Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(2): 1087–1100.
DOI:10.3934/math.2021065
Received: 03 August 2020
Accepted: 04 November 2020
Published: 09 November 2020

Research article

Lie symmetry reductions and exact solutions to a generalized two-component Hunter-Saxton system

Huizhang Yang*, **Wei Liu and Yunmei Zhao**

College of Mathematics, Honghe University, Mengzi, Yunnan, 661199, China

* **Correspondence:** E-mail: yanghuizhangyn@163.com.

Abstract: Based on the classical Lie group method, a generalized two-component Hunter-Saxton system is studied in this paper. All of the its geometric vector fields, infinitesimal generators and the commutation relations of Lie algebra are derived. Furthermore, the similarity variables and symmetry reductions of this new generalized two-component Hunter-Saxton system are derived. Under these Lie symmetry reductions, some exact solutions are obtained by using the symbolic computation. Moreover, a conservation law of this system is presented by using the multiplier approach.

Keywords: generalized two-component Hunter-Saxton system; Lie symmetry analysis; symmetry reductions; exact solutions; conservation law
Mathematics Subject Classification: 37L20, 35C05, 35Q53

1. Introduction

The Hunter-Saxton (HS) equation reads

$$u_{xxt} + uu_{xxx} + 2u_x u_{xx} - 2\kappa u_x = 0, \tag{1.1}$$

where $u(x,t)$ depends on a time variable t and a space variable x, κ is a positive constant. This equation was derived as a model for propagation of orientation waves in a massive nematic liquid crystal director field [1]. In fact, it can be regarded as a short wave limit of the well known Camassa-Holm equation [2, 3].

The two-component Hunter-Saxton (2-HS) equation [1] is

$$\begin{cases} u_{xxt} + uu_{xxx} + 2u_x u_{xx} - 2\kappa u_x = \sigma\rho\rho_x, \\ \rho_t + (\rho u)_x = 0, \end{cases} \tag{1.2}$$

where $u(x,t)$ and $\rho(x,t)$ depend on variables t and x, σ, κ are positive constants. The 2-HS equation has attracted much attention and it has been studied extensively and some results were obtained, we can see [4, 5].

Meanwhile, there is a generalized 2-HS system [6] as follow:

$$\begin{cases} u_{xxt} + uu_{xxx} + (1-\alpha)u_x u_{xx} - \kappa\rho\rho_x = 0, \\ \rho_t + u\rho_x = \alpha u_x \rho, \end{cases} \tag{1.3}$$

where $\alpha(\alpha \neq 1), \kappa$ are constants. The model with $(\alpha, \kappa) = (-1, -1)$ in system (1.3) appeared initially in the work of Lenells [7]. The author showed that system (1.3) is the geodesic equation on a manifold K which admits a *Kähler* structure. The blow-up phenomena of system (1.3) was investigated in [4, 8].

Our goal is to study exact solutions of system (1.3) by applying classical Lie group method [9–14]. Firstly, the vector field for the system (1.3) will be given by Lie symmetry analysis. Secondly, similarity variables and its symmetry reductions equations are obtained. Thirdly, by solving the reduced equations, some exact solutions of the system (1.3) will be presented. Finally, we give a conservation law of system (1.3).

2. Lie symmetry analysis of the system (1.3)

First of all, let us consider a one-parameter Lie group of infinitesimal transformation:

$$x \to x + \epsilon\xi(x,t,u,\rho),$$

$$t \to t + \epsilon\tau(x,t,u,\rho),$$

$$u \to u + \epsilon\phi(x,t,u,\rho),$$

$$\rho \to \rho + \epsilon\psi(x,t,u,\rho),$$

with a small parameter $\epsilon \ll 1$. The vector field associated with the above group of transformations can be written as

$$V = \xi(x,t,u,\rho)\frac{\partial}{\partial x} + \tau(x,t,u,\rho)\frac{\partial}{\partial t} + \phi(x,t,u,\rho)\frac{\partial}{\partial u} + \psi(x,t,u,\rho)\frac{\partial}{\partial \rho}, \tag{2.1}$$

where the coefficient functions $\xi(x,t,u,\rho), \tau(x,t,u,\rho), \phi(x,t,u,\rho)$ and $\psi(x,t,u,\rho)$ of the vector field are to be determined later.

If the vector field (2.1) generates a symmetry of the system (1.3), then V must satisfy the Lie symmetry condition

$$\begin{cases} \mathrm{pr}^{(3)}V(\Delta_1)|_{\Delta_1=0} = 0, \\ \mathrm{pr}^{(1)}V(\Delta_2)|_{\Delta_2=0} = 0, \end{cases} \tag{2.2}$$

where $\mathrm{pr}^{(3)}V, \mathrm{pr}^{(1)}V$ denote the third and the first prolongation of V respectively, and $\Delta_1 = u_{xxt} + uu_{xxx} + (1-\alpha)u_x u_{xx} - \kappa\rho\rho_x$, $\Delta_2 = \rho_t + u\rho_x - \alpha u_x\rho$ for system (1.3). Expanding (2.2), we find that the coefficient functions ξ, τ, ϕ and ψ must satisfy the symmetry condition

$$\begin{cases} \phi^{xxt} + \phi u_{xxx} + u\phi^{xxx} + (1-\alpha)\phi^x u_{xx} + (1-\alpha)u_x\phi^{xx} - \kappa\psi\rho_x - \kappa\rho\psi^x = 0, \\ \psi^t + \phi\rho_x + u\psi^x - \alpha\phi^x\rho - \alpha u_x\psi = 0, \end{cases} \tag{2.3}$$

where $\phi, \psi, \phi^x, \psi^x, \psi^t, \phi^{xx}, \phi^{xxx}, \phi^{xxt}$ are the coefficient functions given by

$$
\begin{aligned}
&\phi^t = D_t\phi - u_x D_t\xi - u_t D_t\tau, \ \ \psi^t = D_t\psi - \rho_x D_t\xi - \rho_t D_t\tau,\\
&\phi^x = D_x\phi - u_x D_x\xi - u_t D_x\tau, \ \ \psi^x = D_x\psi - \rho_x D_x\xi - \rho_t D_x\tau,\\
&\phi^{xx} = D_x^2\phi - u_x D_x^2\xi - u_t D_x^2\tau - 2u_{xx}D_x\xi - 2u_{xt}D_x\tau,\\
&\phi^{xxx} = D_x^3\phi - u_x D_x^3\xi - u_t D_x^3\tau - 3u_{xx}D_x^2\xi - 3u_{xt}D_x^2\tau - 3u_{xxx}D_x\xi - 3u_{xxt}D_x\tau,\\
&\phi^{xxt} = D_t D_x^2\phi - u_x D_t D_x^2\xi - u_{xt}D_x^2\xi - 2u_{xx}D_t D_x\xi - 2u_{xxt}D_x\xi - u_t D_t D_x^2\tau - u_{tt}D_x^2\tau\\
&- 2u_{xt}D_t D_x\tau - 2u_{xtt}D_x\tau - u_{xxx}D_t\xi - u_{xxt}D_t\tau,
\end{aligned}
\tag{2.4}
$$

where D_x, D_t are the total derivatives with respect to x and t respectively.

Substituting (2.4) into (2.3), combined with system (1.3) and setting the coefficients of the various monomials in u and v and their partial derivatives equal to zero one obtains the determining equations for the symmetry group of (1.3) as follows

$$
\begin{aligned}
&\xi_u = 0, \ \xi_\rho = 0, \ \tau_x = 0, \ \tau_u = 0, \ \tau_\rho = 0, \ \phi_\rho = 0, \ \phi_{uu} = 0, \ \phi_{xxu} = 0, \ \psi_u = 0,\\
&\rho\tau_t + \psi = 0, \ \phi_u - \xi_x - \psi_\rho = 0, \ (1-\alpha)(\phi_u - \xi_x + \tau_t) = 0, \ \xi_{xx} - 2\phi_{xu} = 0,\\
&u\phi_{xxx} + \phi_{txx} - \kappa\rho\psi_x = 0, \ -\alpha\rho\phi_x + u\psi_x + \psi_t = 0, \ -u(\xi_x - \tau_t) - \xi_t + \phi = 0,\\
&u\xi_{xxx} - (1-\alpha)\phi_{xx} = 0, \ -2\xi_{tx} + (1-\alpha)\phi_x + 3u(\phi_{xu} - \xi_{xx}) + \phi_{tu} = 0.
\end{aligned}
\tag{2.5}
$$

Solving these determining equations yields

$$
\begin{cases}
\xi = (F_1'(t) + C_1 + C_2)x + F_2(t) + C_3,\\
\tau = -F_1(t)\alpha + C_2 t + C_4,\\
\phi = F_1''(t)x + ((1+\alpha)F_1'(t) + C_1)u + F_2'(t),\\
\psi = (\alpha F_1'(t) - C_2)\rho,
\end{cases}
\tag{2.6}
$$

where $F_1(t), F_2(t)$ are arbitrary functions of t, C_1, C_2, C_3 and C_4 are arbitrary constants.

Thus, the Lie algebra of infinitesimal symmetries of system (1.3) is spanned by the following vector fields

$$
\begin{aligned}
&V_1 = F_1'(t)x\frac{\partial}{\partial x} - \alpha F_1(t)\frac{\partial}{\partial t} + [F_1''(t)x + (1+\alpha)uF_1'(t)]\frac{\partial}{\partial u} + \alpha\rho F_1'(t)\frac{\partial}{\partial \rho},\\
&V_2 = F_2(t)\frac{\partial}{\partial x} + F_2'(t)\frac{\partial}{\partial u}, \ \ V_3 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u},\\
&V_4 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \rho\frac{\partial}{\partial \rho}, \ \ V_5 = \frac{\partial}{\partial x}, \ \ V_6 = \frac{\partial}{\partial t},
\end{aligned}
$$

where V_1 and V_2 are the vector fields corresponding to the arbitrary functions $F_1(t)$ and $F_2(t)$ respectively.

The commutation relations of Lie algebra determined by $V_i(i = 1, 2, \cdots, 6)$, which are shown as

$$
\begin{aligned}
&[V_i, V_i] = 0,\ i = 1, 2, \cdots, 6,\\
&[V_1, V_2] = -[V_2, V_1] = V_6(-F_1' F_2 - \alpha F_1 F_2'),\ \ [V_1, V_3] = -[V_3, V_1] = [V_2, V_5] = -[V_5, V_2] = 0,\\
&[V_3, V_4] = -[V_4, V_3] = [V_3, V_6] = -[V_6, V_3] = [V_5, V_6] = -[V_6, V_5] = 0,\\
&[V_1, V_4] = -[V_4, V_1] = V_1(F_1 - tF_1'),\ \ [V_1, V_5] = -[V_5, V_1] = V_2(-F_1'),\\
&[V_1, V_6] = -[V_6, V_1] = V_1(-F_1'),\ \ [V_2, V_3] = -[V_3, V_2] = V_2(F_2),\\
&[V_2, V_4] = -[V_4, V_2] = V_2(F_2 - tF_2'),\ \ [V_2, V_6] = -[V_6, V_2] = V_2(-F_2'),\\
&[V_3, V_5] = -[V_5, V_3] = -V_5,\ \ [V_4, V_5] = -[V_5, V_4] = -V_5,\ \ [V_4, V_6] = -[V_6, V_4] = -V_6.
\end{aligned}
$$

It is obvious that the vector fields $V_i(i = 1, 2, \cdots, 6)$ are closed under the Lie bracket.

3. Symmetry reductions

In this section, we will get similarity variables and its symmetry reductions. By solving the reduced equations, some exact solutions of the system (1.3) will be presented.

Based on the infinitesimals (2.6), the similarity variables are found by solving the corresponding characteristic equations

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\phi} = \frac{d\rho}{\psi}.$$

Case 1 Let $C_1 = C_2 = F_1(t) = 0$, $C_3(\neq 0)$ and C_4 be arbitrary constants, $F_2(t)$ is an arbitrary functions of t, then by solving the characteristic equation one can get the similarity variables

$$\omega = x - \int \frac{F_2(t) + C_4}{C_3} dt,\ \ f(\omega) = u - \frac{F_2(t)}{C_3},\ \ g(\omega) = \rho,$$

and the group-invariant solution is

$$\begin{cases} u = \dfrac{F_2(t)}{C_3} + f(\omega),\\ \rho = g(\omega). \end{cases} \tag{3.1}$$

Substituting the group-invariant solution (3.1) into system (1.3), we reduce equation (1.3) to the following ODE:

$$\begin{cases} C_4 f''' - C_3 f f''' - (1-\alpha) C_3 f' f'' + C_3 \kappa g g' = 0,\\ C_4 g' + \alpha C_3 f' g - C_3 f g' = 0, \end{cases} \tag{3.2}$$

where $f' = df/d\omega, g' = dg/d\omega$.

Case 2 Let C_1, C_3 be arbitrary non-zero constants, $C_2 = C_4 = F_1(t) = F_2(t) = 0$, then by solving the characteristic equation one can get the similarity variables

$$\omega = x \exp(-\frac{C_1 t}{C_3}),\ \ f(\omega) = u \exp(-\frac{C_1 t}{C_3}),\ \ g(\omega) = \rho,$$

and the group-invariant solution is

$$\begin{cases} u = \exp(\dfrac{C_1 t}{C_3}) f(\omega),\\ \rho = g(\omega). \end{cases} \tag{3.3}$$

Substituting the group-invariant solution (3.3) into system (1.3), we reduce (1.3) to the following ODE:

$$\begin{cases} C_1\omega f''' - C_3 f f''' + C_3(\alpha-1)f'f'' + C_1 f'' + C_3\kappa g g' = 0, \\ C_1\omega g' + \alpha C_3 f' g - C_3 f g' = 0, \end{cases} \tag{3.4}$$

where $f' = df/d\omega, g' = dg/d\omega$.

Case 3 Let $F_1(t) = kt, F_2(t) = 0$, C_1, C_2, C_3, C_4 and k be constants which satisfy $C_2 - \alpha k \neq 0$ and $k + C_1 + C_2 \neq 0$, then by solving the characteristic equation one can get the similarity variables

$$\begin{aligned} \omega &= \frac{[(k+C_1+C_2)x + C_4](-\alpha k t + C_2 t + C_3)^{-\frac{k+C_1+C_2}{C_2-\alpha k}}}{k+C_1+C_2}, \\ f(\omega) &= u[(C_2-\alpha k)t + C_3]^{-\frac{\alpha k + C_1 + k}{C_2-\alpha k}}, \\ g(\omega) &= \rho[(C_2-\alpha k)t + C_3], \end{aligned}$$

and the group-invariant solution is

$$\begin{cases} u = [(C_2-\alpha k)t + C_3]^{\frac{\alpha k + C_1 + k}{C_2-\alpha k}} f(\omega), \\ \rho = \dfrac{g(\omega)}{(C_2-\alpha k)t + C_3}. \end{cases} \tag{3.5}$$

Substituting the group-invariant solution (3.5) into system (1.3), we reduce (1.3) to the following ODE:

$$\begin{cases} -(k+C_1+C_2)\omega f''' + f f''' + (1-\alpha)f'f'' - \kappa g g' = 0, \\ -(k+C_1+C_2)\omega g' - \alpha f' g + f g' = 0, \end{cases} \tag{3.6}$$

where $f' = df/d\omega, g' = dg/d\omega$.

4. Exact solutions

In this section, we will derive the solutions of system (1.3) by using the symbolic computation [15–17]. Suppose that the solution of equation (3.2) is in the form

$$f = a_0 + a_1 F + a_2 F^2, g = b_0 + b_1 F + b_2 F^2 \tag{4.1}$$

where $F(\omega)$ expresses the solution of the following generalized Riccati equation

$$F' = r + pF + qF^2, \tag{4.2}$$

and r, p, q are real constants. Substituting (4.1) along with (4.2) into (3.2) and collecting all terms with the same power in $F^i (i = 0, 1, \cdots, 7)$ and setting the coefficients to zero yields a system of algebraic equations. Solving the algebraic equations and we can have the following results

$$\alpha = 2, a_0 = \pm\frac{\sqrt{2\kappa}b_2 p}{4q^2} + \frac{C_4}{C_3}, a_1 = \pm\frac{\sqrt{2\kappa}b_2}{2q}, a_2 = 0, b_0 = \frac{b_2p^2}{4q^2}, b_1 = \frac{b_2 p}{q}, \tag{4.3}$$

with b_2, p, q, r, C_3, C_4 are constants and κ is a positive constant.

The solutions of equation (4.2) are listed as follows:

(a) When $p^2 - 4qr > 0$ and $pq \neq 0$ ($qr \neq 0$),

$$F_1 = -\frac{1}{2q}\left[p + \sqrt{p^2-4qr}\tanh\left(\frac{\sqrt{p^2-4qr}}{2}\omega\right)\right],$$

$$F_2 = -\frac{1}{2q}\left[p + \sqrt{p^2-4qr}\coth\left(\frac{\sqrt{p^2-4qr}}{2}\omega\right)\right],$$

$$F_3 = -\frac{1}{2q}\left[p + \sqrt{p^2-4qr}\left[\tanh\left(\sqrt{p^2-4qr}\omega\right) \pm i\operatorname{sech}\left(\sqrt{p^2-4qr}\omega\right)\right]\right],$$

$$F_4 = \frac{1}{2q}\left[-p + \frac{\sqrt{p^2-4qr}\left[\sqrt{A^2+B^2} - A\cosh\left(\sqrt{p^2-4qr}\omega\right)\right]}{A\sinh\left(\sqrt{p^2-4qr}\omega\right)+B}\right],$$

$$F_5 = \frac{1}{2q}\left[-p - \frac{\sqrt{p^2-4qr}\left[\sqrt{B^2-A^2} + A\sinh\left(\sqrt{p^2-4qr}\omega\right)\right]}{A\cosh\left(\sqrt{p^2-4qr}\omega\right)+B}\right],\ B^2 - A^2 > 0,$$

$$F_6 = \frac{2r\cosh\left(\frac{\sqrt{p^2-4qr}}{2}\omega\right)}{\sqrt{p^2-4qr}\sinh\left(\frac{\sqrt{p^2-4qr}}{2}\omega\right) - p\cosh\left(\frac{\sqrt{p^2-4qr}}{2}\omega\right)},$$

$$F_7 = \frac{2r\sinh\left(\frac{\sqrt{p^2-4qr}}{2}\omega\right)}{\sqrt{p^2-4qr}\cosh\left(\frac{\sqrt{p^2-4qr}}{2}\omega\right) - p\sinh\left(\frac{\sqrt{p^2-4qr}}{2}\omega\right)},$$

where A, B are arbitrary constants.

(b) When $p^2 - 4qr < 0$ and $pq \neq 0$ ($qr \neq 0$),

$$F_8 = \frac{1}{2q}\left[-p + \sqrt{4qr-p^2}\tan\left(\frac{\sqrt{4qr-p^2}}{2}\omega\right)\right],$$

$$F_9 = -\frac{1}{2q}\left[p + \sqrt{4qr-p^2}\cot\left(\frac{\sqrt{4qr-p^2}}{2}\omega\right)\right],$$

$$F_{10} = \frac{1}{2q}\left[-p + \sqrt{4qr-p^2}\left[\tan\left(\sqrt{4qr-p^2}\omega\right) \pm \sec\left(\sqrt{4qr-p^2}\omega\right)\right]\right],$$

$$F_{11} = \frac{1}{2q}\left[-p + \frac{\sqrt{4qr-p^2}\left[\sqrt{A^2-B^2} - A\cos\left(\sqrt{4qr-p^2}\omega\right)\right]}{A\sin\left(\sqrt{4qr-p^2}\omega\right)+B}\right],\ A^2 - B^2 > 0,$$

$$F_{12} = \frac{1}{2q}\left[-p - \frac{\sqrt{4qr-p^2}\left[\sqrt{A^2-B^2} - A\sin\left(\sqrt{4qr-p^2}\omega\right)\right]}{A\cos\left(\sqrt{4qr-p^2}\omega\right)+B}\right],\ A^2 - B^2 > 0,$$

$$F_{13} = \frac{-2r\cos\left(\frac{\sqrt{4qr-p^2}}{2}\omega\right)}{\sqrt{4qr-p^2}\sin\left(\frac{\sqrt{4qr-p^2}}{2}\omega\right) + p\cos\left(\frac{\sqrt{4qr-p^2}}{2}\omega\right)},$$

$$F_{14} = \frac{2r\sin\left(\frac{\sqrt{4qr-p^2}}{2}\omega\right)}{\sqrt{4qr-p^2}\cos\left(\frac{\sqrt{4qr-p^2}}{2}\omega\right) - p\sin\left(\frac{\sqrt{4qr-p^2}}{2}\omega\right)},$$

where A, B are arbitrary constants.

(c) When $r = 0$ and $pq \neq 0$,

$$F_{15} = -\frac{pC}{q\left[\cosh(p\omega) - \sinh(p\omega) + C\right]},$$
$$F_{16} = -\frac{p\left[\sinh(p\omega) + \cosh(p\omega)\right]}{q\left[\sinh(p\omega) + \cosh(p\omega) + C\right]},$$

where C is an arbitrary constant.

(d) When $p = r = 0$ and $q \neq 0$,

$$F_{17} = -\frac{1}{q\omega + C},$$

where C is an arbitrary constant.

Substituting (4.3) into (4.1) and (3.1), then we can obtain the following different exact solutions of system (1.3):

(a1) If $\Delta = p^2 - 4qr > 0$ and $pq \neq 0$ $(qr \neq 0)$, then the solutions of system (1.3) with $\alpha = 2$ can be derived as

$$\begin{cases} u_1(x,t) = \dfrac{F_2(t) + C_4}{C_3} \pm \dfrac{\sqrt{2\kappa}b_2\sqrt{\Delta}}{4q^2}\tanh\left(\dfrac{\sqrt{\Delta}}{2}\omega\right), \\ \rho_1(x,t) = \dfrac{b_2\Delta}{4q^2}\tanh^2\left(\dfrac{\sqrt{\Delta}}{2}\omega\right), \end{cases} \tag{4.4}$$

where $\omega = x - \int \frac{F_2(t)}{C_3}dt$.

If we take $F(t) = \frac{F_2(t) + C_4}{C_3}$, $\sqrt{\Delta} = 2c_1 (c_1 > 0), b = \frac{b_2}{4q^2}$, then the above solution can be expressed as a simple form as

$$\begin{cases} u_1(x,t) = F(t) \pm 2\sqrt{2\kappa}bc_1\tanh(c_1\omega), \\ \rho_1(x,t) = 4bc_1^2\tanh^2(c_1\omega), \end{cases} \tag{4.5}$$

where $\omega = x - \int F(t)dt$, and $c_1 (> 0), b, \kappa$ are constants.

Similarly, we can derive the other solutions of system (1.3) as

$$\begin{cases} u_2(x,t) = F(t) \pm 2\sqrt{2\kappa}bc_1\coth(c_1\omega), \\ \rho_2(x,t) = 4bc_1^2\coth^2(c_1\omega). \end{cases} \tag{4.6}$$

$$\begin{cases} u_3(x,t) = F(t) \pm 2\sqrt{2\kappa}bc_1\left[\tanh(2c_1\omega) \pm i\,\mathrm{sech}(2c_1\omega)\right], \\ \rho_3(x,t) = 4bc_1^2\left[\tanh(2c_1\omega) \pm i\,\mathrm{sech}(2c_1\omega)\right]^2. \end{cases} \tag{4.7}$$

$$\begin{cases} u_4(x,t) = F(t) \pm 2\sqrt{2\kappa} b c_1 \dfrac{\sqrt{A^2+B^2} - A\cosh(2c_1\omega)}{A\sinh(2c_1\omega)+B}, \\ \rho_4(x,t) = 4bc_1^2\left[\dfrac{\sqrt{A^2+B^2} - A\cosh(2c_1\omega)}{A\sinh(2c_1\omega)+B}\right]^2, \end{cases} \tag{4.8}$$

where A, B are arbitrary constants.

$$\begin{cases} u_5(x,t) = F(t) \pm 2\sqrt{2\kappa} b c_1 \dfrac{\sqrt{B^2-A^2} + A\sinh(2c_1\omega)}{A\cosh(2c_1\omega)+B}, \\ \rho_5(x,t) = 4bc_1^2\left[\dfrac{\sqrt{B^2-A^2} + A\sinh(2c_1\omega)}{A\cosh(2c_1\omega)+B}\right]^2, \end{cases} \tag{4.9}$$

where $B^2 - A^2 > 0$.

$$\begin{cases} u_6(x,t) = F(t) \pm 2\sqrt{2\kappa} b c_1 \left[\dfrac{p\sinh(c_1\omega) - 2c_1\cosh(c_1\omega)}{2c_1\sinh(c_1\omega) - p\cosh(c_1\omega)}\right], \\ \rho_6(x,t) = 4bc_1^2\left[\dfrac{p\sinh(c_1\omega) - 2c_1\cosh(c_1\omega)}{2c_1\sinh(c_1\omega) - p\cosh(c_1\omega)}\right]^2. \end{cases} \tag{4.10}$$

$$\begin{cases} u_7(x,t) = F(t) \pm 2\sqrt{2\kappa} b c_1 \left[\dfrac{p\cosh(c_1\omega) - 2c_1\sinh(c_1\omega)}{2c_1\cosh(c_1\omega) - p\sinh(c_1\omega)}\right], \\ \rho_7(x,t) = 4bc_1^2\left[\dfrac{p\cosh(c_1\omega) - 2c_1\sinh(c_1\omega)}{2c_1\cosh(c_1\omega) - p\sinh(c_1\omega)}\right]^2. \end{cases} \tag{4.11}$$

(a2) When $\Delta = p^2 - 4qr < 0$ and $pq \neq 0$ $(qr \neq 0)$, if we denote $F(t) = \dfrac{F_2(t) + C_4}{C_3}$, $\sqrt{-\Delta} = 2c_1 (c_1 > 0)$, $b = \dfrac{b_2}{4q^2}$, then the solutions of system (1.3) with $\alpha = 2$ can be derived as

$$\begin{cases} u_8(x,t) = F(t) \pm 2\sqrt{2\kappa} b c_1 \tan(c_1\omega), \\ \rho_8(x,t) = 4bc_1^2\tan^2(c_1\omega). \end{cases} \tag{4.12}$$

$$\begin{cases} u_9(x,t) = F(t) \pm 2\sqrt{2\kappa} b c_1 \cot(c_1\omega), \\ \rho_9(x,t) = 4bc_1^2\cot^2(c_1\omega). \end{cases} \tag{4.13}$$

$$\begin{cases} u_{10}(x,t) = F(t) \pm 2\sqrt{2\kappa} b c_1 [\tan(2c_1\omega) \pm i\,\mathrm{sech}(2c_1\omega)], \\ \rho_{10}(x,t) = 4bc_1^2[\tan(2c_1\omega) \pm i\,\mathrm{sech}(2c_1\omega)]^2. \end{cases} \tag{4.14}$$

$$\begin{cases} u_{11}(x,t) = F(t) \pm 2\sqrt{2\kappa} b c_1 \dfrac{\sqrt{A^2-B^2} - A\cos(2c_1\omega)}{A\sin(2c_1\omega)+B}, \\ \rho_{11}(x,t) = 4bc_1^2\left[\dfrac{\sqrt{A^2-B^2} - A\cos(2c_1\omega)}{A\sin(2c_1\omega)+B}\right]^2, \end{cases} \tag{4.15}$$

where A, B are arbitrary constants and $A^2 - B^2 > 0$.

$$\begin{cases} u_{12}(x,t) = F(t) \pm 2\sqrt{2\kappa}bc_1 \dfrac{\sqrt{A^2-B^2} - A\sin(2c_1\omega)}{A\cos(2c_1\omega)+B}, \\ \rho_{12}(x,t) = 4bc_1^2\left[\dfrac{\sqrt{A^2-B^2} - A\sin(2c_1\omega)}{A\cos(2c_1\omega)+B}\right]^2, \end{cases} \tag{4.16}$$

where $A^2 - B^2 > 0$.

$$\begin{cases} u_{13}(x,t) = F(t) \pm 2\sqrt{2\kappa}bc_1\left[\dfrac{p\sin(c_1\omega) - 2c_1\cos(c_1\omega)}{2c_1\sin(c_1\omega) + p\cos(c_1\omega)}\right], \\ \rho_{13}(x,t) = 4bc_1^2\left[\dfrac{p\sin(c_1\omega) - 2c_1\cos(c_1\omega)}{2c_1\sin(c_1\omega) + p\cos(c_1\omega)}\right]^2. \end{cases} \tag{4.17}$$

$$\begin{cases} u_{14}(x,t) = F(t) \pm 2\sqrt{2\kappa}bc_1\left[\dfrac{p\cos(c_1\omega) + 2c_1\sin(c_1\omega)}{2c_1\cos(c_1\omega) - p\sin(c_1\omega)}\right], \\ \rho_{14}(x,t) = 4bc_1^2\left[\dfrac{p\cos(c_1\omega) + 2c_1\sin(c_1\omega)}{2c_1\cos(c_1\omega) - p\sin(c_1\omega)}\right]^2. \end{cases} \tag{4.18}$$

(a3) When $r = 0$ and $pq \neq 0$, if we denote $F(t) = \dfrac{F_2(t) + C_4}{C_3}, b = \dfrac{b_2}{4q^2}$, then the solutions of system (1.3) with $\alpha = 2$ can be derived as

$$\begin{cases} u_{15}(x,t) = F(t) \pm \sqrt{2\kappa}bp\left(1 - \dfrac{2C}{\cosh(p\omega) - \sinh(p\omega) + C}\right), \\ \rho_{15}(x,t) = bp^2\left(1 - \dfrac{2C}{\cosh(p\omega) - \sinh(p\omega) + C}\right)^2, \end{cases} \tag{4.19}$$

where $\omega = x - \int F(t)dt$, $F(t)$ is an arbitrary function and b, C are constants.

$$\begin{cases} u_{16}(x,t) = F(t) \pm \sqrt{2\kappa}bp\left(1 - \dfrac{2C}{\sinh(p\omega) + \cosh(p\omega) + C}\right), \\ \rho_{16}(x,t) = bp^2\left(1 - \dfrac{2C}{\sinh(p\omega) + \cosh(p\omega) + C}\right)^2. \end{cases} \tag{4.20}$$

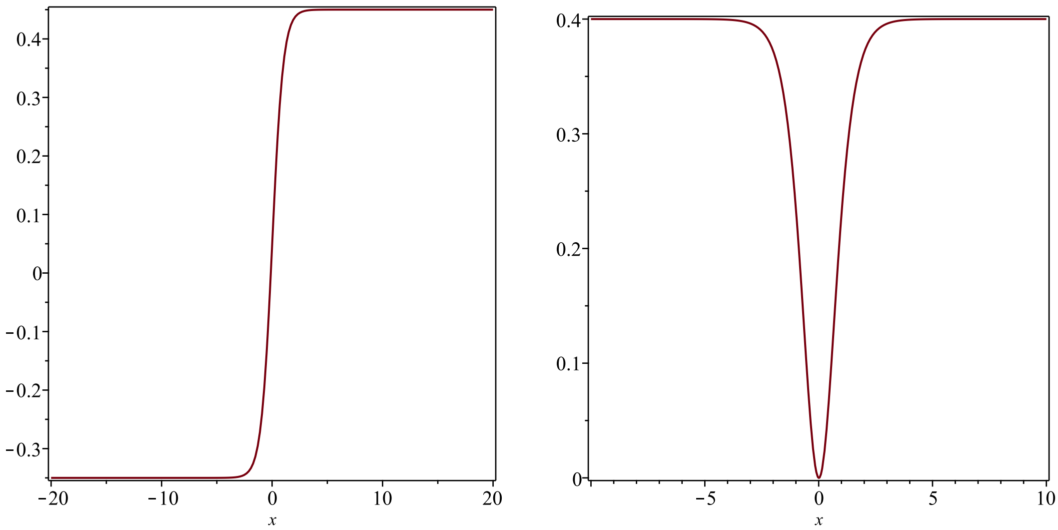
(a4) When $p = r = 0$ and $q \neq 0$, if we denote $F(t) = \dfrac{F_2(t) + C_4}{C_3}, b = \dfrac{b_2}{4q^2}$, then the solution of system (1.3) with $\alpha = 2$ can be derived as

$$\begin{cases} u_{17}(x,t) = F(t) \pm 2\sqrt{2\kappa}b\dfrac{1}{\omega + C}, \\ \rho_{17}(x,t) = 4b\left(\dfrac{1}{\omega + C}\right)^2, \end{cases} \tag{4.21}$$

where $\omega = x - \int F(t)dt$, $F(t)$ is an arbitrary function and b, C are constants.

In order to show the properties of the above solutions visually, we plot the 2D-graphs of some typical solutions. Some wave figures are given as follows (Figures 1–5):

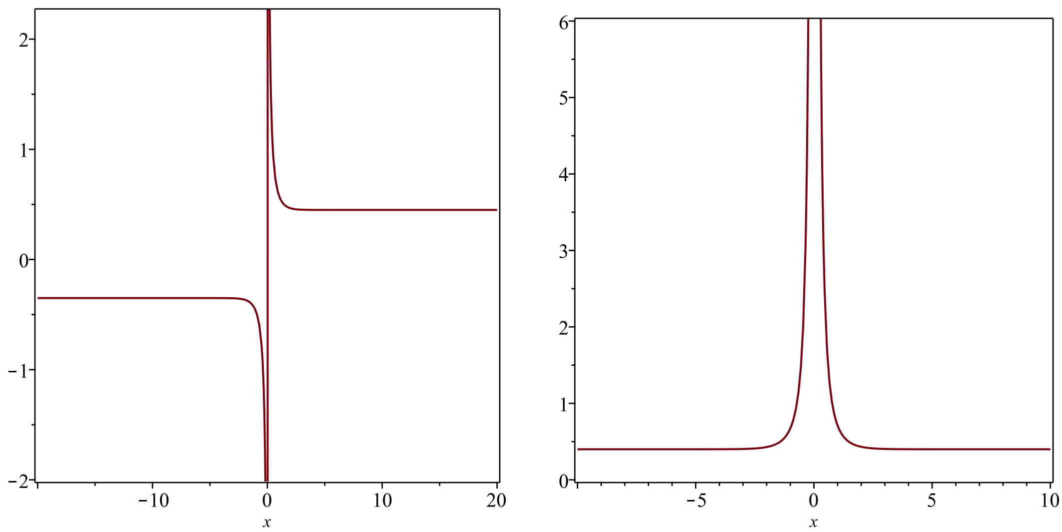
For the solution (4.5), if we take the integration constant as 0 in $\omega = x - \int F(t)dt$, then we plot the solution for the plus sign in u_1 as



(a) α=2, t=0.2 (b) α=2,t=0.2

Figure 1. (a) 2D figure of solution u_1 with $F = 0.05, \kappa = 2, b = 0.1, c_1 = 1$, (b) 2D figure of solution ρ_1 with $F = 0.05, \kappa = 2, b = 0.1, c_1 = 1$.

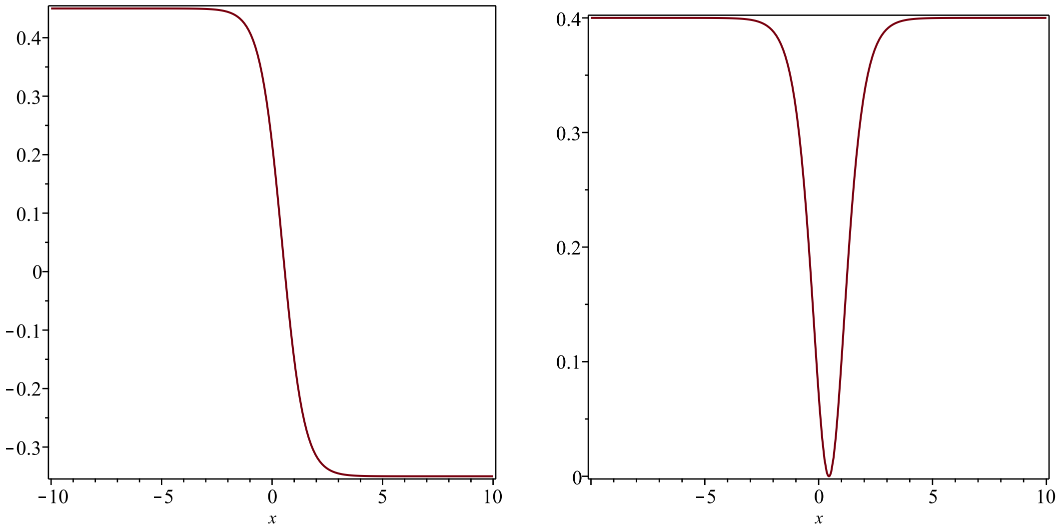
For the solution (4.6), if we take the integration constant as 0 in $\omega = x - \int F(t)dt$, then we plot the solution for the plus sign in u_2 as



(a) α=2, t=0.5 (b) α=2,t=0.5

Figure 2. (a) 2D figure of solution u_2 with $F = 0.05, \kappa = 2, b = 0.1, c_1 = 1$, (b) 2D figure of solution ρ_2 with $F = 0.05, \kappa = 2, b = 0.1, c_1 = 1$.

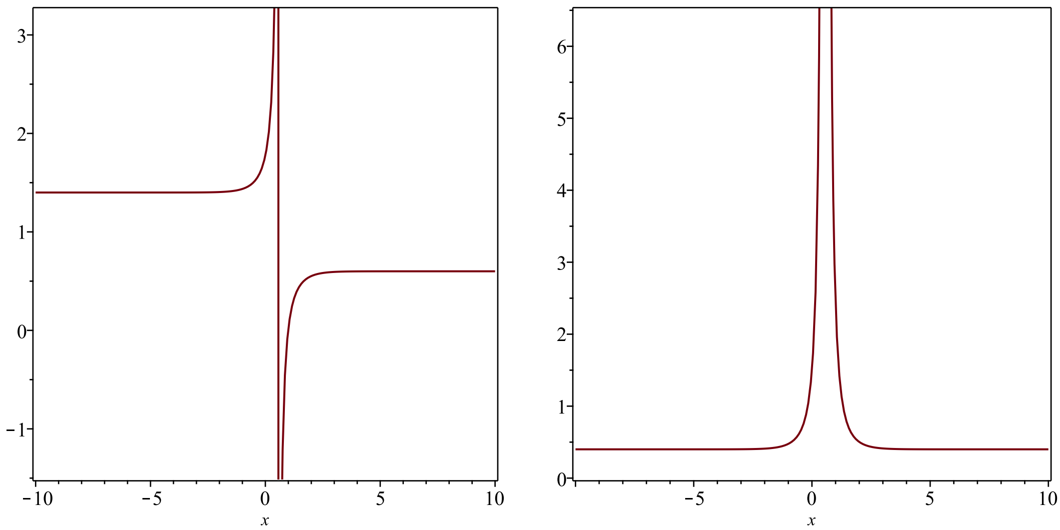
For the solution (4.8), if we take the integration constant as 0 in $\omega = x - \int F(t)dt$, then we plot the solution for the plus sign in u_4 as



(a) α=2, t=0.2 (b) α=2,t=0.2

Figure 3. (a) 2D figure of solution u_4 with $F = 0.05, \kappa = 2, b = 0.1, c_1 = 1, A = 2, B = 2$, (b) 2D figure of solution ρ_4 with $F = 0.05, \kappa = 2, b = 0.1, c_1 = 1, A = 2, B = 2$.

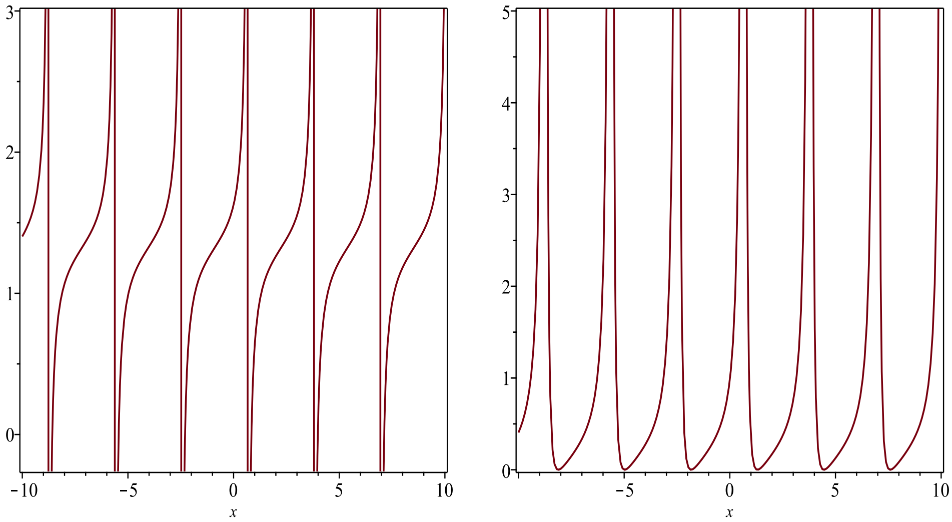
For the solution (4.10), if we take the integration constant as 0 in $\omega = x - \int F(t)dt$, then we plot the solution for the plus sign in u_6 as



(a) α=2, t=0.2 (b) α=2,t=0.2

Figure 4. (a) 2D figure of solution u_6 with $F = 1, \kappa = 2, b = 0.1, c_1 = 1, p = 1$, (b) 2D figure of solution ρ_6 with $F = 1, \kappa = 2, b = 0.1, c_1 = 1, p = 1$.

For the solution (4.17), if we take the integration constant as 0 in $\omega = x - \int F(t)dt$, then we plot the solution for the plus sign in u_{13} as



(a) α=2, t=0.02 (b) α=2,t=0.02

Figure 5. (a) 2D figure of solution u_{13} with $F = 0.02, \kappa = 2, b = 0.1, c_1 = 1, p = 1$, (b) 2D figure of solution ρ_{13} with $F = 0.02, \kappa = 2, b = 0.1, c_1 = 1, p = 1$.

Remark 1 If we take $F(t)$ as a constant, then all of the above solutions of system (1.3) are traveling wave solutions.

Remark 2 For the reduced equations (3.4) and (3.6), there exist a power series solutions [18, 19]. We omit the details here for brevity.

5. Conservation law

In this section, we use the direct multiplier method [20] to derive a conservation law for system (1.3). The zero-order multipliers $\Lambda_1(t, x, u, \rho)$, $\Lambda_2(t, x, u, \rho)$ for the system (1.3) are determined by

$$\begin{cases} \dfrac{\delta}{\delta u}\left[\Lambda_1(u_{xxt} + uu_{xxx} + (1-\alpha)u_x u_{xx} - \kappa\rho\rho_x) + \Lambda_2(\rho_t + u\rho_x - \alpha u_x \rho)\right] = 0 \\ \dfrac{\delta}{\delta \rho}\left[\Lambda_1(u_{xxt} + uu_{xxx} + (1-\alpha)u_x u_{xx} - \kappa\rho\rho_x) + \Lambda_2(\rho_t + u\rho_x - \alpha u_x \rho)\right] = 0, \end{cases} \tag{5.1}$$

where $\frac{\delta}{\delta u}, \frac{\delta}{\delta \rho}$ are Euler-Lagrange operators defined by

$$\begin{aligned} \frac{\delta}{\delta u} &= \frac{\partial}{\partial u} - D_x\frac{\partial}{\partial u_x} + D_x^2\frac{\partial}{\partial u_{xx}} - D_t D_x^2\frac{\partial}{\partial u_{xxt}} - D_x^3\frac{\partial}{\partial u_{xxx}}, \\ \frac{\delta}{\delta \rho} &= \frac{\partial}{\partial \rho} - D_t\frac{\partial}{\partial \rho_t} - D_x\frac{\partial}{\partial \rho_x}. \end{aligned} \tag{5.2}$$

Expanding (5.1) and splitting with respect to derivative of u, ρ, we obtain the following determining equations

$$\begin{aligned} &\Lambda_{1u} = 0,\ \Lambda_{1\rho} = 0,\ \Lambda_{1x} = 0,\ \Lambda_{2u} = 0,\ \Lambda_{2t} = 0,\ \Lambda_{2x} = 0, \\ &\alpha\rho\Lambda_{2\rho} + u\Lambda_{2u} + (1+\alpha)\Lambda_2 = 0. \end{aligned} \tag{5.3}$$

Then we obtain the solution

$$\Lambda_1(t,x,u,\rho) = H(t),\ \Lambda_2(t,x,u,\rho) = A\rho^{-\frac{1+\alpha}{\alpha}}, \tag{5.4}$$

where A is an arbitrary constant, $H(t)$ is an arbitrary functions with respect to t. From the solution (5.4), we can see that system (1.3) has one zero-order multiplier in the form of $\Lambda_1 = H(t),\ \Lambda_2 = \rho^{-\frac{1+\alpha}{\alpha}}$. So a conservation law of system (1.3) is

$$D_t\left(-\alpha\rho^{-\frac{1}{\alpha}}\right) + D_x\left(H(t)u_{xt} + H(t)uu_{xx} - \frac{\alpha}{2}H(t)u_x^2 - \frac{\kappa}{2}H(t)\rho^2 - \alpha u\rho^{-\frac{1}{\alpha}}\right) = 0. \tag{5.5}$$

6. Conclusions

In this paper, a generalized 2-HS system is investigated by using the classical Lie group method. First, Lie symmetry analysis was performed for the generalized 2-HS system, and its infinitesimal generator, geometric vector fields and commutation table of Lie algebra were obtained. Then, all of the similarity variables and its symmetry reductions of this equation are obtained. And by solving the reduced equations, some new exact solutions including traveling wave solutions of this generalized 2-HS system are constructed successfully. These are new solutions for the generalized 2-HS system. Finally, a conservation law of the generalized 2-HS system are shown by using the multiplier method.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (No.11461022) and Applied Basic Research Foundation of Yunnan Province (Nos. 2018FH001-013 and 2018FH001-014), the Science Research Foundation of Yunnan Education Bureau (No. 2018JS479) and the Second Batch of Middle and Young Aged Academic Backbone of Honghe University (No. 2015GG0207).

Conflict of interest

The authors declare that there are no conflict interests regarding the publication of this paper.

References

1. J. K. Hunter, R. Saxton, Dynamics of director fields, *SIAM J. Appl. Math.,* **51** (1991), 1498–1521.
2. R. Camassa, D. D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.,* **71** (1993), 1661–1664.
3. R. Camassa, D. D. Holm, J. M. Hyman, A new integrable shallow water equation, *Adv. Appl. Mech.,* **31** (1994), 1–31.
4. H. Wu, M. Wunsch, Global existence for the generalized two-component Hunter-Saxton system, *J. Math. Fluid Mech.,* **14** (2012), 455–469.
5. B. Moon, Y. Liu, Wave breaking and global existence for the generalized periodic two-component Hunter- Saxton system, *J. Diff. Equ.,* **253** (2012), 319–355.

6. Z. G. Guo, S. R. Liu, W. M. Wang, On a variation of the two-component Hunter-Saxton system, *Appl. Math. Comput.,* **259** (2015), 45–52.
7. J. Lenells, Spheres, Kähler geometry, and the two-component Hunter-Saxton equation, *Proc. R. Soc. A,* **469** (2013), 20120726.
8. M. Wunsch, The generalized Hunter-Saxton system, *SIAM J. Math. Anal.,* **42** (2010), 1286–1304.
9. P. Olver, *Applications of Lie Groups to Differential Equations*, Grad. Texts in Math., vol. 107, Springer, New York, 1993.
10. M. Craddock, K. Lennox, Lie group symmetries as integral transforms of fundamental solutions, *J. Diff. Equ.,* **232** (2007), 652–674.
11. H. Z. Liu, Y. X. Geng, Symmetry reductions and exact solutions to the systems of carbon nanotubes conveying fluid, *J. Diff. Equ.,* **254** (2013), 2289–2303.
12. H. Z. Yang, W. Liu, B. Y. Yang, et al. Lie symmetry analysis and exact explicit solutions of three-dimensional Kudryashov-Sinelshchikov equation, *Commun. Nonlinear Sci. Numer. Simulat.,* **27** (2015), 271–280
13. S. J. Yang, C. C. Hua, Lie symmetry reductions and exact solutions of a coupled KdV-Burgers equation, *Appl. Math. Comput.,* **234** (2014), 579–583.
14. X. Lv, M. S. Peng, Systematic construction of infinitely many conservation laws for certain nonlinear evolution equations in mathematical physics, *Commun. Nonlinear Sci. Numer. Simulat.,* **18** (2013), 2304–2312.
15. Q. S. Liu, R. G. Zhang, L. G. Yang, J. Song, A new model equation for nonlinear Rossby waves and some of its solutions, *Phys. Lett. A* **383** (2019), 514–525.
16. Q. S. Liu, L. G. Chen, Time-space fractional model for complex cylindrical Ion-Acoustic waves in ultrarelativistic plasmas, *Complexity,* **2020** (2020), 9075823.
17. Z. B. Bai, The introduction to the fractional differential equation boundary value problems, *Math. Model. Appl.,* **6** (2017), 1–10
18. H. Z. Liu, J. B. li, Q. X. Zhang, Lie symmetry analysis and exact explicit solutions for general Burgers equation, *J. Comput. Appl. Math.,* **228** (2009), 1–9.
19. H. Z. Liu, J. B. li, Symmetry reductions, dynamical behavior and exact explicit solutions to the Gordon types of equations, *J. Comput. Appl. Math.,* **257** (2014), 144–156.
20. S. C. Anco, G. Bluman, Direct construction method for conservation laws of partial differential equations, Part I: Examples of conservation law classifications, *Eur. J. Appl. Math.,* **13** (2002), 545–566.

AIMS Press