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Research article

Applications of higher-order q-derivatives to the subclass of q-starlike functions associated with the Janowski functions

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Abstract: In this paper, we first investigate some subclasses of q-starlike functions. We then apply higher-order q-derivative operators to introduce and study a new subclass of q-starlike functions, which involves the Janowski functions. Several coefficient inequalities and a sufficient condition are derived. Relevant connections with a number of earlier works on this subject are also pointed out.

Keywords: multivalent functions; univalent functions; convex and q-convex functions; starlike and q-starlike functions; q-Derivative operator

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1. Introduction and definitions

We denote by $\mathcal{A}(p)$, the class of all functions having the following form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \in \mathbb{N} = \{1, 2, 3, \cdots\}), \tag{1.1}$$

which are analytic and multivalent (p-valent) in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

It should be noted that

$$\mathcal{A}(1) = \mathcal{A}.$$

Furthermore, we denote by $\mathcal{S} \subset \mathcal{A}$ the class of univalent functions in $\mathbb{U}$.

The function class $\mathcal{S}^*(p)$ of p-valently starlike functions in $\mathbb{U}$ consists of functions $f \in \mathcal{A}(p)$ along with the following condition:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (\forall\, z \in \mathbb{U}). \tag{1.2}$$

It is easily seen that

$$\mathcal{S}^*(1) = \mathcal{S}^*,$$

where, by $\mathcal{S}^*$, we mean the class of starlike functions with respect to the origin.

Next, by the notation $C(p)$, we mean the class of p-valently convex functions which have the functions $f \in \mathcal{A}(p)$ that satisfy each of the following conditions:

$$f(0) = f'(0) - 1 = 0$$

and

$$\Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0 \quad (\forall\, z \in \mathbb{U}). \tag{1.3}$$

It should be noted that

$$C(1) = C,$$

where, by C, we mean the well-known class of convex functions in $\mathbb{U}$.

For some recent investigations about analytic and multivalent (p-valent) functions, we may refer to [16, 33].

Also we let $\mathcal{P}$ denote the class of Carathéodory functions ψ, which are analytic in the open unit disk $\mathbb{U}$ and normalized by

$$\psi(z) = 1 + \sum_{n=1}^{\infty} \psi_n z^n \tag{1.4}$$

such that

$$\Re(\psi(z)) > 0 \qquad (\forall\, z \in \mathbb{U}).$$

Definition 1.1. For two analytic functions $f, g \in \mathbb{U}$, the function f is said to be subordinate to the function g and written as follows:

$$f \prec g \quad \text{or} \quad f(z) \prec g(z),$$

if there exists a Schwarz function w, which is analytic in $\mathbb{U}$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1,$$

such that

$$f(z) = g(w(z)).$$

Moreover, if the function g is univalent in $\mathbb{U}$, then it follows that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \implies f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Definition 1.2. A function h with $h(0) = 1$ is said to belong to the class $\mathcal{P}[A, B]$ if and only if

$$h(z) \prec \frac{1 + Az}{1 + Bz} \qquad (-1 \leqq B < A \leqq 1).$$

The analytic functions class $\mathcal{P}[A, B]$ was introduced by Janowski [14], he showed that $h(z) \in \mathcal{P}[A, B]$ if and only if there exist a function $\psi \in \mathcal{P}$ such that

$$h(z) = \frac{(A + 1)\psi(z) - (A - 1)}{(B + 1)\psi(z) - (B - 1)} \qquad (-1 \leqq B < A \leqq 1).$$

Definition 1.3. A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{S}^*[A, B]$ if and only if

$$\frac{zf'(z)}{f(z)} = \frac{(A + 1)\psi(z) - (A - 1)}{(B + 1)\psi(z) - (B - 1)} \qquad (-1 \leqq B < A \leqq 1). \tag{1.5}$$

We now recall some concept details and definitions of the q-difference calculus which will play vital role in our presentation. Throughout this article it should be understood that, unless otherwise stated, we presume that $0 < q < 1$ and

$$p \in \mathbb{N} = \{1, 2, 3, \cdots\}.$$

Definition 1.4. Let $q \in (0, 1)$ and define the q-number $[\lambda]_q$ by

$$[\lambda]_q = \begin{cases} \dfrac{1 - q^\lambda}{1 - q} & (\lambda \in \mathbb{C}) \\ \sum\limits_{k=0}^{n-1} q^k = 1 + q + q^2 + \cdots + q^{n-1} & (\lambda = n \in \mathbb{N}). \end{cases}$$

Definition 1.5. Let $q \in (0, 1)$ and define the q-factorial $[n]_q!$ by

$$[n]_q! = \begin{cases} 1 & (n = 0) \\ \prod\limits_{k=1}^{n} [k]_q & (n \in \mathbb{N}). \end{cases}$$

Definition 1.6. (see [12] and [13]) The q-derivative (or the q-difference) operator D_q of a function f is defined, in a given subset of $\mathbb{C}$, by

$$\left(D_q f\right)(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)\,z} & (z \neq 0) \\ \\ f'(0) & (z = 0), \end{cases} \tag{1.6}$$

provided that $f'(0)$ exists.

From Definition 1.6, we can observe that

$$\lim_{q \to 1-} \left(D_q f\right)(z) = \lim_{q \to 1-} \frac{f(z) - f(qz)}{(1-q)\,z} = f'(z)$$

for a differentiable function f in a given subset of $\mathbb{C}$. It is also readily seen from (1.1) and (1.6) that

$$\left(D_q^{(1)} f\right)(z) = [p]_q \, z^{p-1} + \sum_{n=1}^{\infty} [n+p]_q \; a_{n+p} \; z^{n+p-1} \tag{1.7}$$

$$\left(D_q^{(2)} f\right)(z) = [p]_q \, [p-1]_q \, z^{p-2} + \sum_{n=1}^{\infty} [n+p]_q \, [n+p-1]_q \; a_{n+p} \; z^{n+p-2} \tag{1.8}$$

$$\begin{array}{ccc} . & . & . \\ . & . & . \\ . & . & . \end{array}$$

$$\left(D_q^{(p)} f\right)(z) = [p]_q! + \sum_{n=1}^{\infty} \frac{[n+p]_q!}{[n]_q!} \; a_{n+p} \; z^{n}, \tag{1.9}$$

where $\left(D_q^{(p)} f\right)(z)$ is the p-th order q-derivative of $f(z)$.

Recently, the study of the q-calculus has fascinated the intensive devotion of researchers. The great concentration is because of its advantages in many areas of mathematics and physics. The significance of the q-derivative operator D_q is quite obvious by its applications in the study of several subclasses of analytic functions. Initially, in the year 1990, Ismail *et al.* [11] gave the idea of q-starlike functions. Nevertheless, a firm foothold of the usage of the q-calculus in the context of Geometric Function Theory was effectively established, and the use of the generalized basic (or q-) hypergeometric functions in Geometric Function Theory was made by Srivastava (see, for details, [28]). After that, remarkable studies have been done by numerous mathematicians, which offer a momentous part in the advancement of Geometric Function Theory. In particular, Srivastava *et al.* [32] also considered some function classes of q-starlike functions related with conic region. Moreover, Srivastava *et al.* (see, for example, [26, 31, 35, 36]) published a set of articles in which they concentrated upon the classes of q-starlike functions related with the Janowski functions from several different aspects. Additionally, a recently-published survey-cum-expository review article by Srivastava [29] is potentially useful for researchers and scholars working on these topics. In this survey-cum-expository review article [29], the mathematical explanation and applications of the fractional q-calculus and the fractional q-derivative operators in Geometric Function Theory was systematically investigated. For other recent investigations involving the q-calculus, one may refer to [3, 4, 15, 18, 19, 21, 22, 23, 24, 30, 34] see also ([2, 7, 8, 17]).

In this paper, we propose to generalize the work of Srivastava *et al.* [31]. By applying higher-order q-derivative operator, we first define a new general version of the definition presented in [31]. We then derive some coefficient inequalities and a sufficient condition for the general function class which we introduce here. We also indicate a number of other related works on this subject.

Definition 1.7. (see [11]) A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{S}_q^*$ if

$$f(0) = f'(0) - 1 = 0 \tag{1.10}$$

and

$$\left| \frac{z}{f(z)} \left(D_q f\right) z - \frac{1}{1-q} \right| \leqq \frac{1}{1-q}. \tag{1.11}$$

It is readily observed that, as $q \to 1-$, the closed disk

$$\left| w - \frac{1}{1-q} \right| \leqq \frac{1}{1-q}$$

becomes the right-half plane and the class $\mathcal{S}_q^*$ of q-starlike functions reduces to the familiar class $\mathcal{S}^*$. Equivalently, by using the principle of subordination between analytic functions, we can rewrite the conditions in (1.10) and (1.11) as follows (see [37]):

$$\frac{z}{f(z)} \left(D_q f\right)(z) \prec \widehat{p}(z) \qquad \left(\widehat{p}(z) = \frac{1+z}{1-qz} \right).$$

Remark 1.8. For functions f in $\mathcal{A}$, the Alexander theorem [6] was used by Baricz and Swaminathan [5] for defining the class C_q of q-convex functions in the usual way as follows:

$$f(z) \in C_q \iff z\left(D_q f\right)(z) \in \mathcal{S}_q^*.$$

Now, making use of the principle of subordination between analytic functions and the above-mentioned q-calculus, we have the following definition.

Definition 1.9. A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{S}_q^*[p, A, B]$ if and only if

$$\frac{z\left(D_q^{(p)} f\right)(z)}{\left(D_q^{(p-1)} f\right)(z)} = \frac{(1+q)(A+1)(\psi(z)-1) + 2\Big(\psi(z) + 1 - q(\psi(z)-1)\Big)}{(1+q)(B+1)(\psi(z)-1) + 2\Big(\psi(z) + 1 - q(\psi(z)-1)\Big)},$$

which, by using the subordination principle, can be written as follows:

$$\frac{z\left(D_q^{(p)} f\right)(z)}{\left(D_q^{(p-1)} f\right)(z)} \prec \phi(z) \qquad (p \in \mathbb{N}), \tag{1.12}$$

where

$$\phi(z) = \frac{z(A+1) + 2 + zq(A-1)}{z(B+1) + 2 + zq(B-1)} \quad (-1 \leqq B < A \leqq 1) \tag{1.13}$$

and $\left(D_q^{(p)} f\right)(z)$ is the p-th order q-derivative of $f(z)$.

Remark 1.10. First of all, it is easy to see that

$$\lim_{q\to 1-} \mathcal{S}_q^*[1,A,B] = \mathcal{S}^*[A,B],$$

where $\mathcal{S}^*[A,B]$ is the function class introduced and studied by Janowski [14]. Secondly, we have

$$\mathcal{S}_q^*[1,A,B] = \mathcal{S}_q^*[A,B],$$

where $\mathcal{S}_q^*[A,B]$ is function class introduced and studied by Srivastava *et al.* [31]. Thirdly, we can see that

$$\mathcal{S}_q^*[1,1,-1] = \mathcal{S}_q^*,$$

where $\mathcal{S}_q^*$ is the class of q-starlike functions, which is already given in Definition 1.7. Furthermore, in $\mathcal{S}_q^*$, if we let $q \to 1-$ then we get the well-known class of starlike functions. Finally, when

$$p = 1, \quad A = 1 - 2\alpha \quad (0 \leqq \alpha < 1) \quad \text{and} \quad B = -1$$

the function class $\mathcal{S}_q^*[p,A,B]$ reduces to the function class $\mathcal{S}_q^*(\alpha)$, which was introduced and studied by Agrawal and Sahoo [1]. One can also see that

$$\lim_{q\to 1-} \mathcal{S}_q^*(\alpha) = \mathcal{S}^*(\alpha),$$

where $\mathcal{S}^*(\alpha)$ is the function class which was introduced and studied by Silverman (see [27]).

By using the idea of the above-mentioned Alexander theorem, the class $C_q[p,A,B]$ can be defined in the following way.

Definition 1.11. Just as in Remark 1.8, by using the idea of the Alexander theorem [6], the class of $C_q[p,A,B]$ of q-convex functions can be defined by

$$f(z) \in C_q[p,A,B] \iff \frac{z}{[p]_q}\left(D_q f\right)(z) \in \mathcal{S}_q^*[p,A,B].$$

2. Preliminaries

Lemma 2.1. (see [20]) *Let the function $\psi(z)$ given by*

$$\psi(z) = 1 + \psi_1 z + \psi_2 z^2 + \cdots$$

is in the class $\mathcal{P}$ of functions positive real part in $\mathbb{U}$, then, for any complex number υ,

$$\left|\psi_2 - \upsilon\psi_1^2\right| \leqq \begin{cases} -4\upsilon + 2 & (\upsilon \leqq 0) \\ 2 & (0 \leqq \upsilon \leqq 1) \\ 4\upsilon - 2 & (\upsilon \geqq 1). \end{cases} \tag{2.1}$$

When $\upsilon < 0$ or $\upsilon > 1$, the equality holds true in (2.1) *if and only if $\psi(z)$ is given by*

$$\psi(z) = \frac{1+z}{1-z}$$

or one of its rotations. If $0 < \upsilon < 1$, *then the equality holds true in* (2.1) *if and only if*

$$\psi(z) = \frac{1+z^2}{1-z^2}$$

or one of its rotations. If $\upsilon = 0$, *the equality holds true in* (2.1) *if and only if*

$$\psi(z) = \left(\frac{1+\rho}{2}\right)\frac{1+z}{1-z} + \left(\frac{1-\rho}{2}\right)\frac{1-z}{1+z} \qquad (0 \leqq \rho \leqq 1)$$

or one of its rotations. If $\upsilon = 1$, *then the equality in* (2.1) *holds true if* $\psi(z)$ *is a reciprocal of one of the functions such that the equality holds true in the case when* $\upsilon = 0$.

Lemma 2.2. (see [25]) *Let the function* $\psi(z)$ *given by*

$$\psi(z) = 1 + \sum_{n=2}^{\infty} \psi_n z^n$$

be subordinate to the function $H(z)$ *given by*

$$H(z) = 1 + \sum_{n=2}^{\infty} C_n z^n.$$

If $H(z)$ *is univalent in* $\mathbb{U}$ *and* $H(\mathbb{U})$ *is convex, then*

$$|\psi_n| \leqq |C_1| \qquad (n \geqq 1).$$

Lemma 2.3. *Suppose that the sequence* $\{a_k\}_{k=0}^{\infty}$ *is defined by*

$$a_j = 0 \quad (j = 1, 2, 3, \cdots p-1) \qquad \textit{and} \qquad a_p = 1$$

and

$$a_{n+p} = \frac{[n+1]_q!}{[n+p]_q!\left([n+1]_q - 1\right)} \sum_{l=1}^{n} \frac{[n+p-l]_q!}{[n+1-l]_q!}\, a_{n+p-l}\, c_l. \tag{2.2}$$

Then

$$a_{n+p-1} = \prod_{j=2}^{n} \frac{[j]_q \left\{2\left([j-1]_q - 1\right) + (A-B)(q+1)\right\}}{2\left\{[j]_q - 1\right\}[j+p-1]_q}.$$

Proof. By virtue of (2.2), we easily get

$$\frac{[n+p]_q!}{[n+1]_q!}\left([n+1]_q - 1\right) a_{n+p} = \sum_{l=1}^{n} \frac{[n+p-l]_q!}{[n+1-l]_q!}\, a_{n+p-l}\, \psi_l \tag{2.3}$$

and

$$\frac{[n+p-1]_q!}{[n+1]_q!}\left([n+1]_q - 1\right) a_{n+p-1} = \sum_{l=1}^{n} \frac{[n+p-1-l]_q!}{[n+1-l]_q!}\, a_{n+p-1-l}\, \psi_l \tag{2.4}$$

Combining (2.3) and (2.4), we obtain

$$\frac{a_{n+p}}{a_{n+p-1}} = \frac{[n+1]_q \left\{2\left([n]_q - 1\right) + (A-B)(1+q)\right\}}{2\left\{[n+1]_q - 1\right\}[n+p]_q}.$$

Similarly, we can deduce the following result:

$$a_{n+p-1} = \frac{a_{n+p-1}}{a_{n+p-2}} \cdot \frac{a_{n+p-2}}{a_{n+p-3}} \cdots \frac{a_{p+2}}{a_{p+1}} \cdot \frac{a_{p+1}}{a_p} \cdot a_p.$$

The proof of Lemma 2.3 is evidently completed. □

3. Main results

In this section, we will prove our main results. Throughout our discussion, we assume that

$$p \in \mathbb{N} = \{1, 2, 3, \cdots\}, \qquad (-1 \leqq B < A \leqq 1) \quad \text{and} \quad q \in (0,1).$$

Theorem 3.1. *Let the function $f \in \mathcal{S}_q^*[p, A, B]$ be of the form given by* (1.1). *Then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leqq \begin{cases} \left(\frac{A-B}{4q^2}\right)\left(\frac{[2]_q[3]_q}{[p+1]_q[p+2]_q}\right)\Lambda(q) & (\mu < \sigma_1) \\ \\ \left(\frac{A-B}{2q}\right)\left(\frac{[2]_q[3]_q}{[p+1]_q[p+2]_q}\right) & (\sigma_1 \leqq \mu \leqq \sigma_2) \\ \\ \left(\frac{B-A}{4q^2}\right)\left(\frac{[2]_q[3]_q}{[p+1]_q[p+2]_q}\right)\Lambda(q) & (\mu > \sigma_2), \end{cases}$$

where

$$\Lambda(q) = \frac{\left\{(A-B) + (A-2B-1)q + (1-B)q^2\right\}[p+1]_q[3]_q + \mu(A-B)(1+q)^2[p+2]_q}{[p+1]_q[3]_q},$$

$$\sigma_1 = \frac{\{(A-B)(1+q) - q(3-q+(q+1))B\}[p+1]_q[3]_q}{(A-B)(1+q)^3[p+2]_q}$$

and

$$\sigma_2 = \frac{4q[p+1]_q[3]_q - \{q(3-q+(q+1))B - (A-B)(1-q)\}[p+1]_q[3]_q}{(A-B)(1+q)^3[p+2]_q}.$$

Each of the above results is sharp.

Proof. If $f \in \mathcal{S}_q^*[p, A, B]$, then it follows from (1.12) that

$$\frac{zD_q^{(p)}f(z)}{D_q^{(p-1)}f(z)} \prec \phi(z),$$

where the function $\phi(z)$ is given by (1.13).

We now define a function ψ by

$$\psi(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \cdots .$$

It is clear that $\psi \in \mathcal{P}$. This implies that

$$w(z) = \frac{\psi(z) - 1}{\psi(z) + 1}.$$

Thus, by applying (1.12), we have

$$\frac{z D_q^{(p)} f(z)}{D_q^{(p-1)} f(z)} = \phi(w(z))$$

with

$$\phi(w(z)) = \frac{(1+q)(A+1)(\psi(z)-1) + 2\Big(\psi(z) + 1 - q(\psi(z)-1)\Big)}{(1+q)(B+1)(\psi(z)-1) + 2\Big(\psi(z) + 1 - q(\psi(z)-1)\Big)}.$$

Now

$$\frac{z D_q^{(p)} f(z)}{D_q^{(p-1)} f(z)} = \frac{(1+q)(A+1)(\psi(z)-1) + 2\Big(\psi(z) + 1 - q(\psi(z)-1)\Big)}{(1+q)(B+1)(\psi(z)-1) + 2\Big(\psi(z) + 1 - q(\psi(z)-1)\Big)}.$$

Thus, if

$$\psi(z) = 1 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \cdots ,$$

then we find after some simplification that

$$\begin{aligned}
&\frac{(1+q)(A+1)(\psi(z)-1) + 2\Big(\psi(z) + 1 - q(\psi(z)-1)\Big)}{(1+q)(B+1)(\psi(z)-1) + 2\Big(\psi(z) + 1 - q(\psi(z)-1)\Big)} \\
&\quad = 1 + \frac{1}{4}(A-B)(q+1)\psi_1 z + \frac{1}{16}(A-B)(q+1) \\
&\qquad \cdot \Big\{4\psi_2 - (3 - q + (q+1)B)\psi_1^2\Big\} z^2 + \cdots .
\end{aligned}$$

Similarly, we can find that

$$\begin{aligned}
\frac{z D_q^{(p)} f(z)}{D_q^{(p-1)} f(z)} = 1 &+ \frac{q}{1+q} [1+p]_q \, a_{p+1} z \\
&+ \left\{\left(\frac{q[p+2]_q [p+1]_q}{1+q+q^2}\right) a_{p+2} - \frac{q[p+1]_q^2}{(1+q)^2} a_{p+1}^2\right\} z^2 + \cdots .
\end{aligned}$$

Therefore, we have

$$a_{p+1} = \frac{(A-B)(q+1)^2}{4q[p+1]_q} \psi_1 \tag{3.1}$$

and

$$a_{p+2} = \frac{[2]_q[3]_q}{[p+1]_q\,[p+2]_q}\left\{\left(\frac{A-B}{4q}\right)\psi_2 - \left(\frac{A-B}{16q^2}\right)\kappa_1\,(q)\,\psi_1^2\right\}, \tag{3.2}$$

where

$$\kappa_1\,(q) = (B-A) + (2B-A+3)q + (B-1)q^2. \tag{3.3}$$

Thus, clearly, we find that

$$\left|a_{p+2} - \mu a_{p+1}^2\right| = \left(\frac{A-B}{4q}\right)\left(\frac{[2]_q[3]_q}{[p+1]_q[p+2]_q}\right)\left|\psi_2 - \kappa_2\psi_1^2\right|, \tag{3.4}$$

where

$$\kappa_2 = \frac{\kappa_1\,(q)\,[p+1]_q\,[3]_q + \mu\,(A-B)\,(1+q)^3\,[p+2]_q}{4q\,[p+1]_q\,[3]_q}$$

with $\kappa_1\,(q)$ given by (3.3). By an application of Lemma 2.1 in (3.4), we get the result as demonstrated by Theorem 3.1. □

Remark 3.2. If we put $p = 1$ in Theorem 3.1, we arrive at a result which was already proved by Srivastava *et al.* [31].

If, in Theorem 3.1, we set

$$A = -B = p = 1$$

and let $q \to 1-$, we have the following corollary.

Corollary 3.3. (see [10, Corollary 3]) *Let the function f be in the class $\mathcal{S}^*$. Then*

$$\left|a_3 - \mu a_2^2\right| \leqq \begin{cases} 3-4\mu & \left(\mu < \frac{1}{2}\right) \\ 1 & \left(\frac{1}{2} \leqq \mu \leqq 1\right) \\ 4\mu - 3 & (\mu > 1). \end{cases}$$

We next state and prove Theorem 3.4 below.

Theorem 3.4. *Let the function $f \in \mathcal{S}_q^*\,[p, A, B]$ be of the form given by* (1.1). *Then*

$$\left|a_{n+p-1}\right| \leqq \prod_{j=2}^{n} \frac{[j]_q\left\{2\left([j-1]_q - 1\right) + (A-B)\,(q+1)\right\}}{2\left\{[j]_q - 1\right\}[j+p-1]_q}. \tag{3.5}$$

Proof. By definition, for $f \in \mathcal{S}_q^*\,[p, A, B]$, we have

$$\frac{zD_q^{(p)} f\,(z)}{D_q^{(p-1)} f\,(z)} \prec \varphi\,(z), \tag{3.6}$$

where

$$\varphi\,(z) = \frac{z\,(A+1) + 2 + zq\,(A-1)}{z\,(B+1) + 2 + zq\,(B-1)}$$

$$= 1 + \frac{1}{2}(A - B)(q + 1)z + \frac{1}{4}(A - B)(q + 1) \cdot \{(q + 1)B - q + 1\}z^2 + \cdots .$$

Since

$$\psi(z) = 1 + \sum_{n=1}^{\infty} \psi_n z^n$$

then, by Lemma 2.2, we have

$$|\psi_n| \leqq \frac{1}{2}(A - B)(q + 1) \qquad (n \geqq 1). \tag{3.7}$$

Now, from (3.6), we have

$$zD_q^{(p)} f(z) = \psi(z) D_q^{(p-1)} f(z),$$

which implies that

$$[p]_q! z + \sum_{n=1}^{\infty} \frac{[n+p]_q!}{[n]_q!} a_{n+p} z^{n+1} = \left([p]_q! z + \sum_{n=1}^{\infty} \frac{[n+p]_q!}{[n+1]_q!} a_{n+p} z^{n+1}\right) \cdot \left(1 + \sum_{n=1}^{\infty} \psi_n z^n\right).$$

Equating the coefficients of z^{n+1} on both sides, we have

$$\frac{[n+p]_q!}{[n+1]_q!}\left([n+1]_q - 1\right) a_{n+p} = \sum_{l=1}^{n} \frac{[n+p-l]_q!}{[n+1-l]_q!} a_{n+p-l}\psi_l \qquad \left(a_p = 1\right).$$

This last equation implies that

$$\left|a_{n+p}\right| \leqq \frac{[n+1]_q!}{[n+p]_q!\left([n+1]_q - 1\right)} \sum_{l=1}^{n} \frac{[n+p-l]_q!}{[n+1-l]_q!} \left|a_{n+p-l}\right| \cdot |\psi_l| \qquad \left(a_p = 1\right).$$

By using (3.7), we find that

$$\left|a_{n+p}\right| \leqq \frac{(A - B)(q + 1)[n+1]_q!}{2[n+p]_q!\left([n+1]_q - 1\right)} \sum_{l=1}^{n-1} \frac{[p+l]_q!}{[l+1]_q!} \left|a_{p+l}\right| \qquad \left(a_p = 1\right). \tag{3.8}$$

Finally, in order to prove the result asserted by Theorem 3.4, we use Lemma 2.3 and so we get

$$\left|a_{n+p-1}\right| \leqq \prod_{j=2}^{n} \frac{[j]_q \left\{2\left([j]_q - 1\right) + (A - B)(q + 1)\right\}}{2\left\{[j]_q - 1\right\}[j + p - 1]_q}. \tag{3.9}$$

The proof of Theorem 3.4 is now completed. ∎

Remark 3.5. First of all, if we put $p = 1$ in Theorem 3.4, we deduce the result which was already proved by Srivastava *et al.* [31]. Secondly, if we put $p = 1$ and let $q \to 1-$, then we get a result which was proved earlier by Janowski [14]. Thirdly, if we set

$$A = 1 - 2\alpha \quad (0 \leqq \alpha < 1) \qquad \text{and} \qquad -B = 1 = p$$

and let $q \longrightarrow 1-$, then Theorem 3.4 yields the following known result proved by Silverman in [9].

Corollary 3.6. (see [9]) *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{S}^*(\alpha)$. Then, for $n \geqq 2$,*

$$|a_n| \leqq \frac{\prod_{j=2}^{n} |j - 2\alpha|}{(n-1)!} \qquad (0 \leqq \alpha < 1).$$

Theorem 3.7. *Let the function $f(z) \in C_q[p, A, B]$ be of the from* (1.1). *Then*

$$|a_n| \leqq \frac{1}{[n+p]} \prod_{j=2}^{n} \frac{[j]_q \left\{2\left([j-1]_q - 1\right) + (A-B)(q+1)\right\}}{2\left\{[j]_q - 1\right\}[j+p-1]_q} \qquad (n \in \mathbb{N} \setminus \{1\}).$$

Proof. The proof of Theorem 3.7 follows immediately by using Theorem 3.4 and Definition 1.11. □

The following equivalent form of Definition 1.9 is potentially useful in further investigation of the function class $\mathcal{S}_q^*[p, A, B]$:

$$f \in \mathcal{S}_q^*[p, A, B] \iff \left| \frac{(B-1)\dfrac{z\left(D_q^{(p)} f\right)(z)}{\left(D_q^{(p-1)} f\right)(z)} - (A-1)}{(B+1)\dfrac{z\left(D_q^{(p)} f\right)(z)}{\left(D_q^{(p-1)} f\right)(z)} - (A+1)} - \frac{1}{1-q} \right| < \frac{1}{1-q}.$$

Theorem 3.8. *A function $f \in \mathcal{A}(p)$ of the form given by* (1.1) *is in the class $\mathcal{S}_q^*[p, A, B]$ if it satisfies the following condition*:

$$\sum_{n=1}^{\infty} \left(2q[n-1]_q + \frac{[n+p]_q!}{[n+1]_q!} \left|[n+1]_q (B+1) - (A+1)\right|\right) \cdot \left|a_{n+p}\right| < |B - A|. \tag{3.10}$$

Proof. Assuming that (3.10) holds true, it suffices to show that

$$\left| \frac{(B-1)\frac{z\left(D_q^{(p)} f\right)(z)}{\left(D_q^{(p-1)} f\right)(z)} - (A-1)}{(B+1)\frac{z\left(D_q^{(p)} f\right)(z)}{\left(D_q^{(p-1)} f\right)(z)} - (A+1)} - \frac{1}{1-q} \right| < \frac{1}{1-q}.$$

Indeed we have

$$\left| \frac{(B-1)\frac{z\left(D_q^{(p)} f\right)(z)}{\left(D_q^{(p-1)} f\right)(z)} - (A-1)}{(B+1)\frac{z\left(D_q^{(p)} f\right)(z)}{\left(D_q^{(p-1)} f\right)(z)} - (A+1)} - \frac{1}{1-q} \right|$$

$$\leqq \left| \frac{(B-1)\, z \left(D_q^{(p)} f\right)(z) - (A-1) \left(D_q^{(p-1)} f\right)(z)}{(B+1)\, z \left(D_q^{(p)} f\right)(z) - (A+1) \left(D_q^{(p-1)} f\right)(z)} - 1 \right| + \frac{q}{1-q}$$

$$= 2 \left| \frac{\left(D_q^{(p-1)} f\right)(z) - z \left(D_q^{(p)} f\right)(z)}{(B+1)\, z \left(D_q^{(p)} f\right)(z) - (A+1) \left(D_q^{(p-1)} f\right)(z)} \right| + \frac{q}{1-q}$$

$$= 2 \left| \frac{\sum\limits_{n=1}^{\infty} \frac{[n+p]_q!}{[n+1]_q!} \left(1 - [n+1]_q\right) a_{n+p} z^{n+p}}{(B-A)\, [p]_q! + \sum\limits_{n=1}^{\infty} \frac{[n+p]_q!}{[n+1]_q!} \left\{[n+1]_q (B+1) - (A+1)\right\} a_{n+p} z^{n+p}} \right| + \frac{q}{1-q}$$

$$\leqq 2 \frac{\sum\limits_{n=1}^{\infty} \frac{[n+p]_q!}{[n+1]_q!} \left|\left(1 - [n+1]_q\right)\right| \left|a_{n+p}\right|}{|(B-A)|\, [p]_q! - \sum\limits_{n=1}^{\infty} \frac{[n+p]_q!}{[n+1]_q!} \left|\left\{[n+1]_q (B+1) - (A+1)\right\}\right| \left|a_{n+p}\right|} + \frac{q}{1-q}. \tag{3.11}$$

The last expression in (3.11) is bounded above by $\frac{1}{1-q}$ if

$$\sum_{n=1}^{\infty} \left(2q\, [n-1]_q + \frac{[n+p]_q!}{[n+1]_q!} \left|[n+1]_q (B+1) - (A+1)\right|\right) \left|a_{n+p}\right| < |B-A|,$$

which completes the proof of Theorem 3.8. □

Remark 3.9. If we put $p = 1$ in Theorem 3.4, we deduce the result which was already proved by Srivastava *et al.* [31].

Remark 3.10. If we set

$$A = 1 - 2\alpha \quad (0 \leqq \alpha < 1) \qquad \text{and} \qquad -B = 1 = p$$

and let $q \longrightarrow 1-$, then we have the following result proved by Silverman [27].

Corollary 3.11. (see [27]) *A function* $f \in \mathcal{A}$ *of the form* (1.1) *with* $p = 1$ *is in the class* $\mathcal{S}^*(\alpha)$ *if it satisfies the following condition*:

$$\sum_{n=2}^{\infty} (n - \alpha)\, |a_n| < 1 - \alpha \qquad (0 \leqq \alpha < 1).$$

Theorem 3.12. *A function* $f \in \mathcal{A}(p)$ *of the form* (1.1) *is in the class* $C_q\,[A, B]$ *if it satisfies the following condition*:

$$\sum_{n=1}^{\infty} \frac{[n+p]_q}{[p]_q} \left(2q[n-1]_q + \frac{[n+p]_q!}{[n+1]_q!}\ \left|[n+1]_q (B+1) - (A+1)\right|\right) \left|a_{n+p}\right| < |B-A|.$$

Proof. The proof of Theorem 3.12 follows easily when we apply Theorem 3.8 in conjunction with Definition 1.11. □

4. Conclusions

Our present investigation is motivated by the well-established potential for the usages of the basic (or q-) calculus and the fractional basic (or q-) calculus in Geometric Function Theory as described in a recently-published survey-cum-expository review article by Srivastava [29]. Here we have introduced and studied systematically some interesting subclasses of multivalent (p-valent) q-starlike functions in the open unit disk $\mathbb{U}$. We have also provided relevant connections of the various results, which we have demonstrated in this paper, with those derived in many earlier works cited here.

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Conflict of interest

The authors declare that they have no competing interests

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