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Research article

Suzuki type multivalued contractions in C^*-algebra valued metric spaces with an application

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Abstract: In the present paper, we established multivalued fixed point results on C^*-algebra valued metric spaces and utilized the same to prove fixed point results via Suzuki type contraction. An example is also given to exhibit the utility of our main result. We also provided a system of Fredholm integral equations to examine the existence and uniqueness of solutions supporting our main result.

Keywords: C^*-algebra valued metric space; Suzuki type contraction; multivalued fixed point
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1. Introduction

The classical Banach contraction principle is proved in metric spaces. One natural way to improve this result is to enlarge the class of spaces. With this idea in mind, several authors generalized the notion of metric spaces which saw the evolution of some new notions (see [1–11]). In 2008, Suzuki [12] established a new type of contraction mappings and studied the existence and uniqueness of fixed point theorems, which is a genuine extension of the Banach contraction principle. Later on, many researchers have been worked on this contraction mapping for single-valued mappings as well as multivalued mappings. One of the initial results was introduced by Nadler [13] in the line of research of multivalued mappings. Later on, the domain of fixed points of multivalued functions was developed into a very rich and fruitful theory.

In 2014, Ma et al. [14] introduce the class of C^*-algebra valued metric spaces (in short C^*-AVMS), wherein the range set $\mathbb{R}$ is replaced by an unital C^*-algebra, which is more generalized than the class of metric spaces and proved some related fixed point results. Later on, many researchers extended this class by considering some generalized class (see [2, 4, 10, 15–17]).

Inspired by the preceding observations, we prove a fixed point result via Suzuki type contraction multivalued mapping in C^*-algebra valued metric spaces and give the application in Fredholm type integral equation.

Throughout the paper, C denotes an unital C^*-algebra. An element $0_C \in C$ is known as zero element in C and if $0_C \preccurlyeq a \in C$ then a is called positive element in C. Also, $C_+ = \{a \in C; a \succcurlyeq 0_{\mathbb{A}}\}$. Moreover, if $a = a^*$ and $\sigma(a) = \{\lambda \in \mathbb{R} : \lambda I - a \text{ is non-invertible }\} \subseteq [0, \infty)$. The partial ordering on C can be defined as follows: $a \preccurlyeq b$ if and only if $0_C \preccurlyeq b - a$.

In 2014, Ma et al. [14] introduced the following definition:

Definition 1.1. Let $A \neq \emptyset$ and a mapping $d : A \times A \to C$ satisfies the following (for all $a, b, c \in A$):

(i) $d(a, b) \succcurlyeq 0_C$ and $d(a, b) = 0_C$ iff $a = b$;

(ii) $d(a, b) = d(b, a)$;

(iii) $d(a, b) \preccurlyeq d(a, c) + d(c, b)$.

Then the mapping d is known as C^*-algebra valued metric on A and (A, C, d) is known as C^*-algebra valued metric space.

Definition 1.2. [14] Let (A, C, d) be C^*-algebra valued metric space, $a \in A$ and $\{a_n\}$ a sequence in X.

1. $\{a_n\}$ is called convergent (with respect to C), if for given $\epsilon \succ 0_C$, there exists $k \in \mathbb{N}$ such that $d(a_n, a) \prec \epsilon$, for all $n > k$. Equivalently, $\lim\limits_{n\to\infty} d(a_n, a) = 0_C$.
2. $\{a_n\}$ is called Cauchy sequence (with respect to C), if for given $\epsilon \succ 0_C$, there exists $k \in \mathbb{N}$ such that $d(a_n, a_m) \prec \epsilon$, for all $n, m > k$. Equivalently, $\lim\limits_{n\to\infty} d(a_n, a_m) = 0_C$.
3. (A, C, d) is called complete with respect to C, if every Cauchy in A is convergent to some point a in A.

Let (A, C, d) be a C^*-algebra valued metric space. The set

$$B_d(a; \epsilon) = \{\sigma \in A : d(a, b) \prec \epsilon\}.$$

is called open ball of radius $0_C \prec \epsilon \in C$ and at center $a \in A$. Similarly, the set

$$B_d[a; \epsilon] = \{\sigma \in A : d(a, b) \preccurlyeq \epsilon\}.$$

is called closed ball of radius $0_C \prec \epsilon \in C$ and at center $a \in A$. The set of open balls

$$\mathcal{U}_d = \{B_d(a; \epsilon) : a \in A, \epsilon \succ 0_C\},$$

forms a basis of some topology τ_d on A.

Definition 1.3. The max function on $\mathbb{A}$ (C^*-algebra) with the partial order relation '$\preccurlyeq$' is defined by (for all $a, b \in \mathbb{A}_+$):

$$\max\{a, b\} = b \Leftrightarrow \; a \preccurlyeq b \text{ and } \|a\| \leq \|b\|.$$

The family $\mathcal{CB}^C(A)$ stands for all nonempty, closed and bounded subsets of (A, C, d). Moreover, for $M, N \in \mathcal{CB}^C(A)$ and $x \in A$, we define:

$$dist_C(a, M) = \inf\{d(a, b) : b \in M\};$$

$$\delta_C(M, N) = \sup\{dist_C(a, N) : a \in M\};$$

$$\delta_C(N, M) = \sup\{dist_C(b, M) : b \in N\}.$$

Define C^*-algebra valued Hausdorff metric $H_C : \mathcal{CB}^C(A) \times \mathcal{CB}^C(A) \to C$ by:

$$H_C(M, N) = \max\{\delta_C(M, N), \delta_C(N, M)\} \text{ for all } M, N \in \mathcal{CB}^C(A).$$

Remark 1.1. Let (A, C, d) be a C^*-algebra valued metric space and M a nonempty subset of A, then

$$a \in \overline{M} \text{ if and only if } dist_C(a, M) = 0_C,$$

where, $\overline{M}$ denotes the closure of M with respect to C^*-algebra valued metric A. Also, M is closed in (A, C, d) if and only if $M = \overline{M}$.

Proposition 1.1. *Let (A, C, d) be a C^*-algebra valued metric space. For $M, N, L \in \mathcal{CB}^{\mathcal{A}}(A)$, we have the following:*

(i) $\delta_{\mathcal{P}}(M, M) = diam(M)$;

(ii) $\delta_{\mathcal{P}}(M, N) = 0_C \Rightarrow M \subseteq N$;

(iii) $N \subset L \Rightarrow \delta_{\mathcal{P}}(M, L) \preccurlyeq \delta_{\mathcal{P}}(M, N)$;

(iv) $\delta_C(M \cup N, L) = \max\{\delta_C(M, L), \delta_C(N, L)\}$.

Proof. (i) Suppose $M \in \mathcal{CB}^{\mathcal{A}}(A)$. Then by the definition of δ_C, we have

$$\delta_C(M, M) = \sup\{dist_C(a, M) : a \in A\} = diam(M).$$

(ii) Suppose $M, N \in \mathcal{CB}^{\mathcal{A}}(A)$ such that $\delta_C(M, N) = 0_C$. Then

$$\sup\{dist_C(a, N) : a \in M\} = 0_C \Rightarrow dist_C(a, N) = 0_C \text{ for all } a \in M,$$

which implies that $\delta_C(M, N) = 0_C$. Therefore, $dist_C(a, N) = 0_C$ for all $a \in M$ implies that 'a' is in the closure of N for all $a \in M$. Since N is closed, so $M \subseteq N$.
(iii) Suppose $M, N, L \in \mathcal{CB}^{\mathcal{A}}(A)$ such that $N \subseteq L$. Then

$$dist_C(a, N) \preccurlyeq dist_C(a, L) \text{ for all } a \in X.$$

Thus

$$N \subset L \Rightarrow \delta_C(M, L) \preccurlyeq \delta_C(M, N).$$

(iv) Suppose $M, N, L \in \mathcal{CB}^{\mathcal{A}}(A)$. Then

$$\begin{aligned}\delta_C(M \cup N, L) &= \sup\{dist_C(a, L) : a \in M \cup N\}\\ &= \max\{\sup\{dist_C(a, L) : a \in M\}, \sup\{dist_C(b, L) : b \in N\}\}\\ &= \max\{\delta_C(M, L), \delta_C(N, L)\}.\end{aligned}$$

□

2. Fixed point results

In this section, firstly we define following notions which are needed in our subsequent discussions. Consider, $O_C = \{h \in C;\ 0 \leq \|h\| < 1\}$ and $O_C^{'} = \{h \in C;\ 0 \leq \|h\| \leq 1\}$.
Next, let $\xi : O_C \to O_C^{'}$ be the non-increasing function defined by

$$\xi(h) = \begin{cases} I & \text{if } 0 \leq \|h\| < \frac{1}{2} \\ I - h & \text{if } \frac{1}{2} \leq \|h\| < 1. \end{cases} \tag{2.1}$$

Now, we present our main result as follows:

Theorem 2.1. Let (A, C, d) be complete C^*-algebra valued metric space and $f : X \to \mathcal{CB}^{\mathcal{A}}(A)$. Suppose that there exists $h \in O_C$ such that f satisfies the following:

$$\xi(h)^* dist_C(a, fa)\xi(h) \preccurlyeq d(a, b) \Longrightarrow H_C(fa, fb) \preccurlyeq h^* \mathcal{M}(a, b)h, \tag{2.2}$$

for all $a, b \in A$, where ξ is defined by (2.1) and

$$\mathcal{M}(a, b) = \max\left\{d(a, b), dist_C(a, fa), dist_C(b, fb), \frac{dist_C(a, fb) + dist_C(b, fa)}{2}\right\}.$$

Then f has a unique fixed point.

Proof. Consider $h_1 \in O_C$ such that $0 \leq \|h\| < \|h_1\| < 1$. Let $a_1 \in A$ and $a_2 \in fa_1$ be arbitrary points. Since $a_2 \in fa_1$, then $d(a_2, fa_2) \preccurlyeq H_C(fa_1, fa_2)$ and $\|\xi(h)\| < 1$,

$$\xi(h)^* dist_C(a_1, fa_1)\xi(h) \preccurlyeq d(a_1, fa_1) \preccurlyeq d(a_1, a_2)$$

$$\Rightarrow\ \|\xi(h)\|^2 \|dist_C(a_1, fa_1)\| \leq \|d(a_1, fa_1)\| \leq \|d(a_1, a_2)\|.$$

Hence the assumption (2.2) yielding thereby,

$$\begin{aligned} dist_C(a_2, fa_2) &\preccurlyeq H_C(fa_1, a_2) \\ &\preccurlyeq h^* \max\Big\{d(a_1, a_2), dist_C(a_1, fa_1), dist_C(a_2, fa_2), \\ &\qquad \frac{dist_C(a_1, fa_2) + 0_C}{2}\Big\}h \\ &\preccurlyeq h^* \max\Big\{d(a_1, a_2), dist_C(a_2, fa_2), \\ &\qquad \frac{d(a_1, a_2) + dist_C(a_2, fa_2)}{2}\Big\}h \\ &= h^* \max\{d(a_1, a_2), dist_C(a_2, fa_2)\}h. \end{aligned}$$

Assume that $\max\{d(a_1, a_2), dist_C(a_2, fa_2)\} = dist_C(a_2, fa_2)$, then we have

$$\begin{aligned} \|dist_C(a_2, fa_2)\| &\leq \|h^* dist_C(a_2, fa_2)h\| \\ &= \|h\|^2 \|dist_C(a_2, fa_2)h\| \\ &< \|dist_C(a_2, fa_2)h\| \ \text{ as } \ \|h\|^2 < 1, \end{aligned}$$

a contradiction. Thus we have $dist_C(a_2, fa_2) \preccurlyeq h^* d(a_1, a_2)h$. Hence there exists $a_3 \in fa_2$ such that $d(a_2, a_3) \preccurlyeq h_1^* d(a_1, a_2)h_1$. By proceeding with this procedure, we can construct a sequence $\{a_n\}$ in A such that $a_{n+1} \in fa_n$ with

$$d(a_{n+1}, a_{n+2}) \preccurlyeq h_1^* d(a_n, a_{n+1})h_1 \preccurlyeq ... \preccurlyeq (h_1^*)^n d(a_1, a_2)(h_1)^n.$$

and

$$\begin{aligned}
\sum_{n=1}^{\infty} d(a_{n+1}, a_{n+2}) &\preccurlyeq \sum_{n=1}^{\infty} (h_1^*)^n d(a_1, a_2)(h_1)^n \\
&= \sum_{n=1}^{\infty} (h_1^*)^n (d(a_1, a_2))^{\frac{1}{2}} (d(a_1, a_2))^{\frac{1}{2}} (h_1)^n \\
&= \sum_{n=1}^{\infty} \left((d(a_1, a_2))^{\frac{1}{2}} (h_1)^n\right)^* \left((d(a_1, a_2))^{\frac{1}{2}} (h_1)^n\right) \\
&= \sum_{n=1}^{\infty} \left|(d(a_1, a_2))^{\frac{1}{2}} (h_1)^n\right|^2 \\
&\preccurlyeq \left\|\sum_{n=1}^{\infty} \left|(d(a_1, a_2))^{\frac{1}{2}} (h_1)^n\right|^2\right\| I \\
&\preccurlyeq \sum_{n=1}^{\infty} \left\|(d(a_1, a_2))^{\frac{1}{2}}\right\|^2 \left\|h_1^n\right\|^2 I \\
&= \|d(a_1, a_2)\| \sum_{n=1}^{\infty} \|h_1\|^{2n} I \\
&= \|d(a_1, a_2)\| \frac{\|h_1\|^2}{1 - \|h_1\|^2} I.
\end{aligned}$$

Thus

$$\left\|\sum_{n=1}^{\infty} d(a_{n+1}, a_{n+2})\right\| < \infty.$$

Hereby, we presume that $\{a_n\}$ is a Cauchy sequence. Since A is complete C^*-algebra valued metric space, so there is some point $z \in A$ such that $\lim_{n\to\infty} a_n = z$.

Now, we shall show that

$$d(z, fa) \preccurlyeq h^* \max\{d(z, a), d(a, fa)\}h \text{ for all } a \in A \setminus \{z\}. \tag{2.3}$$

As $\lim_{n\to\infty} a_n = z$, there exists $N_0 \in \mathbb{N}$ such that

$$d(z, a_n) \preccurlyeq \frac{1}{3} d(z, a) \text{ for all } n \geq N_0.$$

Therefore, we have

$$\xi(h)^* dist_C(a_n, fa_n)\xi(h) \preccurlyeq dist_C(a_n, fa_n)$$

$$\begin{aligned}
&\preccurlyeq d(a_n, a_{n+1}) \preccurlyeq d(a_n, z) + d(z, a_{n+1}) \\
&\preccurlyeq \frac{2}{3}d(z, x) = d(z, x) - \frac{1}{3}d(z, a) \\
&\preccurlyeq d(z, a) - \frac{1}{3}d(a_n, z) \\
&\preccurlyeq d(a_n, a),
\end{aligned}$$

yielding thereby

$$\begin{aligned}
H_C(fa_n, fa) \preccurlyeq\ & h^* \max\Big\{d(a_n, a), dist_C(a_n, fa_n), dist_C(a, fa), \\
& \frac{dist_C(a_n, fa) + dist_C(a, fa_n)}{2}\Big\}h. \qquad (2.4)
\end{aligned}$$

Since $a_{n+1} \in fa_n$, then

$$dist_C(a_{n+1}, fa) \preccurlyeq H_C(fa_n, fa) \text{ and } dist_C(a_n, fa_n) \preccurlyeq d(a_n, a_{n+1}).$$

Hence from (2.4), we have

$$\begin{aligned}
dist_C(a_{n+1}, fa) \preccurlyeq\ & h^* \max\Big\{d(a_n, a), d(a_n, a_{n+1}), dist_C(a, fa), \\
& \frac{dist_C(a_n, fa) + d(a, a_{n+1})}{2}\Big\}h.
\end{aligned}$$

On making limit as $n \to \infty$, we obtain (2.3).

To show that $z \in fz$. First, we take the case $\xi(h) = I$ for $0 \leq \|h\| \leq \frac{1}{2}$. Let on contrary that $z \notin fz$. Now, let $u \in fz$ such that

$$2\|h\|\|d(u, z)\| < \|dist_C(z, fz)\|.$$

Since $u \in fz$ implies that $u \neq z$ then from (2.3), we have

$$dist_C(z, fu) \preccurlyeq h^* \max\{d(z, u), dist_C(u, fu)\}h.$$

Also, since $\xi(h)^* dist_C(z, fz)\xi(h) \preccurlyeq dist_C(z, fz) \preccurlyeq d(z, u)$, then in view of condition (2.2), we have

$$\begin{aligned}
H_C(fz, fu) &\preccurlyeq h^* \max\Big\{d(z, u), dist_C(z, fz), dist_C(u, fu), \frac{dist_C(z, fu) + 0_C}{2}\Big\}h \\
&\preccurlyeq h^* \max\{d(z, u), dist_C(z, fz), dist_C(u, fu)\}h \\
&\preccurlyeq h^* \max\{d(z, u), dist_C(u, fu)\}h. \qquad (2.5)
\end{aligned}$$

Hence

$$dist_C(u, fu) \preccurlyeq H_C(fz, fu) \preccurlyeq h^* \max\{d(z, u), dist_C(u, fu)\}h.$$

Thus $\|dist_C(u, fu)\| \leq \|h^* d(z, u)h\| < \|d(z, u)\|$. Therefore (2.5) gives arise

$$\begin{aligned}
dist_C(z, fu) &\preccurlyeq h^* d(z, u)h \\
&= h^* d(z, u)^{\frac{1}{2}} d(z, u)^{\frac{1}{2}} h \\
&= \left(d(z, u)^{\frac{1}{2}} h\right)^* \left(d(z, u)^{\frac{1}{2}} h\right)
\end{aligned}$$

$$\begin{aligned} &= \left\| d(z,u)^{\frac{1}{2}} h \right\|^2 I \\ &\leq \|h\|^2 \|d(z,u)\| I. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} dist_C(z, fz) &\preccurlyeq dist_C(z, fu) + H_C(fu, fz) \\ &\preccurlyeq dist_C(z, fu) + h^* \max\{d(z,u), dist_C(u, fu)\} h \\ &\preccurlyeq 2\|h\|^2 \|d(z,u)\| I \\ &\prec \|dist_C(z, fz)\| I \end{aligned}$$

a contradiction. Therefore, $dist_C(z, fz) = 0_C$, which deduce that z is a fixed point f.

Now, we take the case $\frac{1}{2} \leq \|h\| \leq 1$. Now, we prove

$$\begin{aligned} H_C(fa, fz) \preccurlyeq\ & h^* \max \Big\{ d(a,z), dist_C(a, fa), dist_C(z, fz), \\ & \frac{dist_C(a, fz) + dist_C(z, fa)}{2} \Big\} h \end{aligned} \tag{2.6}$$

for all $a \in A$. If $a = z$, then above inequality holds. Hence we let $a \neq z$. Then, for each $n \in \mathbb{N}$, there exists a sequence $b_n \in fa$ such that

$$d(z, b_n) \preccurlyeq dist_C(z, fa) + \frac{1}{n} d(a,z).$$

Now, by using (2.3), for all $n\mathbb{N}$, we have

$$\begin{aligned} dist_C(a, fa) &\preccurlyeq d(a, b_n) \preccurlyeq d(a,z) + d(z, b_n) \\ &\preccurlyeq d(a,z) + dist_C(z, fa) + \frac{1}{n} d(a,z) \\ &\preccurlyeq d(a,z) + h^* \max\{d(a,z), dist_C(a, fa)\} h + \frac{1}{n} d(a,z). \end{aligned}$$

Assume that $d(a,z) \succcurlyeq dist_C(a, fa)$, then

$$\begin{aligned} dist_C(a, fa) &\preccurlyeq d(a,z) + h^* d(a,z) h + \frac{1}{n} d(a,z) \\ &= d(a,z) + \left(d(a,z)^{\frac{1}{2}} h \right)^* \left(d(a,z)^{\frac{1}{2}} h \right) + \frac{1}{n} d(a,z) \\ &= d(a,z) + \left\| d(a,z)^{\frac{1}{2}} h \right\|^2 + \frac{1}{n} d(a,z) \\ &\preccurlyeq d(a,z) + \|h\|^2 \|d(a,z)\| + \frac{1}{n} d(a,z) \\ &= \left(1 + \|h\|^2 + \frac{1}{n} \right) d(a,z). \end{aligned}$$

On making limit as $n \to \infty$, we obtain $dist_C(a, fa) \preccurlyeq \left(1 + \|h\|^2\right) d(a,z)$. Hence

$$\xi(h)^* dist_C(a, fa) \xi(h) = \left(1 - \|h\|^2\right) d(a,z) \preccurlyeq \frac{1}{(1 + \|h\|^2)} d(a,z) \preccurlyeq d(a,z)$$

and from (2.2), we have (2.6). If $d(a,z) \prec dist_C(a, fa)$, then

$$dist_C(a, fa) \preccurlyeq d(a,z) + h^* dist_C(a, fa)h + \frac{1}{n} d(a,z)$$

so that

$$(1 - \|h\|) dist_C(a, fa) \preccurlyeq \left(1 - \|h\|^2\right) dist_C(a, fa) \preccurlyeq \left(1 + \frac{1}{n}\right) d(a,z).$$

On making limit as $n \to \infty$, we have $\xi(h)^* dist_C(a, fa)\xi(h) \preccurlyeq d(a,z)$. Hence from (2.2), again we have (2.6).

Finally, from (2.6), we have

$$\begin{aligned} dist_C(z, fz) &\preccurlyeq \lim_{n\to\infty} dist_C(a_{n+1}, fz) \\ &\preccurlyeq \lim_{n\to\infty} h^* \max\Big\{d(a_n, z), dist_C(a_n, fa_n), dist_C(z, fz), \\ &\qquad \frac{dist_C(a_n, fz) + d(z, fa_n)}{2}\Big\}h \\ &\preccurlyeq \lim_{n\to\infty} h^* \max\Big\{d(a_n, z), d(a_n, a_{n+1}), dist_C(z, fz), \\ &\qquad \frac{dist_C(a_n, fz) + d(z, a_{n+1})}{2}\Big\}h \\ &= h^* dist_C(z, fz)h, \end{aligned}$$

yielding thereby

$$\|dist_C(z, fz)\| \leq \|h^* dist_C(z, fz)h\| \leq \|h\|^2 \|dist_C(z, fz)\| < \|dist_C(z, fz)\|$$

a contraction. Hence $dist_C(z, fz) = 0_C$ implies that $z \in fz$.

For uniqueness, suppose there are $z, w \in A$ so that $z \in fz$ and $w \in fw$. Thus by conditions (2.2), we have

$$\begin{aligned} \|d(z,w)\| &\leq \|H_C(fz, fw)\| \\ &\leq \Big\| h^* \max\Big\{d(z,w), d(z, fz), dist_C(w, fw), \\ &\qquad \frac{dist_C(z, fw) + d(w, fz)}{2}\Big\}h \Big\| \\ &= \|h^* d(z,w)h\| \\ &\leq \|h\|^2 \|\|d(z,w)\| < \|d(z,w)\|, \end{aligned}$$

a contraction. Hence $d(z,w) = 0_C$ implies that $z = w$. This completes the proof. □

Example 2.1. Suppose $A = \{0, \frac{1}{10}, \frac{1}{5}\}$. The C^*-algebra valued metric $d : A \times A \to C$ is defined by

$$d(a,b) = \begin{bmatrix} |a-b| & 0 \\ 0 & \alpha|a-b| \end{bmatrix}, \quad \text{where } \alpha > 0. \tag{2.7}$$

Then (A, C, d) is a complete C^*-algebra valued metric space. Note that $\{0\}$ and $\{\frac{1}{10}\}$ are bounded sets in (A, C, d). In fact, if $a \in \{0, \frac{1}{10}, \frac{1}{5}\}$ then

$$\begin{aligned} a \in \overline{\{0\}} &\Leftrightarrow dist_C(a, \{0\}) = 0_C \\ &\Leftrightarrow \begin{bmatrix} a & 0 \\ 0 & \alpha a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ &\Leftrightarrow a = 0 \Leftrightarrow a \in \{0\}. \end{aligned}$$

Hence $\{0\}$ is closed. Next,

$$\begin{aligned} a \in \overline{\left\{\frac{1}{10}\right\}} &\Leftrightarrow dist_C\left(a, \left\{\frac{1}{10}\right\}\right) = 0_C \\ &\Leftrightarrow \begin{bmatrix} \left|a - \frac{1}{10}\right| & 0 \\ 0 & \alpha\left|a - \frac{1}{10}\right| \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ &\Leftrightarrow \left|a - \frac{1}{10}\right| = 0 \Leftrightarrow a = \frac{1}{10} \\ &\Leftrightarrow a \in \left\{\frac{1}{10}\right\}. \end{aligned}$$

Hence $\{\frac{1}{10}\}$ is also closed. Now, define $f : A \to C\mathcal{B}^C(A)$ by:

$$f0 = f\frac{1}{10} = \{0\} \text{ and } f\frac{1}{5} = \left\{\frac{1}{10}\right\}.$$

To prove the contractive condition (i) of Theorem 2.1, we need the following:
Case 1. Let $a = 0$, then

$$\xi(h)^* dist_C(0, f0)\xi(h) = 0_C \preccurlyeq d(0, b), \text{ for all } b \in A.$$

For $a = 0$ or $b = \frac{1}{10}$, we have

$$H_C(f0, fb) = H_C(\{0\}, \{0\}) = 0_C \preccurlyeq h^* d(0, b)h \preccurlyeq h^* \mathcal{M}(0, b)h.$$

For $a = 0$ or $b = \frac{1}{5}$, we have

$$\begin{aligned} H_C\left(f0, f\frac{1}{5}\right) &= H_C\left(\{0\}, \left\{\frac{1}{10}\right\}\right) \\ &= \begin{bmatrix} \left|0 - \frac{1}{10}\right| & 0 \\ 0 & \alpha\left|0 - \frac{1}{10}\right| \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} \left|0 - \frac{1}{5}\right| & 0 \\ 0 & \alpha\left|0 - \frac{1}{5}\right| \end{bmatrix} \\ &= \frac{1}{2} d\left(0, \frac{1}{5}\right) \\ &\preccurlyeq \frac{1}{2}\mathcal{M}\left(0, \frac{1}{5}\right). \end{aligned}$$

Case 2. Let $a = \frac{1}{5}$. Then

$$\xi(h)^* dist_C\Big(\frac{1}{5}, f\frac{1}{5}\Big)\xi(h) = \xi(h)^* d\Big(\frac{1}{5}, \frac{1}{10}\Big)\xi(h) \preccurlyeq d\Big(\frac{1}{5}, \frac{1}{10}\Big)$$

this implies that

$$H_C\Big(f\frac{1}{5}, f\frac{1}{10}\Big) = H_C(\{0\}, \{0\}) = 0_C \preccurlyeq h^* d\Big(\frac{1}{5}, \frac{1}{10}\Big)h \preccurlyeq h^* \mathcal{M}\Big(\frac{1}{5}, \frac{1}{10}\Big)h.$$

Hence the contractive condition (i) of Theorem 2.1 is satisfied. Observe that, the mapping f has a unique fixed point (namely $a = 0$).

Corollary 2.1. The conclusions of Theorem 2.1 remain true if the contractive condition (2.2) is replaced by any one of the following:
assume that there exists $h \in O_C$ such that $\xi(h)^* dist_C(a, fa)\xi(h) \preccurlyeq d(a, b)$ implies

(i) $H_C(fa, fb) \preccurlyeq h^* d(a, b)h$;

(ii) $H_C(fa, fb) \preccurlyeq h^* \max\{d(a, b), dist_C(a, fa)\}h$;

(iii) $H_C(fa, fb) \preccurlyeq h^* \max\{d(a, b), dist_C(a, fa), dist_C(b, fb)\}\, h$;

(iv) $H_C(fa, fb) \preccurlyeq h^* \max\Big\{d(a, b), \frac{dist_C(a,fa)+dist_C(b,fb)}{2}, \frac{dist_C(a,fb)+dist_C(b,fa)}{2}\Big\}h.$

for all $a, b \in A$, where ξ is defined as in Theorem 2.1. Then f has a unique fixed point.

The following corollary can be obtain from (iii) of Corollary 2.1:

Corollary 2.2. *Let (A, C, d) be complete C^*-algebra valued metric space and $f : A \to C\mathcal{B}^{\mathcal{A}}(A)$. Suppose that there exists $h \in O_C$ such that f satisfies the following:*

$$\xi(h)^* dist_C(a, fa)\xi(h) \preccurlyeq d(a, b)$$

$$\Longrightarrow H_C(fa, fb) \preccurlyeq \gamma^* d(a, b)\gamma + \gamma^* dist_C(a, fa)\gamma + \gamma^* dist_C(b, fb)\gamma,$$

for all $a, b \in A$, where $\gamma = (1/3)h$ and ξ is defined as in Theorem 2.1. Then f has a unique fixed point.

Now, we are presenting following corollary, by considering f as a single-valued mapping:

Corollary 2.3. *Let (A, C, d) be complete C^*-algebra valued metric space and $f : A \to A$. Suppose that there exists $h \in O_C$ such that f satisfies the following:*

$$\xi(h)^* d(a, fa)\xi(h) \preccurlyeq d(a, b)$$

$$\Longrightarrow d(fa, fb) \preccurlyeq h^* \max\left\{d(a, b), d(a, fa), d(b, fb), \frac{d(a, fb) + d(b, fa)}{2}\right\} h,$$

for all $a, b \in A$ and ξ is defined as in Theorem 2.1. Then f has a unique fixed point.

3. Applications

Now, we provide the following system of Fredholm integral equations to examine the existence and uniqueness of solution in support of Corollary 2.3.

$$a(x) = \int_E G(x, y, a(y))dy + l(x),\ \ x, y \in E, \tag{3.1}$$

where, $G : E \times E \times \mathbb{R} \to \mathbb{R}$, $l \in L^\infty(E)$ and E is a measurable set.

Suppose that $A = L^\infty(E)$, $H = L^2(E)$ and $L(H) = C$. Assume that ϕ_u is a multiplicative operator defined on H, that is, $\pi_u : H \to H$ such that

$$\pi_u(\psi) = u.\psi.$$

Define $d : A \times A \to C$ by:

$$d(a, b) = \pi_{|a-b|}\ \text{ for all }\ a, b \in A.$$

Hence (A, C, d) is a complete C^*-algebra valued b-metric space.

Now, we present our following theorem.

Theorem 3.1. *Suppose that (for all $a, b \in A$)*

(1) there exist a continuous function $\psi : E \times E \to \mathbb{R}$ and $k \in (0, 1)$ such that

$$\xi(h)^*|a(y) - fa(y)|\xi(h) \preccurlyeq |a(y) - b(y)|$$

implies that

$$\begin{aligned} | G(x, y, a(y)) - G(x, y, b(y)) | &\le k \mid \psi(x, y) \mid \max\Big\{ | a(y) - b(y) |, | a(y) - fa(y) |, \\ &\quad | b(y) - fb(y) |, \frac{| a(y) - fb(y) | + | b(y) - fa(y) |}{2} \Big\}, \end{aligned}$$

$for\ all$ $x, y \in E$.

(2) $\sup_{x\in E} \int_E | \psi(x, y) | dy \le 1$.

Then the integral equation (3.1) *has a unique solution in A.*

Proof. Define $f : A \to A$ by:

$$fa(x) = \int_E G(x, y, a(y))dy + l(x),\ \ \forall\ x, y \in E.$$

Set $h = kI$, then $h \in C$. For any $u \in H$, we have

$$\begin{aligned} \|d(fa, fb)\| &= \sup_{\|u\|=1} (\pi_{|fa-fb|}u, u) \\ &= \sup_{\|u\|=1} \int_E \Big[\Big| \int_E G(x, y, a(y)) - G(x, y, b(y))dy \Big| \Big] u(x)\bar{u(x)}d\mu \end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{\|u\|=1} \int_E \Big[\int_E \Big| G(x,y,a(y)) - G(x,y,b(y)) \Big| dy \Big] |u(x)|^2 dx \\
&\leq \sup_{\|u\|=1} \int_E \Big[\int_E \Big| k\psi(x,y) \Big| \max\Big\{ \mid a(y) - b(y) \mid, \mid a(y) - fa(y) \mid, \\
&\quad \mid b(y) - fb(y) \mid, \frac{\mid a(y) - fb(y) \mid + \mid b(y) - fa(y) \mid}{2} \Big\} dy \Big] |u(x)|^2 dx \\
&\leq k \sup_{\|u\|=1} \int_E \Big[\int_E |\psi(x,y)| dy \Big] |u(x)|^2 dx \|a - b\|_\infty \\
&\leq k \sup_{x \in E} \int_E |\psi(x,y)| dy \sup_{\|u\|=1} \int_E |u(x)|^2 dx \max\Big\{ \|a(y) - b(y)\|_\infty, \\
&\quad \|a(y) - fa(y)\|_\infty, \|b(y) - fb(y)\|_\infty, \\
&\quad \frac{\|a(y) - fb(y)\|_\infty + \|b(y) - fa(y)\|_\infty}{2} \Big\} \\
&\leq k\|\mathcal{N}(x,y)\| = \|h\| \, \|\mathcal{N}(x,y)\|,
\end{aligned}
$$

where,

$$
\begin{aligned}
\|\mathcal{N}(x,y)\| &= \max\Big\{ \|a(y) - b(y)\|_\infty, \|a(y) - fa(y)\|_\infty, \|b(y) - fb(y)\|_\infty, \\
&\quad \frac{\|a(y) - fb(y)\|_\infty + \|b(y) - fa(y)\|_\infty}{2} \Big\}
\end{aligned}
$$

Since $\|h\| < 1$, so all the requirements of Corollary 2.3 are satisfied. Therefore, f has a unique fixed point, means that Equation (3.1) has a unique solution. □

4. Conclusions

As C^*-algebra valued metric space is a relatively new addition to the existing literature. Many researchers proved fixed point theorems in such space in several directions. This note proved multivalued fixed point theorems in C^*-algebra valued metric spaces wherein we generalized the Suzuki fixed point theorem [12]. An example is also adopted to highlight the realized improvements in our newly proved result. Finally, we apply Theorem 2.1 to examine the existence and uniqueness of the solution for a system of Fredholm integral equation.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. A. Amini-Harandi, Metric-like spaces, partial metric spaces and fixed points. *Fixed Point Theory Appl.*, **2012** (2012), Article ID 204.
2. M. Asim, M. Imdad, C^*-algebra valued extended b-metric spaces and fixed point results with an application, *U.P.B. Sci. Bull., Series A*, **82** (2020), 209–218.
3. M. Asim, M. Imdad, Partial JS-metric spaces and fixed point results, *Indian J. Math.*, **61** (2019), 175–186.
4. M. Asim, M. Imdad, S. Radenovic, Fixed point results in extended rectangular b-metric spaces with an application. *U.P.B. Sci. Bull., Series A*, **81** (2019), 11–20.
5. I. A. Bakhtin, The contraction mapping principle in almost metric spaces. *Funct. Anal., Gos. Ped. Inst. Unianowsk*, **30** (1989), 26–37.
6. Z. Mustafa, J. R. Roshan, V. Parvaneh, Z. Kadelburg, Some common fixed point results in ordered partial b-metric spaces, *J, Inequalities Appl.*, **2013** (2013), 562.
7. N. Hussain, J. R. Roshan, V. Parvaneh, A. Latif, A Unification of G-Metric, Partial Metric, and b-Metric Spaces, *Abstract Appl. Anal.*, **2014** (2014), Article ID 180698.
8. A. Branciari, A fixed point theorem of Banach–Caccioppoli type on a class of generalized metric spaces, *Publ. Math.,* **57** (2000), 31–37.
9. S. Czerwik, Contraction mappings in b-metric spaces, *Acta Mathematica et Informatica Universitatis Ostraviensis*, **1** (1993), 5–11.
10. S. Chandok, D. Kumar, C. Park, C^*-algebra valued partial metric spaces and fixed point theorems, *Proc. Indian Acad. Sci. (Math. Sci.)*, **129** (2019), doi.org/10.1007/s12044-019-0481-0.
11. H. Long-Guang, Z. Xian, Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.*, **332** (2007), 1468—1476.
12. T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, *Proc. Amer. Math. Soc.*, **30** (1969), 475–488,.
13. S. B. Nadler, Multi-valued contraction mappings, *Pac. J. Math.*, **30** (1969), 475–488.
14. Z. H. Ma, L. N. Jiang, H. K. Sun, C^*-algebra valued metric spaces and related fixed point theorems, *Fixed Point Theory Appl.*, **2014** (2014), Article ID 206.
15. Z. H. Ma, L. N. Jiang, C^*-algebra valued b-metric spaces and related fixed point theorems, *Fixed Point Theory Appl.*, **2015** (2015), Article ID 222.
16. N. Mlaiki, M. Asim, M. Imdad, C^*-algebra valued partial b-metric spaces and fixed point results with an application, *MDPI Math.*, **8** (2020), 1381, doi:10.3390/math8081381.
17. M. Samreen, T. Kamran, M. Postolache, Extended b-metric space, extended b-comparison function and nonlinear contractions, *U. Politeh. Buch. Ser. A.*, **80** (2018), 21–28.
18. H. H. Alsulami, E. KarapJnar, M. A. Kutbi, Antonio-Francisco Roldán-López-de-Hierro, An Illusion: "A Suzuki Type Coupled Fixed Point Theorem", *Abstract Appl. Anal.,* **2014** (2014), Article ID 235731, 8.

19. O. Alqahtani, V. M. Himabindu, E. Karapınar, On Pata–Suzuki-Type Contractions, *Mathematics*, **7** (2019), 720.

20. E. Karapınar, V. M. L. Hima Bindu, On Pata–Suzuki-Type Contractions. *Mathematics*, **8** (2020), 389.

21. H. H. Alsulami, E. Karapınar, H. Piri, Fixed Points of Generalized-Suzuki Type Contraction in Complete -Metric Spaces, *Mathematics*, **2015** (2015), Article ID 969726.