Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(2): 1075–1086.
DOI:10.3934/math.2021064
Received: 24 July 2020
Accepted: 02 November 2020
Published: 09 November 2020

Research article

Inertial projection methods for solving general quasi-variational inequalities

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Abstract: In this paper, we consider a new class of quasi-variational inequalities, which is called the general quasi-variational inequality. Using the projection operator technique, we establish the equivalence between the general quasi-variational inequalities and the fixed point problems. We use this alternate formulation to propose some new inertial iterative schemes for solving the general quasi-variational inequalities. The convergence criteria of the new inertial projection methods under some appropriate conditions is investigated. Since the general quasi-variational inequalities include the quasi-variational inequalities, variational inequalities, complementarity problems and the related optimization problems as special cases, our results continue to hold for these problems. It is an interesting problem to compare the efficiency of the proposed methods with other known methods.

Keywords: quasi-variational inequality; inertial term; projection operator; inertial methods; convergence
Mathematics Subject Classification: 49J40, 90C33

1. Introduction

Variational inequality theory contains a wealth of new ideas and techniques. Variational inequality theory, which was introduced and considered in the early 1960s by Stampacchia [35], can be viewed as a natural extension and generalization of the variational principles. It is well known that the minimum $\mu \in K$ of differentiable convex functions on the convex set K can be characterized by an inequality of type :

$$\langle f'(\mu), \nu - \mu \rangle \geq 0, \quad \forall \nu \in K,$$

which is called the variational inequality. Variational inequalities can be viewed as a novel and significant extension of the variational principles, the origin of which can be traced back to Euler, Lagrange, Bernoulli Brothers and Newton. It have been shown that the variational inequalities provide a general, natural, simple, unified and efficient framework for a general treatment of a wide class of unrelated linear and nonlinear problems. This theory combines theoretical and algorithmic advances with novel domain of applications. Analysis of these problems requires a blend of techniques from convex analysis, functional analysis and numerical analysis, There are significant developments of these problems related to non-convex optimization, iterative method and structural analysis. For recent developments and applications of variational inequalities in various fields of pure and applied sciences, see [16, 27, 29, 32, 34] and references therein. If the convex set does depend upon the solution, then a problem in this class of variational inequalities is called a quasi-variational inequality. Quasi-variational inequalities, which were introduced in the early 1970s, are being used to model various problems arising in different branches of pure and applied sciences in a unified and general manner. Bensoussan and Lions [7] have shown that a class of impulse control problems can be formulated as quasi-variational inequality problem. Quasi-variational inequalities continuously benefit from cross-fertilization between functional analysis, convex analysis, numerical analysis, and physics. This interaction between these fields has played a significant and important role in developing several numerical techniques for solving quasi-variational inequalities and related optimization problems, see [6, 9–15, 20–31] and reference therein.

It is well known that variational inequalities and related optimization problems are equivalent to the fixed point problems. This alternative result is used not only to study the existence theory of the solution of the quasi-variational inequalities, but also to develop several iterative methods such as projection method, implicit methods, and their variant modifications. Antipin [2] suggested gradient projection and extra gradient methods for obtaining the solution of quasi-variational inequality, when the involved operator is strongly monotone and Lipschitz continuous. Mijajlovic et al. [19] introduced a more general gradient projection method with strong convergence for solving this inequality in real Hilbert space. This method works well for many useful purposes, so it has tremendous potential.

Polyak [33] was the first author who propose the heavy ball method involving the inertial iteration method to expedite the fast convergence of the method. Alvarez et al. [3] used it to set up a proximal point algorithm. Recently, the inertial method is obtained from the oscillator equation with damping and conservative restoring force. It has become a significant source for improving the performance of the method and has great convergence characteristics. The general foremost features of inertial-type alternatives are that we use previous iterations to construct the next. For constructing inertial methods, many authors have combined the inertial term $\{\Theta_n(\mu_n - \mu_{n-1})\}$ into many kinds of algorithms, such as Halpern, Kranoselski, Mann, Noor, Viscosity, etc. for finding the solution optimization problems and fixed point problems. Here Θ_n is an extrapolating factor that stimulates the convergence rate of the method. Shehu et al. [36] suggested and studied the inertial type projection methods for solving classical quasi-variational inequalities involving the modified projection method Noor [23]. For more details, see [1, 4–6, 8, 18, 33, 36] and reference therein.

Motivated by the ongoing research activities in this direction, we consider a new class of quasi-variational inequalities, which is called the general quasi-variational inequality. It has been shown that several classes of quasi-variational inequalities can be obtained as special cases of general quasi-variational inequalities, which shows that general quasi-variational inequalities are unified ones. It is worth mentioning that the general quasi-variational inequalities considered in this paper are distinctly different from the general quasi-variational inequalities studied by Noor [22, 23, 26] and Noor et al. [27]. We have proved that the general quasi-variational inequalities are equivalent to the fixed point formulation using the projection technique. We use this alternative formulation to propose some new inertial projection methods for solving the general quasi-variational inequalities using the techniques of Noor et al. [27, 31]. We investigate the convergence criteria of the inertial methods under certain conditions. Results obtained in this papers continue to hold for several new and known classes of variational inequalities and related optimization problems. As applications of the main results, some special cases are discussed. We have only studied theoretical aspects of the new algorithms. The implementation and comparison with other methods is an interesting and challenging problem, which needs further efforts.

2. Basic concepts

Let K be a nonempty, closed and convex set in a real Hilbert space H with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let $T, g : H \longrightarrow H$ be nonlinear operators in H. Let $K : H \longrightarrow H$ be a set-valued mapping which, for any element $\mu \in H$, associates a convex-valued and closed set $K(\mu) \subset H$.

We consider the general quasi-variational inequality problem, which consists of finding $\mu \in H : g(\mu) \in K(\mu)$, such that

$$\langle \rho T\mu + \mu - g(\mu), g(\nu) - \mu \rangle \geq 0, \qquad \forall \nu \in H : g(\nu) \in K(\mu), \tag{2.1}$$

where $\rho > 0$ is a constant.

The problem of type (2.1) was introduced and studied by Noor [24, 25]. It is worth mentioning the general quasi-variational inequality (2.1) is quite different than the quasi variational inequality considered and studied by Noor [21, 22]. For more details, see Noor [23] and Noor et al. [27].

Special Cases:

(I). Note that, if $g = I$, the identity operator, then problem (2.1) reduces to the quasi-variational inequality: that is, finding $\mu \in K$, such that

$$\langle T\mu, \nu - \mu \rangle \geq 0, \qquad \forall \nu \in K(\mu). \tag{2.2}$$

It was introduced and studied by Bensoussan et al [7].

(II). $K(\mu) = K$, and $g = I$, then problem (2.1) reduces to the variational inequality: that is, finding $\mu \in K$, such that

$$\langle T\mu, \nu - \mu \rangle \geq 0, \qquad \forall \nu \in K. \tag{2.3}$$

It was introduced and studied by Stampacchia [35] and Lions and Stampacchia [17].

For a different and appropriate choice of the operators and spaces, one can obtain several known and new classes of variational inequalities and related problems. This clearly shows that the problem considered in this paper is more general and unifying.

We need the following basic concepts and results.

Definition 2.1. A mapping $T : H \longrightarrow H$ is called strongly monotone ($\xi \geq 0$), if

$$\langle T\mu - T\nu, \mu - \nu \rangle \geq \xi \|\mu - \nu\|^2, \qquad \forall \mu, \nu \in H. \tag{2.4}$$

Definition 2.2. A mapping $T : H \longrightarrow H$ is called Lipschitz continuous ($\eta > 0$), if

$$\|T\mu - T\nu\| \leq \eta \|\mu - \nu\|, \qquad \forall \mu, \nu \in H. \tag{2.5}$$

From (2.4) and (2.5), it can be noted that $\xi \le \eta$.

We also need the following result, known as Projection Lemma, which plays a significant part in establishing the equivalence between the variational inequalities and the fixed point problem. This result can be used in analyzing the convergence analysis of the projective implicit and explicit methods for solving the variational inequalities and related optimization problems.

Lemma 2.1. *[7] For a given* $\omega \in \mathrm{H}$, *find* $\mu \in \mathrm{K}(\mu)$, *such that*

$$\langle \mu - \omega, \nu - \mu \rangle \ge 0, \quad \forall\, \nu \in \mathrm{K}(\mu),$$

if and only if

$$\mu = \Pi_{\mathrm{K}(\mu)}[\omega],$$

where $\Pi_{\mathrm{K}(\mu)}$ *is the implicit projection of* H *onto the closed convex-valued set* $\mathrm{K}(\mu)$ *in* H.

The implicit projection $\Pi_{\mathrm{K}(\mu)}$ has the following characterization.

Assumpstion 2.1. [28] The implicit projection operator $\Pi_{\mathrm{K}(\mu)}$, satisfies the condition

$$\|\Pi_{\mathrm{K}(\mu)}[\omega] - \Pi_{\mathrm{K}(\nu)}[\omega]\| \le \upsilon \, \|\mu - \nu\| \qquad \forall\, \mu, \nu, \omega \in \mathrm{H}, \tag{2.6}$$

where $\upsilon > 0$, is a constant.

Here we would like to point out that the implicit projection $\Pi_{\mathrm{K}(\mu)}$ is nonexpansive.

The following result is also necessary for investigating our methods.

Lemma 2.2. *[37] Consider a sequence of non negative real numbers* $\{\varrho_n\}$, *satisfying*

$$\varrho_{n+1} \le (1 - \Upsilon_n)\varrho_n + \Upsilon_n \sigma_n + \varsigma_n, \quad \forall\, n \ge 1,$$

where

(i) $\{\Upsilon_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty} \Upsilon_n = \infty$;
(ii) $\limsup \sigma_n \le 0$;
(iii) $\varsigma_n \ge 0 \ (n \ge 1)$, $\sum_{n=1}^{\infty} \varsigma_n < \infty$.

Then, $\varrho_n \longrightarrow 0$ *as* $n \longrightarrow \infty$.

3. Iterative methods

In the following section, we propose some new iterative schemes for solving the general quasi-variational inequality (2.1).

Using Lemma 2.1, one can show that the general quasi-variational inequality (2.1) is equivalent to fixed point problems.

Lemma 3.1. *The function* $\mu \in \mathrm{H} : \mathrm{g}(\mu) \in \mathrm{K}(\mu)$ *is solution of general quasi-variational inequality (2.1) if and only if* $\mu \in \mathrm{H} : \mathrm{g}(\mu) \in \mathrm{K}(\mu)$ *satisfies the relation*

$$\mu = \Pi_{\mathrm{K}(\mu)}[\mathrm{g}(\mu) - \rho \mathrm{T}\mu], \tag{3.1}$$

where $\Pi_{\mathrm{K}(\mu)}$ *is the projection of* H *into* $\mathrm{K}(\mu)$ *and* $\rho > 0$ *is a constant.*

Lemma 3.1 implies that general quasi-variational inequality (2.1) is equivalent to a fixed point problem (3.1). This alternative result is very useful from numerical and theoretical point.

From the relation (3.1), we can defined a mapping $F(\mu)$ associated with the problem (2.1) as:

$$F(\mu) = \Pi_{K(\mu)}\left[g(\mu) - \rho T\mu\right], \tag{3.2}$$

which is used to study the existence of a solution of general quasi-variational inequality (2.1), see [20].

We can rewrite Eq (3.1) using the ideas and technique of Noor et al. [27] as:

$$\mu = \Pi_{K(\mu)}\left[\frac{g(\mu) + g(\mu)}{2} - \rho T\mu\right]. \tag{3.3}$$

This fixed point formulation is used to suggest the implicit method for solving the general quasi-variational inequalities as

Algorithm 3.1. For given $\mu_0 \in H$, compute μ_{n+1} by the recurrence relation

$$\mu_{n+1} = (1 - \alpha_n)\mu_n + \alpha_n \Pi_{K(\mu_{n+1})}\left[\frac{g(\mu_n) + g(\mu_{n+1})}{2} - \rho T\mu_{n+1}\right], \quad n = 0, 1, \ldots,$$

where $\alpha_n \in [0, 1], \quad \forall n \geq 0$.

Algorithm 3.1 is an implicit method. To implement this implicit method, using predictor-corrector technique, we suggest the following inertial-type projection method as:

Algorithm 3.2. For given $\mu_0, \mu_1 \in H$, compute μ_{n+1} by the recurrence relation

$$\omega_n = \mu_n + \Theta_n(\mu_n - \mu_{n-1}), \tag{3.4}$$

$$\mu_{n+1} = (1 - \alpha_n)\mu_n + \alpha_n \Pi_{K(\omega_n)}\left[\frac{g(\mu_n) + g(\omega_n)}{2} - \rho T\omega_n\right], \quad n = 1, 2, \ldots, \tag{3.5}$$

where $\alpha_n, \Theta_n \in [0, 1], \quad \forall n \geq 1$.

Algorithm 3.2 appears to be a new two-step inertial iterative method for solving the general quasi-variational inequality (2.1).

For $\alpha_n = 1$, Algorithm 3.2 reduces to the following inertial method:

Algorithm 3.3. For given $\mu_0, \mu_1 \in H$, compute μ_{n+1} by the recurrence relation

$$\omega_n = \mu_n + \Theta_n(\mu_n - \mu_{n-1}),$$

$$\mu_{n+1} = \Pi_{K(\omega_n)}\left[\frac{g(\mu_n) + g(\omega_n)}{2} - \rho T\omega_n\right], \quad n = 1, 2, \ldots,$$

where $\Theta_n \in [0, 1], \quad \forall n \geq 1$.

If we take g as a linear operator, then Algorithm 3.2 reduces to the following new inertial method:

Algorithm 3.4. For given $\mu_0, \mu_1 \in H$, compute μ_{n+1} by the recurrence relation

$$\omega_n = \mu_n + \Theta_n(\mu_n - \mu_{n-1}),$$

$$\mu_{n+1} = (1 - \alpha_n)\mu_n + \alpha_n \Pi_{K(\omega_n)}\left[\frac{g(\mu_n + \omega_n)}{2} - \rho T\omega_n\right], \quad n = 1, 2, \ldots,$$

where $\alpha_n, \Theta_n \in [0, 1], \quad \forall n \geq 1$.

For g = I, Algorithm 3.2 reduces to the following inertial method:

Algorithm 3.5. For given $\mu_0, \mu_1 \in H$, compute μ_{n+1} by the recurrence relation

$$\omega_n = \mu_n + \Theta_n(\mu_n - \mu_{n-1}),$$
$$\mu_{n+1} = (1-\alpha_n)\mu_n + \alpha_n \Pi_{K(\omega_n)}\left[\frac{\mu_n + \omega_n}{2} - \rho T\omega_n\right], \quad n = 1, 2, \ldots,$$

where $\alpha_n, \Theta_n \in [0, 1], \quad \forall n \geq 1$.

For $K(\mu) = K$, then Algorithm 3.2 reduces to the following inertial method for solving general variational inequality.

Algorithm 3.6. For given $\mu_0, \mu_1 \in H$, compute μ_{n+1} by the recurrence relation

$$\omega_n = \mu_n + \Theta_n(\mu_n - \mu_{n-1}),$$
$$\mu_{n+1} = (1-\alpha_n)\mu_n + \alpha_n \Pi_K\left[\frac{g(\mu_n) + g(\omega_n)}{2} - \rho T\omega_n\right], \quad n = 1, 2, \ldots,$$

where $\alpha_n, \Theta_n \in [0, 1], \quad \forall n \geq 1$.

For a different and suitable choice of operators and spaces in Algorithm (3.2), one can obtain several new and previous iterative methods for solving inequality (2.1) and related problems. This shows that the Algorithm (3.2) is quite general and unifying ones.

4. Convergence analysis

In this section, we analyze the convergence analysis for Algorithm 3.2 under some appropriate conditions.

Theorem 4.1. *Let the following assumptions be fulfilled:*

(i) $K(\mu) \subset H$ *be a nonempty, closed, and convex-valued subset of Hilbert space* H.

(ii) *The operators* $T, g : H \longrightarrow H$ *be strongly monotone and Lipschitz continuous with constants* $\xi_1 > 0, \xi_2 > 0$ *and* $\eta_1 > 0$, $\eta_2 > 0$, *respectively.*

(iii) *Assumption 2.1 holds.*

(iv) *The parameter* $\rho > 0$ *satisfies the conditions*

$$(a).\left|\rho - \frac{\xi_1}{\eta_1^2}\right| < \frac{\sqrt{\xi_1^2 - \eta_1^2\kappa_1(2-\kappa_1)}}{\eta_1^2},\ \xi_1 > \eta_1\sqrt{\kappa_1(2-\kappa_1)},\ \kappa_1 < 2. \tag{4.1}$$

$$(b).\left|\rho - \frac{\xi_1}{\eta_1^2}\right| < \frac{\sqrt{\xi_1^2 - \eta_1^2\kappa_2(2-\kappa_2)}}{\eta_1^2},\ \xi_1 > \eta_1\sqrt{\kappa_2(2-\kappa_2)},\ \kappa_2 < 1. \tag{4.2}$$

where

$$\kappa_1 = \sqrt{4 - 4\xi_2 + \eta_2^2} + 2\upsilon,$$
$$\kappa_2 = \sqrt{1 - 2\xi_2 + \eta_2^2} + \sqrt{4 - 4\xi_2 + \eta_2^2} + 2\upsilon.$$

(v) Let $\alpha_n, \beta_n, \gamma_n, \Theta_n \in [0,1]$, *for all* $n \geq 1$ *such that* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

$\sum_{n=1}^{\infty} \Theta_n \parallel \mu_n - \mu_{n-1} \parallel < \infty$.

Then, for every initial approximation μ_n, *the sequence* $\{\mu_n\}$ *obtained from the iterative scheme defined in Algorithm 3.2 converges to unique solution* $\mu \in H : g(\mu) \in K(\mu)$ *satisfying the general quasi-variational inequality (2.1) as* $n \longrightarrow \infty$.

Proof. Let $\mu \in H : g(\mu) \in K(\mu)$ be a solution of (2.1). Then

$$\mu = (1-\alpha_n)\mu + \alpha_n \Pi_{K(\mu)}\left[\frac{g(\mu)+g(\mu)}{2} - \rho T\mu\right], \tag{4.3}$$

where $0 \leq \alpha_n \leq 1$, for all $n \geq 1$, is a constant.

From (3.5), (4.3), and using Assumption 2.1, we have

$$\begin{aligned}
\|\mu_{n+1} - \mu\| &= \Bigg\|(1-\alpha_n)\mu_n + \alpha_n \Pi_{K(\omega_n)}\left[\frac{g(\mu_n)+g(\omega_n)}{2} - \rho T\omega_n\right] \\
&\quad - (1-\alpha_n)\mu - \alpha_n \Pi_{K(\mu)}\left[\frac{g(\mu)+g(\mu)}{2} - \rho T\mu\right]\Bigg\| \\
&\leq (1-\alpha_n)\|\mu_n - \mu\| + \alpha_n \Bigg\|\Pi_{K(\omega_n)}\left[\frac{g(\mu_n)+g(\omega_n)}{2} - \rho T\omega_n\right] \\
&\quad - \Pi_{K(\mu)}\left[\frac{g(\mu)+g(\mu)}{2} - \rho T\mu\right]\Bigg\| \\
&\leq (1-\alpha_n)\|\mu_n - \mu\| + \alpha_n \Bigg\|\Pi_{K(\omega_n)}\left[\frac{g(\mu_n)+g(\omega_n)}{2} - \rho T\omega_n\right] \\
&\quad - \Pi_{K(\omega_n)}\left[\frac{g(\mu)+g(\mu)}{2} - \rho T\mu\right]\Bigg\| + \alpha_n \Bigg\|\Pi_{K(\omega_n)}\left[\frac{g(\mu)+g(\mu)}{2} - \rho T\mu\right] \\
&\quad - \Pi_{K(\mu)}\left[\frac{g(\mu)+g(\mu)}{2} - \rho T\mu\right]\Bigg\| \\
&\leq (1-\alpha_n)\|\mu_n - \mu\| + \alpha_n \Bigg\|\Big[\frac{g(\mu_n)-g(\mu)}{2} + \frac{g(\omega_n)-g(\mu)}{2} - \rho\,[T\omega_n - T\mu]\,\Big]\Bigg\| \\
&\quad + \alpha_n \upsilon \|\omega_n - \mu\| \\
&= (1-\alpha_n)\|\mu_n - \mu\| + \frac{\alpha_n}{2}\|(\mu_n - \mu) - (\mu_n - \mu) + [g(\mu_n) - g(\mu)]\,\| \\
&\quad + \alpha_n \Bigg\| - (\omega_n - \mu) + \frac{g(\omega_n)-g(\mu)}{2} + (\omega_n - \mu) - \rho\,[T\omega_n - T\mu]\,\Bigg\| \\
&\quad + \alpha_n \upsilon \|\omega_n - \mu\| \\
&\leq (1-\alpha_n)\|\mu_n - \mu\| + \frac{\alpha_n}{2}\|\mu_n - \mu\| + \frac{\alpha_n}{2}\|\mu_n - \mu - [g(\mu_n) - g(\mu)]\,\| \\
&\quad + \alpha_n \|\omega_n - \mu - \frac{1}{2}\,[g(\omega_n) - g(\mu)]\,\| + \|\omega_n - \mu - \rho\,[T\omega_n - T\mu]\,\| \\
&\quad + \alpha_n \upsilon \|\omega_n - \mu\|.
\end{aligned} \tag{4.4}$$

From the strong monotonicity and Lipschitz continuity of operator T, we have

$$\begin{aligned}
&\| \omega_n - \mu - \rho\,[T\omega_n - T\mu]\,\|^2 \\
&= \| \omega_n - \mu \|^2 - 2\rho\langle T\omega_n - T\mu\,,\ \omega_n - \mu\rangle + \rho^2\,\| T\omega_n - T\mu \|^2 \\
&\le (1 - 2\rho\xi_1 + \rho^2\eta_1^2)\,\| \omega_n - \mu \|^2\ . \qquad (4.5)
\end{aligned}$$

Similarly, from the strong monotonicity and Lipschitz continuity of operator g, we have

$$\| \mu_n - \mu - [g\mu_n - g\mu]\,\|^2 \le (1 - 2\xi_2 + \eta_2^2)\,\| \omega_n - \mu \|^2\ . \qquad (4.6)$$

$$\| \omega_n - \mu - \frac{1}{2}\,[g(\omega_n) - g(\mu)]\,\|^2 \le \frac{1}{4}(4 - 4\xi_2 + \eta_2^2)\,\| \omega_n - \mu \|^2\ . \qquad (4.7)$$

From (3.4), we have

$$\begin{aligned}
\| \omega_n - \mu \| &= \| \mu_n - \mu + \Theta_n\,(\mu_n - \mu_{n-1})\ \| \\
&\le \| \mu_n - \mu \| + \Theta_n\,\| \mu_n - \mu_{n-1} \|\ . \qquad (4.8)
\end{aligned}$$

From (4.4)–(4.8), we have

$$\begin{aligned}
&\|\mu_{n+1} - \mu\| \\
&\le (1 - \alpha_n)\|\mu_n - \mu\| + \frac{\alpha_n}{2}\Big(1 + \sqrt{1 - 2\xi_2 + \eta_2^2}\Big)\|\mu_n - \mu\| \\
&\quad + \alpha_n\Big(\frac{1}{2}\sqrt{4 - 4\xi_2 + \eta^2} + \sqrt{1 - 2\rho\xi + \rho^2\eta_2^2} + \upsilon\Big)\|\omega_n - \mu\| \\
&\le \Big[1 - \alpha_n(1 - \vartheta_1)\Big]\|\mu_n - \mu\| + \alpha_n\vartheta_2\,[\|\mu_n - \mu\,\| + \Theta_n\,\|\,\mu_n - \mu_{n-1}\|] \\
&\le \Big[1 - \alpha_n(1 - \vartheta_1)\Big]\|\mu_n - \mu\| + \alpha_n\vartheta_2\|\mu_n - \mu\| + \Theta_n\|\mu_n - \mu_{n-1}\| \\
&= \Big[1 - \alpha_n(1 - (\vartheta_1 + \vartheta_2))\Big]\|\mu_n - \mu\| + \Theta_n\|\mu_n - \mu_{n-1}\|,
\end{aligned}$$

where

$$\begin{aligned}
\vartheta_1 &:= \frac{1}{2}(1 + \sqrt{1 - 2\xi_2 + \eta_2^2}), \\
\vartheta_2 &:= \frac{1}{2}\sqrt{4 - 4\xi_2 + \eta_2^2} + \sqrt{1 - 2\rho\xi + \rho^2\eta^2} + \upsilon < 1, \quad \text{from condition (4.1)}, \\
\vartheta_1 + \vartheta_2 &:= \frac{1}{2}(1 + \sqrt{1 - 2\xi_2 + \eta_2^2} + \sqrt{4 - 4\xi_2 + \eta_2^2}\,) + \sqrt{1 - 2\rho\xi + \rho^2\eta^2} + \upsilon.
\end{aligned}$$

Letting $\vartheta = \vartheta_1 + \vartheta_2$, from condition (4.2), we have $\vartheta < 1$. Since $\sum_{n=1}^{\infty} \alpha_n = \infty$, setting $\sigma_n = 0$ and $\varsigma_n = \sum_{n=1}^{\infty} \Theta_n \,\|\, \mu_n - \mu_{n-1} \,\| < \infty$, by using Lemma 2.2, we have $\mu_n \longrightarrow \mu$ as $n \longrightarrow \infty$. Hence the sequence $\{\mu_n\}$ obtained from Algorithm 3.2 converges to a unique solution $\mu \in H : g(\mu) \in K(\mu)$ satisfying the inequality (2.1), the desired result. □

Similarly convergence analysis for other inertial iterative methods can be estimated.
(I). If $g(\mu) = I$, then following can be obtained result from Theorem 4.1.

Theorem 4.2. *Let the following assumptions be fulfilled:*

- ***(i)*** $K(\mu) \subset H$ *be a nonempty, closed, and convex-valued subset of Hilbert space* H.
- ***(ii)*** *The operators* $T : H \longrightarrow H$ *be strongly monotone and Lipschitz continuous with constant* $\xi_1 > 0$ *and* $\eta_1 > 0$, *respectively.*
- ***(iii)*** *Assumption 2.1 holds.*
- ***(iv)*** *The parameter* $\rho > 0$ *satisfies the conditions*

$$(a).\left|\rho - \frac{\xi_1}{\eta_1^2}\right| < \frac{\sqrt{\xi_1^2 - \eta_1^2\upsilon(2-\upsilon)}}{\eta_1^2},\ \xi_1 > \eta_1\sqrt{\upsilon(2-\upsilon)},\ \upsilon < 1.$$

$$(b).\left|\rho - \frac{\xi_1}{\eta_1^2}\right| < \frac{\sqrt{\xi_1^2 - \eta_1^2\upsilon(2-\upsilon)}}{\eta_1^2},\ \xi_1 > \eta_1\sqrt{\upsilon(2-\upsilon)},\ \upsilon < \frac{1}{2}.$$

- ***(v)*** *Let* $\alpha_n, \beta_n, \gamma_n, \Theta_n \in [0,1]$, *for all* $n \geq 1$ *such that* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

$\sum_{n=1}^{\infty} \Theta_n \parallel \mu_n - \mu_{n-1} \parallel < \infty$.

Then, for every initial approximation μ_n, *the sequence* $\{\mu_n\}$ *obtained from the iterative scheme defined in Algorithm 3.5 converges to unique solution* $\mu \in K(\mu)$ *satisfying the quasi-variational inequality (2.2) as* $n \longrightarrow \infty$.

(II). If $K(\mu) = K$, then following can be obtained result from Theorem 4.1.

Theorem 4.3. *Let the following assumptions be fulfilled:*

- ***(i)*** K *be a nonempty, closed, and convex set in Hilbert space* H.
- ***(ii)*** *The operators* $T, g : H \longrightarrow H$ *be strongly monotone and Lipschitz continuous with constants* $\xi_1 > 0, \xi_2 > 0$, *and* $\eta_1 > 0, \eta_2 > 0$, *respectively.*
- ***(iii)*** *The parameter* $\rho > 0$ *satisfies the conditions*

$$(a).\left|\rho - \frac{\xi_1}{\eta_1^2}\right| < \frac{\sqrt{\xi_1^2 - \eta_1^2\kappa_1(2-\kappa_1)}}{\eta_1^2},\ \xi_1 > \eta_1\sqrt{\kappa_1(2-\kappa_1)},\ \kappa_1 < 2.$$

$$(b).\left|\rho - \frac{\xi_1}{\eta_1^2}\right| < \frac{\sqrt{\xi_1^2 - \eta_1^2\kappa_2(2-\kappa_2)}}{\eta_1^2},\ \xi_1 > \eta_1\sqrt{\kappa_2(2-\kappa_2)},\ \kappa_2 < 1.$$

where

$$\kappa_1 = \sqrt{4 - 4\xi_2 + \eta_2^2},$$
$$\kappa_2 = \sqrt{1 - 2\xi_2 + \eta_2^2} + \sqrt{4 - 4\xi_2 + \eta_2^2}.$$

(iv) *Let* $\alpha_n, \beta_n, \gamma_n, \Theta_n \in [0,1]$, *for all* $n \geq 1$ *such that* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

$\sum_{n=1}^{\infty} \Theta_n \| \mu_n - \mu_{n-1} \| < \infty$.

Then, for every initial approximation μ_n, *the sequence* $\{\mu_n\}$ *obtained from the iterative scheme defined in Algorithm 3.6 converges to unique solution* $\mu \in H : g(\mu) \in K$ *satisfying the general variational inequality as* $n \longrightarrow \infty$.

Remark 4.1. The convergence analysis of other inertial projection algorithms can be analyzed using the above technique.

5. Conclusion

In this paper, we have considered a new class of quasi-variational inequality, which is known as general quasi-variational inequality. We have established the equivalence between the general quasi-variational inequality and the fixed point problem using the projection operator technique. This equivalence is used to suggest and analyze some inertial iterative schemes for solving general quasi-variational inequality using the technique of Noor et al. [27]. Convergence analysis of the inertial projection methods is studied under some suitable conditions. We have only considered the theoretical aspects of inertial projection methods. The implementation and comparison of these new iterative methods with other known methods need further efforts. Also the error estimates and sensitivity analysis for the general quasi-variational inequalities can be considered using the ideas and techniques of Noor [23] and Noor et al. [27]. It is pointed out the general quasi variational inequalities can be extended to n-dimensional functions. It is expected that the results proved in this paper may be starting point further research in this field.

Acknowledgments

The authors would like to thank the Rector, COMSATS University Islamabad, Islamabad, Pakistan for providing excellent academic and research environment. The authors extend their appreciation to King Saud University, Riyadh for funding this research work through Researchers Supporting Project number (RSP-2020/158). Authors are grateful to the referees for their very constructive comments and valuable suggestions.

Conflict of interest

The authors declare that they have no competing interests.

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