Research article

Functional inequalities for several classes of $q$-starlike and $q$-convex type analytic and multivalent functions using a generalized Bernardi integral operator

Pinhong Long$^{1,*}$, Huo Tang$^{2}$ and Wenshuai Wang$^{1}$

$^1$ School of Mathematics and Statistics, Ningxia University, Yinchuan, Ningxia 750021, People’s Republic of China
$^2$ School of Mathematics and Computer Sciences, Chifeng University, Chifeng, Inner Mongolia, 024000, People’s Republic of China

* Correspondence: Email: longph@nxu.edu.cn.

Abstract: In the article we introduce and investigate several new subclasses of $q$-starlike and $q$-convex type analytic and multivalent functions involving a generalized Bernardi integral operator, and establish the Fekete-Szegö type functional inequalities for these function classes. Besides, the corresponding bound estimates of the coefficients $a_{p+1}$ and $a_{p+2}$ are obtained.

Keywords: Fekete-Szegö problem; analytic function; multivalent function; $q$-derivative operator; Bernardi integral operator

Mathematics Subject Classification: 26A33, 30C45, 30C50

1. Introduction

Let $\mathcal{A}_p$ denote the class of analytic and $p$-valent functions $f(z)$ with the next form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p}z^{n+p}, \quad (p \in \mathbb{N} = \{1, 2, \cdots\}) \quad (1.1)$$

in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

For $f \in \mathcal{A}_p$, its $q$-derivative or the $q$-difference $D_q f(z)$ is given by

$$D_q f(z) = [p]_q z^{p-1} + \sum_{n=1}^{\infty} [n+p]_q a_{n+p}z^{n+p-1}, \quad (0 < q < 1),$$
where the $q$-derivative operator $D_q f(z)$ (refer to [13] and [14]) of the function $f$ is defined by

$$D_q f(z) := \begin{cases} \frac{f(qz) - f(z)}{(1-q)z}, & (z \neq 0; 0 < q < 1), \\ f'(0), & (z = 0) \end{cases}$$

provided that $f'(0)$ exists, and the $q$-number $[n]_q$ is just $[\chi]_q$ when $\chi = n \in \mathbb{N}$, here

$$[\chi]_q = \begin{cases} \frac{1-q^\chi}{1-q}, & \text{for } \chi \in \mathbb{C}, \\ \sum_{k=0}^{\chi-1} q^k, & \text{for } \chi = n \in \mathbb{N}. \end{cases}$$

Note that $D_q f(z) \rightarrow f'(z)$ when $q \rightarrow 1_-$, where $f'$ is the ordinary derivative of the function $f$.

Consider the generalized Bernardi integral operator $J_{\eta}^p : \mathcal{A}_p \rightarrow \mathcal{A}_p$ with the next form

$$J_{\eta}^p f(z) = \frac{[p+\eta]q}{z_\eta} \int_0^z t^{p-1} f(t) d_q t, \quad (z \in \Delta, \Re \eta > -1 \text{ and } f \in \mathcal{A}_p). \quad (1.2)$$

Then, for $f \in \mathcal{A}_p$, we obtain that

$$J_{\eta}^p f(z) = z^p + \sum_{n=1}^{\infty} L_{\eta}^p(n) a_n z^{\eta+n}, \quad (z \in \Delta), \quad (1.3)$$

where

$$L_{\eta}^p(n) = \left[ \frac{[p+\eta]q}{[n+\eta]q} \right] := L_n. \quad (1.4)$$

Here we remark that if $p = 1$, it is exactly $q$-Bernardi integral operator $J_q^q$ [21]. Further, if $p = 1$ and $q \rightarrow 1_-$, obviously it is the classical Bernardi integral operator $J_q$ [5]. In fact, Alexander [1] and Libera [18] integral operators are special versions of $J_q$ for $\eta = 0$ and $\eta = 1$, respectively.

For two analytic functions $f$ and $g$, if there exists an analytic function $h$ satisfying $h(0) = 0$ and $|h(z)| < 1$ for $z \in \Delta$ so that $f(z) = g(h(z))$, then $f$ is subordinate to $g$, i.e., $f < g$.

Let $A$ be the class of all analytic function $\phi$ via the form

$$\phi(z) = 1 + \sum_{n=1}^{\infty} A_n z^n, \quad (A_1 > 0, z \in \Delta). \quad (1.5)$$

It is well known that the $q$-calculus [13, 14], even the $(p, q)$-calculus [6], is a generalization of the ordinary calculus without the limit symbol, and its related theory has been applied into mathematical, physical and engineering fields (see [11, 15, 25]). Since Ismail et al. [12] firstly utilized the $q$-derivative operator to investigate the $q$-calculus of the class of starlike functions in disk, there had a great deal of work in this respect; for example, refer to Rehman et al. [24] for partial sums of generalized $q$-Mittag-Leffler functions, Srivastava et al. [31] for Fekete-Szegö inequality for classes of $(p, q)$-starlike and $(p, q)$-convex functions and [27] for close-to-convexity of a certain family of $q$-Mittag-Leffler functions, Seoudy and Aouf [26] for the coefficient estimates of $q$-starlike and $q$-convex functions and Uçar [33] for the coefficient inequality for $q$-starlike functions. Besides, by involving some special functions and operators or increasing the complexity of function classes, many new subclasses of analytic functions associated with $q$-calculus or $(p, q)$-calculus were considered. Here we may refer...
for a new integral operator in $q$ corresponding bound estimates of the coefficients $a$ for the $Fekete$-Szeg"{o} type functional inequalities for these function classes. Besides, the analytic and multivalent functions involving a generalized Bernardi integral operator, and establish the analysis and fractional $q$-calculus and associated $Fekete$-Szeg"{o} problems for certain integral operators, and Khan et al. [17] for a new integral operator in $q$-analog for multivalent functions. Stimulated by the previous results, in the paper we intend to introduce and investigate several new subclasses of functions, [31] for $Fekete$-Szeg"{o} inequality for classes of $(p,q)$-starlike and $(p,q)$-convex type analytic and multivalent functions involving a generalized Bernardi integral operator, and establish the corresponding $Fekete$-Szeg"{o} type functional inequalities for these function classes. Besides, the corresponding bound estimates of the coefficients $a_{p+1}$ and $a_{p+2}$ are provided.

From now on we introduce some general subclasses of analytic and multivalent functions associated with the $q$-derivative operator and the generalized Bernardi integral operator.

**Definition 1.1.** Let $f(z) \in A_p$ and $\mu, \lambda \geq 0$. If the following subordination

$$
(1 - \lambda) \left( \frac{J_{p,q}^\eta f(z)}{z^p} \right) + \lambda \frac{D_q \left( J_{p,q}^\eta f(z) \right)}{[p]_q z^{p-1}} - \alpha \leq \phi(z)
$$

is satisfied for $z \in \Delta$, then we call that $f(z)$ belongs to the class $\mathcal{L}N_{p,q}^\eta (\mu, \lambda; \phi)$.

**Definition 1.2.** Let $f(z) \in A_p$ and $0 \leq \lambda \leq 1$. If the following subordination

$$
(1 - \lambda) \frac{z D_q \left( J_{p,q}^\eta f(z) \right)}{[p]_q J_{p,q}^\eta f(z)} + \lambda \left( 1 + q z D_q \left( J_{p,q}^\eta f(z) \right) \right) < \phi(z)
$$

is satisfied for $z \in \Delta$, then we call that $f(z)$ belongs to the class $\mathcal{L}M_{p,q}^\eta (\lambda; \phi)$.

**Definition 1.3.** Let $f(z) \in A_p$ and $\mu \geq 0$. If the following subordination

$$
\left( \frac{z D_q \left( J_{p,q}^\eta f(z) \right)}{[p]_q J_{p,q}^\eta f(z)} \right) - \alpha \leq \phi(z)
$$

is satisfied for $z \in \Delta$, then we call that $f(z)$ belongs to the class $\mathcal{NS}_{p,q}^\eta (\mu; \phi)$.

**Remark 1.4.** If we put

$$
\phi(z) = \left( \frac{1 + z}{1 - z} \right)^\alpha \text{ for } 0 < \alpha \leq 1
$$

or

$$
\phi(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \text{ for } 0 \leq \beta < 1
$$
in Definition (1.1–1.3), then the class $\mathcal{LN}^0_{p,q}(\mu,\lambda;\phi)$ (res. $\mathcal{LM}^0_{p,q}(\lambda;\phi)$ and $\mathcal{NS}^0_{p,q}(\mu;\phi)$) reduces to $\mathcal{LN}^0_{p,q}(\mu,\lambda;\alpha)$ (res. $\mathcal{LM}^0_{p,q}(\lambda;\alpha)$ and $\mathcal{NS}^0_{p,q}(\mu;\alpha)$) or $\mathcal{LN}^0_{p,q}(\mu,\lambda;\beta)$ (res. $\mathcal{LM}^0_{p,q}(\lambda;\beta)$ and $\mathcal{NS}^0_{p,q}(\mu;\beta)$). Without the generalized Bernardi integral operator, the class $\mathcal{LN}^0_{p,q}(\mu,\lambda;\phi)$ (res. $\mathcal{LM}^0_{p,q}(\lambda;\phi)$ and $\mathcal{NS}^0_{p,q}(\mu;\phi)$) is the classical function class $\mathcal{LN}(\mu,\lambda;\phi)$ (res. $\mathcal{LM}(\lambda;\phi)$ and $\mathcal{NS}(\mu;\phi)$) when $p = 1$ and $q \to 1$.

Let $\Omega$ be the class of functions $\omega(z)$ denoted by

$$\omega(z) = \sum_{n=1}^{\infty} E_n z^n, \quad (z \in \Delta)$$

via the inequality $|\omega(z)| < 1 (z \in \Delta)$. Now we recall some necessary Lemmas below.

**Lemma 1.5** ([16]). Let the function $\omega \in \Omega$. Then

$$|E_2 - \tau E_1^2| \leq \max\{1, |\tau|\}, \quad (\tau \in \mathbb{C}).$$

Specially, the sharp result holds for the next function

$$\omega(z) = z \quad \text{or} \quad \omega(z) = z^2, \quad (z \in \Delta).$$

**Lemma 1.6** ([7, 10]). Let $\mathcal{P}$ be the class of all analytic functions $h(z)$ of the following form

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad (z \in \Delta)$$

satisfying $\Re h(z) > 0$ and $h(0) = 1$. Then there exist the sharp coefficient estimates $|c_n| \leq 2 (n \in \mathbb{N})$. In particular, the equality holds for all $n$ for the next function

$$h(z) = \frac{1 + z}{1 - z} = 1 + \sum_{n=1}^{\infty} 2z^n.$$

**Lemma 1.7** ([3, 19]). Let the function $\omega \in \Omega$. Then

$$|E_2 - \kappa E_1^2| \leq \begin{cases} -\kappa & \text{if } \kappa \leq -1, \\ 1 & \text{if } -1 \leq \kappa \leq 1, \\ \kappa & \text{if } \kappa \geq 1. \end{cases}$$

For $\kappa < -1$ or $\kappa > 1$, the inequality holds literally if and only if $\omega(z) = z$ or one of its rotations. If $\kappa < 1$, the inequality holds literally if and only if $\omega(z) = z^2$ or one of its rotations. In particular, if $\kappa = -1$, then the sharp result holds for the next function

$$\omega(z) = \frac{z(z + \xi)}{1 + \xi z}, \quad (0 \leq \xi \leq 1)$$

or one of its rotations. If $\kappa = 1$, then the sharp result holds for the next function

$$\omega(z) = -\frac{z(z + \xi)}{1 + \xi z}, \quad (0 \leq \xi \leq 1)$$

or one of its rotations. If $-1 < \kappa < 1$, then the upper bound is sharp as the followings

$$|E_2 - \kappa E_1^2| + (\kappa + 1)|E_1|^2 \leq 1, \quad (-1 < \kappa \leq 0)$$

and

$$|E_2 - \kappa E_1^2| + (1 - \kappa)|E_1|^2 \leq 1, \quad (0 < \kappa < 1).$$
2. Functional estimates for the class $\mathcal{LN}_{n,q}^n(\mu, \lambda; \phi)$

By (1.9) we give that

$$
\phi(\omega(z)) = 1 + A_1E_1z + (A_1E_2 + A_2E_1^2)z^2 + (A_1E_3 + 2A_2E_1E_2 + A_3E_1^3)z^3 + \ldots \quad (2.1)
$$

In the section, with Lemma 1.5 we study Fekete-Szegö functional problem for the class $\mathcal{LN}_{n,q}^n(\mu, \lambda; \phi)$ and provide the following theorem.

**Theorem 2.1.** Let $\delta \in \mathbb{C}$. If $f(z) \in \mathcal{A}_p$ belongs to the class $\mathcal{LN}_{n,q}^n(\mu, \lambda; \phi)$, then

$$
|a_{p+1} - \delta_a^2| \leq \frac{A_1[p]_q}{|L_2|\mu[p]_q + \lambda([p + 2]_q - [p]_q)} \max \left\{ 1; \frac{A_1[p]_q\Theta}{2L_1^2[p]_q + \lambda([p + 1]_q - [p]_q)} - A_2 \right\},
$$

where

$$
\Theta = 2\delta L_2[p]_q + \lambda([p + 2]_q - [p]_q)] + L_2^2([\mu - 1][\mu - \lambda(U - \mu + 1)]p]_q + 2\lambda(2\mu - \lambda \mu - 1)[p + 1]_q.
$$

Moreover, the sharp result holds for the next function

$$
\omega(z) = z \quad \text{or} \quad \omega(z) = z^2, \quad (z \in \Delta).
$$

**Proof.** Assume that $f(z) \in \mathcal{LN}_{n,q}^n(\mu, \lambda; \phi)$. Then, from Definition 1.1, there exists an analytic function $\omega(z) \in \Omega$ such that

$$
(1 - \lambda) \left( \frac{J_{p,q}^n f(z)}{z^p} \right)^{\mu} + \frac{\lambda D_q \left( \frac{J_{p,q}^n f(z)}{z^p} \right)(z)}{[p]_q z^{p-1}} \left( \frac{J_{p,q}^n f(z)}{z^p} \right)^{\mu-1} = \phi(\omega(z)). \quad (2.2)
$$

Part case

$$
\left( \frac{J_{p,q}^n f(z)}{z^p} \right)^{\mu-1} = 1 + (\mu - 1)L_1a_{p+1}z + \left( \mu - 1\right)L_2a_{p+2} + \frac{(\mu - 1)(\mu - 2)}{2}L_2^2a_{p+1}^2 \right]z^2 + \left( \mu - 1\right)L_3a_{p+3} + \frac{(\mu - 1)(\mu - 2)}{2}L_4L_2a_{p+1}a_{p+2} + \frac{(\mu - 1)(\mu - 2)(\mu - 3)}{6}L_4^2a_{p+1}^3 \right]z^3 + \ldots,
$$

$$
\frac{\lambda D_q \left( \frac{J_{p,q}^n f(z)}{z^p} \right)(z)}{[p]_q z^{p-1}} = \lambda + \frac{\lambda L_1a_{p+1}[p + 1]_q}{[p]_q}z + \frac{\lambda L_2a_{p+2}[p + 2]_q}{[p]_q}z^2 + \frac{\lambda L_3a_{p+3}[p + 3]_q}{[p]_q}z^3 + \ldots,
$$
\[
\frac{\lambda D_q \left( \mathcal{I}_{p,q}^q f(z) \right) (z)}{[p]_q z^{p-1}} \left( \frac{\mathcal{I}_{p,q}^q f(z)}{z^p} \right)^{\mu-1} \\
= \lambda + \lambda \left\{ \mu + \left( \frac{[p+1]_q}{[p]_q} - 1 \right) \right\} L_1 a_{p+1} z \\
+ \lambda \left\{ \left[ \mu + \left( \frac{[p+2]_q}{[p]_q} - 1 \right) \right] L_2 a_{p+2} + \left( \mu - 1 \right) \left[ \frac{\mu}{2} + \left( \frac{[p+1]_q}{[p]_q} - 1 \right) \right] \right\} L_1^2 a_{p+1} z^2 + \ldots ,
\]

\[
(1 - \lambda) \left( \frac{\mathcal{I}_{p,q}^q f(z)}{z^p} \right)^{\mu} \\
= (1 - \lambda) + (1 - \lambda) \mu L_1 a_{p+1} z + (1 - \lambda) \left[ \mu L_2 a_{p+2} + \frac{\mu(\mu - 1)}{2} L_1^2 a_{p+1}^2 \right] z^2 \\
+ (1 - \lambda) \left[ \mu L_3 a_{p+3} + \frac{\mu(\mu - 1)(\mu - 2)}{6} L_1 L_2 a_{p+1} a_{p+2} + \frac{\mu(\mu - 1)(\mu - 2)}{6} L_1^3 a_{p+1}^3 \right] z^3 + \ldots .
\]

Since

\[
(1 - \lambda) \left( \frac{\mathcal{I}_{p,q}^q f(z)}{z^p} \right)^{\mu} + \frac{\lambda D_q \left( \mathcal{I}_{p,q}^q f(z) \right) (z)}{[p]_q z^{p-1}} \left( \frac{\mathcal{I}_{p,q}^q f(z)}{z^p} \right)^{\mu-1} \\
= 1 + \left\{ \mu + \lambda \left( \frac{[p+1]_q}{[p]_q} - 1 \right) \right\} L_1 a_{p+1} z + \left\{ \mu + \lambda \left( \frac{[p+2]_q}{[p]_q} - 1 \right) \right\} L_2 a_{p+2} \\
+ \left\{ \frac{\mu - 1}{2} \left[ \mu - 2 \lambda (\lambda \mu - \mu + 1) \right] + \lambda (2 \mu - \lambda \mu - 1) \frac{[p+1]_q}{[p]_q} \right\} L_1^2 a_{p+1}^2 z^2 + \ldots ,
\]

by (2.1) and (2.2) we see that

\[
A_1 E_1 = \left[ \mu + \lambda \left( \frac{[p+1]_q}{[p]_q} - 1 \right) \right] L_1 a_{p+1},
\]

\[
A_1 E_2 + A_2 E_1^2 = \left[ \mu + \lambda \left( \frac{[p+2]_q}{[p]_q} - 1 \right) \right] L_2 a_{p+2} \\
+ \left[ \frac{\mu - 1}{2} \left[ \mu - 2 \lambda (\lambda \mu - \mu + 1) \right] + \lambda (2 \mu - \lambda \mu - 1) \frac{[p+1]_q}{[p]_q} \right] L_1^2 a_{p+1}^2 .
\]

Thereby

\[
a_{p+1} = \frac{A_1 E_1 [p]_q}{L_1 [\mu [p]_q + \lambda ([p+1]_q - [p]_q)]} \tag{2.3}
\]

and

\[
a_{p+2} = \frac{(A_1 E_2 + A_2 E_1^2) [p]_q}{L_2 [\mu [p]_q + \lambda ([p+2]_q - [p]_q)]} \\
- \frac{A_1^2 E_1^2 [p]_q^2 ([\mu - 2 \lambda (\mu - \mu + 1)] [p]_q + 2 \lambda (2 \mu - \lambda \mu - 1) [p+1]_q)}{2L_2 [\mu [p]_q + \lambda ([p+1]_q - [p]_q)]^2 [\mu [p]_q + \lambda ([p+2]_q - [p]_q)]} \tag{2.4}
\]
Further, with (2.3) and (2.4) we obtain that

\[ a_{p+2} - \delta a_{p+1}^2 = \frac{A_1[p]q}{L_2[\mu(p[q] + \lambda([p + 2]_q - [p]_q))] [E_2 - hE_2^2]}, \]

where

\[ h = \left( 2\delta L_2[\mu(p[q] + \lambda([p + 2]_q - [p]_q))] + L_1^2([\mu - 1][\mu - \lambda(\mu - \mu + 1)]/[p]_q) 
+ \frac{2\lambda(2\mu - \lambda - 1)[p + 1]_q]}{2L_1^2[\mu(p[q] + \lambda([p + 1]_q - [p]_q))]^2 - \frac{A_2}{A_1}.} \]

Therefore, according to Lemma 1.5 we finish the proof of Theorem 2.1. \( \Box \)

**Corollary 2.2.** If \( f(z) \in \mathcal{A}_p \) belongs to the class \( \mathcal{LN}_{p,q}^0(\mu, \lambda; \phi) \), then

\[ |a_{p+2}| \leq \frac{A_1[p]q}{L_2[\mu(p[q] + \lambda([p + 2]_q - [p]_q))]} \times \max \left\{ 1; \left( \frac{[\mu - 1][\mu - \lambda(\mu - \mu + 1)]/[p]_q + 2\lambda(2\mu - \lambda - 1)[p + 1]_q}{2\mu(p[q] + \lambda([p + 1]_q - [p]_q))]^2 - \frac{A_2}{A_1} \right) \right\}. \]

Moreover, the sharp result holds for the next function

\[ \omega(z) = z \quad \text{or} \quad \omega(z) = z^2, \quad (z \in \Delta). \]

When \( \phi \in \mathcal{P} \), combining (2.3) and (2.4) with Lemma 1.6 we instantly establish the next corollary for the coefficient bounds of \( a_{p+1} \) and \( a_{p+2} \).

**Corollary 2.3.** If \( f(z) \in \mathcal{A}_p \) belongs to the class \( \mathcal{LN}_{p,q}^0(\mu, \lambda; \phi) \), then

\[ |a_{p+1}| \leq \frac{2[E_1[p]q}{L_1[\mu(p[q] + \lambda([p + 1]_q - [p]_q))] \]

and

\[ |a_{p+2}| \leq \frac{2(E_2 + E_1^2)[p]_q}{L_2[\mu(p[q] + \lambda([p + 2]_q - [p]_q))] \times \left( \frac{2E_2^2[p]_q[\mu - 1][\mu - \lambda(\mu - \mu + 1)]/[p]_q + 2\lambda(2\mu - \lambda - 1)[p + 1]_q}{|L_2[\mu(p[q] + \lambda([p + 1]_q - [p]_q))]^2[\mu(p[q] + \lambda([p + 2]_q - [p]_q))] - \frac{A_2}{A_1}.} \right) \]

If we choose real \( \delta \) and \( \eta \), then by Lemma 1.7 we derive the next result for Fekete-Szegö problem.

**Theorem 2.4.** Let \( \delta, \eta \in \mathbb{R} \) and \( \phi \in \Lambda \) satisfying

\[ \phi(z) = 1 + \sum_{n=1}^{\infty} A_n z^n, \quad (A_1, A_2 > 0, z \in \Delta). \]

If \( f(z) \in \mathcal{A}_p \) belongs to the class \( \mathcal{LN}_{p,q}^n(\mu, \lambda; \phi) \), then

\[ |a_{p+2} - \delta a_{p+1}^2| \leq \begin{cases} \frac{A_1[p]q}{L_2[\mu(p[q] + \lambda([p + 2]_q - [p]_q))] \times \left( \frac{A_2}{2L_2^2[\mu(p[q] + \lambda([p + 1]_q - [p]_q))]^2} + \delta \leq \gamma_1 \right); \\
\frac{A_1[p]q}{L_2[\mu(p[q] + \lambda([p + 2]_q - [p]_q))] \times \left( \frac{A_1[p]q}{2L_2^2[\mu(p[q] + \lambda([p + 1]_q - [p]_q))]^2} + \delta \leq \gamma_2 \right); \\
\frac{A_1[p]q}{L_2[\mu(p[q] + \lambda([p + 2]_q - [p]_q))] \times \left( \frac{A_1[p]q}{2L_2^2[\mu(p[q] + \lambda([p + 1]_q - [p]_q))]^2} - \delta \geq \gamma_2 \right); \end{cases} \]
where

\[
\gamma_1 = \frac{(A_2 - A_1)L_1^2[\mu[p]_q + \lambda([p + 1]_q - [p]_q)]^2}{A_1^2L_1[p]_q[\mu[p]_q + \lambda([p + 2]_q - [p]_q)] - L_1^2([\mu - 1][\mu - \lambda(\mu - \mu + 1)][p]_q + 2\lambda(2\mu - \lambda \mu - 1)[p + 1]_q)]}{2L_2[\mu[p]_q + \lambda([p + 2]_q - [p]_q)]}
\]

and

\[
\gamma_2 = \frac{(A_2 + A_1)L_1^2[\mu[p]_q + \lambda([p + 1]_q - [p]_q)]^2}{A_1^2L_1[p]_q[\mu[p]_q + \lambda([p + 2]_q - [p]_q)] - L_1^2([\mu - 1][\mu - \lambda(\mu - \mu + 1)][p]_q + 2\lambda(2\mu - \lambda \mu - 1)[p + 1]_q)]}{2L_2[\mu[p]_q + \lambda([p + 2]_q - [p]_q)]}
\]

Moreover, we take

\[
\gamma_3 = \frac{A_2L_1^2[\mu[p]_q + \lambda([p + 1]_q - [p]_q)]^2}{A_1^2L_1[p]_q[\mu[p]_q + \lambda([p + 2]_q - [p]_q)] - L_1^2([\mu - 1][\mu - \lambda(\mu - \mu + 1)][p]_q + 2\lambda(2\mu - \lambda \mu - 1)[p + 1]_q)]}{2L_2[\mu[p]_q + \lambda([p + 2]_q - [p]_q)]}
\]

Then, each of the following results is true:

(A) For \(\delta \in [\gamma_1, \gamma_3]\),

\[
|a_{p+2} - \delta a_{p+1}| + \left\{ 2(A_1 - A_2)L_1^2[\mu[p]_q + \lambda([p + 1]_q - [p]_q)]^2 + A_1^2[p]_q\Theta \right\} \times \frac{|a_{p+1}|^2}{2A_1^2L_1[p]_q[\mu[p]_q + \lambda([p + 2]_q - [p]_q)]} \leq \frac{A_1[p]_q}{L_2[\mu[p]_q + \lambda([p + 2]_q - [p]_q)]};
\]

(B) For \(\delta \in [\gamma_3, \gamma_2]\),

\[
|a_{p+2} - \delta a_{p+1}| + \left\{ 2(A_1 + A_2)L_1^2[\mu[p]_q + \lambda([p + 1]_q - [p]_q)]^2 - A_1^2[p]_q\Theta \right\} \times \frac{|a_{p+1}|^2}{2A_1^2L_1[p]_q[\mu[p]_q + \lambda([p + 2]_q - [p]_q)]} \leq \frac{A_1[p]_q}{L_2[\mu[p]_q + \lambda([p + 2]_q - [p]_q)]};
\]

where

\[
\Theta = 2\delta L_2[\mu[p]_q + \lambda([p + 2]_q - [p]_q)] - L_1^2([\mu - 1][\mu - \lambda(\mu - \mu + 1)][p]_q + 2\lambda(2\mu - \lambda \mu - 1)[p + 1]_q].
\]

Remark 2.5. Fixing the parameter \(p = 1\) in Theorems 2.1 and 2.4, we can state the new results for the univalent function classes \(\mathcal{LN}^{\alpha}_{1,q}(\mu, \lambda; \phi)\) or \(\mathcal{LN}^{\alpha}_{p,q}(\mu, \lambda; \phi)\). As Remark 1.4, we may consider \(\mathcal{LN}^{\alpha}_{p,q}(\mu, \lambda; \alpha)\) or \(\mathcal{LN}^{\alpha}_{p,q}(\mu, \lambda; \beta)\) to establish latest results. On the other hand, for the different parameters \(\mu\) and \(\lambda\), we can deduce new results for \(\mathcal{LN}^{\alpha}_{p,q}(\mu, \lambda; \phi)\).
3. Functional estimates for the class $\mathcal{LM}_{p,q}^q(\lambda; \phi)$

In the section we mainly consider Fekete-Szegő functional problem for the class $\mathcal{LM}_{p,q}^q(\lambda; \phi)$ and establish the theorem as follows.

**Theorem 3.1.** Let $\delta \in \mathbb{C}$. If $f(z) \in \mathcal{A}_p$ belongs to the class $\mathcal{LM}_{p,q}^q(\lambda; \phi)$, then

$$|a_{p+2} - \delta a_{p+1}| \leq \frac{A_1[p]_q^2}{|L_2([p+2]_q - [p]_q)\lambda([p+2]_q - [p]_q) + [p]_q|} \times \max \left\{ 1; \left| \frac{A_1[p]_q \Phi}{L_2([p+1]_q - [p]_q)\lambda([p+1]_q - [p]_q) + [p]_q} - \frac{A_2}{A_1} \right| \right\},$$

where

$$\Phi = \delta L_2(p_2([p+2]_q - [p]_q)\lambda([p+2]_q - [p]_q) + [p]_q) + L_2([p+1]_q - [p]_q)\lambda([p+1]_q^2 - [p]_q^2) + [p]_q^2).$$

Moreover, the sharp result holds for the next function

$$\omega(z) = z \quad \text{or} \quad \omega(z) = z^2, \quad (z \in \Delta).$$

**Proof.** If $f \in \mathcal{LM}_{p,q}^q(\lambda; \phi)$, from Definition 1.2 there exists an analytic function $\omega(z) \in \Omega$ such that

$$\frac{(1 - \lambda)zD_q(\mathcal{J}^q_{p,q} f)(z)}{|p|_q \mathcal{J}^q_{p,q} f(z)} + \frac{\lambda}{|p|_q} \left( 1 + \frac{qzD_q[D_q(\mathcal{J}^q_{p,q} f)](z)}{D_q(\mathcal{J}^q_{p,q} f)(z)} \right) = \phi(\omega(z)). \quad (3.1)$$

Part case

$$zD_q(\mathcal{J}^q_{p,q} f)(z)$$

$$\frac{1}{|p|_q} \left( 1 + \frac{qzD_q[D_q(\mathcal{J}^q_{p,q} f)](z)}{D_q(\mathcal{J}^q_{p,q} f)(z)} \right) = 1 + \frac{L_1[p+1]_q}{|p|_q} \left( \frac{1}{a_{p+1}^2} \right) + \frac{L_2[p+2]_q}{|p|_q} \left( \frac{1}{a_{p+1}^2} \right) ...$$

Since

$$\frac{(1 - \lambda)zD_q(\mathcal{J}^q_{p,q} f)(z)}{|p|_q \mathcal{J}^q_{p,q} f(z)} + \frac{\lambda}{|p|_q} \left( 1 + \frac{qzD_q[D_q(\mathcal{J}^q_{p,q} f)](z)}{D_q(\mathcal{J}^q_{p,q} f)(z)} \right) = 1 + \left( \frac{1}{a_{p+1}^2} \right)$$

AIMS Mathematics
Furthermore, in accordance with (3.2) and (3.3) we gain that

\[ A_1E_1 = \left[ \frac{[p + 1]_q}{[p]_q} - 1 \right] \left[ \lambda \left( \frac{[p + 1]_q}{[p]_q} - 1 \right) + 1 \right] L_1a_{p+1} \]

and

\[ A_1E_2 + A_2E_1^2 = \left[ \frac{[p + 2]_q}{[p]_q} - 1 \right] \left[ \lambda \left( \frac{[p + 2]_q}{[p]_q} - 1 \right) + 1 \right] L_2a_{p+2} \]

\[ - \left[ \frac{[p + 1]_q}{[p]_q} - 1 \right] \left[ \lambda \left( \frac{[p + 1]_q^2}{[p]_q^2} - 1 \right) + 1 \right] L_1^2a_{p+1}^2. \]

Then, it leads to

\[ a_{p+1} = \frac{A_1E_1[p]_q^2}{L_1([p + 1]_q - [p]_q)[\lambda([p + 1]_q - [p]_q) + [p]_q]} \]  \hspace{1cm} (3.2)

and

\[ a_{p+2} = \frac{[p]_q^2}{L_2([p + 2]_q - [p]_q)[\lambda([p + 2]_q - [p]_q) + [p]_q]} \times \left[ (A_1E_1 + A_2E_1^2) + \frac{A_1^2E_1^2[p]_q^2\lambda([p + 1]_q^2 - [p]_q^2) + [p]_q^2}{([p + 1]_q - [p]_q)[\lambda([p + 1]_q - [p]_q) + [p]_q]^2} \right]. \]  \hspace{1cm} (3.3)

Furthermore, in accordance with (3.2) and (3.3) we gain that

\[ a_{p+2} - \delta a_{p+1}^2 = \frac{A_1[p]_q^2}{L_2([p + 2]_q - [p]_q)[\lambda([p + 2]_q - [p]_q) + [p]_q]} \{ E_2 - \varphi E_1^2 \}, \]

where

\[ \varphi = \frac{A_1[p]_q^2}{L_1^2([p + 1]_q - [p]_q)^2[\lambda([p + 1]_q - [p]_q) + [p]_q]^2} \frac{A_2}{A_1}. \]

Thus, from Lemma 1.5 we give the Fekete-Szegö functional inequality in Theorem 3.1.  \hfill \Box

\textbf{Corollary 3.2.} If \( f(z) \in \mathcal{A}_p \) belongs to the class \( \mathcal{LM}_{p,q}^q(\lambda; \phi) \), then

\[ |a_{p+2}| \leq \frac{A_1[p]_q^2}{L_2([p + 2]_q - [p]_q)[\lambda([p + 2]_q - [p]_q) + [p]_q]} \times \max \left\{ 1; \left( \frac{A_1[p]_q^2\lambda([p + 1]_q^2 - [p]_q^2) + [p]_q^2}{([p + 1]_q - [p]_q)[\lambda([p + 1]_q - [p]_q) + [p]_q]^2} - \frac{A_2}{A_1} \right) \right\}. \]

Moreover, the sharp result holds for the next function

\[ \omega(z) = z \quad \text{or} \quad \omega(z) = z^2, \quad (z \in \Delta). \]
If \( \phi \in \mathcal{P} \), by (3.2) and (3.3) we take Lemma 1.6 to prove the next corollary for the coefficient bounds of \( a_{p+1} \) and \( a_{p+2} \).

**Corollary 3.3.** If \( f(z) \in \mathcal{A}_p \) belongs to the class \( \mathcal{LM}_p^{q}(\lambda; \phi) \), then

\[
|a_{p+1}| \leq \frac{2|E_1[p]_q^2}{|L_1|[([p + 1]_q - [p]_q)[\lambda([p + 1]_q - [p]_q)] + [p]_q]}
\]

and

\[
|a_{p+2}| \leq \frac{2[p]_q^2}{|L_2|[([p + 2]_q - [p]_q)[\lambda([p + 2]_q - [p]_q)] + [p]_q]} \times \left[ |E_2| + |E_1|^2 + \frac{2[E_1[p]_q^2][\lambda([p + 1]_q - [p]_q)] + [p]_q^2}{(|L_1|[([p + 1]_q - [p]_q)[\lambda([p + 1]_q - [p]_q)] + [p]_q])^2} \right].
\]

On the other hand, if we take real \( \delta \) and \( \eta \), then by Lemma 1.7 we give the next result for Fekete-Szegö problem.

**Theorem 3.4.** Let \( \delta, \eta \in \mathbb{R} \) and \( \phi \in \Lambda \) satisfying

\[
\phi(z) = 1 + \sum_{n=1}^{\infty} A_n z^n, \quad (A_1, A_2 > 0, z \in \Delta).
\]

If \( f(z) \in \mathcal{A}_p \) belongs to the class \( \mathcal{LM}_p^{q}(\lambda; \phi) \), then

\[
|a_{p+2} - \delta a_{p+1}^2| \leq \left\{
\begin{array}{l}
\frac{|p]_q^2}{|L_2|[([p + 2]_q - [p]_q)[\lambda([p + 2]_q - [p]_q)] + [p]_q]} \left(A_2 - L_1^2([p + 1]_q - [p]_q)[\lambda([p + 1]_q - [p]_q)] + [p]_q^2] \right), \ (\delta \leq \Gamma_1);
\frac{|p]_q^2}{|L_2|[([p + 2]_q - [p]_q)[\lambda([p + 2]_q - [p]_q)] + [p]_q]} \left(-A_2 + A_1[p]_q^2 \Phi \right), \ (\delta \geq \Gamma_2),
\end{array}
\right.
\]

where

\[
\Gamma_1 = \{(A_2 - A_1)\{[p + 1]_q - [p]_q)[\lambda([p + 1]_q - [p]_q)] + [p]_q^2 - A_1[p]_q^2[\lambda([p + 1]_q - [p]_q)] + [p]_q^2]\}
\]

\[
\times \frac{L_1^2([p + 1]_q - [p]_q)}{L_2 A_1^2[p]_q^2([p + 2]_q - [p]_q)[\lambda([p + 2]_q - [p]_q)] + [p]_q^2}.
\]

and

\[
\Gamma_2 = \{(A_2 + A_1)\{[p + 1]_q - [p]_q)[\lambda([p + 1]_q - [p]_q)] + [p]_q^2 - A_1[p]_q^2[\lambda([p + 1]_q - [p]_q)] + [p]_q^2]\}
\]

\[
\times \frac{L_1^2([p + 1]_q - [p]_q)}{L_2 A_1^2[p]_q^2([p + 2]_q - [p]_q)[\lambda([p + 2]_q - [p]_q)] + [p]_q^2}.
\]

Moreover, we choose

\[
\Gamma_3 = \{A_2([p + 1]_q - [p]_q)[\lambda([p + 1]_q - [p]_q)] + [p]_q^2 - A_1[p]_q^2[\lambda([p + 1]_q - [p]_q)] + [p]_q^2}\}
\]

Aims Mathematics
\[ \times \frac{L_1^2((p + 1)_q - [p]_q)}{L_2A_1^2[p]_q^2((p + 2)_q - [p]_q)[\lambda((p + 2)_q - [p]_q) + [p]_q]}. \]

Then, each of the following results is true:

(A) For \( \delta \in [\Gamma_1, \Gamma_3] \),

\[ |a_{p+2} - \delta a_{p+1}^2| + \frac{(A_1 - A_2)L_1^2((p + 1)_q - [p]_q)[\lambda((p + 1)_q - [p]_q) + [p]_q]^2 + A_1^2[p]_q \Phi}{A_1^2L_2[p]_q^2((p + 2)_q - [p]_q)[\lambda((p + 2)_q - [p]_q) + [p]_q]}|a_{p+1}|^2 \leq \frac{L_2((p + 2)_q - [p]_q)[\lambda((p + 2)_q - [p]_q) + [p]_q]}. \]

(B) For \( \delta \in [\Gamma_3, \Gamma_2] \),

\[ |a_{p+2} - \delta a_{p+1}^2| + \frac{(A_1 + A_2)L_1^2((p + 1)_q - [p]_q)[\lambda((p + 1)_q - [p]_q) + [p]_q]^2 - A_1^2[p]_q \Phi}{A_1^2L_2[p]_q^2((p + 2)_q - [p]_q)[\lambda((p + 2)_q - [p]_q) + [p]_q]}|a_{p+1}|^2 \leq \frac{L_2((p + 2)_q - [p]_q)[\lambda((p + 2)_q - [p]_q) + [p]_q]}. \]

\[ \Phi = \delta L_2[p]_q((p + 2)_q - [p]_q)[\lambda((p + 2)_q - [p]_q) + [p]_q] + L_1^2[p + 1]_q((p + 1)_q - [p]_q)[\lambda((p + 1)_q - [p]_q) + [p]_q]. \]

**Remark 3.5.** Similarly, by taking the parameter \( p = 1 \) in Theorems 3.1 and 3.4, we can obtain the new results for the univalent function classes \( \mathcal{LM}^\eta_{q, \lambda}(\alpha; \phi) = \mathcal{LM}^\eta_{q, \lambda}(\alpha; \phi) \). As Remark 1.4, we may consider \( \mathcal{LM}^\eta_{p, \lambda}(\alpha; \phi) \) or \( \mathcal{LM}^\eta_{p, \lambda}(\beta; \phi) \) to establish latest results. Clearly, for special parameter \( \lambda \), we can still imply new results for \( \mathcal{LM}^\eta_{p, \lambda}(\lambda; \phi) \).

### 4. Functional estimates for the class \( NS_{p, q}^\eta(\mu; \phi) \)

In the section we investigate Fekete-Szegö functional problem for the class \( NS_{p, q}^\eta(\mu; \phi) \) and obtain the corresponding theorem below.

**Theorem 4.1.** Let \( \delta \in \mathbb{C} \). If \( f(z) \in \mathcal{A}_p \) belongs to the class \( NS_{p, q}^\eta(\mu; \phi) \), then

\[ |a_{p+2} - \delta a_{p+1}^2| \leq \frac{|A_1[p]_q|}{|L_2[\mu[p]_q + [p + 2]_q - [p]_q]|} \times \max \left\{ 1; \frac{|A_1[p]_q \Psi|}{2L_2^2[\mu[p]_q + [p + 1]_q - [p]_q]^2 - A_2^2} \right\}, \]

where

\[ \Psi = 2\delta L_2[p]_q[\mu[p]_q + [p + 2]_q - [p]_q] + (\mu - 1)[\mu[p]_q + 2([p + 1]_q - [p]_q)]L_1^2. \]

Moreover, the sharp result holds for the next function

\[ \omega(z) = z \quad \text{or} \quad \omega(z) = z^2, \quad (z \in \Delta). \]
Proof. Since \( f \in NS_{p,q}^\eta(\mu, \phi) \), from Definition 1.3 there exists an analytic function \( \omega(z) \in \Omega \) such that

\[
\left( \frac{z D_q \left( J_{p,q}^\eta f \right)(z)}{[p]_q J_{p,q}^\eta f(z)} \right) \left( \frac{J_{p,q}^\eta f(z)}{z^p} \right)^\mu = \phi(\omega(z)).
\] (4.1)

Part case

\[
\frac{z D_q \left( J_{p,q}^\eta f \right)(z)}{[p]_q J_{p,q}^\eta f(z)} = 1 + \left( \frac{[p + 1]_q}{[p]_q} - 1 \right) L_1 a_{p+1} z + \left( \left( \frac{[p + 2]_q}{[p]_q} - 1 \right) L_2 a_{p+2} - \left( \frac{[p + 1]_q}{[p]_q} - 1 \right) L_1^2 a_{p+1}^2 \right) z^2 + \ldots,
\]

\[
\left( \frac{J_{p,q}^\eta f(z)}{z^p} \right)^\mu = 1 + \mu L_1 a_{p+1} z + \left( \mu L_2 a_{p+2} + \frac{\mu(\mu - 1)}{2} L_1^2 a_{p+1}^2 \right) z^2 + \ldots + \left( \left( \frac{[p + 2]_q}{[p]_q} - 1 + \mu \right) L_2 a_{p+2} + (\mu - 1) \left( \left( \frac{[p + 1]_q}{[p]_q} - 1 + \frac{\mu}{2} \right) L_1^2 a_{p+1}^2 \right) \right) z^2 + \ldots.
\]

Since

\[
\frac{z D_q \left( J_{p,q}^\eta f \right)(z)}{[p]_q J_{p,q}^\eta f(z)} \left( \frac{J_{p,q}^\eta f(z)}{z^p} \right)^\mu = 1 + \left( \frac{[p + 1]_q}{[p]_q} - 1 + \mu \right) L_1 a_{p+1} z
\]

\[
+ \left( \left( \frac{[p + 2]_q}{[p]_q} - 1 + \mu \right) L_2 a_{p+2} + (\mu - 1) \left( \left( \frac{[p + 1]_q}{[p]_q} - 1 + \frac{\mu}{2} \right) L_1^2 a_{p+1}^2 \right) \right) z^2 + \ldots,
\]

from (2.1) and (4.1) we know that

\[
A_1 E_1 = \left( \frac{[p + 1]_q}{[p]_q} - 1 + \mu \right) L_1 a_{p+1}
\]

and

\[
A_1 E_2 + A_2 E_1^2 = \left( \frac{[p + 2]_q}{[p]_q} - 1 + \mu \right) L_2 a_{p+2}
\]

\[
+ (\mu - 1) \left( \left( \frac{[p + 1]_q}{[p]_q} - 1 + \frac{\mu}{2} \right) L_1^2 a_{p+1}^2 \right).
\]

Thus it deduces that

\[
a_{p+1} = \frac{A_1 E_1 [p]_q}{L_1 (\mu [p]_q + [p + 1]_q - [p]_q)}
\] (4.2)

and

\[
a_{p+2} = \frac{(A_1 E_2 + A_2 E_1^2) [p]_q}{L_2 (\mu [p]_q + [p + 2]_q - [p]_q)}.
\]
Moreover, in the light of (4.2) and (4.3) we know that

$$\alpha_{p+1} - \delta \alpha_{p+1} = \frac{A_1[p]_q}{L_2(\mu[p]_q + [p + 2]_q - [p]_q)} [E_2 - \mathcal{N} E_1^2],$$

where

$$\mathcal{N} = \frac{2\delta L_2[p]_q(\mu[p]_q + [p + 2]_q - [p]_q) + (\mu - 1)[\mu[p]_q + 2([p + 1]_q - [p]_q)]L_1^2}{2L_1^2(\mu[p]_q + [p + 1]_q - [p]_q)^2} A_1[p]_q - A_2 A_1. $$

Hence, in view of Lemma 1.5 we get the Fekete-Szegő functional inequality in Theorem 4.1.

**Corollary 4.2.** If \( f(z) \in \mathcal{A}_p \) belongs to the class \( \mathcal{NS}_{p,q}^\mu(\mu; \phi), \) then

$$|\alpha_{p+2}| \leq \frac{A_1[p]_q}{|L_2|([\mu[p]_q + [p + 2]_q - [p]_q) \times \max \left\{ 1; \frac{A_1(\mu - 1)[\mu[p]_q + 2([p + 1]_q - [p]_q)]}{2(\mu[p]_q + [p + 1]_q - [p]_q)^2} - \frac{A_2}{A_1} \right\}. $$

Moreover, the sharp result holds for the next function

$$\omega(z) = z \quad \text{or} \quad \omega(z) = z^2, \quad (z \in \Delta).$$

Once \( \phi \in \mathcal{P} \), together with (4.2) and (4.3) we apply Lemma 1.6 to prove the next corollary for the coefficient bounds of \( \alpha_{p+1} \) and \( \alpha_{p+2} \).

**Corollary 4.3.** If \( f(z) \in \mathcal{A}_p \) belongs to the class \( \mathcal{NS}_{p,q}^\mu(\mu; \phi), \) then

$$|\alpha_{p+1}| \leq \frac{2E_1[p]_q}{|L_1|([\mu[p]_q + [p + 1]_q - [p]_q)$$

and

$$|\alpha_{p+2}| \leq \frac{2(|E_2| + |E_1|^2)[p]_q}{|L_2|([\mu[p]_q + [p + 2]_q - [p]_q) \times \frac{2(\mu - 1)|E_1|^2[p]_q + 2([p + 1]_q - [p]_q)]}{|L_2|([\mu[p]_q + [p + 2]_q - [p]_q) + (\mu[p]_q + [p + 2]_q - [p]_q)^2}. $$

Clearly, if we let \( \delta \) and \( \eta \) be real, then from Lemma 1.7 we also show the following result for Fekete-Szegő problem.

**Theorem 4.4.** Let \( \delta, \eta \in \mathbb{R} \) and \( \phi \in \Lambda \) satisfying

$$\phi(z) = 1 + \sum_{n=1}^{\infty} A_n z^n, \quad (A_1, A_2 > 0, z \in \Delta).$$
If \( f(z) \in A_p \) belongs to the class \( \mathcal{NS}^n_{p,q} (\mu; \phi) \), then

\[
|a_{p+2} - \delta a_{p+1}^2| \leq \begin{cases}
\frac{|\phi|_q}{L_2(\mu[p]_q + |p+2| - |p|_q)} \left( A_2 - \frac{A_1^2[p]_{a \Psi}}{2L_1^2(\mu[p]_q + |p+1| - |p|_q)^2} \right), & (\delta \leq \Pi_1); \\
\frac{|\phi|_q}{L_2(\mu[p]_q + |p+2| - |p|_q)} \left( -A_2 + \frac{A_1^2[p]_{a \Psi}}{2L_1^2(\mu[p]_q + |p+1| - |p|_q)^2} \right), & (\delta \geq \Pi_2),
\end{cases}
\]

where

\[
\Pi_1 = \frac{2(A_2 - A_1)L_1^2(\mu[p]_q + |p+1| - |p|_q)^2 - A_1^2(\mu - 1)L_2^2[p]_{q}[\mu[p]_q + 2(|p+1| - |p|_q)]}{2L_2A_1^2[p]_{q}[\mu[p]_q + |p+2| - |p|_q]}
\]

and

\[
\Pi_2 = \frac{2(A_2 + A_1)L_1^2(\mu[p]_q + |p+1| - |p|_q)^2 - A_1^2(\mu - 1)L_2^2[p]_{q}[\mu[p]_q + 2(|p+1| - |p|_q)]}{2L_2A_1^2[p]_{q}[\mu[p]_q + |p+2| - |p|_q]}
\]

Moreover, we put

\[
\Pi_1 = \frac{2A_2L_1^2(\mu[p]_q + |p+1| - |p|_q)^2 - A_1^2(\mu - 1)L_2^2[p]_{q}[\mu[p]_q + 2(|p+1| - |p|_q)]}{2L_2A_1^2[p]_{q}[\mu[p]_q + |p+2| - |p|_q]}
\]

Then, each of the following results is true:

(A) For \( \delta \in [\Pi_1, \Pi_3] \),

\[
|a_{p+2} - \delta a_{p+1}^2| + \frac{2(A_1 - A_2)L_1^2(\mu[p]_q + |p+1| - |p|_q)^2 + A_1^2[p]_{q} \Psi |a_{p+1}|^2}{2L_2A_1^2[p]_{q}[\mu[p]_q + (|p+2| - |p|_q)]} \leq \frac{A_1[p]_{q}}{L_2(\mu[p]_q + |p+2| - |p|_q)};
\]

(B) For \( \delta \in [\Pi_3, \Pi_2] \),

\[
|a_{p+2} - \delta a_{p+1}^2| + \frac{2(A_1 + A_2)L_1^2(\mu[p]_q + |p+1| - |p|_q)^2 - A_1^2[p]_{q} \Psi |a_{p+1}|^2}{2L_2A_1^2[p]_{q}[\mu[p]_q + (|p+2| - |p|_q)]} \leq \frac{A_1[p]_{q}}{L_2(\mu[p]_q + |p+2| - |p|_q)};
\]

where

\[
\Psi = 2\delta L_2[p]_{q}(\mu[p]_q + |p+2| - |p|_q) + (\mu - 1)|\mu[p]_q + 2(|p+1| - |p|_q)]L_1^2.
\]

**Remark 4.5.** Similarly, by choose the parameter \( p = 1 \) in Theorems 4.1 and 4.4, we can provide the new results for the univalent function classes \( \mathcal{NS}^n_{1,q} (\mu; \phi) = \mathcal{NS}^n_q (\mu; \phi) \). As Remark 1.4, we may consider \( \mathcal{NS}^n_{p,q} (\mu; \alpha) \) or \( \mathcal{NS}^n_{p,q} (\mu; \beta) \) to establish latest results. Besides, for the fixed parameter \( \mu \), we can still infer new results for \( \mathcal{NS}^n_{p,q} (\mu; \phi) \).
5. Conclusion

By involving a generalized Bernardi integral operator, several new subclasses of $q$-starlike and $q$-convex type analytic and multivalent functions are introduced to generalize the classical starlike and convex functions. Meanwhile, for these classes we may know integral operator and $q$-derivative as well as multivalency how to change the coefficients of functions. In our main results, we establish the Fekete-Szegő type functional inequalities for these function classes. Further, the corresponding bound estimates of the coefficients $a_{p+1}$ and $a_{p+2}$ are interpreted. In fact, if we use the other integral operators or take $(p, q)$-operator when certain function is univalent but not multivalent, we may get many similar results as in this article.

Acknowledgments

We thank the referees for their careful readings and using comments so that this manuscript is greatly improved. This work is supported by Institution of Higher Education Scientific Research Project in Ningxia of the People's Republic of China under Grant NGY2017011, Natural Science Foundation of Ningxia of the People’s Republic of China under Grant 2020AAC03066, the Program for Young Talents of Science and Technology in Universities of Inner Mongolia Autonomous Region under Grant NJYT-18-A14, the Natural Science Foundation of Inner Mongolia of the People’s Republic of China under Grant 2018MS01026, the Natural Science Foundation of the People’s Republic of China under Grant 11561001, 42064004 and 11762016.

Conflict of interest

The authors declare no conflict of interest.

References


