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Research article

Fault-tolerant edge metric dimension of certain families of graphs

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Abstract: Let $W_E = \{w_1, w_2, \ldots, w_k\}$ be an ordered set of vertices of graph G and let e be an edge of G. Suppose $d(x, e)$ denotes distance between edge e and vertex x of G, defined as $d(e, x) = d(x, e) = \min\{d(x, a), d(x, b)\}$, where $e = ab$. A vertex x distinguishes two edges e_1 and e_2, if $d(e_1, x) \neq d(e_2, x)$. The representation $r(e \mid W_E)$ of e with respect to W_E is the k-tuple $(d(e, w_1), d(e, w_2), \ldots, d(e, w_k))$. If distinct edges of G have distinct representation with respect to W_E, then W_E is called edge metric generator for G. An edge metric generator of minimum cardinality is an edge metric basis for G, and its cardinality is called edge metric dimension of G, denoted by edim(G).
In this paper, we initiate the study of fault-tolerant edge metric dimension. Let $\acute{W}_E$ be edge metric generator of graph G, then $\acute{W}_E$ is called fault-tolerant edge metric generator of G if $\acute{W}_E \setminus \{v\}$ is also an edge metric generator of graph G for every $v \in \acute{W}_E$. A fault-tolerant edge metric generator of minimum cardinality is a fault-tolerant edge metric basis for graph G, and its cardinality is called fault-tolerant edge metric dimension of G. We also computed the fault-tolerant edge metric dimension of path, cycle, complete graph, cycle with chord graph, tadpole graph and kayak paddle graph.

Keywords: fault-tolerant edge metric dimension; edge metric generator; cycle with chord graphs; tadpole graphs; kayak paddle graphs
Mathematics Subject Classification: 68R01, 68R05, 68R10

1. Introduction and preliminaries

Suppose that G is connected, simple and undirected graph having edge set $E(G)$ and vertex set $V(G)$, respectively. The order of graph G is $|V(G)|$ and size of graph G is $|E(G)|$. Moreover, $\Delta(G)$ and $\delta(G)$ represent the maximum and minimum degree of graph G respectively. Let $W = \{v_1, v_2, \ldots, v_k\}$ be an ordered set of $V(G)$ and let u be a vertex of G. The representation $r(u \mid W)$ of u with respect to W is the k-tuple $(d(u, v_1), d(u, v_2), \ldots, d(u, v_k))$. If distinct vertices of G have distinct representation with respect to W, then W is called metric generator for G. A metric generator of minimum cardinality is metric basis for G, and its cardinality is called metric dimension of G, denoted by $\dim(G)$ (see [1]). A metric generator $\acute{W}$ for G is called fault-tolerant metric generator if $\acute{W} \setminus \{v\}$ is also a metric generator, for each $v \in \acute{W}$. The fault-tolerant metric dimension of G is the minimum cardinality of this set $\acute{W}$ and is denoted by $fdim(G)$ (see [2]).

Let $d(x, e)$ denotes distance between edge e and vertex x, defined as $d(x, e) = \min\{d(x, a), d(x, b)\}$, where $e = ab$ (see [3]). A vertex x distinguishes two edges e_1 and e_2, if $d(e_1, x) \neq d(e_2, x)$. Let $W_E = \{w_1, w_2, \ldots, w_k\}$ be an ordered set of vertices of G and let e be an edge of G. The representation $r(e \mid W_E)$ of e with respect to W_E is the k-tuple $(d(e, w_1), d(e, w_2), \ldots, d(e, w_k))$. If distinct edges of G have distinct representation with respect to W_E, then W_E is called edge metric generator for G (see [3]). An edge metric generator of minimum cardinality is an edge metric basis for G, and its cardinality is called edge metric dimension of G, denoted by $\mathrm{edim}(G)$ [4–7].

Slater proposed the idea of metric dimension to find the location of intruder in a network (see [1,8]). The proposed idea was further extended by Melter and Harary in [9]. Metric dimension is important in robot navigation, chemistry, problems of image processing and pattern recognition etc. (see [10–15]). The use of metric dimension of graphs was also observed in games like mastermind and coin weighing (see [16]).

Kelenc in [3] extended the idea of metric dimension to edge metric dimension and make a comparison between them. He also discussed some useful results for paths P_n, cycles C_n, complete graphs K_n and wheel graphs. In [8], Zubrilina classified the graphs on n vertices for which edge metric dimension is $n - 1$. In [17], Kratica computed the edge metric dimension of generalized petersen graphs $GP(n, k)$ for $k = 1$ and 2 while for the other values of k the lower bound is given. In [18], Ahsan computed the edge metric dimension of convex polytopes related graphs [19–21].

In 2008, Hernando, Slater, Mora and Wood introduced the new idea of fault-tolerant metric dimension in [2]. Further in 2017, Voronov calculated the fault-tolerant metric dimension of the king's graph (see in [22]). In 2018, Raza et al. computed the fault-tolerant metric dimension of generalized convex polytopes [23]. Recently in 2019, Liu, Munir, Ali, Hussain and Ahmed have computed the fault-tolerant metric dimension of wheel related graphs like gear graphs [24]. Basak has computed the fault-tolerant metric dimension of circulant graphs [25].

A framework where failure of any single unit, another chain of units not containing the defective unit can substitute the initially utilized chain is called fault-tolerant self-stable framework. These graphs can tolerate the failure of one part (vertex) keeping consistent execution (see [24,26]). For this purpose we propose the concept of fault-tolerant edge metric dimension. Let $\acute{W}_E$ be edge metric generator of graph G, then $\acute{W}_E$ is called fault-tolerant edge metric generator of G if $\acute{W}_E \setminus \{v\}$ is also an edge metric generator of graph G for each $v \in \acute{W}_E$. A fault-tolerant edge metric generator of minimum cardinality is a fault-tolerant edge metric basis for graph G, and its cardinality is called fault-tolerant edge metric dimension of G, we are denoting it by $fedim(G)$ [27,28]. In this concept, we will extend the work of edge metric dimension to fault-tolerant edge metric dimension.

The lemmas given below are very helpful for calculating the fault-tolerant edge metric dimension of graphs:

Lemma 1.1. *[3] For any $n \geq 2$, $edim(P_n) = \dim(P_n) = 1$, $edim(C_n) = \dim(C_n) = 2$, $edim(K_n) = \dim(K_n) = n - 1$. Moreover, $edim(G) = 1$ if and only if G is path.*

Lemma 1.2. *[3] For a connected graph G, $edim(G) \geq log_2(\triangle(G))$.*

Lemma 1.3. *[3] For a connected graph G of order n,* $edim(G) \geq 1 + \lceil \log_2 \delta(G) \rceil$.

From the definition of fault-tolerant edge metric dimension, it can be seen that

Lemma 1.4. *For a connected graph G,*

1. $fedim(G) \geq 1 + edim(G)$.
2. $2 \leq fedim(G) \leq n$.

The rest of paper is structured as follows: In the second section, we will study the fault-tolerant edge metric dimension of family of path, cycle and complete graphs. In third section, we will investigate the fault-tolerant edge metric dimension of family of cycle with chord graphs C_n^m. In fourth section, fault-tolerant edge metric dimension of family of tadpole graphs G_n^l will be determined. In last section, we will compute the fault-tolerant edge metric dimension of family of kayak paddle graphs $G_{n,m}^l$.

2. Fault-tolerant edge metric dimension of family of path, cycle and complete graphs

In this section, we will investigate the fault-tolerant edge metric dimension of family of paths, cycles and complete graphs. The family P_n have $V(P_n) = \{u_1, u_2, \ldots, u_n\}$ and $E(P_n) = \{u_iu_{i+1} : 1 \leq i \leq n-1\}$. The family P_n for $n = 10$ is shown in Figure 1. The following theorem tells us the edge metric dimension of P_n.

Theorem 2.1. *[3] For any integer* $n \geq 2$, $edim(P_n) = 1$.

Figure 1. Path graph P_{10}.

Now, we will compute the fault-tolerant edge metric dimension of P_n.

Theorem 2.2. *For any integer* $n \geq 2$, $fedim(P_n) = 2$.

Proof. In order to compute fault-tolerant edge metric dimension of P_n, we have $\acute{W}_E = \{u_1, u_n\} \subset V(P_n)$, we have to show that $\acute{W}_E$ is a fault-tolerant edge metric generator of P_n. For this, we give representations of each edge of P_n.

$r(u_iu_{i+1}|\acute{W}_E) = (i-1, n-i-1)$, where $1 \leq i \leq n-1$.

We see that there are no two tuples having the same representations. This shows that fault-tolerant edge metric dimension of P_n is less than or equal to 2. Since by Lemma 1.4, P_n has fault-tolerant edge metric dimension greater than or equal to 2. Hence fault-tolerant edge metric dimension is equal to 2. □

The family C_n have $V(C_n) = \{u_1, u_2, \ldots, u_n\}$ and $E(C_n) = \{u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{u_nu_1\}$. The family C_n for $n = 15$ is shown in Figure 2. The following theorem tells us the edge metric dimension of C_n.

Theorem 2.3. *[3] For any integer $n \geq 3$, $edim(C_n) = 2$.*

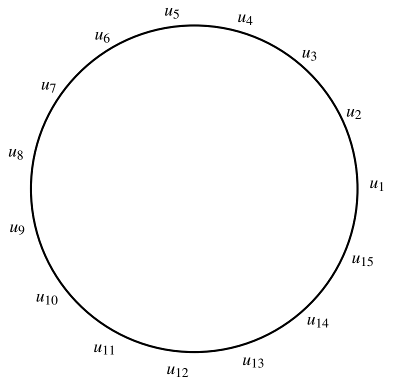

Figure 2. Cycle graph C_{15}.

Now, we will compute the fault-tolerant edge metric dimension of C_n.

Theorem 2.4. *For any integer $n \geq 3$, $fedim(C_n) = 3$.*

Proof. In order to compute fault-tolerant edge metric dimension of C_n, we have the following cases.
Case (i). n is odd. Take $\acute{W}_E = \{u_1, u_2, u_3\} \subset V(C_n)$, we have to show that $\acute{W}_E$ is a fault-tolerant edge metric generator of C_n. For this, we give representations of each edge of C_n.

$$r(u_i u_{i+1}|\acute{W}_E) = \begin{cases} (0,0,1), & \text{if } i = 1; \\ (1,0,0), & \text{if } i = 2; \\ (i-1, i-2, i-3), & \text{if } 3 \leq i \leq \frac{n+1}{2}; \\ (\frac{n-3}{2}, \frac{n-1}{2}, \frac{n-3}{2}), & \text{if } i = \frac{n+1}{2} + 1; \\ (n-i, n-i+1, n-i+2), & \text{if } \frac{n+1}{2} + 2 \leq i \leq n-1; \end{cases}$$

$r(u_n u_1|\acute{W}_E) = (0,1,2)$.

Case (ii). n is even. Take $\acute{W}_E = \{u_1, u_2, u_3\} \subset V(C_n)$, we have to show that $\acute{W}_E$ is a fault-tolerant edge metric generator of C_n. For this, we give representations of each edge of C_n.

$$r(u_i u_{i+1}|\acute{W}_E) = \begin{cases} (0,0,1), & \text{if } i = 1; \\ (1,0,0), & \text{if } i = 2; \\ (i-1, i-2, i-3), & \text{if } 3 \leq i \leq \frac{n}{2}; \\ (\frac{n-2}{2}, \frac{n-2}{2}, \frac{n-4}{2}), & \text{if } i = \frac{n}{2} + 1; \\ (\frac{n-4}{2}, \frac{n-2}{2}, \frac{n-2}{2}), & \text{if } i = \frac{n}{2} + 2; \\ (n-i, n-i+1, n-i+2), & \text{if } \frac{n}{2} + 3 \leq i \leq n-1; \end{cases}$$

$r(u_n u_1|\acute{W}_E) = (0,1,2)$.

We see that there are no two tuples having the same representations. This shows that fault-tolerant edge metric dimension of C_n is less than or equal to 3. Since by Lemma 1.4, C_n has fault-tolerant edge metric dimension greater than or equal to 3. Hence fault-tolerant edge metric dimension of C_n is equal to 3. □

Theorem 2.5. *For any integer $n \geq 2$, $fedim(K_n) = n$.*

Proof. The proof is straight forward from Lemma 1.1 and Lemma 1.4.

□

3. Fault-tolerant edge metric dimension of family of cycle with chord graphs C_n^m

In this section, we will investigate the fault-tolerant edge metric dimension of family of cycle with chord graphs C_n^m. The family C_n^m have $V(C_n^m) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n^m) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_n v_1, v_1 v_m\}$. It suffices to consider $2 < m \leq \lfloor \frac{n}{2} \rfloor$. The family C_n^m for $n = 20$ and $m = 9$ is shown in Figure 3. The following theorem tells us the edge metric dimension of C_n^m.

Theorem 3.1. *[29] For all $n \geq 4$, $edim(C_n^m) = 2$.*

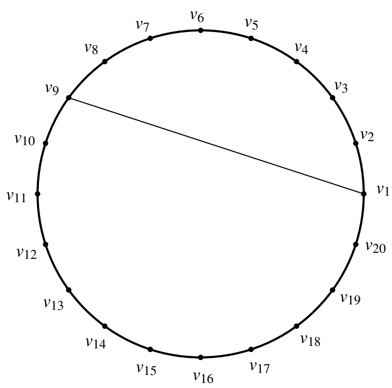

Figure 3. Cycle with Chord graph C_{20}^9.

Now, we will compute the fault-tolerant edge metric dimension of C_n^m.

Theorem 3.2. *For all $n \geq 4$, $fedim(C_n^m) = 3$.*

Proof. In order to compute fault-tolerant edge metric dimension of C_n^m, we have the following cases.
Case (i). Both n and m are even. Let $\acute{W}_E = \{v_2, v_{\frac{m}{2}+1}, v_{m+1}\} \subset V(C_n^m)$, we have to show that $\acute{W}_E$ is a fault-tolerant edge metric generator of C_n^m. For this, we give representations of each edge of C_n^m.

$$r(v_iv_{i+1}|\acute{W}_E) = \begin{cases} (0, \frac{m}{2}-1, 2), & \text{if } i = 1; \\ (i-2, \frac{m}{2}-i, i+1), & \text{if } 2 \le i \le \frac{m}{2}-1; \\ (\frac{m}{2}-2, 0, \frac{m}{2}), & \text{if } i = \frac{m}{2}; \\ (\frac{m}{2}-1, 0, \frac{m}{2}-1), & \text{if } i = \frac{m}{2}+1; \\ (m-i+1, i-\frac{m}{2}-1, m-i), & \text{if } \frac{m}{2}+2 \le i \le m-1; \\ (2, \frac{m}{2}-1, 0), & \text{if } i = m; \\ (i-m+2, i-\frac{m}{2}-1, i-m-1), & \text{if } m+1 \le i \le \frac{n}{2}+\frac{m}{2}-1; \\ (\frac{n}{2}-\frac{m}{2}+1, \frac{n}{2}-1, \frac{n}{2}-\frac{m}{2}-1), & \text{if } i = \frac{n}{2}+\frac{m}{2}; \\ (\frac{n}{2}-\frac{m}{2}, \frac{n}{2}-1, \frac{n}{2}-\frac{m}{2}), & \text{if } i = \frac{n}{2}+\frac{m}{2}+1; \\ (n-i+1, n+\frac{m}{2}-i, n-i+2), & \text{if } \frac{n}{2}+\frac{m}{2}+2 \le i \le n; \end{cases}$$

$r(v_nv_1|\acute{W}_E) = (1, \frac{m}{2}, 2)$ and $r(v_1v_m|\acute{W}_E) = (1, \frac{m}{2}-1, 1)$.

Case (ii). n is odd and m is even. Let $\acute{W}_E = \{v_2, v_{\frac{m}{2}+1}, v_{m+1}\} \subset V(C_n^m)$, we have to show that $\acute{W}_E$ is a fault-tolerant edge metric generator of C_n^m. For this, we give representations of each edge of C_n^m.

$$r(v_iv_{i+1}|\acute{W}_E) = \begin{cases} (0, \frac{m}{2}-1, 2), & \text{if } i = 1; \\ (i-2, \frac{m}{2}-i, i+1), & \text{if } 2 \le i \le \frac{m}{2}-1; \\ (\frac{m}{2}-2, 0, \frac{m}{2}), & \text{if } i = \frac{m}{2}; \\ (\frac{m}{2}-1, 0, \frac{m}{2}-1), & \text{if } i = \frac{m}{2}+1; \\ (m-i+1, i-\frac{m}{2}-1, m-i), & \text{if } \frac{m}{2}+2 \le i \le m-1; \\ (2, \frac{m}{2}-1, 0), & \text{if } i = m; \\ (i-m+2, i-\frac{m}{2}-1, i-m-1), & \text{if } m+1 \le i \le \frac{n-1}{2}+\frac{m}{2}; \\ (\frac{n+1}{2}-\frac{m}{2}, \frac{n-1}{2}, \frac{n-1}{2}-\frac{m}{2}), & \text{if } i = \frac{n-1}{2}+\frac{m}{2}+1; \\ (n-i+1, n+\frac{m}{2}-i, n-i+2), & \text{if } \frac{n-1}{2}+\frac{m}{2}+2 \le i \le n-1; \end{cases}$$

$r(v_nv_1|\acute{W}_E) = (1, \frac{m}{2}, 2)$ and $r(v_1v_m|\acute{W}_E) = (1, \frac{m}{2}-1, 1)$.

Case (iii). n is even and m is odd. Let $\acute{W}_E = \{v_2, v_{\frac{m+1}{2}+1}, v_{\frac{n}{2}+\frac{m+1}{2}}\} \subset V(C_n^m)$, we have to show that $\acute{W}_E$ is a fault-tolerant edge metric generator of C_n^m. For this, we give representations of each edge of C_n^m.

$$r(v_iv_{i+1}|\acute{W}_E) = \begin{cases} (0, \frac{m-1}{2}, \frac{n}{2} - \frac{m-1}{2}), & \text{if } i = 1; \\ (i-2, \frac{m+1}{2} - i, m+i-2), & \text{if } 2 \le i \le \frac{m-1}{2}; \\ (\frac{m-1}{2} - 1, 0, \frac{n}{2} - 1), & \text{if } i = \frac{m+1}{2}; \\ (m-i+1, i - \frac{m+1}{2} - 1, \frac{n}{2} + \frac{m-1}{2} - i), & \text{if } \frac{m+1}{2} + 1 \le i \le m-1; \\ (i-m+2, i - \frac{m+1}{2} - 1, \frac{n}{2} + \frac{m-1}{2} - i), & \text{if } m \le i \le \frac{n}{2} + \frac{m-1}{2}; \\ (n-i+1, n + \frac{m-1}{2} - i, \\ i+m-n-2), & \text{if } \frac{n}{2} + \frac{m-1}{2} + 1 \le i \le n-1; \end{cases}$$

$r(v_nv_1|\acute{W}_E) = (1, \frac{m-1}{2}, m-2)$ and $r(v_1v_m|\acute{W}_E) = (1, \frac{m-1}{2} - 1, m-1)$.

Case (iv). Both n and m are odd. Let $\acute{W}_E = \{v_{\frac{m+1}{2}}, v_{m+1}, v_n\} \subset V(C_n^m)$, we have to show that $\acute{W}_E$ is a fault-tolerant edge metric generator of C_n^m. For this, we give representations of each edge of C_n^m.

$$r(v_iv_{i+1}|\acute{W}_E) = \begin{cases} (\frac{m-1}{2} - i, i+1, i), & \text{if } 1 \le i \le \frac{m-1}{2} \\ (i - \frac{m+1}{2}, m-i, m-i+1), & \text{if } \frac{m+1}{2} \le i \le m-1; \\ (\frac{m-1}{2}, 0, 2), & \text{if } i = m; \\ (i - \frac{m+1}{2}, i-m-1, \\ i-m+2), & \text{if } m+1 \le i \le \frac{n-1}{2} + \frac{m-1}{2} - 1; \\ (i - \frac{m+1}{2}, i-m-1, \\ n-i-1), & \text{if } \frac{n-1}{2} + \frac{m-1}{2} \le i \le \frac{n-1}{2} + \frac{m+1}{2}; \\ (\frac{n-3}{2}, \frac{n+1}{2} - \frac{m+1}{2}, \frac{n-1}{2} - \frac{m+1}{2} - 1), & \text{if } i = \frac{n+1}{2} + \frac{m+1}{2}; \\ (n + \frac{m-1}{2} - i, n-i+2, n-i-1), & \text{if } \frac{n+1}{2} + \frac{m+1}{2} \le i \le n-1; \end{cases}$$

$r(v_nv_1|\acute{W}_E) = (\frac{m-1}{2}, 2, 0)$ and $r(v_1v_m|\acute{W}_E) = (\frac{m-1}{2}, 1, 1)$.

We see that there are no two tuples having the same representations in all the four cases. This shows that fault-tolerant edge metric dimension of C_n^m is less than or equal to 3. Since by Lemma 1.4, C_n^m is not a path so fault-tolerant edge metric dimension of C_n^m is greater than or equal to 3. Hence fault-tolerant edge metric dimension of C_n^m is 3.

□

4. Fault-tolerant edge metric dimension of family of tadpole graphs G_n^l

In this section, we will compute the fault-tolerant edge metric dimension of family of tadpole graphs G_n^l. The family G_n^l have $V(G_n^l) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_l\}$ and $E(G_n^l) = \{v_iv_{i+1} : 1 \le i \le n-1\} \cup \{u_su_{s+1}, : 1 \le s \le l-1\} \cup \{v_nu_1, u_1v_1\}$. The graph G_n^l for $n = 9$ and $l = 7$ is shown in Figure 4. The following theorem tells us the edge metric dimension of G_n^l.

Theorem 4.1. *[29] For all $n \ge 2$, $l \ge 3$, $edim(G_n^l) = 2$.*

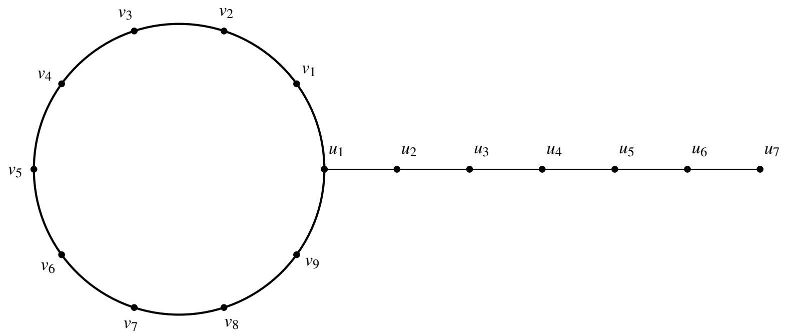

Figure 4. Tadpole graph G_9^7.

Now, we will compute the fault-tolerant edge metric dimension of G_n^l.

Theorem 4.2. *For all $n \geq 2$, $l \geq 3$, $fedim(G_n^l) = 3$.*

Proof. In order to compute fault-tolerant edge metric dimension of G_n^l, we have the following cases.
Case (i). n is odd. Let $\acute{W}_E = \{v_1, v_n, u_m\} \subset V(G_n^l)$, we have to show that $\acute{W}_E$ is a fault-tolerant edge metric generator of G_n^l. For this, we give representations of each edge of G_n^l.

$$r(v_iv_{i+1}|\acute{W}_E) = \begin{cases} (i-1, i+1, i+\frac{n-1}{2}), & \text{if } 1 \leq i \leq \frac{n-1}{2}-1; \\ (\frac{n-1}{2}-1, \frac{n-1}{2}, \frac{n-1}{2}+m-1), & \text{if } i = \frac{n-1}{2}; \\ (\frac{n-1}{2}, \frac{n-1}{2}-1, \frac{n-1}{2}+m-1), & \text{if } i = \frac{n-1}{2}+1; \\ (n-i+1, n-i-1, n+m-i-1), & \text{if } \frac{n-1}{2}+2 \leq i \leq n-1; \end{cases}$$

$r(u_iu_{i+1}|\acute{W}_E) = (i, i, m-i-1)$ where $1 \leq i \leq m-1$,
$r(v_nu_1|\acute{W}_E) = (1, 0, m-1)$ and $r(u_1v_1|\acute{W}_E) = (0, 1, m-1)$.

Case (ii). n is even. Let $\acute{W}_E = \{v_1, v_n, u_m\} \subset V(G_n^l)$, we have to show that $\acute{W}_E$ is a fault-tolerant edge metric generator of G_n^l. For this, we give representations of each edge of G_n^l.

$$r(v_iv_{i+1}|\acute{W}_E) = \begin{cases} (i-1, i+1, i+\frac{n}{2}), & \text{if } 1 \leq i \leq \frac{n}{2}-1; \\ (\frac{n}{2}-1, \frac{n}{2}-1, \frac{n}{2}+m-1), & \text{if } i = \frac{n}{2}; \\ (n-i+1, n-i-1, n+m-i-1), & \text{if } \frac{n}{2}+1 \leq i \leq n-1; \end{cases}$$

$r(u_iu_{i+1}|\acute{W}_E) = (i, i, m-i-1)$ where $1 \leq i \leq m-1$,
$r(v_nu_1|\acute{W}_E) = (1, 0, m-1)$ and $r(u_1v_1|\acute{W}_E) = (0, 1, m-1)$.

We see that there are no two tuples having the same representations. This shows that fault-tolerant edge metric dimension of G_n^l is less than or equal to 3 and now we try to show that fault-tolerant edge metric dimension of G_n^l is greater than or equal to 3. Since by Lemma 1.4, G_n^l is not a path so fault-tolerant edge metric dimension of G_n^l is greater than or equal to 3. Hence fault-tolerant edge metric dimension of G_n^l is equal to 3.

□

5. Fault-tolerant edge metric dimension of family of kayak paddle graphs $G^l_{n,m}$

In this section, we will compute the edge metric dimension of family of kayak paddle graphs $G^l_{n,m}$. The family $G^l_{n,m}$ have $V(G^l_{n,m}) = \{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_l\}$ and $E(G^l_{n,m}) = \{v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{w_jw_{j+1} : 1 \leq j \leq l-1\} \cup \{u_su_{s+1} : 1 \leq s \leq m-1\} \cup \{v_nw_1, w_1v_1, w_lu_1, u_mw_l\}$. The family $G^l_{n,m}$ for $n = 8$, $m = 5$ and $l = 4$ is shown in Figure 5. The following theorem tells us the edge metric dimension of $G^l_{n,m}$.

Theorem 5.1. *[29] For every $n \geq 2$, $m \geq 2$ and $l \geq 4$, $edim(G^l_{n,m}) = 2$.*

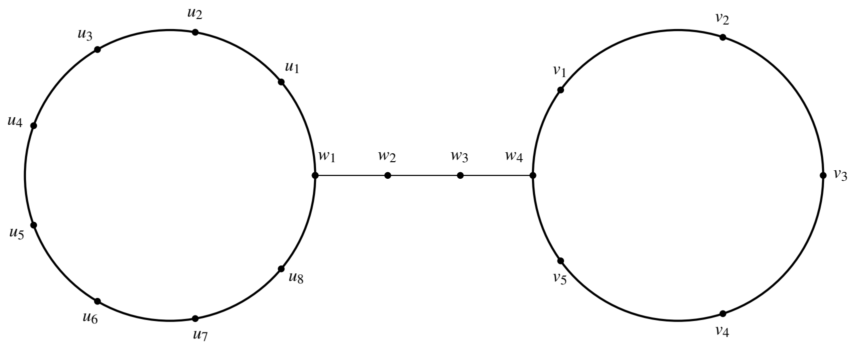

Figure 5. Kayak Paddle graph $G^4_{8,5}$.

Now, we will compute the fault-tolerant edge metric dimension of $G^l_{n,m}$.

Theorem 5.2. *For $n \geq 2$, $m \geq 2$ and $l \geq 4$, $fedim(G^l_{n,m}) = 4$.*

Proof. In order to compute fault-tolerant edge metric dimension of $G^l_{n,m}$, we have the following cases.
Case (i). n is odd and m is even. Let $\acute{W}_E = \{v_1, v_2, u_1, u_2\} \subset V(G^l_{n,m})$, we have to show that $\acute{W}_E$ is a fault-tolerant edge metric generator of $G^l_{n,m}$. For this, we give representations of each edge of $G^l_{n,m}$.

$$r(v_iv_{i+1}|\acute{W}_E) = \begin{cases} (0, 0, l+1, l+2), & \text{if } i = 1; \\ (i-1, i-2, , l+i, l+i+1), & \text{if } 2 \leq i \leq \frac{n-1}{2}; \\ (\frac{n-1}{2}, \frac{n-1}{2}-1, \frac{n-1}{2}+l, \frac{n-1}{2}+l+1), & \text{if } i = \frac{n+1}{2}; \\ (\frac{n-1}{2}, \frac{n-1}{2}, \frac{n-1}{2}+l-1, \frac{n-1}{2}+l), & \text{if } i = \frac{n+1}{2}+1; \\ (n-i+1, n-i+2, n+l-i, & \\ n+l-i+1), & \text{if } \frac{n+1}{2}+2 \leq i \leq n-1; \end{cases}$$

$r(w_iw_{i+1}|\acute{W}_E) = (i, i+1, l-i, l-i+1)$ where $1 \leq i \leq l-1$,

$$r(u_iu_{i+1}|\acute{W}_E) = \begin{cases} (l+1, l+2, 0, 0), & \text{if } i = 1; \\ (l+i, l+i+1, i-1, i-2), & \text{if } 2 \leq i \leq \frac{m}{2}; \\ (l+\frac{m}{2}-1, l+\frac{m}{2}, \frac{m}{2}, \frac{m}{2}-1), & \text{if } i = \frac{m}{2}+1; \\ (m+l-i, m+l-i+1, m-i+1, & \\ m-i+2), & \text{if } \frac{m}{2}+2 \leq i \leq m-1; \end{cases}$$

$r(v_nw_1|\acute{W}_E) = (1, 2, l, l+1)$, $r(w_1v_1|\acute{W}_E) = (0, 1, l, l+1)$, $r(w_lu_1|\acute{W}_E) = (l, l+1, 0, 1)$ and $r(u_mw_l|\acute{W}_E) = (l, l+1, 1, 2)$.

Case (ii). Both n and m are even. Let $\acute{W}_E = \{v_1, v_2, u_1, u_2\} \subset V(G^l_{n,m})$, we have to show that $\acute{W}_E$ is a fault-tolerant edge metric generator of $G^l_{n,m}$. For this, we give representations of each edge of $G^l_{n,m}$.

$$r(v_iv_{i+1}|\acute{W}_E) = \begin{cases} (0, 0, l+1, l+2), & \text{if } i = 1; \\ (i-1, i-2, , l+i, l+i+1), & \text{if } 2 \le i \le \frac{n}{2}; \\ (\frac{n}{2}, \frac{n}{2}-1, \frac{n}{2}+l-1, \frac{n}{2}+l), & \text{if } i = \frac{n}{2}+1; \\ (\frac{n}{2}-1, \frac{n}{2}, \frac{n}{2}+l-2, \frac{n}{2}+l-1), & \text{if } i = \frac{n}{2}+2; \\ (n-i+1, n-i+2, n+l-i, \\ n+l-i+1), & \text{if } \frac{n}{2}+3 \le i \le n-1; \end{cases}$$

$r(w_iw_{i+1}|\acute{W}_E) = (i, i+1, l-i, l-i+1)$ where $1 \le i \le l-1$,

$$r(u_iu_{i+1}|\acute{W}_E) = \begin{cases} (l+1, l+2, 0, 0), & \text{if } i = 1; \\ (l+i, l+i+1, i-1, i-2), & \text{if } 2 \le i \le \frac{m}{2}; \\ (l+\frac{m}{2}-1, l+\frac{m}{2}, \frac{m}{2}, \frac{m}{2}-1), & \text{if } i = \frac{m}{2}+1; \\ (m+l-i, m+l-i+1, m-i+1, \\ m-i+2), & \text{if } \frac{m}{2}+2 \le i \le m-1; \end{cases}$$

$r(v_nw_1|\acute{W}_E) = (1, 2, l, l+1)$, $r(w_1v_1|\acute{W}_E) = (0, 1, l, l+1)$, $r(w_lu_1|\acute{W}_E) = (l, l+1, 0, 1)$ and $r(u_mw_l|\acute{W}_E) = (l, l+1, 1, 2)$.

Case (iii). Both n and m are odd. Let $\acute{W}_E = \{v_1, v_2, u_1, u_2\} \subset V(G^l_{n,m})$, we have to show that $\acute{W}_E$ is a fault-tolerant edge metric generator of $G^l_{n,m}$. For this, we give representations of each edge of $G^l_{n,m}$.

$$r(v_iv_{i+1}|\acute{W}_E) = \begin{cases} (0, 0, l+1, l+2), & \text{if } i = 1; \\ (i-1, i-2, , l+i, l+i+1), & \text{if } 2 \le i \le \frac{n-1}{2}; \\ (\frac{n-1}{2}, \frac{n-1}{2}-1, \frac{n-1}{2}+l, \frac{n+1}{2}+l), & \text{if } i = \frac{n+1}{2}; \\ (\frac{n-1}{2}, \frac{n-1}{2}, \frac{n-1}{2}+l-1, \frac{n-1}{2}+l), & \text{if } i = \frac{n+1}{2}+1; \\ (n-i+1, n-i+2, n+l-i, \\ n+l-i+1), & \text{if } \frac{n+1}{2}+2 \le i \le n-1; \end{cases}$$

$r(w_iw_{i+1}|\acute{W}_E) = (i, i+1, l-i, l-i+1)$ where $1 \le i \le l-1$,

$$r(u_iu_{i+1}|\acute{W}_E) = \begin{cases} (l+1, l+2, 0, 0), & \text{if } i = 1; \\ (l+i, l+i+1, i-1, i-2), & \text{if } 2 \le i \le \frac{m-1}{2}; \\ (l+\frac{m-1}{2}, l+\frac{m+1}{2}, \frac{m-1}{2}, \frac{m-1}{2}-1), & \text{if } i = \frac{m+1}{2}; \\ (l+\frac{m-1}{2}-1, l+\frac{m-1}{2}, \frac{m-1}{2}, \frac{m-1}{2}), & \text{if } i = \frac{m+1}{2}+1; \\ (m+l-i, m+l-i+1, m-i+1, \\ m-i+2), & \text{if } \frac{m+1}{2}+2 \le i \le m-1; \end{cases}$$

$r(v_nw_1|\acute{W}_E) = (1, 2, l, l+1)$, $r(w_1v_1|\acute{W}_E) = (0, 1, l, l+1)$, $r(w_lu_1|\acute{W}_E) = (l, l+1, 0, 1)$ and $r(u_mw_l|\acute{W}_E) = (l, l+1, 1, 2)$.

We see that there are no two tuples having the same representations. This shows that fault-tolerant edge metric dimension of $G^l_{n,m}$ is less than or equal to 4 and now we try to show that fault-tolerant edge metric dimension of $G^l_{n,m}$ is grater than or equal to 4.

For this purpose, we have to show that there is no fault-tolerant edge metric generator having cardinality 3, we suppose on contrary that fault-tolerant edge metric dimension of $G^l_{n,m}$ is 3 and let $\acute{W}_E = \{v_i, v_j, v_k\}$. Then the Table 1 shows all order pairs of edges (e, f) for which $r(e|\acute{W}_E) = r(f|\acute{W}_E)$.

Table 1. (e, f) for which $r(e|\acute{W}_E) = r(f|\acute{W}_E)$.

Conditions on i, j and k	(e, f)
$1 \le i, j, k \le n$	(u_1w_l, u_mw_l)
$1 \le i, j \le n, 1 \le k \le l$	(u_1w_l, u_mw_l)
$1 \le i \le n$ and $1 \le j, k \le l$	(u_1w_l, u_mw_l)
$1 \le i, j, k \le l$	(u_1w_l, u_mw_l)
$1 \le i \le n, 1 \le j \le l$ and $1 \le k \le m$ If we take $\acute{W}_E \setminus \{v_k\}$	(u_1w_l, u_mw_l)
$1 \le i, j \le n,$ and $1 \le k \le m$ If we take $\acute{W}_E \setminus \{v_k\}$	(w_1w_2, w_1v_1) or (w_1w_2, w_1v_n)

In all possibilities, we conclude that there is no fault-tolerant edge metric generator of 3 vertices. Hence fault-tolerant edge metric dimension of $G^l_{n,m}$ is 4.

□

6. Conclusions

In this paper, we have computed the fault-tolerant edge metric dimension of some planar graphs path, cycle, complete, cycle with chord, tadpole and kayak paddle. It is observed that the fault-tolerant edge metric dimension of these graphs is constant and does not depend on the number of vertices. It is concluded that the fault-tolerant edge metric dimension of families of path graphs is two, the fault-tolerant edge metric dimension of families of cycle graphs, cycle with chord graphs, tadpole graphs is three and the fault-tolerant edge metric dimension of kayak paddle graphs is found to be four. Here we end with an open problem.

Open Problem

Characterize all families of graphs for which difference of fault-tolerant metric dimension and edge metric dimension is one.

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Conflict of interest

The authors declare that no competing interests exist.

References

1. P. J. Slater, Leaves of trees, *Congr. Numer.,* **14** (1975), 549–559.
2. M. C. Hernando, M. Mora, P. J. Slater, D. R. Wood, Fault-tolerant metric dimension of graphs, *Convexity Discrete Struct.,* **5** (2008), 81–85.
3. A. Kelenc, N. Tratnik, I. G. Yero, Uniquely identifying the edges of a graph: The edge metric dimension, *Discrete Appl. Math.,* **251** (2018), 204–220.
4. A. Tabassum, M. A. Umar, M. Perveen, A. Raheem, Antimagicness of subdivided fans, *Open J. Math. Sci.,* **4** (2020), 18–22.
5. J. B. Liu, Z. Zahid, Z. Nasir, W. Nazeer, Edge version of metric dimension and doubly resolving sets of the necklace graph, *Mathematics,* **6** (2018), 243.
6. H. F. M. Salih, S. M. Mershkhan, Generalized the Liouville's and Mobius functions of graph, *Open J. Math. Sci.,* **4** (2020), 186–194.
7. W. Gao, B. Muzaffar, W. Nazeer, K-Banhatti and K-hyper Banhatti indices of dominating David derived network, *Open J. Math. Anal.,* **1** (2017), 13–24.
8. N. Zubrilina, On the edge dimension of a graph, *Discrete Math.,* **341** (2018), 2083–2088.
9. F. Haray, R. A. Melter, On the metric dimension of a graph, *Ars Comb.,* **2** (1976), 191–195.
10. G. Chartrand, L. Eroh, M. A. Johnson, O. R. Oeller-mann, Resolvability in graphs and the metric dimension of a graph, *Discrete Appl. Math.,* **105** (2000), 99–113.
11. G. Chartrand, C. Poisson, P. Zhang, Resolvability and the upper dimension of graphs, *Comput. Math. Appl.,* **39** (2000), 19–28.
12. M. Imran, A. Q. Baig, A. Ahmad, Families of plane graphs with constant metric dimension, *Utilitas Math.,* **88** (2012), 43–57.
13. M. Johnson, Structure-activity maps for visualizing the graph variables arising in drug design, *J. Biopharm. Stat.,* **3** (1993), 203–236.

14. S. Khuller, B. Raghavachari, A. Rosenfeld, Landmarks in graphs, *Discrete Appl. Math.,* **70** (1996), 217–229.

15. R. A. Melter, I. Tomescu, Metric bases in digital geometry, *Comput. Vision, Graphics, Image Process.,* **28** (1984), 113–121.

16. J. Caceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, et al., On the metric dimension of cartesian products of graphs, *SIAM J. Discrete Math.,* **21** (2008), 423–441.

17. J. Kratica, V. Filipovii, A. Kartelj, Edge metric dimension of some generalized Petersen graphs, 2018. Available from: `http://arxiv.org/abs/1807.00580v1`.

18. M. Ahsan, Z. Zahid, S. Zafar, A. Rafiq, M. S. Sindhu, M. Umar, Computing the edge metric dimension of convex polytopes related graphs, *J. Math. Computer Sci.,* **22** (2021), 174–188.

19. U. Ali, Y. Ahmad, M. S. Sardar, On 3-total edge product cordial labeling of tadpole, book and flower graphs, *Open J. Math. Sci.,* **4** (2020), 48–55.

20. S. M. Kang, M. A. Zahid, W. Nazeer, W. Gao, Calculating the degree-based topological indices of dendrimers, *Open Chem.,* **16** (2018), 681–688.

21. Y. C. Kwun, A. U. R. Virk, W. Nazeer, M. A. Rehman, S. M. Kang, On the multiplicative degree-based topological indices of silicon-carbon Si2C3-I [p, q] and Si2C3-II [p, q], *Symmetry,* **10** (2018), 320.

22. R. V. Voronov, The fault-tolerant metric dimension of the king's graph, *Vestnik Saint Petersburg Univ. Appl. Math., Comput. Sci. Control Processes,* **13** (2017), 241–249.

23. H. Raza, S. Hayat, X. F. Pan, On the fault-tolerant metric dimension of convex polytopes, *Appl. Math. Comput.,* **339** (2018), 172–185.

24. J. B. Liu, M. Munir, I. Ali, Z. Hussain, A. Ahmed, Fault-Tolerant metric dimension of Wheel related graphs, 2019. Available from:
`https://hal.archives-ouvertes.fr/hal-01857316v2`.

25. M. Basak, L. Saha, G. K. Das, K. Tiwary, Fault-tolerant metric dimension of circulant graphs $C_n(1, 2, 3)$, *Theor. Comput. Sci.,* 2019. DOI: 10.1016/j.tcs.2019.01.011.

26. A. Shabbir, T. Zamfirescu, Fault-tolerant designs in triangular lattice networks, *Appl. Anal.,* **10** (2016), 447–456.

27. W. Nazeer, A. Farooq, M. Younas, M. Munir, S. M. Kang, On molecular descriptors of carbon nanocones, *Biomolecules,* **8** (2018), 92.

28. Y. C. Kwun, M. Munir, W. Nazeer, S. Rafique, S. M. Kang, Computational analysis of topological indices of two Boron Nanotubes, *Sci. Rep.,* **8** (2018), 1–14.

29. M. Ahsan, S. Zafar, Edge metric dimension of certain families of graphs, *Utilitas Math.,* In Press.

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