\begin{aligned}\int_0^T (\phi_p(u'(s)))' u^m ds &= \int_0^T u^m d\phi_p(u'(s)) \\ &= -m \int_0^T u^{m-1} |u'|^p ds \quad \leq 0.\end{aligned} \tag{3.5}$$

In view of (3.4) and (3.5), we have

$$\int_0^T f(u) u^m ds \geq \int_0^T h(s) ds = T\overline{h}$$

and

$$f_M \int_0^T u^m ds \geq T\overline{h}.$$

By mean value theorem of integrals, there exists a point $\xi_2 \in [0, T]$ such that

$$u^m(\xi_2) \geq \frac{\overline{h}}{f_M},$$

i.e.,

$$u(\xi_2) \geq \left(\frac{\overline{h}}{f_M}\right)^{\frac{1}{m}} := A_2.$$

Theorem 3.1. Suppose that conditions of Lemma 3.1 hold. Further assume that some assumptions on $f(u)$ and $h(t)$:
(H_1) Suppose that $f(u) \leq \frac{1}{u^{m+1}}$ for $u > 0$ and $m > 0$, and $h(t) > 0$ for $t \in \mathbb{R}$.
Then Eq (1.3) has at least one $T-$periodic solution, i.e., Eq (1.1) has at least one periodic wave solution, if $c < 0$ and $A_2^{m+1} - \frac{m+1}{|c|} T f_M B_1^m - \frac{m+1}{|c|} T\overline{|h|} > 0$, where A_2 is defined by Lemma 3.1, B_1 is defined by (3.6).
Proof. We complete the proof by three steps.

Step 1. For $t_1 < t_2$, let

$$u(t_1) = \max_{t \in [0,T]} u(t), \quad u(t_2) = \min_{t \in [0,T]} u(t).$$

By Eq (3.1) and $(\phi_p(u'(t)))'|_{t=t_1} \leq 0$, we have

$$f(u(t_1)) \geq \frac{h(t_1)}{u^m(t_1)}.$$

Thus, by assumption (H_1) we have

$$u(t_1) \leq \frac{1}{h_L} := B_1, \tag{3.6}$$

where $h_l = \min_{t \in [0,T]} |h(t)|$. Multiplying both sides of (2.2) by u^m and integrating them on $[t_1, t_2]$, we have

$$\int_{t_1}^{t_2} (\phi_p(u'(s)))' u^m ds = c\lambda \int_{t_1}^{t_2} u'(s) u^m ds - \lambda \int_{t_1}^{t_2} f(u) u^m ds + \lambda \int_{t_1}^{t_2} h(s) ds, \quad \lambda \in (0, 1]. \tag{3.7}$$

Obviously, $\int_{t_1}^{t_2}(\phi_p(u'(s)))'u^m ds \leq 0$. Then, from (3.7), Lemma 3.1 and assumptions of Theorem 3.1, we have

$$c\int_{t_1}^{t_2}u'(s)u^m ds-\int_{t_1}^{t_2}f(u)u^m ds+\int_{t_1}^{t_2}h(s)ds\leq 0$$

and

$$\begin{aligned}u^{m+1}(t_2)&\geq u^{m+1}(t_1)+\frac{m+1}{c}\int_{t_1}^{t_2}f(u)u^m ds-\frac{m+1}{c}\int_{t_1}^{t_2}h(s)ds\\&\geq A_2^{m+1}-\frac{m+1}{|c|}Tf_M B_1^m-\frac{m+1}{|c|}T\overline{|h|}.\end{aligned}$$

Since $A_2^{m+1}-\frac{m+1}{|c|}Tf_M B_1^m-\frac{m+1}{|c|}T\overline{|h|}>0$, we get

$$u(t_2)\geq\left(A_2^{m+1}-\frac{m+1}{|c|}Tf_M B_1^m-\frac{m+1}{|c|}T\overline{|h|}\right)^{\frac{1}{m+1}}:=B_2. \tag{3.8}$$

Multiplying both sides of (3.1) by $u'(t)$ and integrating them on $[0,T]$, we have

$$\int_0^T(\phi_p(u'(s)))'u'(s)ds=c\lambda\int_0^T|u'(s)|^2ds-\lambda\int_0^T f(u)u'(s)ds+\lambda\int_0^T\frac{h(s)}{u^m}u'(s)ds. \tag{3.9}$$

Obviously,

$$\int_0^T(\phi_p(u'(s)))'u'(s)ds=\int_0^T u'(s)d\phi_p(u'(s))=0 \tag{3.10}$$

and

$$\int_0^T f(u)u'(s)ds=0. \tag{3.11}$$

From (3.9)–(3.11), we get

$$\begin{aligned}\|u'\|_2^2&\leq\frac{1}{|c|B_1^m}\int_0^T|h(s)||u'(s)|ds\\&\leq\frac{1}{|c|B_1^m}\|h\|_2\|u'\|_2,\end{aligned}$$

i.e.,

$$\|u'\|_2\leq\frac{1}{|c|B_1^m}\|h\|_2. \tag{3.12}$$

In view of (3.1), (3.12) and Hölder inequality , we have

$$
\begin{aligned}
|u'(t)|^{p-1} &= \left|\phi_p(u'(t_1) + \int_{t_1}^{t} (\phi_p(u'(s))'ds\right| \\
&\le \int_0^T |(\phi_p(u'(s))'|ds \\
&\le \int_0^T |c||u'(s)|ds + \int_0^T |f(u(s))|ds + \int_0^T \frac{|h(s)|}{u^m(s)}ds \\
&\le |c|T^{\frac{1}{2}}\|u'(s)\|_2 + Tf_M + \frac{T\overline{|h|}}{B_2^m} \\
&\le \frac{T^{\frac{1}{2}}}{B_1^m}\|h\|_2 + Tf_M + \frac{T\overline{|h|}}{B_2^m} \\
&:= M_1,
\end{aligned}
$$

i.e.,

$$|u'|_0 \le M_1^{\frac{1}{p-1}}.$$

Choose positive constants δ_1, δ_2 and M such that $\delta_1 < B_2 < B_1 < \delta_2$ and $M > M_1^{\frac{1}{p-1}}$. Let

$$\Omega_1 = \{u \in C_T^1 : \delta_1 < u(t) < \delta_2,\ |u'(t)| < M\}.$$

For each $\lambda \in (0, 1)$, Eq (3.1) has no solution on $\partial\Omega_1$. Hence, condition (1) of Lemma 2.2 is satisfied.

Step 2. We will show that condition (2) of Lemma 2.2 is satisfied. On the contrary, assume that there exists $u = a \in \partial\Omega_1$ such that $\mathcal{F}(a) = 0$, then $a \in \mathbb{R}$ is a constant and

$$\mathcal{F}(a) = \frac{1}{T}\int_0^T \Big[- f(a) + \frac{h(s)}{a^m}\Big] = 0.$$

We have

$$B_2 \le \left(\frac{\overline{h}}{f_M}\right)^{\frac{1}{m}} \le a \le \left(\frac{h_M}{f_L}\right)^{\frac{1}{m}} \le B_1$$

which contradicts to $a \in \partial\Omega_1$. Hence, condition (2) of Lemma 2.2 is satisfied.

Step 3. We will show that condition (3) of Lemma 2.2 is satisfied. Due to the proof of Step 2, if $u \in \Omega_1 \cap \mathbb{R}$ such that $\mathcal{F}(u) = 0$, the $u = a \in [B_2, B_1]$. It is easy to see that a is unique by using $f(u)$ is strictly monotonically increasing for $u \in [B_2, B_1]$. Hence,

$$d_B(\mathcal{F}, \Omega \cap \mathbb{R}, 0) = 1 \neq 0.$$

Applying Lemma 2.2, we reach the conclusion.

Lemma 3.2. Assume that the function f such that

$$f(0) = \lim_{u \to 0^+} f(u) > 0,\ f(u) > 0,\ f'(u) < 0,\ \ \forall u > 0.$$

Furthermore, assume $\overline{h} > 0$. Then for each $u \in \Omega$, there are constants $\eta_1, \eta_2 \in [0, T]$ such that

$$u(\eta_1) \le \left(\frac{\overline{h^+}}{\overline{f}}\right)^{\frac{1}{m}} := A_3$$

and

$$u(\eta_2) \geq \left(\frac{\overline{h}}{f(0)}\right)^{\frac{1}{m}} := A_4.$$

Proof. Integrating (3.1) on $[0, T]$, we have

$$\int_0^T f(u)ds = \int_0^T \frac{h(s)}{u^m}ds$$

and

$$T\overline{f} \leq \int_0^T \frac{h^+(s)}{u^m}ds. \tag{3.13}$$

By (3.13) and mean value theorem of integrals, there exists a point $\eta_1 \in [0, T]$ such that

$$u^m(\eta_1) \leq \frac{\overline{h^+}}{\overline{f}},$$

i.e.,

$$u(\eta_1) \leq \left(\frac{\overline{h^+}}{\overline{f}}\right)^{\frac{1}{m}} := A_3.$$

By $f'(u) < 0$ for $u > 0$, we have $f(0) > f(u)$ for $u > 0$. Similar to the proof of (3.4) and (3.5) in Lemma 3.1, we have

$$\int_0^T f(u)u^m ds \geq T\overline{h}$$

and

$$f(0)\int_0^T u^m ds \geq T\overline{h}. \tag{3.14}$$

By (3.14) and mean value theorem of integrals, there exists a point $\eta_2 \in [0, T]$ such that

$$u^m(\eta_2) \geq \frac{\overline{h}}{f(0)},$$

i.e.,

$$u(\eta_2) \geq \left(\frac{\overline{h}}{f(0)}\right)^{\frac{1}{m}} := A_4.$$

Theorem 3.2. Suppose that conditions of Lemma 3.2 hold. Then Eq (1.3) has at least one $T-$periodic solution, i.e., Eq (1.1) has at least one periodic wave solution.

Proof. Let $u(t_0) = \min_{t\in[0,T]} u(t)$. By Eq (3.1), we have

$$f(u(t_0)) = \frac{h(t_0)}{u^m(t_0)}.$$

Thus,

$$u(t_0) \geq \left(\frac{h_L}{f(0)}\right)^{\frac{1}{m}} := B_0, \tag{3.15}$$

where $h_L = \min_{t\in[0,T]} |h(t)|$. For $u \in \Omega$, by Lemma 3.2 and Hölder inequality we have

$$|u|_0 \le A_3 + T^{\frac{1}{q}}\Big(\int_0^T |u'(s)|^p ds\Big)^{\frac{1}{p}}, \tag{3.16}$$

where $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Multiply (3.1) with $u(t)$, and integrate it over the interval $[0, T]$, then

$$\begin{aligned} \int_0^T |u'(s)|^p ds &= \lambda \int_0^T f(u)uds - \lambda \int_0^T \frac{h(s)}{u^m} uds \\ &\le \int_0^T f(u)uds + \int_0^T \frac{h^-(s)}{u^m} uds \\ &\le |u|_0 \int_0^T f(u)ds + |u|_0 \int_0^T \frac{h^-(s)}{u^m} ds. \end{aligned} \tag{3.17}$$

Integrating (3.1) over the interval $[0, T]$, we gain

$$\int_0^T f(u)ds = \int_0^T \frac{h(s)}{u^m} ds. \tag{3.18}$$

By (3.17) and (3.18), we have

$$\begin{aligned} \int_0^T |u'(s)|^p ds &\le |u|_0 \int_0^T \frac{h^+(s)}{u^m} ds \\ &\le \frac{T|u|_0 \overline{h^+}}{B_0^m}. \end{aligned} \tag{3.19}$$

In view of (3.16) and (3.19), we gain

$$|u|_0 \le A_3 + T^{\frac{1}{q}}\Big(\frac{T|u|_0\overline{h^+}}{B_0^m}\Big)^{\frac{1}{p}}$$

which implies that there is a constant $\rho > 0$ such that

$$|u|_0 < \rho,$$

i.e.,

$$\max_{t\in[0,T]} u(t) < \rho.$$

From (3.9)–(3.11) and (3.15), we have

$$\|u'\|_2 \le \frac{1}{|c|B_0^m}\|h\|_2. \tag{3.20}$$

In view of (3.1), (3.20) and (3.15), we have

$$\begin{aligned}
|u'(t)|^{p-1} &= \left|\phi_p(u'(t_0) + \int_{t_0}^{t} (\phi_p(u'(s))'ds\right| \\
&\le \int_0^T |(\phi_p(u'(s))'|ds \\
&\le \int_0^T |c||u'(s)|ds + \int_0^T f(u(s))ds + \int_0^T \frac{|h(s)|}{u^m(s)}ds \\
&\le |c|T^{\frac{1}{2}}\|u'(s)\|_2 + T\overline{f} + \frac{T\overline{|h|}}{B_0^m} \\
&\le \frac{T^{\frac{1}{2}}}{B_0^m}\|h\|_2 + T\overline{f} + \frac{T\overline{|h|}}{B_0^m} \\
&:= N,
\end{aligned}$$

i.e.,

$$|u'|_0 \le N^{\frac{1}{p-1}}.$$

The following proof is similar to the proof of Step 2 and Step 3 in Theorem 3.1, we omit it.

Remark 3.1. In Theorems 3.1 and 3.2, nonlinear term $f(u)$ has no singularity at $u = 0$. For example, in Eq (1.1), let $f(u) = \frac{1}{u^2}$ or $f(u) = -\frac{1}{u^2}$. Then, nonlinear term $f(u)$ has singularity at $u = 0$. We naturally ask the following question: if nonlinear term $f(u)$ has singularity at $u = 0$. i.e., $\lim_{u\to 0^+} f(u) = \pm\infty$, are there periodic wave solutions for Eq (1.1)? We very hope that the researchers will be able to solve the above problems.

Remark 3.2. In [10], the authors studied the existence of periodic wave solutions for Eq (1.2) which nonlinear term $f(q)$ is a strictly monotone function. Since monotonicity of $f(q)$ is very critical for *prior* bounds of solutions, in the present paper, we also assume that $f(q)$ is a strictly monotone function. When $f(q)$ is not a monotone function, whether Eq (1.1) has periodic wave solutions which is a open problem. The above issue is our research topic.

Remark 3.3. In [8], Eq (1.2) is changed into the following equation:

$$-cu'(s) = (\phi_p(u'(s)))' + f(u(s)) + e(s).$$

Under the following assumptions:

(H_1) there exist constants $m_0 > 0,\ m_1 > 1$ such that

$$uf(u) \le -m_0u^m,\quad \forall u \in \mathbb{R},$$

(H_2) $e \in C(\mathbb{R}, \mathbb{R})$ is a continuous $2T$−periodic function with $e(s) \ne 0$, and

$$\left(\int_{-T}^{t} |e(s)|^{\frac{m}{m-1}}\right)^{\frac{m-1}{m}} + \sup_{s\in[-T,T]} |e(s)| < +\infty,$$

then Eq (1.2) has at least one $2T$−periodic wave solution. In the present paper, since Eq (1.1) has an indefinite singularity, we add the stronger conditions for nonlinear term f, i.e., assume that the function f such that

$$f_L < f(u) < f_M,\ f'(u) > 0,\quad \forall u > 0,$$

where f_L and f_M are positive constants.

In this section, we will give two examples to illustrate the theoretical results in the present paper.

Example 4.1. Consider the following non-Newtonian filtration equations with an indefinite singularity:

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial x}\left(\left|\frac{\partial y}{\partial x}\right|^{p-2}\frac{\partial y}{\partial x}\right) + f(y) + \frac{h(t,x)}{y^m}. \tag{4.1}$$

Let $h(t,x) = -h(x+ct) = -h(s)$, where $c \in \mathbb{R}$. Let $y(t,x) = u(s)$ with $s = x + ct$ be the solution of Eq (4.1), then Eq (4.1) is changed into the following equation:

$$cu'(s) = (\phi_p(u'(s)))' + f(u) - \frac{h(s)}{u^m}. \tag{4.2}$$

Let $p = 3,\ m = 1,\ T = 2\pi,\ f(u) = 1 + arctanu,\ h(s) = 1 + \sin s,\ c = -260$. Obviously, $f'(u) = \frac{1}{1+u^2} > 0$ is a strictly monotone increasing function. After a simple calculation, we have

$$f_L = 1,\ f_M = 1 + \frac{\pi}{2},\ \overline{h} = 1,\ h_M = 2,\ A_2 = \left(\frac{\overline{h}}{f_M}\right)^{\frac{1}{m}} \doteq 0.39,$$

$$B_1 = \left(\frac{h_M}{f_L}\right)^{\frac{1}{m}} = 1,\ A_2^{m+1} - \frac{m+1}{|c|}TB_1^m - \frac{m+1}{|c|}T\overline{|h|} = 0.05538 > 0.$$

Thus, all conditions of Theorems 3.1 hold. Therefore, Theorems 3.1 guarantees the existence of at least one one periodic solution for Eq (4.2), i.e., Eq (4.1) has least one one periodic wave solution.

Example 4.2. In Eq (4.2), let $p = 3,\ m = 1,\ T = 2\pi,\ f(u) = 3 - arctanu,\ h(s) = 1 + \sin s$. Obviously,

$$\overline{h} = 1 > 0,\ f(0) = 3 > 0,\ f(u) > 0,\ f'(u) = -\frac{1}{1+u^2} < 0 \text{ for } u > 0.$$

Then $f(u)$ is a strictly monotone decreasing function. Thus, all conditions of Theorems 3.2 hold. and Theorems 3.2 guarantees the existence of at least one one periodic solution for Eq (4.2), i.e., Eq (4.1) has least one one periodic wave solution.

4. Conclusion

In this article, we study a non-Newtonian filtration equations with an indefinite singularity. By using an generalization of Mawhin's continuation theorem and some mathematic analysis methods, we obtain some existence results of periodic wave solutions for the considered equation. Two examples are used to demonstrate the usefulness of our theoretical results. The novelty of the present paper is that it is the first time to discuss the existence of periodic wave solutions for the indefinite singular non-Newtonian filtration equations. Our results improve and extend some corresponding results in the literature. However, many important questions about indefinite singular non-Newtonian filtration equations remain to be studied, such as oscillation problems, exponential stability and asymptotic stability problems, non-Newtonian filtration equations with impulse effects and stochastic effects, etc. We hope to focus on the above issues in future studies.

Acknowledgments

The authors would like to express the sincere appreciation to the editor and reviewers for their helpful comments in improving the presentation and quality of the paper.

Conflict of interest

The author confirms that they have no conflict of interest.

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