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Research article

On generalizations of quantum Simpson's and quantum Newton's inequalities with some parameters

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Abstract: In this paper, we prove two identities concerning quantum derivatives, quantum integrals, and some parameters. Using the newly proved identities, we prove new Simpson's and Newton's type inequalities for quantum differentiable convex functions with two and three parameters, respectively. We also look at the special cases of our key findings and find some new and old Simpson's type inequalities, Newton's type inequalities, midpoint type inequalities, and trapezoidal type inequalities.

Keywords: Simpson's inequalities; Newton's inequalities; quantum calculus; convex functions
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1. Introduction

Simpson's rules are well-known methods for numerical integration and numerical estimation of definite integral. Thomas Simpson is credited with inventing this process (1710–1761). However, about 100 years earlier, Johannes Kepler used the same approximation, so this form is also known as Kepler's law. The three-point Newton-Cotes quadrature rule is included in Simpson's rule, so estimation based on three steps quadratic kernel is often referred to as Newton type results.

1) Simpson's quadrature formula (Simpson's 1/3 rule)

$$\int_{\pi_1}^{\pi_2} \Pi(x)dx \approx \frac{\pi_2 - \pi_1}{6}\left[\Pi(\pi_1) + 4\Pi\left(\frac{\pi_1 + \pi_2}{2}\right) + \Pi(\pi_2)\right].$$

2) Simpson's second formula or Newton-Cotes quadrature formula (Simpson's 3/8 rule).

$$\int_{\pi_1}^{\pi_2} \Pi(x)dx \approx \frac{\pi_2 - \pi_1}{8}\left[\Pi(\pi_1) + 3\Pi\left(\frac{2\pi_1 + \pi_2}{3}\right) + 3\Pi\left(\frac{\pi_1 + 2\pi_2}{3}\right) + \Pi(\pi_2)\right].$$

In the literature, there are several estimations linked to these quadrature laws, one of which is known as Simpson's inequality:

Theorem 1.1. *Suppose that* $\Pi : [\pi_1, \pi_2] \to \mathbb{R}$ *is a four times continuously differentiable mapping on* (π_1, π_2), *and let* $\left\|\Pi^{(4)}\right\|_\infty = \sup_{x\in(\pi_1,\pi_2)} \left|\Pi^{(4)}(x)\right| < \infty$. *Then, one has the inequality*

$$\left|\frac{1}{3}\left[\frac{\Pi(\pi_1) + \Pi(\pi_2)}{2} + 2\Pi\left(\frac{\pi_1 + \pi_2}{2}\right)\right] - \frac{1}{\pi_2 - \pi_1}\int_{\pi_1}^{\pi_2} \Pi(x)dx\right| \le \frac{1}{2880}\left\|\Pi^{(4)}\right\|_\infty (\pi_2 - \pi_1)^4 .$$

Many authors have concentrated on Simpson's type inequalities for different classes of functions in recent years. Since convexity theory is an effective and efficient method for solving a large number of problems that exist within various branches of pure and applied mathematics, some mathematicians have worked on Simpson's and Newton's type results for convex mappings. Dragomir et al. [1], presented new Simpson's type inequalities and their applications to numerical integration quadrature formulas. Furthermore, Alomari et al. in [2] derive some Simpson's type inequalities for s-convex functions. Following that, in [3], Sarikaya et al. discovered variants of Simpson's type inequalities dependent on convexity. The authors given some Newton's type inequalities for harmonic and p-harmonic convex functions in [4, 5]. Iftikhar et al. also have new Newton's type inequalities for functions whose local fractional derivatives are generalized convex in [6].

On the other hand, in the domain of q analysis, many works are being carried out as initiated by Euler in order to attain adeptness in mathematics that constructs quantum computing q calculus considered as a relationship between physics and mathematics. In different areas of mathematics, it has numerous applications such as combinatorics, number theory, basic hypergeometric functions, orthogonal polynomials, and other sciences, as well as mechanics, the theory of relativity, and quantum theory [7, 8]. Quantum calculus also has many applications in quantum information theory, which is an interdisciplinary area that encompasses computer science, information theory, philosophy, and cryptography, among other areas [9, 10]. Apparently, Euler invented this important branch of mathematics. He used the q parameter in Newton's work on infinite series. Later, in a methodical manner, the q-calculus, calculus without limits, was firstly given by Jackson [11, 12]. In 1966, Al-Salam [13] introduced a q-analogue of the q-fractional integral and q-Riemann–Liouville fractional. Since then, related research has gradually increased. In particular, in 2013, Tariboon [14] introduced the ${}_{\pi_1}D_q$-difference operator and q_{π_1}-integral. In 2020, Bermudo et al. [15] introduced the notion of ${}^{\pi_2}D_q$ derivative and q^{π_2}-integral. Sadjang [16] generalized to quantum calculus and introduced the notions of post-quantum calculus, or briefly (p, q)-calculus. Soontharanon et al. [17] introduced the fractional (p, q)-calculus later on. In [18], Tunç and Göv gave the post-quantum variant of ${}_{\pi_1}D_q$-difference operator and q_{π_1}-integral. Recently, in 2021, Chu et al. [19] introduced the notions of ${}^{\pi_2}D_{p,q}$ derivative and $(p, q)^{\pi_2}$-integral.

Many integral inequalities have been studied using quantum and post-quantum integrals for various types of functions. For example, in [15, 20–27], the authors used ${}_{\pi_1}D_q, {}^{\pi_2}D_q$-derivatives and q_{π_1}, q^{π_2}-integrals to prove Hermite–Hadamard integral inequalities and their left–right estimates for convex and coordinated convex functions. In [28], Noor et al. presented a generalized version of quantum integral inequalities. For generalized quasi-convex functions, Nwaeze et al. proved certain parameterized quantum integral inequalities in [29]. Khan et al. proved quantum Hermite–Hadamard inequality using the green function in [30]. Budak et al. [31], Ali et al. [32, 33], and Vivas-Cortez et al. [34] developed new quantum Simpson's and quantum Newton's type inequalities for convex and coordinated convex functions. For quantum Ostrowski's inequalities for convex and co-ordinated convex functions, one can consult [35–38]. Kunt et al. [39] generalized the results of [22] and proved Hermite–Hadamard-type inequalities and their left estimates using ${}_{\pi_1}D_{p,q}$ difference operator and $(p,q)_{\pi_1}$ integral. Recently, Latif et al. [40] found the right estimates of Hermite–Hadamard type inequalities proved by Kunt et al. [39]. To prove Ostrowski's inequalities, Chu et al. [19] used the concepts of ${}^{\pi_2}D_{p,q}$ difference operator and $(p,q)^{\pi_2}$ integral.

Inspired by this ongoing studies, we offer some new quantum parameterized Simpson's and Newton's type inequalities for convex functions using the notions of quantum derivatives and integrals.

The structure of this paper is as follows: Section 2 provides a quick review of the ideas of q-calculus, as well as some related works. In Section 3, we present two integral identities that aid in the proof of the key conclusions. We prove quantum Simpson's and quantum Newton's inequalities in sections 4 and 5, respectively. Section 6 finishes with a few suggestions for future research.

2. Preliminaries of q-calculus and some inequalities

In this section, we first present some known definitions and related inequalities in q-calculus. Set the following notation(see, [8]):

$$[n]_q = \frac{1-q^n}{1-q} = \sum_{k=0}^{n-1} q^k, \quad q \in (0,1).$$

Jackson [11] defined the q-integral of a given function Π from 0 to π_2 as follows:

$$\int_0^{\pi_2} \Pi(x) \ d_q x = (1-q)\pi_2 \sum_{n=0}^{\infty} q^n \Pi(\pi_2 q^n), \text{ where } 0 < q < 1 \tag{2.1}$$

provided that the sum converges absolutely. Moreover, he defined the q-integral of a given function over the interval $[\pi_1, \pi_2]$ as follows:

$$\int_{\pi_1}^{\pi_2} \Pi(x) \ d_q x = \int_0^{\pi_2} \Pi(x) \ d_q x - \int_0^{\pi_1} \Pi(x) \ d_q x .$$

Definition 2.1. *[14] We consider the mapping $\Pi : [\pi_1, \pi_2] \to \mathbb{R}$. Then, the q_{π_1}-derivative of Π at $x \in [\pi_1, \pi_2]$ is defined by the the following expression*

$$ {}_{\pi_1}D_q \Pi(x) = \frac{\Pi(x) - \Pi(qx + (1-q)\pi_1)}{(1-q)(x-\pi_1)}, \ x \neq \pi_1. \tag{2.2}$$

If $x=\pi_1$, we define ${}_{\pi_1}D_q\Pi(\pi_1)=\lim_{x\to\pi_1}\ {}_{\pi_1}D_q\Pi(x)$ if it exists and it is finite.

Definition 2.2. *[15] We consider the mapping $\Pi:[\pi_1,\pi_2]\to\mathbb{R}$. Then, the q^{π_2}-derivative of Π at $x\in[\pi_1,\pi_2]$ is defined by*

$$ {}^{\pi_2}D_q\Pi(x) = \frac{\Pi(qx+(1-q)\pi_2)-\Pi(x)}{(1-q)(\pi_2-x)},\ x\neq\pi_2. \tag{2.3}$$

If $x=\pi_2$, we define ${}^{\pi_2}D_q\Pi(\pi_2)=\lim_{x\to\pi_2}\ {}^{\pi_2}D_q\Pi(x)$ if it exists and it is finite.

Definition 2.3. *[14] We consider the mapping $\Pi:[\pi_1,\pi_2]\to\mathbb{R}$. Then, the q_{π_1}-definite integral on $[\pi_1,\pi_2]$ is defined by*

$$\begin{aligned}\int_{\pi_1}^{\pi_2}\Pi(x)\ {}_{\pi_1}d_qx &= (1-q)(\pi_2-\pi_1)\sum_{n=0}^{\infty}q^n\Pi(q^n\pi_2+(1-q^n)\pi_1) \\ &= (\pi_2-\pi_1)\int_0^1\Pi((1-\tau)\pi_1+\tau\pi_2)\ d_q\tau\ .\end{aligned}\tag{2.4}$$

Remark 2.1. *If we set $\pi_1=0$ in Definition 2.3, then we obtain q-Jackson integral, which is given in expression (2.1).*

In [22, 27], the authors proved quantum Hermite-Hadamard type inequalities and their estimations by using the notions of q_{π_1}-derivative and q_{π_1}-integral.

On the other hand, in [15], Bermudo et al. gave the following definition and obtained the related Hermite-Hadamard type inequalities:

Definition 2.4. *[15] We consider the mapping $\Pi:[\pi_1,\pi_2]\to\mathbb{R}$. Then, the q^{π_2}-definite integral on $[\pi_1,\pi_2]$ is defined by*

$$\begin{aligned}\int_{\pi_1}^{\pi_2}\Pi(x)\ {}^{\pi_2}d_qx &= (1-q)(\pi_2-\pi_1)\sum_{n=0}^{\infty}q^n\Pi(q^n\pi_1+(1-q^n)\pi_2) \\ &= (\pi_2-\pi_1)\int_0^1\Pi(\tau\pi_1+(1-\tau)\pi_2)\ d_q\tau\ .\end{aligned}$$

Theorem 2.1. *[15] Let $\Pi:[\pi_1,\pi_2]\to\mathbb{R}$ be a convex function on $[\pi_1,\pi_2]$ and $0<q<1$. Then, q^{π_2}-Hermite-Hadamard inequalities are given as follows:*

$$\Pi\left(\frac{\pi_1+q\pi_2}{[2]_q}\right)\leq\frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}\Pi(x)\ {}^{\pi_2}d_qx\ \leq\frac{\Pi(\pi_1)+q\Pi(\pi_2)}{[2]_q}. \tag{2.5}$$

In [24], Budak proved the left and right bounds of the inequality (2.5).

3. Crucial identities

To obtain the key results of this paper, we prove three separate identities in this section.
Let's begin with the following crucial Lemma.

Lemma 3.1. *If* $\Pi : [\pi_1, \pi_2] \subset \mathbb{R} \to \mathbb{R}$ *is a* q_{π_1}*-differentiable function on* (π_1, π_2) *such that* ${}_{\pi_1}D_q\Pi$ *is continuous and integrable on* $[\pi_1, \pi_2]$*, then we have the following identity:*

$$q\lambda\Pi(\pi_1) + (1-\mu q)\Pi(\pi_2) + q(\mu-\lambda)\Pi\left(\frac{\pi_1 q + \pi_2}{[2]_q}\right) - \frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}\Pi(x)\ {}_{\pi_1}d_qx \tag{3.1}$$
$$= q(\pi_2-\pi_1)$$
$$\times\left[\int_0^{\frac{1}{[2]_q}}(t-\lambda)\ {}_{\pi_1}D_q\Pi(t\pi_2+(1-t)\pi_1)\,d_qt + \int_{\frac{1}{[2]_q}}^1(t-\mu)\ {}_{\pi_1}D_q\Pi(t\pi_2+(1-t)\pi_1)\,d_qt\right]$$

where $q \in (0,1)$.

Proof. From Definition 2.1, we have

$${}_{\pi_1}D_q\Pi(t\pi_2+(1-t)\pi_1) = \frac{\Pi(t\pi_2+(1-t)\pi_1) - \Pi(qt\pi_2+(1-qt)\pi_1)}{(1-q)(\pi_2-\pi_1)t}. \tag{3.2}$$

By utilizing the properties of quantum integrals, we obtain

$$\int_0^{\frac{1}{[2]_q}}(t-\lambda)\ {}_{\pi_1}D_q\Pi(t\pi_2+(1-t)\pi_1)\,d_qt + \int_{\frac{1}{[2]_q}}^1(t-\mu)\ {}_{\pi_1}D_q\Pi(t\pi_2+(1-t)\pi_1)\,d_qt \tag{3.3}$$
$$= \int_0^{\frac{1}{[2]_q}}(\mu-\lambda)\ {}_{\pi_1}D_q\Pi(t\pi_2+(1-t)\pi_1)\,d_qt + \int_0^1(t-\mu)\ {}_{\pi_1}D_q\Pi(t\pi_2+(1-t)\pi_1)\,d_qt$$
$$= (\mu-\lambda)\int_0^{\frac{1}{[2]_q}}\frac{\Pi(t\pi_2+(1-t)\pi_1)-\Pi(qt\pi_2+(1-qt)\pi_1)}{(1-q)(\pi_2-\pi_1)t}d_qt$$
$$+\int_0^1\frac{\Pi(t\pi_2+(1-t)\pi_1)-\Pi(qt\pi_2+(1-qt)\pi_1)}{(1-q)(\pi_2-\pi_1)}d_qt$$
$$-\mu\int_0^1\frac{\Pi(t\pi_2+(1-t)\pi_1)-\Pi(qt\pi_2+(1-qt)\pi_1)}{(1-q)(\pi_2-\pi_1)t}d_qt.$$

By Definition 2.3, we have the following equalities

$$\int_0^{\frac{1}{[2]_q}}\frac{\Pi(t\pi_2+(1-t)\pi_1)-\Pi(qt\pi_2+(1-qt)\pi_1)}{(1-q)(\pi_2-\pi_1)t}d_qt \tag{3.4}$$
$$= \frac{1}{\pi_2-\pi_1}\left[\sum_{n=0}^{\infty}\Pi\left(\frac{q^n}{[2]_q}\pi_2+\left(1-\frac{q^n}{[2]_q}\right)\pi_1\right) - \sum_{n=0}^{\infty}\Pi\left(\frac{q^{n+1}}{[2]_q}\pi_2+\left(1-\frac{q^{n+1}}{[2]_q}\right)\pi_1\right)\right]$$
$$= \frac{1}{\pi_2-\pi_1}\left[\Pi\left(\frac{\pi_1 q+\pi_2}{[2]_q}\right)-\Pi(\pi_1)\right],$$

$$\int_0^1\frac{\Pi(t\pi_2+(1-t)\pi_1)-\Pi(qt\pi_2+(1-qt)\pi_1)}{(1-q)(\pi_2-\pi_1)t}d_qt \tag{3.5}$$

$$= \frac{1}{\pi_2 - \pi_1}\left[\Pi(\pi_2) - \Pi(\pi_1)\right]$$

and

$$\int_0^1 \frac{\Pi(t\pi_2 + (1-t)\pi_1) - \Pi(qt\pi_2 + (1-qt)\pi_1)}{(1-q)(\pi_2 - \pi_1)} d_q t \tag{3.6}$$
$$= \frac{1}{\pi_2 - \pi_1}\left[\sum_{n=0}^{\infty} q^n \Pi(q^n \pi_2 + (1-q^n)\pi_1) - \sum_{n=0}^{\infty} q^n \Pi\left(q^{n+1}\pi_2 + \left(1 - q^{n+1}\right)\pi_1\right)\right]$$
$$= \frac{1}{\pi_2 - \pi_1}\left[\sum_{n=0}^{\infty} q^n \Pi(q^n \pi_2 + (1-q^n)\pi_1) - \frac{1}{q}\sum_{n=1}^{\infty} q^n \Pi(q^n \pi_2 + (1-q^n)\pi_1)\right]$$
$$= \frac{1}{\pi_2 - \pi_1}\left[\sum_{n=0}^{\infty} q^n \Pi(q^n \pi_2 + (1-q^n)\pi_1) - \frac{1}{q}\sum_{n=0}^{\infty} q^n \Pi(q^n \pi_2 + (1-q^n)\pi_1) + \frac{1}{q}\Pi(\pi_2)\right]$$
$$= \frac{1}{\pi_2 - \pi_1}\left[\frac{1}{q}\Pi(\pi_2) - \frac{1}{q(\pi_2 - \pi_1)}\int_{\pi_1}^{\pi_2} \Pi(x) \ {}_{\pi_1}d_q x\right].$$

If we substitute the computed integrals (3.4)–(3.6) in (3.3), we establish the required identity (3.1). □

Remark 3.1. *In Lemma 3.1, if we choose* $\lambda = \frac{1}{[6]_q}$ *and* $\mu = \frac{[5]_q}{[6]_q}$*, then we have the following identity:*

$$\frac{1}{[6]_q}\left[q\Pi(\alpha) + q^2 [4]_q \Pi\left(\frac{q\pi_1 + \pi_2}{[2]_q}\right) + \Pi(\pi_2)\right] - \frac{1}{\pi_2 - \pi_1}\int_{\pi_1}^{\pi_2} \Pi(s) \ {}_{\pi_1}d_q s$$
$$= q(\pi_2 - \pi_1)$$
$$\times\left[\int_0^{\frac{1}{[2]_q}}\left(t - \frac{1}{[6]_q}\right) {}_{\pi_1}D_q \Pi(t\pi_2 + (1-t)\pi_1)\, d_q t + \int_{\frac{1}{[2]_q}}^1 \left(t - \frac{[5]_q}{[6]_q}\right) {}_{\pi_1}D_q \Pi(t\pi_2 + (1-t)\pi_1)\, d_q t\right]$$

which is proved by Iftikhar et al. in [41].

Remark 3.2. *In Lemma 3.1, if we choose* $\lambda = \mu = \frac{1}{[2]_q}$*, then we obtain [42, Lemma 3.1].*

Remark 3.3. *In Lemma 3.1, if we choose* $\lambda = 0$ *and* $\mu = \frac{1}{q}$*, then Lemma 3.1 reduces to [22, Lemma 11].*

Remark 3.4. *In Lemma 3.1, if we take the limit* $q \to 1^-$*, then we have [43, Lemma 2.1 for* $m = 1$*].*

Lemma 3.2. *If* $\Pi : [\pi_1, \pi_2] \subset \mathbb{R} \to \mathbb{R}$ *is a* q_{π_1}*-differentiable function on* (π_1, π_2) *such that* ${}_{\pi_1}D_q \Pi$ *is continuous and integrable on* $[\pi_1, \pi_2]$*, then we have the following identity:*

$$q\lambda \Pi(\pi_1) + q(\mu - \lambda)\Pi\left(\frac{\pi_1 q [2]_q + \pi_2}{[3]_q}\right) \tag{3.7}$$
$$+ q(\nu - \mu)\Pi\left(\frac{\pi_1 q^2 + \pi_2 [2]_q}{[3]_q}\right) + (1 - \nu q)\Pi(\pi_2) - \frac{1}{\pi_2 - \pi_1}\int_{\pi_1}^{\pi_2} \Pi(x) \ {}_{\pi_1}d_q x$$
$$= (\pi_2 - \pi_1) q \left[\int_0^{\frac{1}{[3]_q}} (t - \lambda) \ {}_{\pi_1}D_q \Pi(t\pi_2 + (1-t)\pi_1)\, d_q t\right.$$

$$+\int_{\frac{1}{[3]_q}}^{\frac{[2]_q}{[3]_q}} (t-\mu)\ {}_{\pi_1}D_q\Pi(t\pi_2+(1-t)\pi_1)\,d_qt$$
$$\left.+\int_{\frac{[2]_q}{[3]_q}}^{1} (t-\nu)\ {}_{\pi_1}D_q\Pi(t\pi_2+(1-t)\pi_1)\,d_qt\right]$$

where $q\in(0,1)$.

Proof. By the fundamental properties of quantum integrals, we have

$$\begin{aligned}
&\int_0^{\frac{1}{[3]_q}} (t-\lambda)\ {}_{\pi_1}D_q\Pi(t\pi_2+(1-t)\pi_1)\,d_qt+\int_{\frac{1}{[3]_q}}^{\frac{[2]_q}{[3]_q}} (t-\mu)\ {}_{\pi_1}D_q\Pi(t\pi_2+(1-t)\pi_1)\,d_qt\\
&+\int_{\frac{[2]_q}{[3]_q}}^{1} (t-\nu)\ {}_{\pi_1}D_q\Pi(t\pi_2+(1-t)\pi_1)\,d_qt\\
=\ &\int_0^{\frac{1}{[3]_q}} (\mu-\lambda)\ {}_{\pi_1}D_q\Pi(t\pi_2+(1-t)\pi_1)\,d_qt+\int_0^{\frac{[2]_q}{[3]_q}} (\nu-\mu)\ {}_{\pi_1}D_q\Pi(t\pi_2+(1-t)\pi_1)\,d_qt\\
&+\int_0^{1} (t-\nu)\ {}_{\pi_1}D_q\Pi(t\pi_2+(1-t)\pi_1)\,d_qt.
\end{aligned}$$

By applying the same steps in the proof of Lemma 3.1 for rest of this proof, one can obtain the desired identity (3.7). □

Remark 3.5. *If we take* $\lambda=\frac{1}{[8]_q}$, $\mu=\frac{1}{[2]_q}$, *and* $\nu=\frac{[7]_q}{[8]_q}$ *in Lemma 3.2, then we obtain the following identity*

$$\begin{aligned}
&\frac{1}{[8]_q}\left[q\Pi(\pi_1)+\frac{q^3[6]_q}{[2]_q}\Pi\left(\frac{\pi_1q[2]_q+\pi_2}{[3]_q}\right)+\frac{q^2[6]_q}{[2]_q}\Pi\left(\frac{\pi_1q^2+\pi_2[2]_q}{[3]_q}\right)+\Pi(\pi_2)\right]\\
&-\frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}\Pi(x)\ {}_{\pi_1}d_qx\\
=\ &q(\pi_2-\pi_1)\left[\int_0^{\frac{1}{[3]_q}}\left(t-\frac{1}{[8]_q}\right)\ {}_{\pi_1}D_q\Pi(t\pi_2+(1-t)\pi_1)\,d_qt\right.\\
&+\int_{\frac{1}{[3]_q}}^{\frac{[2]_q}{[3]_q}}\left(t-\frac{1}{[2]_q}\right)\ {}_{\pi_1}D_q\Pi(t\pi_2+(1-t)\pi_1)\,d_qt\\
&\left.+\int_{\frac{[2]_q}{[3]_q}}^{1}\left(t-\frac{[7]_q}{[8]_q}\right)\ {}_{\pi_1}D_q\Pi(t\pi_2+(1-t)\pi_1)\,d_qt\right]
\end{aligned}$$

which is proved by Erden et al. in [44].

Remark 3.6. *If we take* $\lambda=\mu=\nu=\frac{1}{[2]_q}$, *in Lemma 3.2, then we obtain [42, Lemma 3.1].*

Corollary 3.1. *If we take the limit* $q\to1^-$ *in Lemma 3.2, then we obtain the following new identity*

$$\lambda\Pi(\pi_1)+(\mu-\lambda)\Pi\left(\frac{2\pi_1+\pi_2}{3}\right)+(\nu-\mu)\Pi\left(\frac{\pi_1+2\pi_2}{3}\right)+(1-\nu)\Pi(\pi_2)$$

$$
\begin{aligned}
& -\frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}\Pi(x)\,dx \\
= \; & (\pi_2-\pi_1)\left[\int_0^{\frac{1}{3}}(t-\lambda)\,\Pi'(t\pi_2+(1-t)\,\pi_1)\,dt+\int_{\frac{1}{3}}^{\frac{2}{3}}(t-\mu)\,\Pi'(t\pi_2+(1-t)\,\pi_1)\,dt\right. \\
& \left.+\int_{\frac{2}{3}}^{1}(t-\nu)\,\Pi'(t\pi_2+(1-t)\,\pi_1)\,dt\right]
\end{aligned}
$$

For brevity, let us prove another lemma that will be used frequently in the main results.

Lemma 3.3. *The following quantum integrals holds for $\lambda,\mu,\nu\geq 0$:*

$$
\Omega_{11}=\int_0^{\frac{1}{[2]_q}}|t-\lambda|\,d_qt=\frac{2\lambda^2q}{[2]_q}+\frac{1}{\left([2]_q\right)^3}-\frac{\lambda}{[2]_q} \tag{3.8}
$$

$$
\Omega_{12}=\int_{\frac{1}{[2]_q}}^{1}|t-\mu|\,d_qt=\frac{2\mu^2q}{[2]_q}+\frac{\left([2]_q\right)^2+1}{\left([2]_q\right)^3}-\frac{\mu\left([2]_q+1\right)}{[2]_q} \tag{3.9}
$$

$$
\Omega_{13}=\int_0^{\frac{1}{[3]_q}}|t-\lambda|\,d_qt=\frac{2\lambda^2q}{[2]_q}+\frac{1}{[2]_q\left([3]_q\right)^2}-\frac{\lambda}{[3]_q} \tag{3.10}
$$

$$
\Omega_{14}=\int_{\frac{1}{[3]_q}}^{\frac{[2]_q}{[3]_q}}|t-\mu|\,d_qt=\frac{2\mu^2q}{[2]_q}-\frac{\mu\left([2]_q+1\right)}{[3]_q}+\frac{\left([2]_q\right)^2+1}{[2]_q\left([3]_q\right)^2} \tag{3.11}
$$

$$
\Omega_{15}=\int_{\frac{[2]_q}{[3]_q}}^{1}|t-\nu|\,d_qt=\frac{2\nu^2q}{[2]_q}-\frac{\nu\left([2]_q+[3]_q\right)}{[3]_q}+\frac{[2]_q}{\left([3]_q\right)^2}+\frac{1}{[2]_q} \tag{3.12}
$$

$$
\Omega_{1}=\int_0^{\frac{1}{[2]_q}}t\,|t-\lambda|\,d_qt=\frac{2\lambda^3q^2}{[2]_q\,[3]_q}+\frac{1}{\left([2]_q\right)^3[3]_q}-\frac{\lambda}{\left([2]_q\right)^3} \tag{3.13}
$$

$$
\begin{aligned}
\Omega_2 &= \int_0^{\frac{1}{[2]_q}}(1-t)\,|t-\lambda|\,d_qt \qquad (3.14)\\
&= \Omega_{11}-\Omega_1 \\
&= \frac{2\lambda^2q}{[2]_q}-\frac{2\lambda^3q^2}{[2]_q\,[3]_q}-\frac{\lambda\left(\left([2]_q\right)^2-1\right)}{\left([2]_q\right)^3}+\frac{[3]_q-1}{\left([2]_q\right)^3[3]_q}
\end{aligned}
$$

$$
\Omega_3=\int_{\frac{1}{[2]_q}}^{1}t\,|t-\mu|\,d_qt \tag{3.15}
$$

$$= \frac{2\mu^3 q^2}{[2]_q [3]_q} + \frac{1 + \left([2]_q\right)^3}{\left([2]_q\right)^3 [3]_q} - \frac{\mu\left(\left([2]_q\right)^2 + 1\right)}{\left([2]_q\right)^3}$$

$$\begin{aligned} \Omega_4 &= \int_{\frac{1}{[2]_q}}^{1} (1-t)\,|t-\mu|\,d_q t = \\ &= \Omega_{12} - \Omega_3 \\ &= \frac{2\mu^2 q}{[2]_q} - \frac{2\mu^3 q^2}{[2]_q [3]_q} - \frac{\mu\left(\left([2]_q\right)^3 - 1\right)}{\left([2]_q\right)^3} + \frac{[3]_q\left(1 + \left([2]_q\right)^2\right) - \left([2]_q\right)^3 - 1}{\left([2]_q\right)^3 [3]_q} \end{aligned} \tag{3.16}$$

$$\Omega_5 = \int_0^{\frac{1}{[3]_q}} t\,|t-\lambda|\,d_q t = \frac{2\lambda^3 q^2}{[2]_q [3]_q} + \frac{1}{\left([3]_q\right)^4} - \frac{\lambda}{\left([3]_q\right)^2 [2]_q} \tag{3.17}$$

$$\begin{aligned} \Omega_6 &= \int_0^{\frac{1}{[3]_q}} (1-t)\,|t-\lambda|\,d_q t = \\ &= \Omega_{13} - \Omega_5 \\ &= \frac{2\lambda^2 q}{[2]_q} - \frac{2\lambda^3 q^2}{[2]_q [3]_q} + \frac{\lambda\left(1 - [2]_q [3]_q\right)}{\left([3]_q\right)^2 [2]_q} + \frac{\left([3]_q\right)^2 - [2]_q}{\left([3]_q\right)^4 [2]_q} \end{aligned} \tag{3.18}$$

$$\Omega_7 = \int_{\frac{1}{[3]_q}}^{\frac{[2]_q}{[3]_q}} t\,|t-\mu|\,d_q t = \frac{2\mu^3 q^2}{[2]_q [3]_q} + \frac{1 + \left([2]_q\right)^3}{\left([3]_q\right)^4} - \frac{\mu\left(\left([2]_q\right)^2 + 1\right)}{\left([3]_q\right)^2 [2]_q} \tag{3.19}$$

$$\begin{aligned} \Omega_8 &= \int_{\frac{1}{[3]_q}}^{\frac{[2]_q}{[3]_q}} (1-t)\,|t-\mu|\,d_q t \\ &= \Omega_{14} - \Omega_7 \\ &= \frac{2\mu^2 q}{[2]_q} - \frac{2\mu^3 q^2}{[2]_q [3]_q} - \frac{\mu\left(\left([2]_q\right)^2\left([3]_q - 1\right) + [2]_q [3]_q\right)}{\left([3]_q\right)^2 [2]_q} \\ &\quad + \frac{\left(\left([2]_q\right)^2 + 1\right)\left([3]_q\right)^3 - [2]_q - \left([2]_q\right)^4}{\left([3]_q\right)^4 [2]_q} \end{aligned} \tag{3.20}$$

$$\Omega_9 = \int_{\frac{[2]_q}{[3]_q}}^{1} t\,|t-\nu|\,d_qt = \frac{2\nu^3q^2}{[2]_q\,[3]_q} - \frac{\nu\left(\left([2]_q\right)^2+\left([3]_q\right)^2\right)}{[2]_q\left([3]_q\right)^2} + \frac{\left([2]_q\right)^3+\left([3]_q\right)^3}{\left([3]_q\right)^4} \tag{3.21}$$

$$\Omega_{10} = \int_{\frac{[2]_q}{[3]_q}}^{1} (1-t)\,|t-\nu|\,d_qt \tag{3.22}$$

$$= \Omega_{15} - \Omega_9$$

$$= \frac{2\upsilon^2 q}{[2]_q} - \frac{2\upsilon^3q^2}{[2]_q\,[3]_q} - \frac{\upsilon\left(\left([3]_q\right)^2\left([2]_q-1\right)+\left([2]_q\right)^2\left([3]_q-1\right)\right)}{\left([3]_q\right)^2[2]_q} \tag{3.23}$$

$$+\frac{\left([3]_q\right)^2\left([2]_q-[3]_q\right)-\left([2]_q\right)^3}{\left([3]_q\right)^4} \tag{3.24}$$

Proof. By the definition of q-integral, we have

$$\begin{aligned}
\Omega_1 &= \int_0^{\frac{1}{[2]_q}} t\,|t-\lambda|\,d_qt \\
&= \int_0^{\lambda} t(\lambda-t)\,d_qt + \int_{\lambda}^{\frac{1}{[2]_q}} t(t-\lambda)\,d_qt \\
&= 2\int_0^{\lambda} t(\lambda-t)\,d_qt + \int_0^{\frac{1}{[2]_q}} t(t-\lambda)\,d_qt \\
&= \frac{2\lambda^3q^2}{[2]_q\,[3]_q} + \frac{1}{\left([2]_q\right)^3[3]_q} - \frac{\lambda}{\left([2]_q\right)^3}
\end{aligned}$$

and so

$$\Omega_1 = \frac{2\lambda^3q^2}{[2]_q\,[3]_q} + \frac{1}{\left([2]_q\right)^3[3]_q} - \frac{\lambda}{\left([2]_q\right)^3}.$$

This gives the proof of the equality (3.13). The others can be calculated in similar way. □

4. Generalizations of Simpson's type inequalities for quantum integrals with two parameters

In this section, we prove a new generalization of quantum Simpson's, Midpoint and Trapezoid type inequalities for quantum differentiable convex functions.

Theorem 4.1. *We assume that the given conditions of Lemma 3.1 hold. If the mapping $\left|{}_{\pi_1}D_q\Pi\right|$ is convex on $[\pi_1, \pi_2]$, then the following inequality holds:*

$$\left|q\lambda\Pi(\pi_1) + (1-\mu q)\Pi(\pi_2) + q(\mu-\lambda)\Pi\left(\frac{\pi_1 q+\pi_2}{[2]_q}\right) - \frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}\Pi(x)\ {}_{\pi_1}d_qx\right| \tag{4.1}$$

$$\leq \quad q\left(\pi_2-\pi_1\right)\left[\left(\Omega_1+\Omega_3\right)\left|{}_{\pi_1}D_q\Pi\left(\pi_2\right)\right|+\left(\Omega_2+\Omega_4\right)\left|{}_{\pi_1}D_q\Pi\left(\pi_1\right)\right|\right]$$

where Ω_1-Ω_4 are given in (3.13)-(3.16), respectively.

Proof. By taking the modulus in Lemma 3.1 and using the convexity of $\left|{}_{\pi_1}D_q\Pi\right|$, we obtain

$$\begin{aligned}
& q\lambda\Pi\left(\pi_1\right)+\left(1-\mu q\right)\Pi\left(\pi_2\right)+q\left(\mu-\lambda\right)\Pi\left(\frac{\pi_1 q+\pi_2}{[2]_q}\right)-\frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}\Pi\left(x\right)\ {}_{\pi_1}d_qx \\
\leq\ & q\left(\pi_2-\pi_1\right) \\
& \times\left[\int_0^{\frac{1}{[2]_q}}\left|t-\lambda\right|\left|{}_{\pi_1}D_q\Pi\left(t\pi_2+\left(1-t\right)\pi_1\right)\right|d_qt+\int_{\frac{1}{[2]_q}}^1\left|t-\mu\right|\left|{}_{\pi_1}D_q\Pi\left(t\pi_2+\left(1-t\right)\pi_1\right)\right|d_qt\right] \\
\leq\ & \left(\pi_2-\pi_1\right)q\left[\left|{}_{\pi_1}D_q\Pi\left(\pi_2\right)\right|\left\{\int_0^{\frac{1}{[2]_q}}t\left|t-\lambda\right|d_qt+\int_{\frac{1}{[2]_q}}^1 t\left|t-\mu\right|d_qt\right\}\right. \\
& \left.+\left|{}_{\pi_1}D_q\Pi\left(\pi_1\right)\right|\left\{\int_0^{\frac{1}{[2]_q}}\left(1-t\right)\left|t-\lambda\right|d_qt+\int_{\frac{1}{[2]_q}}^1\left(1-t\right)\left|t-\mu\right|d_qt\right\}\right] \\
=\ & \left(\pi_2-\pi_1\right)q\left[\left(\Omega_1+\Omega_3\right)\left|{}_{\pi_1}D_q\Pi\left(\pi_2\right)\right|+\left(\Omega_2+\Omega_4\right)\left|{}_{\pi_1}D_q\Pi\left(\pi_1\right)\right|\right]
\end{aligned}$$

which is the desired inequality. □

Remark 4.1. *If we take the limit $q\to 1^-$ in Theorem 4.1, then we have [43, Theorem 2.1 for $s=m=1$].*

Remark 4.2. *If we assume $\lambda=\mu=\frac{1}{[2]_q}$ in Theorem 4.1, then we obtain [42, Theorem 4.1].*

Remark 4.3. *In Theorem 4.1, if we choose $\lambda=0$ and $\mu=\frac{1}{q}$, then Theorem 4.1 reduces to [22, Theorem 13].*

Remark 4.4. *If we assume $\lambda=\frac{1}{[6]_q}$ and $\mu=\frac{[5]_q}{[6]_q}$ in Theorem 4.1, then we obtain the following inequality*

$$\begin{aligned}
& \left|\frac{1}{[6]_q}\left[q\Pi\left(\alpha\right)+q^2\left[4\right]_q\Pi\left(\frac{q\pi_1+\pi_2}{[2]_q}\right)+\Pi\left(\pi_2\right)\right]-\frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}\Pi\left(s\right)\ {}_{\pi_1}d_qs\right| \\
\leq\ & q\left(\pi_2-\pi_1\right)\left\{\left|{}_{\pi_1}D_q\Pi\left(\pi_2\right)\right|\left[A_1\left(q\right)+A_2\left(q\right)\right]+\left|{}_{\pi_1}D_q\Pi\left(\pi_1\right)\right|\left[B_1\left(q\right)+B_2\left(q\right)\right]\right\},
\end{aligned}$$

where

$$\begin{aligned}
A_1\left(q\right) &= \frac{2q^2\left[2\right]_q^2+\left[6\right]_q^2\left(\left[6\right]_q-\left[3\right]_q\right)}{\left[2\right]_q^3\left[3\right]_q\left[6\right]_q^3}, \\
B_1\left(q\right) &= 2\frac{q\left[3\right]_q\left[6\right]_q-q^2}{\left[2\right]_q\left[3\right]_q\left[6\right]_q^3}+\frac{1}{\left[2\right]_q^3}\left(\frac{q+q^2}{\left[3\right]_q}-\frac{q^2+2q}{\left[6\right]_q}\right), \\
A_2\left(q\right) &= \frac{2q^2\left[5\right]_q^3}{\left[2\right]_q\left[3\right]_q\left[6\right]_q^3}+\frac{\left[6\right]_q\left(1+\left[2\right]_q^3\right)-\left[3\right]_q\left[5\right]_q\left(1+\left[2\right]_q^2\right)}{\left[2\right]_q^3\left[3\right]_q\left[6\right]_q}, \\
B_2\left(q\right) &= 2\frac{q\left[5\right]_q^2\left[6\right]_q\left[3\right]_q-q^2\left[5\right]_q^3}{\left[2\right]_q\left[3\right]_q\left[6\right]_q^3}+\frac{q^2}{\left[2\right]_q\left[3\right]_q}-\frac{q\left[5\right]_q}{\left[2\right]_q\left[6\right]_q}
\end{aligned}$$

$$-\frac{1}{[2]_q^3}\left[\frac{[5]_q\left(2q+q^2\right)}{[6]_q}-\frac{q+q^2}{[3]_q}\right]$$

which is proved by Ifitikhar et al. [41].

Theorem 4.2. *We assume that the given conditions of Lemma 3.1 hold. If the mapping* $\left|{}_{\pi_1}D_q\Pi\right|^{p_1}$, $p_1 \geq 1$ *is convex on* $[\pi_1, \pi_2]$*, then the following inequality holds:*

$$\left|\lambda q\Pi(\pi_1)+(1-\mu q)\Pi(\pi_2)+q(\mu-\lambda)\Pi\left(\frac{\pi_1 q+\pi_2}{[2]_q}\right)-\frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}\Pi(x)\ {}_{\pi_1}d_qx\right| \tag{4.2}$$
$$\leq (\pi_2-\pi_1)q\left[\Omega_{11}^{1-\frac{1}{p_1}}\left(\Omega_1\left|{}_{\pi_1}D_q\Pi(\pi_2)\right|^{p_1}+\Omega_2\left|{}_{\pi_1}D_q\Pi(\pi_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right.$$
$$\left.+\Omega_{12}^{1-\frac{1}{p_1}}\left(\Omega_3\left|{}_{\pi_1}D_q\Pi(\pi_2)\right|^{p_1}+\Omega_4\left|{}_{\pi_1}D_q\Pi(\pi_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right]$$

where Ω_{11}, Ω_{12} *and* Ω_1-Ω_4 *are given in (3.8), (3.9), and (3.13)–(3.16), respectively.*

Proof. By taking the modulus in Lemma 3.1 and using the power mean inequality, we have

$$\left|\lambda q\Pi(\pi_1)+(1-\mu q)\Pi(\pi_2)+q(\mu-\lambda)\Pi\left(\frac{\pi_1 q+\pi_2}{[2]_q}\right)-\frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}\Pi(x)\ {}_{\pi_1}d_qx\right|$$
$$\leq (\pi_2-\pi_1)q\left[\left(\int_0^{\frac{1}{[2]_q}}|t-\lambda|\,d_qt\right)^{1-\frac{1}{p_1}}\left(\int_0^{\frac{1}{[2]_q}}|t-\lambda|\left|{}_{\pi_1}D_q\Pi(t\pi_2+(1-t)\pi_1)\right|^{p_1}d_qt\right)^{\frac{1}{p_1}}\right.$$
$$\left.+\left(\int_{\frac{1}{[2]_q}}^1|t-\mu|\,d_qt\right)^{1-\frac{1}{p_1}}\left(\int_{\frac{1}{[2]_q}}^1|t-\mu|\left|{}_{\pi_1}D_q\Pi(t\pi_2+(1-t)\pi_1)\right|^{p_1}d_qt\right)^{\frac{1}{p_1}}\right].$$

By using the convexity of $\left|{}_{\pi_1}D_q\Pi\right|^{p_1}$, we have

$$\left|\lambda q\Pi(\pi_1)+(1-\mu q)\Pi(\pi_2)+q(\mu-\lambda)\Pi\left(\frac{\pi_1 q+\pi_2}{[2]_q}\right)-\frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}\Pi(x)\ {}_{\pi_1}d_qx\right|$$
$$\leq (\pi_2-\pi_1)q\left[\left(\int_0^{\frac{1}{[2]_q}}|t-\lambda|\,d_qt\right)^{1-\frac{1}{p_1}}\right.$$
$$\times\left(\left|{}_{\pi_1}D_q\Pi(\pi_2)\right|^{p_1}\int_0^{\frac{1}{[2]_q}}t|t-\lambda|\,d_qt+\left|{}_{\pi_1}D_q\Pi(\pi_1)\right|^{p_1}\int_0^{\frac{1}{[2]_q}}(1-t)|t-\lambda|\,d_qt\right)^{\frac{1}{p_1}}$$
$$+\left(\int_{\frac{1}{[2]_q}}^1|t-\mu|\,d_qt\right)^{1-\frac{1}{p_1}}$$
$$\left.\times\left(\left|{}_{\pi_1}D_q\Pi(\pi_2)\right|^{p_1}\int_{\frac{1}{[2]_q}}^1 t|t-\mu|\,d_qt+\left|{}_{\pi_1}D_q\Pi(\pi_1)\right|^{p_1}\int_{\frac{1}{[2]_q}}^1(1-t)|t-\mu|\,d_qt\right)^{\frac{1}{p_1}}\right]$$

$$= (\pi_2-\pi_1)\, q \left[\Omega_{11}^{1-\frac{1}{p_1}} \left(\Omega_1 \left| {}_{\pi_1}D_q \Pi(\pi_2) \right|^{p_1} + \Omega_2 \left| {}_{\pi_1}D_q \Pi(\pi_1) \right|^{p_1} \right)^{\frac{1}{p_1}} \right.$$
$$\left. + \Omega_{12}^{1-\frac{1}{p_1}} \left(\Omega_3 \left| {}_{\pi_1}D_q \Pi(\pi_2) \right|^{p_1} + \Omega_4 \left| {}_{\pi_1}D_q \Pi(\pi_1) \right|^{p_1} \right)^{\frac{1}{p_1}} \right]$$

and the proof is completed. □

Remark 4.5. *If we take the limit $q \to 1^-$ in Theorem 4.2, then we have [43, Theorem 2.3 for $s = m = 1$].*

Remark 4.6. *If we assume $\lambda = \mu = \frac{1}{[2]_q}$ in Theorem 4.2, then we obtain [42, Theorem 4.2].*

Remark 4.7. *If we assume $\lambda = \frac{1}{[6]_q}$ and $\mu = \frac{[5]_q}{[6]_q}$ in Theorem 4.2, then we obtain the following inequality*

$$\left| \frac{1}{[6]_q} \left[q\Pi(\alpha) + q^2 [4]_q \Pi\left(\frac{q\pi_1 + \pi_2}{[2]_q} \right) + \Pi(\pi_2) \right] - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi(s) \ {}_{\pi_1}d_q s \right|$$
$$\leq q(\pi_2-\pi_1) \left[\left(\frac{2q}{[2]_q [6]_q^2} + \frac{q^3 [3]_q - q}{[6]_q [2]_q^3} \right)^{1-\frac{1}{p_1}} \right.$$
$$\times \left(A_1(q) \left| {}_{\pi_1}D_q \Pi(\pi_2) \right|^{p_1} + B_1(q) \left| {}_{\pi_1}D_q \Pi(\pi_1) \right|^{p_1} \right)^{\frac{1}{p_1}}$$
$$+ \left(2q \frac{[5]_q^2}{[2]_q [6]_q^2} + \frac{1}{[2]_q} - \frac{[5]_q}{[6]_q} - \frac{[5]_q [2]_q^2 - [6]_q}{[6]_q [2]_q^3} \right)^{1-\frac{1}{p_1}}$$
$$\times \left(A_2(q) \left| {}_{\pi_1}D_q \Pi(\pi_2) \right|^{p_1} + B_2(q) \left| {}_{\pi_1}D_q \Pi(\pi_1) \right|^{p_1} \right)^{\frac{1}{p_1}}$$

where $A_1(q)$, $A_2(q)$, $B_1(q)$ and $B_2(q)$ are defined in Remark 4.4. The above inequality is proved by Ifitikhar et al. [41].

Remark 4.8. *In Theorem 4.2, if we choose $\lambda = 0$ and $\mu = \frac{1}{q}$, then Theorem 4.2 reduces to [22, Theorem 16].*

Theorem 4.3. *We assume that the given conditions of Lemma 3.1 hold. If the mapping $\left| {}_{\pi_1}D_q \Pi \right|^{p_1}$, $p_1 > 1$ is convex on $[\pi_1, \pi_2]$, then the following inequality holds:*

$$\left| \lambda q \Pi(\pi_1) + (1-\mu q) \Pi(\pi_2) + q(\mu - \lambda) \Pi\left(\frac{\pi_1 q + \pi_2}{[2]_q} \right) - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi(x) \ {}_{\pi_1}d_q x \right| \quad (4.3)$$
$$\leq (\pi_2-\pi_1)\, q \left[\Omega_{16}^{\frac{1}{r_1}} \left(\frac{\left| {}_{\pi_1}D_q \Pi(\pi_2) \right|^{p_1}}{\left([2]_q\right)^3} + \frac{\left(\left([2]_q\right)^2 - 1 \right) \left| {}_{\pi_1}D_q \Pi(\pi_1) \right|^{p_1}}{\left([2]_q\right)^3} \right)^{\frac{1}{p_1}} \right.$$
$$\left. + \Omega_{17}^{\frac{1}{r_1}} \left(\frac{\left(\left([2]_q\right)^2 - 1 \right) \left| {}_{\pi_1}D_q \Pi(\pi_2) \right|^{p_1}}{\left([2]_q\right)^3} + \frac{\left(\left([2]_q\right)^3 - 2\left([2]_q\right)^2 + 1 \right) \left| {}_{\pi_1}D_q \Pi(\pi_1) \right|^{p_1}}{\left([2]_q\right)^3} \right)^{\frac{1}{p_1}} \right]$$

where $p_1^{-1} + r_1^{-1} = 1$ *and*

$$\Omega_{16} = \int_0^{\frac{1}{[2]_q}} |t-\lambda|^{r_1} \, d_q t, \ \Omega_{17} = \int_{\frac{1}{[2]_q}}^1 |t-\mu|^{r_1} \, d_q t$$

Proof. By taking the modulus in Lemma 3.1 and using the Hölder inequality, we have

$$\begin{aligned}
&\left|\lambda q \Pi(\pi_1) + (1-\mu q)\Pi(\pi_2) + q(\mu-\lambda)\Pi\left(\frac{\pi_1 q + \pi_2}{[2]_q}\right) - \frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2} \Pi(x) \ {}_{\pi_1}d_q x\right| \\
\leq \ & (\pi_2-\pi_1) q \left[\left(\int_0^{\frac{1}{[2]_q}} |t-\lambda|^{r_1} \, d_q t\right)^{\frac{1}{r_1}} \left(\int_0^{\frac{1}{[2]_q}} \left|{}_{\pi_1}D_q \Pi(t\pi_2 + (1-t)\pi_1)\right|^{p_1} d_q t\right)^{\frac{1}{p_1}}\right. \\
&\left. + \left(\int_{\frac{1}{[2]_q}}^1 |t-\mu|^{r_1} \, d_q t\right)^{\frac{1}{r_1}} \left(\int_{\frac{1}{[2]_q}}^1 \left|{}_{\pi_1}D_q \Pi(t\pi_2 + (1-t)\pi_1)\right|^{p_1} d_q t\right)^{\frac{1}{p_1}}\right].
\end{aligned}$$

Since $\left|{}_{\pi_1}D_q \Pi\right|^{p_1}$ is convex on $[\pi_1, \pi_2]$, we have

$$\begin{aligned}
&\left|\lambda q \Pi(\pi_1) + (1-\mu q)\Pi(\pi_2) + q(\mu-\lambda)\Pi\left(\frac{\pi_1 q + \pi_2}{[2]_q}\right) - \frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2} \Pi(x) \ {}_{\pi_1}d_q x\right| \\
\leq \ & q(\pi_2-\pi_1) \\
&\times\left[\left(\int_0^{\frac{1}{[2]_q}} |t-\lambda|^{r_1} \, d_q t\right)^{\frac{1}{r_1}} \left(\left|{}_{\pi_1}D_q \Pi(\pi_2)\right|^{p_1} \int_0^{\frac{1}{[2]_q}} t d_q t + \left|{}_{\pi_1}D_q \Pi(\pi_1)\right|^{p_1} \int_0^{\frac{1}{[2]_q}} (1-t) \, d_q t\right)^{\frac{1}{p_1}}\right. \\
&\left. + \left(\int_{\frac{1}{[2]_q}}^1 |t-\mu|^{r_1} \, d_q t\right)^{\frac{1}{r_1}} \left(\left|{}_{\pi_1}D_q \Pi(\pi_2)\right|^{p_1} \int_{\frac{1}{[2]_q}}^1 t d_q t + \left|{}_{\pi_1}D_q \Pi(\pi_1)\right|^{p_1} \int_{\frac{1}{[2]_q}}^1 (1-t) \, d_q t\right)^{\frac{1}{p_1}}\right] \\
= \ & (\pi_2-\pi_1) q \left[\Omega_{16}^{\frac{1}{r_1}} \left(\frac{\left|{}_{\pi_1}D_q \Pi(\pi_2)\right|^{p_1}}{\left([2]_q\right)^3} + \frac{\left(\left([2]_q\right)^2 - 1\right)\left|{}_{\pi_1}D_q \Pi(\pi_1)\right|^{p_1}}{\left([2]_q\right)^3}\right)^{\frac{1}{p_1}}\right. \\
&\left. + \Omega_{17}^{\frac{1}{r_1}} \left(\frac{\left(\left([2]_q\right)^2 - 1\right)\left|{}_{\pi_1}D_q \Pi(\pi_2)\right|^{p_1}}{\left([2]_q\right)^3} + \frac{\left(\left([2]_q\right)^3 - 2\left([2]_q\right)^2 + 1\right)\left|{}_{\pi_1}D_q \Pi(\pi_1)\right|^{p_1}}{\left([2]_q\right)^3}\right)^{\frac{1}{p_1}}\right].
\end{aligned}$$

This completes the proof. □

Remark 4.9. *If we take the limit* $q \to 1^-$ *in Theorem 4.3, then Theorem 4.3 becomes [43, Theorem 2.2 for* $s = m = 1$*].*

Remark 4.10. *If we assume* $\lambda = \mu = \frac{1}{[2]_q}$ *in Theorem 4.3, then we obtain [27, Theorem 3.3].*

Remark 4.11. *If we assume* $\lambda = \frac{1}{[6]_q}$ *and* $\mu = \frac{[5]_q}{[6]_q}$ *in Theorem 4.3, then we obtain the following inequality*

$$\left|\frac{1}{[6]_q}\left[q\Pi(\alpha) + q^2 [4]_q \Pi\left(\frac{q\pi_1 + \pi_2}{[2]_q}\right) + \Pi(\pi_2)\right] - \frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2} \Pi(s) \ {}_{\pi_1}d_q s\right|$$

$$\leq \; q(\pi_2-\pi_1)\left[\left(\frac{q^{2r_1}[4]_q^{r_1}}{[2]_q^{r_1+1}[6]_q^{r_1}}\right)^{\frac{1}{r_1}}\right.$$
$$+\left(\frac{[2]_q^{r_1+1}[5]_q^{r_1}-q^{r_1}[4]_q^{r_1}}{[2]_q^{r_1+1}[6]_q^{r_1}}\right)^{\frac{1}{r_1}}$$
$$\left.\times\left(\frac{q^2+2q}{[2]_q^3}\left|{}_{\pi_1}D_q\Pi(\pi_2)\right|^{p_1}+\frac{q^3+q^2-q}{[2]_q^3}\left|{}_{\pi_1}D_q\Pi(\pi_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right]$$

which is established by Iftikhar et al. in [41].

Remark 4.12. *In Theorem 4.2, if we choose $\lambda = 0$ and $\mu = \frac{1}{q}$, then Theorem 4.3 reduces to [22, Theorem 18].*

5. Generalizations of Newton's type inequalities for quantum integrals with three parameters

Some new generalized versions of quantum Newton's and Trapezoid type inequalities for quantum differentiable convex functions are offered in this section.

Theorem 5.1. *We assume that the given conditions of Lemma 3.2 hold. If the mapping $\left|{}_{\pi_1}D_q\Pi\right|$ is convex on $[\pi_1,\pi_2]$, then the following inequality holds:*

$$\left|q\lambda\Pi(\pi_1)+q(\mu-\lambda)\Pi\left(\frac{\pi_1 q[2]_q+\pi_2}{[3]_q}\right)+q(\nu-\mu)\Pi\left(\frac{\pi_1 q^2+\pi_2[2]_q}{[3]_q}\right)\right. \tag{5.1}$$
$$\left.+(1-\nu q)\Pi(\pi_2)-\frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}\Pi(x)\;{}_{\pi_1}d_qx\right|$$
$$\leq \; (\pi_2-\pi_1)q\left[(\Omega_5+\Omega_7+\Omega_9)\left|{}_{\pi_1}D_q\Pi(\pi_2)\right|+(\Omega_6+\Omega_8+\Omega_{10})\left|{}_{\pi_1}D_q\Pi(\pi_1)\right|\right]$$

where Ω_5-Ω_{10} are given in (3.17)-(3.22), respectively.

Proof. By considering Lemma 3.2 and applying the same method that used in the proof of Theorem 4.1, then we can obtain the desired inequality (5.1). □

Remark 5.1. *If we assume $\lambda=\mu=\nu=\frac{1}{[2]_q}$ in Theorem 5.1, then we obtain [42, Theorem 4.1].*

Corollary 5.1. *If we take the limit $q\to 1^-$ in Theorem 5.1, then we obtain the following inequality*

$$\left|\lambda\Pi(\pi_1)+(\mu-\lambda)\Pi\left(\frac{2\pi_1+\pi_2}{3}\right)+(\nu-\mu)\Pi\left(\frac{\pi_1+2\pi_2}{3}\right)\right.$$
$$\left.+(1-\nu)\Pi(\pi_2)-\frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}\Pi(x)\,dx\right|$$
$$\leq \; (\pi_2-\pi_1)q\left[(\Omega_5^*+\Omega_7^*+\Omega_9^*)\left|{}_{\pi_1}D_q\Pi(\pi_2)\right|+(\Omega_6^*+\Omega_8^*+\Omega_{10}^*)\left|{}_{\pi_1}D_q\Pi(\pi_1)\right|\right]$$

where

$$\Omega_5^*=\int_0^{\frac{1}{3}}t\,|t-\lambda|\,dt=\frac{\lambda^3}{3}+\frac{1}{81}-\frac{\lambda}{18},$$

$$\Omega_6^* = \int_0^{\frac{1}{3}} (1-t)\,|t-\lambda|\,dt = \frac{18\lambda^2 - 5\lambda + 1}{18} - \frac{1}{81} - \frac{\lambda^3}{3},$$

$$\Omega_7^* = \int_{\frac{1}{3}}^{\frac{2}{3}} t\,|t-\mu|\,dt = \frac{\mu^3}{3} - \frac{5\mu}{18} + \frac{1}{9}$$

$$\Omega_8^* = \int_{\frac{1}{3}}^{\frac{2}{3}} (1-t)\,|t-\mu|\,dt = \frac{18\mu^2 + 5 + 5\mu}{18} - \mu - \frac{1}{9} - \frac{\mu^3}{3}$$

$$\Omega_9^* = \int_{\frac{2}{3}}^{1} t\,|t-\nu|\,dt = \frac{\nu^3}{3} - \frac{13\nu}{18} + \frac{35}{81},$$

$$\Omega_{10}^* = \int_{\frac{2}{3}}^{1} (1-t)\,|t-\nu|\,dt = \frac{18\nu^2 + 13 + 13\nu}{18} - \frac{5\nu}{3} - \frac{35}{81} - \frac{\nu^3}{3}$$

Remark 5.2. *If we take* $\lambda = \frac{1}{[8]_q}$, $\mu = \frac{1}{[2]_q}$, *and* $\nu = \frac{[7]_q}{[8]_q}$ *in Theorem 5.1, then we obtain the following inequality*

$$\left|\frac{1}{[8]_q}\left[q\Pi(\pi_1) + \frac{q^3[6]_q}{[2]_q}\Pi\left(\frac{\pi_1 q[2]_q + \pi_2}{[3]_q}\right) + \frac{q^2[6]_q}{[2]_q}\Pi\left(\frac{\pi_1 q^2 + \pi_2[2]_q}{[3]_q}\right) + \Pi(\pi_2)\right]\right.$$
$$\left. - \frac{1}{\pi_2 - \pi_1}\int_{\pi_1}^{\pi_2} \Pi(x)\ {}_{\pi_1}d_q x\right|$$
$$\leq q(\pi_2 - \pi_1)\left[\left|{}_{\pi_1}D_q\Pi(\pi_2)\right|\left[A_3(q) + A_4(q) + A_5(q)\right] + \left|{}_{\pi_1}D_q\Pi(\pi_1)\right|\left[B_3(q) + B_4(q) + B_5(q)\right]\right]$$

where

$$A_3(q) = \frac{2q^2[3]_q^3 + [8]_q^2\left([8]_q[2]_q - [3]_q^2\right)}{[8]_q^3[3]_q^4[2]_q},$$

$$B_3(q) = 2\frac{q[8]_q[3]_q - q^2}{[8]_q^3[2]_q[3]_q} + \frac{[3]_q^2 - [2]_q}{[3]_q^4[2]_q} + \frac{1 - [3]_q[2]_q}{[8]_q[3]_q^2[2]_q},$$

$$A_4(q) = \frac{2q^2}{[2]_q^4[3]_q} + \frac{[2]_q^2\left(1 + [2]_q^3\right) - [3]_q^2\left(1 + [2]_q^2\right)}{[3]_q^4[2]_q^2},$$

$$B_4(q) = \frac{2q}{[2]_q^3} - \frac{q}{[3]_q^2} - \frac{q^2}{[3]_q^2} - A_4(q),$$

$$A_5(q) = \frac{2q^2[7]_q^3}{[8]_q^3[2]_q[3]_q} + \frac{[2]_q[8]_q\left([2]_q^3 + [3]_q^3\right) - [7]_q[3]_q^2\left([2]_q^2 + [3]_q^2\right)}{[3]_q^4[8]_q[2]_q},$$

and

$$B_5(q) = 2\frac{q[7]_q^2[8]_q[3]_q - q^2[7]_q^3}{[8]_q^3[2]_q[3]_q} + \frac{q^2}{[2]_q[3]_q} - \frac{q[7]_q}{[2]_q[8]_q}$$

$$+\frac{[2]_q\left([3]_q^2-[2]_q^2\right)}{[3]_q^4}-\frac{\left(q+q^2\right)[7]_q\,[2]_q}{[3]_q^2\,[8]_q}.$$

Theorem 5.2. *We assume that the given conditions of Lemma 3.2 hold. If the mapping $\left|_{\pi_1}D_q\Pi\right|^{p_1}$, $p_1 \geq 1$ is convex on $[\pi_1, \pi_2]$, then the following inequality holds:*

$$\begin{aligned}
&\left|q\lambda\Pi(\pi_1)+q(\mu-\lambda)\Pi\left(\frac{\pi_1 q[2]_q+\pi_2}{[3]_q}\right)+q(\nu-\mu)\Pi\left(\frac{\pi_1 q^2+\pi_2[2]_q}{[3]_q}\right)\right. \\
&\left.+(1-\nu q)\Pi(\pi_2)-\frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}\Pi(x)\ {}_{\pi_1}d_qx\right| \\
\leq\ &(\pi_2-\pi_1)q\left[\Omega_{13}^{1-\frac{1}{p_1}}\left(\Omega_5\left|_{\pi_1}D_q\Pi(\pi_2)\right|^{p_1}+\Omega_6\left|_{\pi_1}D_q\Pi(\pi_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right. \\
&+\Omega_{14}^{1-\frac{1}{p_1}}\left(\left(\Omega_7\left|_{\pi_1}D_q\Pi(\pi_2)\right|^{p_1}+\Omega_8\left|_{\pi_1}D_q\Pi(\pi_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right) \\
&\left.+\Omega_{15}^{1-\frac{1}{p_1}}\left(\Omega_9\left|_{\pi_1}D_q\Pi(\pi_2)\right|^{p_1}+\Omega_{10}\left|_{\pi_1}D_q\Pi(\pi_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right]
\end{aligned} \tag{5.2}$$

where Ω_5-Ω_{10} and Ω_{13}-Ω_{15} are given in (3.17)–(3.22) and (3.10)–(3.12), respectively. The above inequality established by Erden et al. in [44].

Proof. By applying the steps used in the proof of Theorem 4.2 and taking into account Lemma 3.2, we can obtain the required inequality (5.2). □

Corollary 5.2. *If we take the limit $q \to 1^-$ in Theorem 5.2, then we obtain the following inequality*

$$\begin{aligned}
&\left|\lambda\Pi(\pi_1)+(\mu-\lambda)\Pi\left(\frac{2\pi_1+\pi_2}{3}\right)+(\nu-\mu)\Pi\left(\frac{\pi_1+2\pi_2}{3}\right)+(1-\nu)\Pi(\pi_2)\right. \\
&\left.-\frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}\Pi(x)\,dx\right| \\
\leq\ &(\pi_2-\pi_1)q\left[\Theta_{11}^{1-\frac{1}{p_1}}\left(\Omega_5^*\left|_{\pi_1}D_q\Pi(\pi_2)\right|^{p_1}+\Omega_6^*\left|_{\pi_1}D_q\Pi(\pi_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right. \\
&+\Theta_{12}^{1-\frac{1}{p_1}}\left(\left(\Omega_7^*\left|_{\pi_1}D_q\Pi(\pi_2)\right|^{p_1}+\Omega_8^*\left|_{\pi_1}D_q\Pi(\pi_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right) \\
&\left.+\Theta_{13}^{1-\frac{1}{p_1}}\left(\Omega_9^*\left|_{\pi_1}D_q\Pi(\pi_2)\right|^{p_1}+\Omega_{10}^*\left|_{\pi_1}D_q\Pi(\pi_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right]
\end{aligned}$$

where Ω_5^*-Ω_{10}^* are defined in Corollary 5.1 and

$$\Theta_{11}=\int_0^{\frac{1}{3}}|t-\lambda|\,dt=\lambda^2+\frac{1}{9[2]_q}-\frac{\lambda}{3},$$

$$\Theta_{12}=\int_{\frac{1}{3}}^{\frac{2}{3}}|t-\mu|\,dt=\frac{18\mu^2+5}{18}-\mu,$$

$$\Theta_{13}=\int_{\frac{2}{3}}^{1}|t-\nu|\,dt=\frac{18\nu^2+13}{18}-\frac{5\nu}{3}.$$

Remark 5.3. *If we take* $\lambda = \frac{1}{[8]_q}$, $\mu = \frac{1}{[2]_q}$, *and* $\nu = \frac{[7]_q}{[8]_q}$ *in Theorem 5.2, then we obtain the following inequality*

$$\begin{aligned}
&\left|\frac{1}{[8]_q}\left[q\Pi(\pi_1) + \frac{q^3[6]_q}{[2]_q}\Pi\left(\frac{\pi_1 q[2]_q + \pi_2}{[3]_q}\right) + \frac{q^2[6]_q}{[2]_q}\Pi\left(\frac{\pi_1 q^2 + \pi_2[2]_q}{[3]_q}\right) + \Pi(\pi_2)\right]\right. \\
&\left.-\frac{1}{\pi_2 - \pi_1}\int_{\pi_1}^{\pi_2}\Pi(x)\ {}_{\pi_1}d_qx\right| \\
\leq\ & q(\pi_2 - \pi_1)\left[\left(\frac{2q}{[8]_q^2[2]_q} + \frac{[8]_q - [3]_q[2]_q}{[3]_q^2[2]_q[8]_q}\right)^{1-\frac{1}{p_1}}\right. \\
&\times\left(A_3(q)\left|{}_{\pi_1}D_q\Pi(\pi_2)\right|^{p_1} + B_3(q)\left|{}_{\pi_1}D_q\Pi(\pi_1)\right|^{p_1}\right)^{\frac{1}{p_1}} \\
&+\left(\frac{2q}{[2]_q^3} + \frac{q}{[3]_q^2[2]_q} + \frac{1 - [3]_q[2]_q}{[3]_q^2[2]_q}\right)^{1-\frac{1}{p_1}} \\
&\times\left(A_4(q)\left|{}_{\pi_1}D_q\Pi(\pi_2)\right|^{p_1} + B_4(q)\left|{}_{\pi_1}D_q\Pi(\pi_1)\right|^{p_1}\right)^{\frac{1}{p_1}} \\
&+\left(2\frac{q[7]_q^2}{[8]_q^2[2]_q} + \frac{[3]_q^2 + [2]_q^2}{[2]_q[3]_q^2} - \frac{[7]_q\left([3]_q + [2]_q\right)}{[8]_q[3]_q}\right)^{1-\frac{1}{p_1}} \\
&\times\left(A_5(q)\left|{}_{\pi_1}D_q\Pi(\pi_2)\right|^{p_1} + B_5(q)\left|{}_{\pi_1}D_q\Pi(\pi_1)\right|^{p_1}\right)^{\frac{1}{p_1}}
\end{aligned}$$

where $A_3(q) - A_5(q)$ *and* $B_3(q) - B_5(q)$ *are given in Remark 5.2. The above inequality established by Erden et al. in [44].*

Remark 5.4. *If we assume* $\lambda = \mu = \nu = \frac{1}{[2]_q}$ *in Theorem 5.2, then we obtain [42, Theorem 4.2].*

Theorem 5.3. *We assume that the given conditions of Lemma 3.2 hold. If the mapping* $\left|{}_{\pi_1}D_q\Pi\right|^{p_1}$, $p_1 > 1$ *is convex on* $[\pi_1, \pi_2]$, *then the following inequality holds:*

$$\begin{aligned}
&\left|q\lambda\Pi(\pi_1) + q(\mu - \lambda)\Pi\left(\frac{\pi_1 q[2]_q + \pi_2}{[3]_q}\right) + q(\nu - \mu)\Pi\left(\frac{\pi_1 q^2 + \pi_2[2]_q}{[3]_q}\right)\right. \qquad (5.3) \\
&\left.+(1 - \nu q)\Pi(\pi_2) - \frac{1}{\pi_2 - \pi_1}\int_{\pi_1}^{\pi_2}\Pi(x)\ {}_{\pi_1}d_qx\right| \\
\leq\ & (\pi_2 - \pi_1)q\left[\Omega_{18}^{\frac{1}{r_1}}\left(\frac{\left|{}_{\pi_1}D_q\Pi(\pi_2)\right|^{p_1}}{\left([3]_q\right)^2[2]_q} + \frac{\left([2]_q[3]_q - 1\right)\left|{}_{\pi_1}D_q\Pi(\pi_1)\right|^{p_1}}{\left([3]_q\right)^2[2]_q}\right)^{\frac{1}{p_1}}\right. \\
&+\Omega_{19}^{\frac{1}{r_1}}\left(\frac{\left(\left([2]_q\right)^2 - 1\right)\left|{}_{\pi_1}D_q\Pi(\pi_2)\right|^{p_1}}{\left([3]_q\right)^2[2]_q} + \frac{\left(\left([2]_q\right)^2\left([3]_q - 1\right) - [3]_q[2]_q + 1\right)\left|{}_{\pi_1}D_q\Pi(\pi_1)\right|^{p_1}}{3[2]_q}\right)^{\frac{1}{p_1}} \\
&\left.+\Omega_{20}^{\frac{1}{r_1}}\left(\begin{array}{c}\frac{\left(\left([3]_q\right)^2 - \left([2]_q\right)^2\right)\left|{}_{\pi_1}D_q\Pi(\pi_2)\right|^{p_1}}{\left([3]_q\right)^2[2]_q} \\ +\frac{\left(\left([3]_q\right)^2\left([2]_q - 1\right) - \left([2]_q\right)^2\left([3]_q - 1\right)\right)\left|{}_{\pi_1}D_q\Pi(\pi_1)\right|^{p_1}}{\left([3]_q\right)^2[2]_q}\end{array}\right)^{\frac{1}{p_1}}\right]
\end{aligned}$$

where $p_1^{-1} + r_1^{-1} = 1$ *and*

$$\Omega_{18} = \int_0^{\frac{1}{[3]_q}} |t-\lambda|^{r_1}\, d_qt,\ \Omega_{19} = \int_{\frac{1}{[3]_q}}^{\frac{[2]_q}{[3]_q}} |t-\mu|^{r_1}\, d_qt,\ \Omega_{20} = \int_{\frac{[2]_q}{[3]_q}}^{1} |t-\nu|^{r_1}\, d_qt.$$

Proof. By applying the steps used in the proof of Theorem 4.3 and taking into account Lemma 3.2, we can obtain the required inequality (5.3). □

Remark 5.5. *If we assume* $\lambda = \mu = \frac{1}{[2]_q}$ *in Theorem 5.3, then we obtain [27, Theorem 3.3].*

Remark 5.6. *If we take* $\lambda = \frac{1}{[8]_q}$, $\mu = \frac{1}{[2]_q}$, *and* $\nu = \frac{[7]_q}{[8]_q}$ *in Theorem 5.3, then we obtain the following inequality*

$$\begin{aligned}
&\left|\frac{1}{[8]_q}\left[q\Pi(\pi_1) + \frac{q^3[6]_q}{[2]_q}\Pi\left(\frac{\pi_1 q[2]_q + \pi_2}{[3]_q}\right) + \frac{q^2[6]_q}{[2]_q}\Pi\left(\frac{\pi_1 q^2 + \pi_2[2]_q}{[3]_q}\right) + \Pi(\pi_2)\right]\right.\\
&\left. -\frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}\Pi(x)\ {}_{\pi_1}d_qx\right|\\
\leq\ & q(\pi_2-\pi_1)\left[\left(\frac{q^{3r_1}[5]_q^{r_1}}{[3]_q^{r_1+1}[8]_q^{r_1}}\right)^{\frac{1}{r_1}}\right.\\
&\times\left(\frac{1}{[3]_q^2[2]_q}\left|{}_{\pi_1}D_q\Pi(\pi_1)\right|^{p_1} + \frac{[3]_q[2]_q-1}{[3]_q^2[2]_q}\left|{}_{\pi_1}D_q\Pi(\pi_2)\right|^{p_1}\right)^{\frac{1}{p_1}}\\
&+\left(\frac{q^{r_1}[2]_q - q^{2r_1}}{[3]_q^{r_1+1}[2]_q^{r_1}}\right)^{\frac{1}{r_1}}\\
&\times\left(\frac{q^2+2}{[3]_q^2[2]_q}\left|{}_{\pi_1}D_q\Pi(\pi_1)\right|^{p_1} + \frac{q[3]_q[2]_q-\left(q^2+2q\right)}{[3]_q^2[2]_q}\left|{}_{\pi_1}D_q\Pi(\pi_2)\right|^{p_1}\right)^{\frac{1}{p_1}}\\
&+\left(\frac{q^{7r_1}}{[8]_q^{r_1}} - \frac{[2]_q\left([7]_q[3]_q-[8]_q[2]_q\right)^{r_1}}{[8]_q^{r_1}[3]_q^{r_1+1}}\right)^{\frac{1}{r_1}}\\
&\left.\times\left(\frac{[3]_q^2-[2]_q^2}{[3]_q^2[2]_q}\left|{}_{\pi_1}D_q\Pi(\pi_1)\right|^{p_1} + \frac{q^2[3]_q[2]_q+[2]_q^2-[3]_q^2}{[3]_q^2[2]_q}\left|{}_{\pi_1}D_q\Pi(\pi_2)\right|^{p_1}\right)^{\frac{1}{p_1}}\right]
\end{aligned}$$

which is proved by Iftikhar et al. in [41].

6. Conclusions

To sum up, we provided some generalisations of quantum Simpson's and quantum Newton's inequalities for quantum differentiable convex functions with two and three parameters, respectively. It is important to note that by considering the limit $q \to 1^-$ and different special choices of the involved parameters in our key results, our results transformed into some new and well-known results. We believe that it is an interesting and innovative problem for future researchers who can obtain similar inequalities for different types of convexity and quantum integrals.

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Conflict of interest

The authors declare no conflict of interest.

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