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Research article

Blow-up analysis for a reaction-diffusion equation with gradient absorption terms

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Abstract: This paper deals with the blow-up phenomena of solution to a reaction-diffusion equation with gradient absorption terms under nonlinear boundary flux. Based on the technique of modified differential inequality and comparison principle, we establish some conditions on nonlinearities to guarantee the solution exists globally or blows up at finite time. Moreover, some bounds for blow-up time are derived under appropriate measure in higher dimensional spaces ($N \geq 2$).

Keywords: reaction-diffusion equation; gradient absorption term; nonlinear boundary flux; bounds for blow-up time
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1. Introduction

We study the following reaction-diffusion equation with gradient absorption terms:

$$u_t = \Delta u - f(|\nabla u|), \quad (x,t) \in \Omega \times (0, t^*), \tag{1.1}$$

under nonlinear boundary flux and initial conditions

$$\frac{\partial u}{\partial \nu} = g(u), \quad (x,t) \in \partial\Omega \times (0, t^*), \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded star-shaped domain with smooth boundary $\partial\Omega$, ν is the unit outward normal vector on $\partial\Omega$, and t^* is a possible blow-up time when blow-up occurs, otherwise $t^* = +\infty$. Nonlinear functions f and g are assumed to be nonnegative continuous functions and satisfy appropriate conditions. Moreover, initial data $u_0(x)$ is a positive C^1-function and meets an appropriate compatibility condition. Therefore, as it is well-known from the standard parabolic theory, we deduce that the problem (1.1)–(1.3) has unique non-negative classical solutions.

The gradient model (1.1) is often referred to as a viscous Hamilton-Jacobi equation. And it is closely related to the Kardar-Parisi-Zhang equation describing growth and roughening of surfaces in the physical theory, see [1,2] and references therein for details. Furthermore, the nonlinear boundary flux (1.2) satisfies the nonlinear radial law from the physical point of view (cf. [3,4]).

During the past decades, many scholars have dealt with existence and nonexistence of global solutions, blow-up of solutions, blow-up rates, life span, and asymptotic behavior of the solutions to reaction-diffusion equations (systems), see monographs [5,6] and review literature [7–9]. In particular, the monograph [5, Chapters 2 and 4] illustrates a series of research progresses on reaction-diffusion equations with nonlinear terms $f(u)$ and $f(u, \nabla u)$. Among them, it is important to investigate whether the solution of the reaction-diffusion equation blows up and when blow-up occurs in the sense of appropriate measure.

In this paper, we will investigate bounds for blow-up time of the solution to a gradient diffusion model under nonlinear boundary flux. Levine [10] used a variety of methods to study blow-up phenomena and, in many cases, the methods used to show blow-up of solutions often provide an upper bound for blow-up time. However, lower bounds for blow-up time may be harder to be determined. Recently, since researchers, such as Payne, Schaefer and Philippin, made pioneering works on determining lower bounds for blow-up time, there have been many new progresses on the issue of lower bounds for blow-up time in models without gradient term under nonlinear boundary flux. One can refer to papers [11–14] for constant coefficients and [15–18] for variable coefficients. Note that the lower bounds for the blow-up time are mostly derived in three-dimensional space and the main difficulty lies in determination of Sobolev optimal constant.

However, there are few works on bounds for blow-up time for the gradient diffusion model. The salient feature of the gradient model is that boundary or internal gradient blow-up may or may not occur under some conditions (cf. [19–21]). In particular, literatures [19,20] studied the following reaction-diffusion equation with inner source and gradient absorption terms

$$u_t = \Delta u + \lambda u^p - |\nabla u|^q, \quad (x,t) \in \Omega \times (0, t^*), \tag{1.4}$$

under Dirichlet boundary condition. They pointed out that gradient blow-up never occurs, while L^∞ blow-up does. Payne and Song [22] firstly derived the lower bounds of blow-up time for the gradient damping model (1.4) in three-dimensional space when blow-up occurs. For a high-dimensional case ($N \geq 3$), we refer to [23]. Recently, Liu et al. [24] studied lower bounds of blow-up time for the reaction-diffusion equation (1.4) with gradient absorption terms in a bounded convex domain in three-dimensional space under nonlinear boundary flux. For the studies on reaction-diffusion equations (systems) with time-dependent or space-dependent coefficients and non-divergence form quasilinear equations with inner gradient terms, one can refer to [25–28].

To the best of our knowledge, no research on blow-up analysis to problem (1.1)–(1.3) with gradient absorption terms under nonlinear boundary flux has been done. The main difficulty lies in finding an effect of the competitive relationship between the inner gradient absorption terms and the nonlinear boundary flux on the blow-up solutions. In particular, comparing with the studies, in the aforementioned literatures, on non-gradient problems under nonlinear boundary flux, we consider the gradient damped model, which can be considered as one of the difficult and interesting research problems. Motivated by these observations, using the auxiliary function method, the technique of modified differential inequality and the method of constructing the sub-solution, we establish some conditions for which the solution of (1.1)–(1.3) exists globally or blows up and derive some bounds for blow-up time in high-dimensional spaces ($N \geq 2$).

The remainder of this paper is organized as follows. In Section 2, we present some conditions on nonlinearities f and g for which the solution of problem (1.1)–(1.3) exists globally. In Section 3, we construct a suitable sub-solution to show the solution blows up at finite time. In Section 4, we are devoted to deriving the lower bounds for blow-up time when blow-up occurs.

2. The global existence

In this section, we present some conditions on nonlinearities $f(|\nabla\xi|)$ and $g(\xi)$ for which a global solution of problem (1.1)–(1.3) exists. In order to prove our main results, we introduce the following lemma:

Lemma 2.1. *Suppose that $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a bounded domain assumed to be star-shaped and convex in $N-1$ orthogonal directions with smooth boundary $\partial\Omega$. Then for any nonnegative C^1-function u and constant $l \geq 1$, we have the inequality*

$$\int_\Omega u^{\left(1+\frac{1}{2N-2}\right)l} dx \leq C(N,d)\left[\frac{n_1}{2n_0}\int_\Omega u^l dx + \frac{l}{2}\left(1+\frac{n_2}{n_0}\right)\int_\Omega u^{l-1}|\nabla u| dx\right]^{1+\frac{1}{2N-2}},$$

where

$$C(N,d) = \begin{cases} (1+2d)^{N-3}, & N \geq 3, \\ 1, & N = 2, \end{cases}$$

$d = \max_{x\in\bar{\Omega}} |x|$, and $n_0,\ n_1,\ n_2 > 0$ are constants given in the proof.

Proof. Define a function h_i on $\bar{\Omega}$ such that

$$\sum_{i=1}^N h_i \nu_i \geq n_0 > 0,\ x \in \partial\Omega;\ \ (h_i)_{x_i} \leq \frac{n_1}{N}, h_i \leq \frac{n_2}{N},\ x \in \Omega,$$

where ν_i is the unit outward normal on $\partial\Omega$. By divergence theorem, one can have

$$\begin{aligned} n_0 \int_{\partial\Omega} u^l ds &\leq \sum_{i=1}^N \int_{\partial\Omega} h_i \nu_i u^l ds = \sum_{i=1}^N \int_\Omega \left(h_i u^l\right)_{x_i} dx \\ &= \sum_{i=1}^N \int_\Omega (h_i)_{x_i} u^l dx + l \sum_{i=1}^N \int_\Omega \left(h_i u^{l-1}\right) u_{x_i} dx \\ &\leq n_1 \int_\Omega u^l dx + n_2 l \int_\Omega u^{l-1}|\nabla u| dx \end{aligned}$$

and

$$\int_{\partial\Omega} u^l ds \leq \frac{n_1}{n_0}\int_\Omega u^l dx + \frac{n_2 l}{n_0}\int_\Omega u^{l-1}|\nabla u| dx. \tag{2.1}$$

When $N \geq 3$, a similar argument as in the proof of Lemma 4.1 given in [18] can be used to obtain the desired result by replacing the integral $\int_{\partial\Omega} \omega^{\sigma} ds$ contained in (4.7) of [18, pp. 9] with (2.1), and hence, we omit the proof. However, we cannot use the argument for the case that $N = 2$ and so we give a detailed proof.

Let $P = (\bar{x}_1, \bar{x}_2)$ be an arbitrary point in $\Omega \subset R^2$, and let $P_k = (\xi_k, \bar{x}_2)$ and $Q_k = (\bar{x}_1, \eta_k)$ be the points on the boundary $\partial\Omega$ associated with P, where $k = 1, 2$, and $\xi_1 < \xi_2$ and $\eta_1 < \eta_2$. Then we have

$$u^l(P) = u^l(P_1) + l \int_{P_1}^{P} u^{l-1} u_{x_1} dx_1,$$

$$u^l(P) = u^l(P_2) - l \int_{P_2}^{P} u^{l-1} u_{x_1} dx_1,$$

and then

$$u^l(P) \leq \frac{1}{2}\left[u^l(P_1) + u^l(P_2)\right] + \frac{l}{2} \int_{P_1}^{P_2} u^{l-1} \left|u_{x_1}\right| dx_1. \tag{2.2}$$

Similarly, one can have the inequality

$$u^l(P) \leq \frac{1}{2}\left[u^l(Q_1) + u^l(Q_2)\right] + \frac{l}{2} \int_{Q_1}^{Q_2} u^{l-1} \left|u_{x_2}\right| dx_2. \tag{2.3}$$

By multiplying (2.2) and (2.3) and integrating the result over Ω, we obtain the inequalities

$$\begin{aligned}
\int_{\Omega} u^{2l} dx &\leq \left\{\frac{1}{2}\int_{(x_2)_m}^{(x_2)_M} \left[u^l(P_1) + u^l(P_2)\right] dx_2 + \frac{l}{2}\int_{\Omega} u^{l-1}\left|u_{x_1}\right| dx\right\} \\
&\quad \times \left\{\frac{1}{2}\int_{(x_1)_m}^{(x_1)_M} \left[u^l(Q_1) + u^l(Q_2)\right] dx_1 + \frac{l}{2}\int_{\Omega} u^{l-1}\left|u_{x_2}\right| dx\right\} \\
&\leq \left[\frac{1}{2}\int_{\partial\Omega} u^l |\nu_1| ds + \frac{l}{2}\int_{\Omega} u^{l-1}\left|u_{x_1}\right| dx\right] \\
&\quad \times \left[\frac{1}{2}\int_{\partial\Omega} u^l |\nu_2| ds + \frac{l}{2}\int_{\Omega} u^{l-1}\left|u_{x_2}\right| dx\right] \\
&\leq \left[\frac{1}{2}\int_{\partial\Omega} u^l ds + \frac{l}{2}\int_{\Omega} u^{l-1} |\nabla u| dx\right]^2,
\end{aligned} \tag{2.4}$$

where $(x_k)_m = \min_{\overline{\Omega}} x_k$, $(x_k)_M = \max_{\overline{\Omega}} x_k$, $k = 1, 2$, and ν_i is the unit outward normal on $\partial\Omega$, $i = 1, 2$. We then have the inequality

$$\int_{\Omega} u^{2l} dx \leq \left[\frac{n_1}{2n_0}\int_{\Omega} u^l dx + \frac{l}{2}\left(1 + \frac{n_2}{n_0}\right)\int_{\Omega} u^{l-1} |\nabla u| dx\right]^2,$$

by inserting (2.1) into (2.4). □

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a bounded star-shaped domain assumed to be convex in $N-1$ orthogonal directions with smooth boundary $\partial\Omega$. Assume that the nonnegative function f and positive function g satisfy the following conditions:*

$$f(\xi)\begin{cases} \geq a_1 \xi^p, & \xi > 0, \\ = 0, & \xi \leq 0, \end{cases} \qquad g(\xi)\begin{cases} \leq a_2 \xi^q, & \xi > 0, \\ > 0, & \xi \leq 0, \end{cases} \tag{2.5}$$

where $a_1, a_2 > 0$, $p, q > 1$, and $2q < p + 1$. Then the nonnegative classical solution $u(x,t)$ of problem (1.1)–(1.3) does not blow up; that is, $u(x,t)$ exists for all $t > 0$.

Remark 2.1. *Because $p, q > 1$ and $2q < p + 1$, it can be easily seen that $p > q$. From a physical point of view, the absorption term is dominant. Therefore, the nonnegative classical solution of problem (1.1)–(1.3) does not blow up.*

Proof. Define an auxiliary function

$$\Phi(t) := \int_\Omega u^{2n} dx, \quad n \geq 1. \tag{2.6}$$

Using (1.1), (1.2), (2.5), and Green's formula, it can be seen that

$$\begin{aligned} \Phi'(t) &= 2n \int_\Omega u^{2n-1} u_t dx = 2n \int_\Omega u^{2n-1} \left(\Delta u - f\left(|\nabla u|\right)\right) dx \\ &\leq 2na_2 \int_{\partial\Omega} u^{2n+q-1} ds - 2n\left(2n-1\right) \int_\Omega u^{2(n-1)} \left|\nabla u\right|^2 dx - 2na_1 \int_\Omega u^{2n-1} \left|\nabla u\right|^p dx \\ &= 2na_2 \int_{\partial\Omega} u^{2n+q-1} ds - \frac{2\left(2n-1\right)}{n} \int_\Omega |\nabla u^n|^2 dx - \frac{2na_1 p^p}{(2n+p-1)^p} \int_\Omega \left|\nabla u^{\frac{2n+p-1}{p}}\right|^p dx. \end{aligned} \tag{2.7}$$

We begin with estimating the first term on the right side of (2.7). From [12, (2.7)], one can see that

$$\int_{\partial\Omega} u^{2n+q-1} ds \leq \frac{N}{\rho_0} \int_\Omega u^{2n+q-1} dx + \frac{\left(2n+q-1\right)d}{\rho_0} \int_\Omega u^{2n+q-2} \left|\nabla u\right| dx, \tag{2.8}$$

where $\rho_0 = \min\limits_{x\in\partial\Omega} \left(x \cdot \nu\right) > 0$ and $d = \max\limits_{x\in\bar{\Omega}} |x|$. Note that if Ω is a bounded star-shaped domain containing the origin, then d clearly exists, while if Ω is a bounded star-shaped domain containing x_0 with $x_0 \neq 0$, we can also have the inequality (2.8) with

$$\rho_0 = \min_{x\in\partial\Omega} \left(\left(x - x_0\right) \cdot \nu\right), \quad d = \max_{x\in\bar{\Omega}} \left|x - x_0\right|,$$

by using the technique of translation. It follows from Hölder's and Young's inequalities that

$$\frac{N}{\rho_0} \int_\Omega u^{2n+q-1} dx \leq \frac{1}{2} \int_\Omega u^{2n} dx + \frac{1}{2} \left(\frac{N}{\rho_0}\right)^2 \int_\Omega u^{2n+2q-2} dx, \tag{2.9}$$

$$\frac{\left(2n+q-1\right)d}{\rho_0} \int_\Omega u^{2n+q-2} \left|\nabla u\right| dx \frac{\left(2n+q-1\right)^2 d^2}{2\rho_0^2 \delta_1} \int_\Omega u^{2n+2q-2} dx + \frac{\delta_1}{2n^2} \int_\Omega |\nabla u^n|^2 dx, \tag{2.10}$$

where δ_1 is a positive constant to be determined later. Hence, we get the inequality

$$\begin{aligned} 2na_2 \int_{\partial\Omega} u^{2n+q-1} ds \leq & na_2 \int_\Omega u^{2n} dx + \frac{\delta_1 a_2}{n} \int_\Omega |\nabla u^n|^2 dx \\ & + na_2 \left[\left(\frac{N}{\rho_0}\right)^2 + \frac{\left(2n+q-1\right)^2 d^2}{\rho_0^2 \delta_1}\right] \int_\Omega u^{2n+2q-2} dx. \end{aligned} \tag{2.11}$$

Next, we estimate the last term on the right side of (2.7). For simplification, we set $v = u^{\frac{2n+p-1}{p}}$. Then the last term can be written as

$$-\frac{2na_1p^p}{(2n+p-1)^p}\int_\Omega \left|\nabla u^{\frac{2n+p-1}{p}}\right|^p dx = -\frac{2na_1p^p}{(2n+p-1)^p}\int_\Omega |\nabla v|^p dx.$$

Now, we consider the following two cases:

Case 1. $N \geq 3$: By Lemma 2.1 and Young's inequality, it can be seen that

$$\begin{aligned}\int_\Omega v^p dx &\leq \left(\int_\Omega v^{\left(1+\frac{1}{2^{N-2}}\right)p} dx\right)^{\frac{2^{N-2}}{2^{N-2}+1}} |\Omega|^{\frac{1}{2^{N-2}+1}} \\ &\leq (1+2d)^{\frac{2^{N-2}(N-3)}{2^{N-2}+1}} \left\{\left[\frac{n_1}{2n_0}\int_\Omega v^p dx + \frac{p}{2}\left(1+\frac{n_2}{n_0}\right)\int_\Omega v^{p-1}|\nabla v|\,dx\right]^{1+\frac{1}{2^{N-2}}}\right\}^{\frac{2^{N-2}}{2^{N-2}+1}} |\Omega|^{\frac{1}{2^{N-2}+1}} \\ &= (1+2d)^{\frac{2^{N-2}(N-3)}{2^{N-2}+1}} |\Omega|^{\frac{1}{2^{N-2}+1}} \left[\frac{n_1}{2n_0}\int_\Omega v^p dx + \frac{p}{2}\left(1+\frac{n_2}{n_0}\right)\int_\Omega v^{p-1}|\nabla v|\,dx\right] \\ &\leq D\left[\frac{n_1}{2n_0}\int_\Omega v^p dx + \frac{(p-1)\delta_2}{2}\left(1+\frac{n_2}{n_0}\right)\int_\Omega v^p dx + \frac{1}{2\delta_2^{p-1}}\left(1+\frac{n_2}{n_0}\right)\int_\Omega |\nabla v|^p\,dx\right],\end{aligned} \tag{2.12}$$

where

$$D = (1+2d)^{\frac{2^{N-2}(N-3)}{2^{N-2}+1}} |\Omega|^{\frac{1}{2^{N-2}+1}} > 0,$$

and δ_2 is a positive constant to be determined later. It then follows from (2.12) that

$$\begin{aligned}&\left[1 - \frac{n_1 D}{2n_0} - \frac{D\,(p-1)\,\delta_2}{2}\left(1+\frac{n_2}{n_0}\right)\right]\int_\Omega v^p dx \\ &\leq \frac{D}{2\delta_2^{p-1}}\left(1+\frac{n_2}{n_0}\right)\int_\Omega |\nabla v|^p dx.\end{aligned} \tag{2.13}$$

For suitable constants $n_j\,(j = 0, 1, 2)$ and $\delta_2 > 0$ small enough such that

$$1 - \frac{n_1 D}{2n_0} - \frac{D\,(p-1)\,\delta_2}{2}\left(1+\frac{n_2}{n_0}\right) > 0,$$

inequality (2.13) can be reduced to

$$\int_\Omega |\nabla v|^p dx \geq B_1 \int_\Omega v^p dx,$$

where

$$B_1 = \frac{\left[1 - \frac{n_1 D}{2n_0} - \frac{D(p-1)\delta_2}{2}\left(1+\frac{n_2}{n_0}\right)\right]}{\frac{D}{2\delta_2^{p-1}}\left(1+\frac{n_2}{n_0}\right)} > 0.$$

Hence, we obtain the inequality

$$-\frac{2na_1p^p}{(2n+p-1)^p}\int_\Omega \left|\nabla u^{\frac{2n+p-1}{p}}\right|^p dx \leq -\frac{2na_1p^pB_1}{(2n+p-1)^p}\int_\Omega u^{2n+p-1} dx. \tag{2.14}$$

Case 2. $N = 2$: By Lemma 2.1 and Young's inequality, it can be seen that

$$\begin{aligned}\int_\Omega v^p dx &\le |\Omega|^{\frac{1}{2}}\left(\int_\Omega v^{2p}dx\right)^{\frac{1}{2}}\\ &\le |\Omega|^{\frac{1}{2}}\left[\frac{n_1}{2n_0}\int_\Omega v^p dx+\frac{p}{2}\left(1+\frac{n_2}{n_0}\right)\int_\Omega v^{p-1}|\nabla u|\,dx\right]\\ &\le |\Omega|^{\frac{1}{2}}\left[\frac{n_1}{2n_0}\int_\Omega v^p dx+\frac{(p-1)\delta_3}{2}\left(1+\frac{n_2}{n_0}\right)\int_\Omega v^p dx\right.\\ &\quad\left.+\frac{1}{2\delta_3^{p-1}}\left(1+\frac{n_2}{n_0}\right)\int_\Omega|\nabla v|^p\,dx\right],\end{aligned}\tag{2.15}$$

where δ_3 is a positive constant to be determined later.

For suitable constants $n_j\,(j=0,1,2)$ and $\delta_3>0$ small enough such that

$$1-|\Omega|^{\frac{1}{2}}\left[\frac{n_1}{2n_0}+\frac{(p-1)\,\delta_3}{2}\left(1+\frac{n_2}{n_0}\right)\right]>0,$$

inequality (2.15) can be reduced to

$$\int_\Omega|\nabla v|^p dx\ge B_2\int_\Omega v^p dx,$$

where

$$B_2=\frac{1-|\Omega|^{\frac{1}{2}}\left[\frac{n_1}{2n_0}+\frac{(p-1)\delta_3}{2}\left(1+\frac{n_2}{n_0}\right)\right]}{\frac{1}{2\delta_3^{p-1}}\left(1+\frac{n_2}{n_0}\right)|\Omega|^{\frac{1}{2}}}>0.$$

We then have the inequality

$$-\frac{2na_1p^p}{(2n+p-1)^p}\int_\Omega\left|\nabla u^{\frac{2n+p-1}{p}}\right|^p dx\le-\frac{2na_1p^pB_2}{(2n+p-1)^p}\int_\Omega u^{2n+p-1}dx.\tag{2.16}$$

Setting $B=\min\{B_1,B_2\}>0$ and combining (2.14) with (2.16), we obtain the inequality

$$-\frac{2na_1p^p}{(2n+p-1)^p}\int_\Omega\left|\nabla u^{\frac{2n+p-1}{p}}\right|^p dx\le-\frac{2na_1p^pB}{(2n+p-1)^p}\int_\Omega u^{2n+p-1}dx.\tag{2.17}$$

Then one can see that

$$\begin{aligned}\Phi'(t)\le{}& na_2\int_\Omega u^{2n}dx+na_2\left[\left(\frac{N}{\rho_0}\right)^2+\frac{(2n+q-1)^2\,d^2}{\rho_0^2\delta_1}\right]\int_\Omega u^{2n+2q-2}dx\\ &+\left(\frac{\delta_1a_2}{n}-\frac{2\,(2n-1)}{n}\right)\int_\Omega|\nabla u^n|^2dx-\frac{2na_1p^pB}{(2n+p-1)^p}\int_\Omega u^{2n+p-1}dx,\end{aligned}$$

by substituting (2.11) and (2.17) into (2.7). It can be also seen that

$$\begin{aligned}\Phi'(t)\le{}& na_2\int_\Omega u^{2n}dx+na_2\left[\left(\frac{N}{\rho_0}\right)^2+\frac{(2n+q-1)^2d^2}{\rho_0^2\delta_1}\right]\int_\Omega u^{2n+2q-2}dx\\ &-\frac{2na_1p^pB}{(2n+p-1)^p}\int_\Omega u^{2n+p-1}dx,\end{aligned}\tag{2.18}$$

by selecting $\delta_1 = \frac{2(2n-1)}{a_2} > 0$ such that

$$\frac{\delta_1 a_2}{n} - \frac{2(2n-1)}{n} = 0.$$

We now focus on the first and third terms on the right side of (2.18). From Hölder's inequality, we get

$$\int_\Omega u^{2n+2q-2}dx \le \left(\int_\Omega u^{2n+p-1}dx\right)^{\frac{2n+2q-2}{2n+p-1}} |\Omega|^{\frac{p-2q+1}{2n+p-1}} \tag{2.19}$$

and

$$\int_\Omega u^{2n}dx \le \left(\int_\Omega u^{2n+2q-2}dx\right)^{\frac{2n}{2n+2q-2}} |\Omega|^{\frac{2q-2}{2n+2q-2}}. \tag{2.20}$$

Substituting (2.19) and (2.20) into (2.18), one can see that

$$\begin{aligned}\Phi'(t) &\le na_2 |\Omega|^{\frac{2q-2}{2n+2q-2}} \left(\int_\Omega u^{2n+2q-2}dx\right)^{\frac{2n}{2n+2q-2}} + na_2\left[\left(\frac{N}{\rho_0}\right)^2 + \frac{(2n+q-1)^2 d^2}{\rho_0^2\delta_1}\right] \\ &\quad\times \int_\Omega u^{2n+2q-2}dx - \frac{2na_1p^pB}{(2n+p-1)^p}|\Omega|^{-\frac{p-2q+1}{2n+2q-2}}\left(\int_\Omega u^{2n+2q-2}dx\right)^{\frac{2n+p-1}{2n+2q-2}} \\ &= \int_\Omega u^{2n+2q-2}dx\left[I_1\left(\int_\Omega u^{2n+2q-2}dx\right)^{\frac{2-2q}{2n+2q-2}} + I_2 - I_3\left(\int_\Omega u^{2n+2q-2}dx\right)^{\frac{p-2q+1}{2n+2q-2}}\right],\end{aligned} \tag{2.21}$$

where

$$\begin{aligned} I_1 &= na_2|\Omega|^{\frac{2q-2}{2n+2q-2}} > 0, \\ I_2 &= na_2\left[\left(\frac{N}{\rho_0}\right)^2 + \frac{(2n+q-1)^2 d^2}{\rho_0^2\delta_1}\right] > 0, \\ I_3 &= \frac{2na_1p^pB}{(2n+p-1)^p}|\Omega|^{-\frac{p-2q+1}{2n+2q-2}} > 0. \end{aligned}$$

Finally, it follows from (2.20) and (2.21) that

$$\Phi'(t) \le \int_\Omega u^{2n+2q-2}dx\left[I_1|\Omega|^{\frac{(q-1)^2}{n(n+q-1)}}\Phi^{\frac{1-q}{n}} + I_2 - I_3|\Omega|^{\frac{(1-q)(p-2q+1)}{n(2n+2q-2)}}\Phi^{\frac{p-2q+1}{2n}}\right], \tag{2.22}$$

where $\frac{1-q}{n} < 0$ and $\frac{p-2q+1}{2n} > 0$.

We conclude from (2.22) that $\Phi(t)$ remains bounded for all time under the conditions stated in Theorem 2.1. In fact, if $u(x,t)$ blows up at finite time t^*, then $\Phi(t)$ is unbounded near t^*, which implies $\Phi(t)$ is decreasing in some interval $[t_0, t^*)$, from (2.22). Hence, we have $\Phi(t) \le \Phi(t_0)$ in $[t_0, t^*)$, which means that $\Phi(t)$ is bounded in $[t_0, t^*)$, which is a contradiction. Therefore, $u(x,t)$ exists for all $t > 0$, which completes the proof. □

Remark 2.2. *If the boundary is adiabatic; that is,* $g(u) = 0$*. From (2.6) and (2.7), we know the energy functional* $\Phi(t)$ *is decreasing, and hence, the nonnegative classical solution* $u(x,t)$ *of problem (1.1)–(1.3) exists for all* $t > 0$*.*

Remark 2.3. *If we use L^2-norm $\Phi(t) := \int_\Omega u^2 dx$, the condition $q > 1$ is replaced by $p > q$, and other conditions remain unchanged, then the conclusion of Theorem 2.1 is still valid. In fact, using (2.7), (2.8) and (2.17), we have*

$$\begin{aligned}\Phi'(t) \leq& \frac{2a_2N}{\rho_0}\int_\Omega u^{q+1}dx + \frac{2(q+1)a_2d}{\rho_0}\int_\Omega u^q\,|\nabla u|dx - 2\int_\Omega |\nabla u|^2dx\\ &- \frac{2a_1p^pB}{(p+1)^p}\int_\Omega u^{p+1}dx.\end{aligned}$$

We now apply Young's inequality to $\int_\Omega u^q\,|\nabla u|dx$ to obtain the inequality

$$\int_\Omega u^q\,|\nabla u|dx \leq \frac{1}{2\varsigma}\int_\Omega |\nabla u|^2dx + \frac{\varsigma}{2}\int_\Omega u^{2q}dx.$$

Choosing $\varsigma = \frac{(q+1)a_2d}{2\rho_0}$, we obtain the inequality

$$\begin{aligned}\Phi'(t) \leq& \frac{2a_2N}{\rho_0}\int_\Omega u^{q+1}dx + 2\varsigma^2\int_\Omega u^{2q}dx - \frac{2a_1p^pB}{(p+1)^p}\int_\Omega u^{p+1}dx\\ =& \int_\Omega\left(\frac{2a_2N}{\rho_0}\frac{u^q}{u^p} - \frac{a_1p^pB}{(p+1)^p}\right)u^{p+1}dx + \int_\Omega\left(2\varsigma^2\frac{u^{2q}}{u^{p+1}} - \frac{a_1p^pB}{(p+1)^p}\right)u^{p+1}dx.\end{aligned}$$

Since $p > q$ and $2q < p + 1$, we can conclude that the nonnegative classical solution $u(x,t)$ of problem (1.1)–(1.3) exists for all $t > 0$. In fact, if $u(x,t)$ blows up at finite time t^*, then $u(x,t)$ is unbounded near t^*. And it is easy to know there exists an interval $[t_0, t^*)$, such that

$$\frac{2a_2N}{\rho_0}u^{-(p-q)} - \frac{a_1p^pB}{(p+1)^p} < 0, \quad 2\varsigma^2u^{-(p+1-2q)} - \frac{a_1p^pB}{(p+1)^p} < 0,$$

which implies $\Phi(t)$ is decreasing in some interval $[t_0, t^*)$. So we have $\Phi(t) \leq \Phi(t_0)$ in $[t_0, t^*)$, which means that $\Phi(t)$ is bounded in $[t_0, t^*)$, which is a contradiction. Therefore, $u(x,t)$ exists for all $t > 0$.

3. Blow-up criterion

In this section, the domain Ω only needs to be a bounded region with smooth boundary, instead of star-shaped one. We construct a suitable sub-solution to show the solution blows up at finite time. Our result can be summarized as follows:

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a bounded domain with smooth boundary $\partial\Omega$. Suppose that $u(x,t)$ is a nonnegative classical solution of problem (1.1)–(1.3) and the nonnegative functions f and g are such that*

$$f(|\xi|) = |\xi|^p, \quad g(\xi) = \xi^q, \quad \forall \xi \geq 0, \tag{3.1}$$

where $2q > p+1$ and $p > 1$. Then the solution of (1.1)–(1.3) blows up in a finite time for some suitably large initial data.

Proof. We construct a sub-solution of the form

$$\underline{u}(x,t) = A\left[A^{\alpha}\varphi(x) + (1-\beta t)^{k}\right]^{-\frac{2}{p-1}}, \quad (x,t) \in \bar{\Omega} \times \left[0, \frac{1}{\beta}\right),$$

where $k \geq 1$, $\beta > 0$, $\frac{p-1}{2} < \alpha < q-1$, $A \geq 1$ are constants to be determined and $\varphi(x)$ be the positive normalized eigenfunction, i.e., $\max_{x\in\bar{\Omega}} \varphi(x) = 1$, corresponding to the first eigenvalue λ_0 of the problem

$$-\Delta\varphi(x) = \lambda\varphi(x), \quad x \in \Omega,$$

$$\varphi(x) = 0, \quad x \in \partial\Omega.$$

It is well known that $\lambda_0 > 0$, $\varphi(x) > 0$ in Ω, and $\frac{\partial\varphi}{\partial\nu} < 0$ on $\partial\Omega$. Moreover, there exist positive constants R_1, R_2, R_3 such that $|\nabla\varphi(x)| \leq R_1$ for all $x \in \bar{\Omega}$ and $R_2 \leq -\frac{\partial\varphi}{\partial\nu} \leq R_3$ on $\partial\Omega$.

By direct calculation, one can see that

$$\begin{aligned}
\underline{u}_t &= \frac{2\beta Ak}{p-1}\left[A^{\alpha}\varphi(x) + (1-\beta t)^{k}\right]^{-\frac{p+1}{p-1}}(1-\beta t)^{k-1} \\
&\leq \frac{2\beta Ak}{p-1}\left[A^{\alpha}\varphi(x) + (1-\beta t)^{k}\right]^{-\frac{p+1}{p-1}}, \\
\nabla\underline{u} &= \frac{2}{p-1}A^{\alpha+1}\left[A^{\alpha}\varphi(x) + (1-\beta t)^{k}\right]^{-\frac{p+1}{p-1}}(-\nabla\varphi), \\
\Delta\underline{u} &= \frac{2}{p-1}A^{\alpha+1}\left[A^{\alpha}\varphi(x) + (1-\beta t)^{k}\right]^{-\frac{p+1}{p-1}}(-\Delta\varphi) \\
&\quad + \frac{2(p+1)}{(p-1)^2}A^{2\alpha+1}\left[A^{\alpha}\varphi(x) + (1-\beta t)^{k}\right]^{-\frac{2p}{p-1}}|\nabla\varphi|^2.
\end{aligned}$$

If $x \in \Omega_{\varepsilon} := \{x \in \Omega | \operatorname{dist}(x, \partial\Omega) \geq \varepsilon\}$ for $\varepsilon > 0$, then there exists a positive constant R_4 such that $\varphi(x) \geq R_4$ and

$$\begin{aligned}
\underline{u}_t + \left|\nabla\underline{u}\right|^p &\leq \frac{2\beta Ak}{p-1}\left[\varphi(x)A^{\alpha} + (1-\beta t)^{k}\right]^{-\frac{p+1}{p-1}} \\
&\quad + \left(\frac{2}{p-1}\right)^p A^{p(\alpha+1)}\left[A^{\alpha}\varphi(x) + (1-\beta t)^{k}\right]^{-\frac{p(p+1)}{p-1}}|\nabla\varphi|^p \\
&\leq \frac{2\beta Ak}{p-1}(R_4A^{\alpha})^{-\frac{p+1}{p-1}} + \left(\frac{2}{p-1}\right)^p R_1^p A^{p(\alpha+1)}(R_4A^{\alpha})^{-\frac{p(p+1)}{p-1}} \\
&= \frac{2\beta k}{p-1}R_4^{-\frac{p+1}{p-1}}A^{1-\frac{\alpha(p+1)}{p-1}} + \left(\frac{2}{p-1}\right)^p R_1^p A^{\frac{p(p-1-2\alpha)}{p-1}} R_4^{-\frac{p(p+1)}{p-1}}, \\
\Delta\underline{u} &\geq \frac{2\lambda_0}{p-1}A^{\alpha+1}R_4(A^{\alpha}+1)^{-\frac{p+1}{p-1}} \geq 2^{-\frac{2}{p-1}}\frac{\lambda_0R_4}{p-1}A^{\frac{p-1-2\alpha}{p-1}}.
\end{aligned}$$

From the inequalities above, it can be seen that

$$\underline{u}_t \leq \Delta\underline{u} - \left|\nabla\underline{u}\right|^p, \quad (x,t) \in \Omega_{\varepsilon} \times \left(0, \frac{1}{\beta}\right),$$

provided that

$$2^{-\frac{2}{p-1}}\frac{\lambda_0 R_4}{p-1}A^{\frac{p-1-2\alpha}{p-1}}$$
$$\geq \frac{2\beta k}{p-1}R_4^{-\frac{p+1}{p-1}}A^{1-\frac{\alpha(p+1)}{p-1}}+\left(\frac{2}{p-1}\right)^p R_1^p A^{\frac{p(p-1-2\alpha)}{p-1}}R_4^{-\frac{p(p+1)}{p-1}}. \tag{3.2}$$

If $x\in\Omega\backslash\Omega_\varepsilon := \{x\in\Omega | dist\,(x,\partial\Omega)<\varepsilon\}$, it is easy to know $|\nabla\varphi(x)|\geq\frac{R_2}{2}$ and

$$\begin{aligned}\underline{u}_t+\left|\nabla\underline{u}\right|^p &\leq \frac{2\beta Ak}{p-1}\left[A^\alpha\varphi(x)+(1-\beta t)^k\right]^{-\frac{p+1}{p-1}} \\ &\quad+\left(\frac{2}{p-1}\right)^p A^{p(\alpha+1)}\left[A^\alpha\varphi(x)+(1-\beta t)^k\right]^{-\frac{p(p+1)}{p-1}}|\nabla\varphi|^p \\ &\leq \left[A^\alpha\varphi(x)+(1-\beta t)^k\right]^{-\frac{2p}{p-1}}\left\{\frac{2\beta Ak}{p-1}\left[A^\alpha\varphi(x)+(1-\beta t)^k\right]\right. \\ &\quad\left.+\left(\frac{2R_1}{p-1}\right)^p A^{p(\alpha+1)}\left[A^\alpha\varphi(x)+(1-\beta t)^k\right]^{-p}\right\} \\ &\leq \left[A^\alpha\varphi(x)+(1-\beta t)^k\right]^{-\frac{2p}{p-1}}\left[\frac{4\beta kA^{\alpha+1}}{p-1}+\left(\frac{R_1}{p-1}\right)^p A^p\right],\end{aligned}$$

$$\Delta\underline{u}\geq\frac{(p+1)R_2^2}{2(p-1)^2}A^{2\alpha+1}\left[A^\alpha\varphi(x)+(1-\beta t)^k\right]^{-\frac{2p}{p-1}}.$$

Hence,

$$\underline{u}_t\leq\Delta\underline{u}-\left|\nabla\underline{u}\right|^p,\quad (x,t)\in\Omega\backslash\Omega_\varepsilon\times\left(0,\frac{1}{\beta}\right),$$

if

$$\frac{(p+1)R_2^2}{2(p-1)^2}A^{2\alpha+1}\geq\frac{4\beta kA^{\alpha+1}}{p-1}+\left(\frac{R_1}{p-1}\right)^p A^p, \tag{3.3}$$

In addition, for $(x,t)\in\partial\Omega\times\left(0,\frac{1}{\beta}\right)$,

$$\frac{\partial\underline{u}}{\partial\nu}=\frac{2}{p-1}A^{\alpha+1}\left[(1-\beta t)^k\right]^{-\frac{p+1}{p-1}}\left(-\frac{\partial\varphi}{\partial\nu}\right),$$

$$\underline{u}^q=A^q\left[(1-\beta t)^k\right]^{-\frac{2q}{p-1}}.$$

Since $\frac{p+1}{p-1}<\frac{2q}{p-1}$ and $\alpha<q-1$, we have

$$\frac{\partial\underline{u}}{\partial\nu}\leq\underline{u}^q,\quad (x,t)\in\partial\Omega\times\left(0,\frac{1}{\beta}\right)$$

provided that

$$\frac{2R_3}{p-1}A^{\alpha+1}\leq A^q. \tag{3.4}$$

Thanks to $p > 1$ and $\frac{p-1}{2} < \alpha < q - 1$, then $\alpha > 0$ and $2\alpha - p + 1 > 0$. The inequalities (3.2)–(3.4) hold for A such that

$$\begin{aligned} A \geq \max \Bigg\{ & 1, \left(\frac{2R_3}{p-1}\right)^{\frac{1}{q-(\alpha+1)}}, \left[\frac{2^{\frac{p+1}{p-1}}(p-1)}{\lambda_0} \right. \\ & \left. \times \left(\frac{2\beta k}{p-1} R_4^{-\frac{2p}{p-1}} + \left(\frac{2}{p-1}\right)^p R_1^p R_4^{-\frac{p^2+2p-1}{p-1}}\right)\right]^\theta \\ & \left[\frac{4(p-1)^2}{R_2^2(p+1)} \left(\frac{4\beta k}{p-1} + \frac{R_1^p}{(p-1)^p}\right)\right]^\theta \Bigg\}, \end{aligned}$$

where $\frac{1}{\theta} = \min\{\alpha, 2\alpha - p + 1\}$.

Therefore, if we take $u_0(x)$ suitably large for which

$$\underline{u}(x,0) = A\left[A^\alpha \varphi(x) + 1\right]^{-\frac{2}{p-1}} \leq u_0(x)$$

for every $x \in \Omega$, then the comparison principle shows that

$$\underline{u}(x,t) = A\left[A^\alpha \varphi(x) + (1-\beta t)^k\right]^{-\frac{2}{p-1}}$$

is a sub-solution of (1.1)–(1.3). Moreover, we easy to see that $\underline{u}$ occurs boundary blow-up in a finite time $t^* = \frac{1}{\beta}$, and hence, the solution of (1.1)–(1.3) blows up in a finite time t^* with upper bound $\frac{1}{\beta}$ for suitably large initial data. □

4. Lower bounds for t^*

In this section, we assume some conditions on the nonlinearities f and g to find lower bounds for the blow-up time t^* in high-dimensional spaces ($N \geq 2$).

4.1. The case that $N \geq 3$

In this subsection, the domain $\Omega \subset R^N$ ($N \geq 3$) is assumed to be a bounded star-shaped domain and convex in $N - 1$ orthogonal directions with smooth boundary.

Theorem 4.1. *Suppose that $u(x,t)$ is the nonnegative classical solution of problem (1.1)–(1.3), $u(x,t)$ blows up at t^*, and that the nonnegative functions f and g satisfy the following conditions:*

$$f(|\xi|) \geq a_1 |\xi|^p, \quad g(\xi) \leq a_2 \xi^q, \quad \forall \xi \geq 0, \tag{4.1}$$

where $a_1, a_2 > 0$, $p, q > 1$, and $2q \geq p + 1$. Define a function

$$\phi(t) := \int_\Omega u^{2n} dx,$$

where

$$n > \max\{2(N-2)(q-1), 1\}.$$

Then the blow-up time t^* is bounded below, i.e.,

$$t^* \geq \int_{\phi(0)}^{+\infty} \frac{d\xi}{Q_1\xi^{\frac{3(N-2)}{3N-8}} + Q_2\xi + Q_3},$$

where $\phi(0) = \int_\Omega u_0^{2n}dx$ and Q_1–Q_3 are some positive constants given in the proof.

Proof. Using (1.1), (1.2), (4.1), and Green's formula, we have

$$\begin{aligned}
\phi'(t) &= 2n\int_\Omega u^{2n-1}u_t dx \\
&= 2n\int_\Omega u^{2n-1}\left(\Delta u - f\left(|\nabla u|\right)\right)dx \\
&\leq 2na_2\int_{\partial\Omega} u^{2n+q-1}ds \quad - 2n\left(2n-1\right)\int_\Omega u^{2(n-1)}\left|\nabla u\right|^2 dx \\
&\quad - 2na_1\int_\Omega u^{2n-1}\left|\nabla u\right|^p dx \\
&= 2na_2\int_{\partial\Omega} u^{2n+q-1}ds \quad - \frac{2\left(2n-1\right)}{n}\int_\Omega \left|\nabla u^n\right|^2 dx \\
&\quad - \frac{2na_1p^p}{(2n+p-1)^p}\int_\Omega \left|\nabla u^{\frac{2n+p-1}{p}}\right|^p dx.
\end{aligned} \tag{4.2}$$

By (2.8), we get

$$\int_{\partial\Omega} u^{2n+q-1}ds \leq \frac{N}{\rho_0}\int_\Omega u^{2n+q-1}dx + \frac{\left(2n+q-1\right)d}{\rho_0}\int_\Omega u^{2n+q-2}\left|\nabla u\right|dx. \tag{4.3}$$

Applying Hölder's and Young's inequalities to the terms on the right side of (4.3), respectively, one can see that

$$\frac{N}{\rho_0}\int_\Omega u^{2n+q-1}dx \leq \frac{1}{2}\int_\Omega u^{2n}dx + \frac{1}{2}\left(\frac{N}{\rho_0}\right)^2\int_\Omega u^{2n+2q-2}dx, \tag{4.4}$$

$$\begin{aligned}
&\frac{\left(2n+q-1\right)d}{\rho_0}\int_\Omega u^{2n+q-2}\left|\nabla u\right|dx \\
\leq\; &\frac{\left(2n+q-1\right)^2 d^2}{2\rho_0^2\epsilon_1}\int_\Omega u^{2n+2q-2}dx + \frac{\epsilon_1}{2n^2}\int_\Omega |\nabla u^n|^2 dx,
\end{aligned} \tag{4.5}$$

where ϵ_1 is a positive constant to be determined later.

Next, we estimate the last term on the right side of (4.2). It follows from (2.17) that

$$-\frac{2na_1p^p}{(2n+p-1)^p}\int_\Omega \left|\nabla u^{\frac{2n+p-1}{p}}\right|^p dx \leq -\frac{2na_1p^pB}{(2n+p-1)^p}\int_\Omega u^{2n+p-1}dx, \tag{4.6}$$

where $B > 0$ is the constant given in the proof of Theorem 2.1. By using Hölder's inequality, we have

$$\int_\Omega u^{2n+p-1}dx \geq |\Omega|^{\frac{-p+1}{2n}}\left(\int_\Omega u^{2n}dx\right)^{\frac{2n+p-1}{2n}}. \tag{4.7}$$

Substituting (4.3)–(4.7) into (4.2), one can obtain the inequality

$$\begin{aligned}\phi'(t) \leq & na_2 \int_\Omega u^{2n} dx + na_2 \left[\left(\frac{N}{\rho_0} \right)^2 + \frac{(2n+q-1)^2 d^2}{\rho_0^2 \varepsilon_1} \right] \int_\Omega u^{2n+2q-2} dx \\ & + \left(\frac{\varepsilon_1 a_2}{n} - \frac{2(2n-1)}{n} \right) \int_\Omega |\nabla u^n|^2 dx \\ & - \frac{2na_1 p^p B}{(2n+p-1)^p} |\Omega|^{\frac{-p+1}{2n}} \left(\int_\Omega u^{2n} dx \right)^{\frac{2n+p-1}{2n}} .\end{aligned} \tag{4.8}$$

We now consider the second term on the right side of (4.8). By using Hölder's and Young's inequalities, we get

$$\begin{aligned}\int_\Omega u^{2n+2q-2} dx &= \int_\Omega u^{\frac{n(2N-3)}{N-2} \cdot \frac{(N-2)(2n+2q-2)}{n(2N-3)}} dx \\ &\leq |\Omega|^{1-m_1} \left(\int_\Omega u^{\frac{n(2N-3)}{N-2}} dx \right)^{m_1} \\ &\leq (1-m_1) |\Omega| + m_1 \int_\Omega u^{\frac{n(2N-3)}{N-2}} dx,\end{aligned} \tag{4.9}$$

where

$$m_1 = \frac{(N-2)(2n+2q-2)}{n(2N-3)} \in (0,1).$$

Substituting (4.9) into (4.8), we obtain the inequality

$$\begin{aligned}\phi'(t) \leq & na_2 \int_\Omega u^{2n} dx + P_1 \int_\Omega u^{\frac{n(2N-3)}{N-2}} dx \\ & + P_2 \int_\Omega |\nabla u^n|^2 dx - P_3 \left(\int_\Omega u^{2n} dx \right)^{\frac{2n+p-1}{2n}} + P_4,\end{aligned} \tag{4.10}$$

where

$$\begin{aligned}P_1 &= na_2 m_1 \left[\left(\frac{N}{\rho_0} \right)^2 + \frac{(2n+q-1)^2 d^2}{\rho_0^2 \varepsilon_1} \right] > 0, \\ P_2 &= \frac{\varepsilon_1 a_2}{n} - \frac{2(2n-1)}{n}, \\ P_3 &= \frac{2na_1 p^p B}{(2n+p-1)^p} |\Omega|^{\frac{-p+1}{2n}} > 0, \\ P_4 &= na_2 (1-m_1) \left[\left(\frac{N}{\rho_0} \right)^2 + \frac{(2n+q-1)^2 d^2}{\rho_0^2 \varepsilon_1} \right] |\Omega| > 0.\end{aligned}$$

By applying Schwarz's inequality to the second term on the right side of (4.10), we have

$$\begin{aligned}\int_\Omega u^{\frac{n(2N-3)}{(N-2)}} dx &\leq \left(\int_\Omega u^{2n} dx \right)^{\frac{1}{2}} \left(\int_\Omega u^{\frac{2n(N-1)}{N-2}} dx \right)^{\frac{1}{2}} \\ &\leq \left(\int_\Omega u^{2n} dx \right)^{\frac{1}{2}} \left[\left(\int_\Omega u^{2n} dx \right)^{\frac{1}{2}} \left(\int_\Omega (u^n)^{\frac{2N}{N-2}} dx \right)^{\frac{1}{2}} \right]^{\frac{1}{2}} \\ &= \left(\int_\Omega u^{2n} dx \right)^{\frac{3}{4}} \left(\int_\Omega (u^n)^{\frac{2N}{N-2}} dx \right)^{\frac{1}{4}} .\end{aligned} \tag{4.11}$$

To bound $\int_\Omega (u^n)^{\frac{2N}{N-2}}\,dx$, we use the Sobolev inequality ($N \geq 3$) given in [29] and then obtain the inequalities

$$\begin{aligned} \|u^n\|_{L^{\frac{2N}{N-2}}(\Omega)}^{\frac{N}{2(N-2)}} &\leq (c_s)^{\frac{N}{2(N-2)}}\,\|u^n\|_{W^{1,2}(\Omega)}^{\frac{N}{2(N-2)}} \\ &\leq c\left(\|\nabla u^n\|_{L^2(\Omega)}^{\frac{N}{2(N-2)}} + \|u^n\|_{L^2(\Omega)}^{\frac{N}{2(N-2)}}\right), \end{aligned} \tag{4.12}$$

where c_s is a constant depending on Ω and N, and

$$c = \begin{cases} 2^{\frac{1}{2}}\,(c_s)^{\frac{3}{2}}, & N = 3, \\ (c_s)^{\frac{N}{2(N-2)}}, & N > 3. \end{cases}$$

Substituting (4.12) into (4.11) and using Young's inequality, one can see that

$$\begin{aligned} \int_\Omega u^{\frac{n(2N-3)}{N-2}}\,dx &\leq c\left(\int_\Omega u^{2n}dx\right)^{\frac{3}{4}}\left(\int_\Omega |\nabla u^n|^2\,dx\right)^{\frac{N}{4(N-2)}} + c\left(\int_\Omega u^{2n}dx\right)^{\frac{2N-3}{2(N-2)}} \\ &\leq \frac{c^{\frac{4(N-2)}{3N-8}}\,(3N-8)}{4\,(N-2)}\varepsilon_2^{-\frac{N}{3N-8}}\left(\int_\Omega u^{2n}dx\right)^{\frac{3(N-2)}{3N-8}} \\ &\quad + \frac{N\varepsilon_2}{4\,(N-2)}\int_\Omega |\nabla u^n|^2\,dx + c\left(\int_\Omega u^{2n}dx\right)^{\frac{2N-3}{2(N-2)}}, \end{aligned} \tag{4.13}$$

where ϵ_2 is a positive constant to be determined later. It follows from Young's inequality that

$$\left(\int_\Omega u^{2n}dx\right)^{\frac{2N-3}{2(N-2)}} \leq m_2\varepsilon_3^{-\frac{m_3}{m_2}}\left(\int_\Omega u^{2n}dx\right)^{\frac{3(N-2)}{3N-8}} + m_3\varepsilon_3\left(\int_\Omega u^{2n}dx\right)^{\frac{2n+p-1}{2n}}, \tag{4.14}$$

where

$$\begin{aligned} m_2 &= \frac{(3N-8)\,[2n\,(2N-3) - 2\,(N-2)\,(2n+p-1)]}{2\,(N-2)\,[6n\,(N-2) - (3N-8)\,(2n+p-1)]} \in (0,1)\,, \\ m_3 &= \frac{2n\left[6\,(N-2)^2 - (2N-3)\,(3N-8)\right]}{2\,(N-2)\,[6n\,(N-2) - (3N-8)\,(2n+p-1)]} \in (0,1)\,, \end{aligned}$$

and ϵ_3 is a positive constant to be determined later. Substituting (4.13) and (4.14) into (4.10), we obtain the inequality

$$\begin{aligned} \phi'(t) &\leq Q_1\left(\int_\Omega u^{2n}dx\right)^{\frac{3(N-2)}{3N-8}} + Q_2\int_\Omega u^{2n}dx + Q_3 \\ &\quad + Q_4\int_\Omega |\nabla u^n|^2\,dx + Q_5\left(\int_\Omega u^{2n}dx\right)^{\frac{2n+p-1}{2n}}, \end{aligned} \tag{4.15}$$

where

$$Q_1 = P_1\left[\frac{c^{\frac{4(N-2)}{3N-8}}(3N-8)}{4(N-2)}\varepsilon_2^{-\frac{N}{3N-8}} + cm_2\varepsilon_3^{-\frac{m_3}{m_2}}\right] > 0,$$
$$Q_2 = na_2 > 0,$$
$$Q_3 = P_4 = na_2(1-m_1)\left[\left(\frac{N}{\rho_0}\right)^2 + \frac{(2n+q-1)^2 d^2}{2\rho_0^2\varepsilon_1}\right]|\Omega| > 0,$$
$$Q_4 = \frac{P_1 N\varepsilon_2}{4(N-2)} + P_2,$$
$$Q_5 = P_1 cm_3\varepsilon_3 - P_3.$$

With appropriate constants $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ for which Q_4 and $Q_5 = 0$, inequality (4.15) can be written as

$$\phi'(t) \le Q_1\left(\int_\Omega u^{2n}dx\right)^{\frac{3(N-2)}{3N-8}} + Q_2\int_\Omega u^{2n}dx + Q_3. \tag{4.16}$$

Integrating (4.16) from 0 to t, we obtain the inequlity

$$\int_{\phi(0)}^{\phi(t)} \frac{d\xi}{Q_1\xi^{\frac{3(N-2)}{3N-8}} + Q_2\xi + Q_3} \le t.$$

Letting $t \to t^{*-}$, we can obtain the desired result

$$\int_{\phi(0)}^{+\infty} \frac{d\xi}{Q_1\xi^{\frac{3(N-2)}{3N-8}} + Q_2\xi + Q_3} \le t^*.$$

□

4.2. The case that $N = 2$

In this subsection, the domain $\Omega \subset R^2$ is assumed to be a bounded star-shaped domain with smooth boundary.

Theorem 4.2. *Suppose that $u(x,t)$ is the nonnegative classical solution of problem (1.1)–(1.3), $u(x,t)$ blows up at t^*, and that the nonnegative functions f and g satisfy the following conditions:*

$$f(|\xi|) \ge a_3|\xi|^p, \quad g(\xi) \le a_4\xi^{1+\frac{\sigma}{2}}, \quad \forall\xi \ge 0, \tag{4.17}$$

where $a_3, a_4 > 0,\ \sigma \ge 1,\ p > 1$, and $p \le \sigma + 1$. Define a function

$$\psi(t) := \int_\Omega u^{2\sigma}dx.$$

Then the blow-up time t^* is bounded below, i.e.,

$$t^* \ge \int_{\psi(0)}^{+\infty}\frac{d\eta}{\Lambda(\eta)} = \int_{\psi(0)}^{+\infty}\frac{d\eta}{H_1\eta + H_2\eta^{\frac{3}{2}} + H_3\eta^2},$$

where

$$H_1 = K_1,\ H_2 = \begin{cases} \tilde{K}_2, & p = \sigma+1 \\ 0, & p < \sigma+1 \end{cases},\ H_3 = \begin{cases} K_3, & p = \sigma+1 \\ \tilde{K}_3, & p < \sigma+1 \end{cases},$$

and $K_1, \tilde{K}_2, K_3$, and $\tilde{K}_3$ are some positive constants defined in the proof.

Proof. Using (1.1), (1.2), (4.17), Green's formula and an adapted version of (2.8), it can be shown that

$$\begin{aligned}\psi'(t) &= 2\sigma\int_\Omega u^{2\sigma-1}(\Delta u - f(|\nabla u|))dx\\ &= 2\sigma\int_{\partial\Omega} u^{2\sigma-1}\frac{\partial u}{\partial \nu}ds - 2\sigma(2\sigma-1)\int_\Omega u^{2\sigma-2}|\nabla u|^2dx - 2\sigma\int_\Omega u^{2\sigma-1}f(|\nabla u|)dx\\ &\le 2a_4\sigma\int_{\partial\Omega} u^{\frac{5\sigma}{2}}ds - \frac{2(2\sigma-1)}{\sigma}\int_\Omega|\nabla u^\sigma|^2dx - 2a_3\sigma\int_\Omega u^{2\sigma-1}|\nabla u|^pdx\\ &\le 2a_4\sigma\left(\frac{2}{\rho_0}\int_\Omega u^{\frac{5\sigma}{2}}dx + \frac{5\sigma d}{2\rho_0}\int_\Omega u^{\frac{5\sigma}{2}-1}|\nabla u|dx\right) - \frac{2(2\sigma-1)}{\sigma}\int_\Omega|\nabla u^\sigma|^2dx\\ &\quad - \frac{2a_3\sigma p^p}{(2\sigma+p-1)^p}\int_\Omega\left|\nabla u^{\frac{2\sigma+p-1}{p}}\right|^pdx.\end{aligned} \tag{4.18}$$

Using Hölder's and Young's inequalities to the first term on the right side of (4.18), we have

$$\int_\Omega u^{\frac{5\sigma}{2}}dx \le \left(\int_\Omega u^{3\sigma}dx\int_\Omega u^{2\sigma}dx\right)^{\frac12} \le \frac12\int_\Omega u^{3\sigma}dx + \frac12\int_\Omega u^{2\sigma}dx \tag{4.19}$$

and

$$\begin{aligned}\int_\Omega u^{\frac{5\sigma}{2}-1}|\nabla u|dx &= \frac{1}{\sigma}\int_\Omega u^{\frac{3\sigma}{2}}|\nabla u^\sigma|dx \le \frac1\sigma\left(\int_\Omega u^{3\sigma}dx\int_\Omega|\nabla u^\sigma|^2dx\right)^{\frac12}\\ &\le \frac{1}{2\mu_1}\int_\Omega u^{3\sigma}dx + \frac{\mu_1}{2\sigma^2}\int_\Omega|\nabla u^\sigma|^2dx,\end{aligned} \tag{4.20}$$

where μ_1 is a positive constant to be determined later.

We estimate the last term on the right side of (4.18). From (2.17), one can see that

$$-\frac{2a_3\sigma p^p}{(2\sigma+p-1)^p}\int_\Omega\left|\nabla u^{\frac{2\sigma+p-1}{p}}\right|^pdx \le -\frac{2a_3\sigma p^pB}{(2\sigma+p-1)^p}\int_\Omega u^{2\sigma+p-1}dx, \tag{4.21}$$

where $B>0$ is the constant given in the proof of Theorem 2.1. By Hölder's inequality, one can have the inequality

$$\int_\Omega u^{2\sigma+p-1}dx \ge |\Omega|^{\frac{-p+1}{2\sigma}}\left(\int_\Omega u^{2\sigma}dx\right)^{\frac{2\sigma+p-1}{2\sigma}}. \tag{4.22}$$

Substituting (4.19)–(4.22) into (4.18), we obtain the inequality

$$\begin{aligned}\psi'(t) \le& \frac{2a_4\sigma}{\rho_0}\psi(t) + \frac{a_4\sigma}{\rho_0}\left(2+\frac{5\sigma d}{2\mu_1}\right)\int_\Omega u^{3\sigma}dx + \left(\frac{5a_4d\mu_1}{2\rho_0} - \frac{2(2\sigma-1)}{\sigma}\right)\\ &\times\int_\Omega|\nabla u^\sigma|^2dx - \frac{2a_3\sigma p^pB}{(2\sigma+p-1)^p}|\Omega|^{\frac{-p+1}{2\sigma}}\psi(t)^{\frac{2\sigma+p-1}{2\sigma}}.\end{aligned} \tag{4.23}$$

By applying Hölder's inequality, (2.4) and [12, (2.7)] to the second term on the right side of (4.23), it can be seen that

$$\begin{aligned}\int_\Omega u^{3\sigma}dx &\le \left(\int_\Omega u^{2\sigma}dx\right)^{\frac12}\left(\int_\Omega u^{4\sigma}dx\right)^{\frac12}\\ &\le \left(\int_\Omega u^{2\sigma}dx\right)^{\frac12}\left[\frac{1}{\rho_0}\int_\Omega u^{2\sigma}dx + \sigma\left(1+\frac{d}{\rho_0}\right)\int_\Omega u^{2\sigma-1}|\nabla u|dx\right].\end{aligned} \tag{4.24}$$

It follows from Hölder's inequality that

$$\begin{aligned}\sigma\int_\Omega u^{2\sigma-1}|\nabla u|\,dx &= \sigma\int_\Omega u^{\sigma-1}|\nabla u|\cdot u^\sigma dx\\ &\le \sigma\left(\int_\Omega u^{2(\sigma-1)}|\nabla u|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega u^{2\sigma}dx\right)^{\frac{1}{2}}\\ &= \left(\int_\Omega |\nabla u^\sigma|^2dx\int_\Omega u^{2\sigma}dx\right)^{\frac{1}{2}}.\end{aligned}\tag{4.25}$$

Substituting (4.25) into (4.24) and using the Cauchy inequality, we obtain

$$\begin{aligned}\int_\Omega u^{3\sigma}dx &\le \left(\int_\Omega u^{2\sigma}dx\right)^{\frac{1}{2}}\left[\frac{1}{\rho_0}\int_\Omega u^{2\sigma}dx+\left(1+\frac{d}{\rho_0}\right)\left(\int_\Omega|\nabla u^\sigma|^2dx\int_\Omega u^{2\sigma}dx\right)^{\frac{1}{2}}\right]\\ &= \frac{1}{\rho_0}\left(\int_\Omega u^{2\sigma}dx\right)^{\frac{3}{2}}+\left(1+\frac{d}{\rho_0}\right)\left(\int_\Omega|\nabla u^\sigma|^2dx\right)^{\frac{1}{2}}\int_\Omega u^{2\sigma}dx\\ &\le \frac{1}{\rho_0}\psi^{\frac{3}{2}}(t)+\left(1+\frac{d}{\rho_0}\right)\left[\mu_2\psi^2(t)+\frac{1}{4\mu_2}\int_\Omega|\nabla u^\sigma|^2dx\right],\end{aligned}\tag{4.26}$$

where μ_2 is a positive constant to be determined later. Substituting (4.26) into (4.23), one can obtain the inequality

$$\psi'(t)\le K_1\psi(t)+K_2\psi^{\frac{3}{2}}(t)+K_3\psi^2(t)+K_4\int_\Omega|\nabla u^\sigma|^2dx-K_5|\Omega|^{\frac{-p+1}{2\sigma}}\psi^{\frac{2\sigma+p-1}{2\sigma}}(t),$$

where

$$\begin{aligned}K_1 &= \frac{2a_4\sigma}{\rho_0}>0,\\ K_2 &= \frac{a_4\sigma}{\rho_0^2}\left(2+\frac{5\sigma d}{2\mu_1}\right)>0,\\ K_3 &= \frac{a_4\sigma\mu_2}{\rho_0}\left(2+\frac{5\sigma d}{2\mu_1}\right)\left(1+\frac{d}{\rho_0}\right)>0,\\ K_4 &= \frac{5a_4d\mu_1}{2\rho_0}-\frac{2(2\sigma-1)}{\sigma}+\frac{a_4\sigma}{4\rho_0\mu_2}\left(2+\frac{5\sigma d}{2\mu_1}\right)\left(1+\frac{d}{\rho_0}\right),\\ K_5 &= \frac{2a_3\sigma p^pB}{(2\sigma+p-1)^p}>0.\end{aligned}$$

With $\mu_2>0$ such that $K_4=0$, the above inequality becomes

$$\psi'(t)\le K_1\psi(t)+K_2\psi^{\frac{3}{2}}(t)+K_3\psi^2(t)-K_5|\Omega|^{\frac{-p+1}{2\sigma}}\psi^{\frac{2\sigma+p-1}{2\sigma}}(t):=\Lambda(\psi).\tag{4.27}$$

We now consider the following two cases that $p=\sigma+1$ and $p<\sigma+1$:
Case 1. If $p=\sigma+1$, then (4.27) can be written as

$$\psi'(t)\le K_1\psi(t)+\tilde{K}_2\psi^{\frac{3}{2}}(t)+K_3\psi^2(t),\tag{4.28}$$

where

$$\tilde{K}_2 = K_2 - K_5 |\Omega|^{-\frac{1}{2}} > 0$$

for $\mu_1 > 0$ small enough. Integrating (4.28) from 0 to t, we get the inequality

$$t \geq \int_{\psi(0)}^{\psi(t)} \frac{d\eta}{\Lambda(\eta)} = \int_{\psi(0)}^{\psi(t)} \frac{d\eta}{K_1\eta + \tilde{K}_2\eta^{\frac{3}{2}} + K_3\eta^2},$$

which implies that

$$t^* \geq \int_{\psi(0)}^{+\infty} \frac{d\eta}{\Lambda(\eta)} = \int_{\psi(0)}^{+\infty} \frac{d\eta}{K_1\eta + \tilde{K}_2\eta^{\frac{3}{2}} + K_3\eta^2},$$

since $\lim_{t\to t^*} \psi(t) = +\infty$.

Case 2. If $p < \sigma + 1$, we use Young's inequality to obtain

$$\begin{aligned}\psi^{\frac{3}{2}}(t) &= \left(\mu_3\psi^{\frac{2\sigma+p-1}{2\sigma}}(t)\right)^{\frac{\sigma}{2\sigma-p+1}} \left(\mu_3^{\frac{-\sigma}{\sigma-p+1}}\psi^2(t)\right)^{\frac{\sigma-p+1}{2\sigma-p+1}} \\ &\leq \frac{\sigma\mu_3}{2\sigma-p+1}\psi^{\frac{2\sigma+p-1}{2\sigma}}(t) + \frac{\sigma-p+1}{2\sigma-p+1}\mu_3^{\frac{-\sigma}{\sigma-p+1}}\psi^2(t),\end{aligned}$$

for all $\mu_3 > 0$. Choosing $\mu_3 > 0$ such that $\frac{K_2\sigma\mu_3}{2\sigma-p+1} - K_5 |\Omega|^{\frac{-p+1}{2\sigma}} = 0$, one can have the inequality

$$\psi'(t) \leq K_1\psi(t) + \tilde{K}_3\psi^2(t),$$

where

$$\tilde{K}_3 = K_3 + K_2\frac{\sigma-p+1}{2\sigma-p+1}\mu_3^{\frac{-\sigma}{\sigma-p+1}} > 0.$$

By a similar argument as in Case 1, we obtain

$$t^* \geq \int_{\psi(0)}^{+\infty} \frac{d\eta}{\Lambda(\eta)} = \int_{\psi(0)}^{+\infty} \frac{d\eta}{K_1\eta + \tilde{K}_3\eta^2}.$$

□

Remark 4.1. *If $p > \sigma + 1$, then $p + 1 > 2\left(1 + \frac{\sigma}{2}\right)$, and hence, the nonnegative classical solution $u(x, t)$ of problem (1.1)–(1.3) exists globally by Theorem 2.1.*

Remark 4.2. *The derivation of (4.24)–(4.27) in the proof of Theorem 4.2 can also adopt the embedded idea, and the lower bound for blow-up time can be obtained. Indeed, using Hölder's inequality, we have*

$$\int_\Omega u^{3\sigma}dx \leq \left(\int_\Omega u^{2\sigma}dx\right)^{\frac{2}{3}} \left(\int_\Omega u^{5\sigma}dx\right)^{\frac{1}{3}} \tag{4.29}$$

and from [29, Corollary 9.14], one can easily see that $W^{1,2}(\Omega) \subset L^5(\Omega)$, $N = 2$; that is,

$$\left(\int_\Omega u^{5\sigma}dx\right)^{\frac{1}{5}} \leq C\left(\int_\Omega u^{2\sigma}dx + \int_\Omega |\nabla u^\sigma|^2 dx\right)^{\frac{1}{2}}, \tag{4.30}$$

where C is a constant depending on Ω.

Substituting (4.30) into (4.29) and using the inequality $(a+b)^p \le a^p + b^p, (a, b \ge 0,\ 0 < p \le 1)$ *and Young's inequality, one can have*

$$\begin{aligned}
\int_\Omega u^{3\sigma} dx &\le C^{\frac{5}{3}} \left(\int_\Omega u^{2\sigma} dx\right)^{\frac{2}{3}} \left(\int_\Omega u^{2\sigma} dx + \int_\Omega |\nabla u^\sigma|^2 dx\right)^{\frac{5}{6}} \\
&\le C^{\frac{5}{3}} \left(\int_\Omega u^{2\sigma} dx\right)^{\frac{2}{3}} \left[\left(\int_\Omega u^{2\sigma} dx\right)^{\frac{5}{6}} + \left(\int_\Omega |\nabla u^\sigma|^2 dx\right)^{\frac{5}{6}}\right] \\
&= C^{\frac{5}{3}} \left[\left(\int_\Omega u^{2\sigma} dx\right)^{\frac{3}{2}} + \left(\int_\Omega u^{2\sigma} dx\right)^{\frac{2}{3}} \left(\int_\Omega |\nabla u^\sigma|^2 dx\right)^{\frac{5}{6}}\right] \\
&\le C^{\frac{5}{3}} \left(\int_\Omega u^{2\sigma} dx\right)^{\frac{3}{2}} + \frac{1}{6} C^{\frac{5}{3}} \mu_4 \left(\int_\Omega u^{2\sigma} dx\right)^4 + \frac{5}{6} C^{\frac{5}{3}} \mu_4^{-\frac{1}{5}} \int_\Omega |\nabla u^\sigma|^2 dx,
\end{aligned} \tag{4.31}$$

where μ_4 *is a positive constant to be determined later. Substituting (4.31) into (4.23), we obtain the inequality*

$$\psi'(t) \le L_1 \psi(t) + L_2 \psi^{\frac{3}{2}}(t) + L_3 \psi^4(t) + L_4 \int_\Omega |\nabla u^\sigma|^2 dx - L_5 |\Omega|^{\frac{-p+1}{2\sigma}} \psi^{\frac{2\sigma+p-1}{2\sigma}}(t),$$

where

$$\begin{aligned}
L_1 &= \frac{2a_4\sigma}{\rho_0} > 0, \\
L_2 &= \frac{a_4 \sigma C^{\frac{5}{3}}}{\rho_0} \left(2 + \frac{5\sigma d}{2\mu_1}\right) > 0, \\
L_3 &= \frac{a_4 \sigma \mu_4 C^{\frac{5}{3}}}{6\rho_0} \left(2 + \frac{5\sigma d}{2\mu_1}\right) > 0, \\
L_4 &= \frac{5a_4 d \mu_1}{2\rho_0} - \frac{2(2\sigma - 1)}{\sigma} + \frac{5a_4\sigma C^{\frac{5}{3}}}{6\rho_0 \mu_4^{\frac{1}{5}}} \left(2 + \frac{5\sigma d}{2\mu_1}\right), \\
L_5 &= \frac{2a_3 \sigma p^p B}{(2\sigma + p - 1)^p} > 0.
\end{aligned}$$

Choosing appropriate $\mu_4 > 0$ *such that* $L_4 = 0$*, the above inequality becomes*

$$\psi'(t) \le L_1 \psi(t) + L_2 \psi^{\frac{3}{2}}(t) + L_3 \psi^4(t) - L_5 |\Omega|^{\frac{-p+1}{2\sigma}} \psi^{\frac{2\sigma+p-1}{2\sigma}}(t). \tag{4.32}$$

Similarly, we now consider the following two cases:

Case 1. If $p = \sigma + 1$*, then (4.32) can be written as*

$$\psi'(t) \le L_1 \psi(t) + \tilde{L}_2 \psi^{\frac{3}{2}}(t) + L_3 \psi^4(t), \tag{4.33}$$

where

$$\tilde{L}_2 = L_2 - L_5 |\Omega|^{-\frac{1}{2}} > 0$$

for $\mu_1 > 0$ *small enough. Integrating (4.33) from 0 to* t^**, we get the inequality*

$$t^* \ge \int_{\psi(0)}^{+\infty} \frac{d\eta}{L_1 \eta + \tilde{L}_2 \eta^{\frac{3}{2}} + L_3 \eta^4}.$$

Case 2. If $p < \sigma + 1$, we use Young's inequality to get

$$\begin{aligned}\psi^{\frac{3}{2}}(t) &= \left(\mu_5 \psi^{\frac{2\sigma+p-1}{2\sigma}}(t)\right)^{\frac{5\sigma}{6\sigma-p+1}} \left(\mu_5^{\frac{-5\sigma}{\sigma-p+1}} \psi^4(t)\right)^{\frac{\sigma-p+1}{6\sigma-p+1}} \\ &\le \frac{5\sigma\mu_5}{6\sigma-p+1}\psi^{\frac{2\sigma+p-1}{2\sigma}}(t) + \frac{\sigma-p+1}{6\sigma-p+1}\mu_5^{\frac{-5\sigma}{\sigma-p+1}}\psi^4(t),\end{aligned}$$

for all $\mu_5 > 0$. Choosing $\mu_5 > 0$ such that $\frac{5L_2\sigma\mu_5}{6\sigma-p+1} - L_5|\Omega|^{\frac{-p+1}{2\sigma}} = 0$, one can have the inequality

$$\psi'(t) \le L_1\psi(t) + \tilde{L}_3\psi^4(t),$$

where

$$\tilde{L}_3 = L_3 + L_2\frac{\sigma-p+1}{6\sigma-p+1}\mu_5^{\frac{-5\sigma}{\sigma-p+1}} > 0.$$

By a similar argument as in Case 1, we obtain

$$t^* \ge \int_{\psi(0)}^{+\infty} \frac{d\eta}{L_1\eta + \tilde{L}_3\eta^4}.$$

Remark 4.3. *In fact, the results of Theorems 4.1 and 4.2 can be generalized to the following more general divergence form parabolic equations with nonlinear boundary flux:*

$$u_t = \sum_{i,j=1}^{N}\left(a_{ij}(x)u_{x_i}\right)_{x_j} - f(|\nabla u|),$$

where $\left(a_{ij}(x)\right)_{N\times N}$ is a positive definite matrix; that is, there exists a $\theta > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\eta_i\eta_j \ge \theta|\eta|^2.$$

for all $\eta \in R^N$.

5. Conclusions

In this paper, by using the modified differential inequality and comparison principle, we study the blow-up phenomena for a reaction-diffusion equation with gradient absorption terms under nonlinear boundary flux. Our results cover the relevant blow-up and life span results of gradient model in existing literature. Meanwhile, its analytical method can be used in other gradient models.

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

1. M. Ben-Artzi, P. Souplet, F. B. Weissler, The local theory for viscous Hamilton-Jacobi equations in Lebesgue spaces, *J. Math. Pure. Appl.,* **81** (2002), 343–378.
2. B. H. Gilding, M. Guedda, R. Kersner, The Cauchy problem for $u_t = \Delta u + |\nabla u|^p$, *J. Math. Anal. Appl.,* **284** (2003), 733–755.
3. H. A. Levine, L. E. Payne, Nonexistence theorems for the heat equation with nonlinear boundary conditions and for the porous medium equation backward in time, *J. Differ. Equations,* **16** (1974), 319–334.
4. J. Filo, Diffusivity versus absorption through the boundary, *J. Differ. Equations,* **99** (1992), 281–305.
5. P. Quittner, P. Souplet, Blow-up, global existence and steady states, In: *Superlinear parabolic problems*, Basel: Birkhauser, 2007.
6. B. Hu, *Blow up theories for semilinear parabolic equations*, Berlin: Springer, 2011.
7. H. A. Levine, The role of critical exponents in blow-up theorems, *SIAM Rev.,* **32** (1990), 262–288.
8. J. L. Gomez, V. Marquez, N. Wolanski, Blow up results and localization of blow up points for the heat equation with a nonlinear boundary condition, *J. Differ. Equations,* **92** (1991), 384–401.
9. A. Rodriguez-Bernal, A. Tajdine, Nonlinear balance for reaction-diffusion equations under nonlinear boundary conditions: Dissipativity and blow-up, *J. Differ. Equations,* **169** (2001), 332–372.
10. H. A. Levine, Nonexistence of global weak solutions to some properly and improperly posed problems of mathematical physics: The method of unbounded fourier coefficients, *Math. Ann.,* **214** (1975), 205–220.
11. L. E. Payne, P. W. Schaefer, Bounds for blow-up time for the heat equation under nonlinear boundary conditions, *Proc. Roy. Soc. Edinb. A,* **139** (2009), 1289–1296.
12. L. E. Payne, G. A. Philippin, S. Vernier Piro, Blow-up phenomena for a semilinear heat equation with nonlinear boundary condition, I, *Z. Angew. Math. Phys.,* **61** (2010), 999–1007.
13. F. S. Li, J. L. Li, Global existence and blow-up phenomena for nonlinear divergence form parabolic equations with inhomogeneous Neumann boundary conditions, *J. Math. Anal. Appl.,* **385** (2012), 1005–1014.
14. K. Baghaei, M. Hesaaraki, Lower bounds for the blow-up time in the higher-dimensional nonlinear divergence form parabolic equations, *C. R. Math.,* **351** (2013), 731–735.
15. Z. B. Fang, Y. X. Wang, Blow-up analysis for a semilinear parabolic equation with time-dependent coefficients under nonlinear boundary flux, *Z. Angew. Math. Phys.,* **66** (2015), 2525–2541.
16. L. W. Ma, Z. B. Fang, Blow-up analysis for a reaction-diffusion equation with weighted nonlocal inner absorptions under nonlinear boundary flux, *Nonlinear Anal.-Real,* **32** (2016), 338–354.

17. L. W. Ma, Z. B. Fang, Blow-up phenomena for a semilinear parabolic equation with weighted inner absorption under nonlinear boundary flux, *Math. Method. Appl. Sci.,* **40** (2017), 115–128.

18. J. Z. Zhang, F. S. Li, Global existence and blow-up phenomena for divergence form parabolic equation with time-dependent coefficient in multidimensional space, *Z. Angew. Math. Phys.,* **70** (2019), 1–16.

19. P. Quittner, On global existence and stationary solutions for two classes of semilinear parabolic equations, *Comment. Math. Univ. Ca.,* **34** (1993), 105–124.

20. P. Souplet, Gradient blow-up for multidimensional nonlinear parabolic equations with general boundary conditions, *Differ. Integral Equ.,* **15** (2002), 237–256.

21. M. Hesaaraki, A. Moameni, Blow-up of positive solutions for a family of nonlinear parabolic equations in general domain in R^N, *Mich. Math. J.,* **52** (2004), 375–389.

22. L. E. Payne, J. C. Song, Lower bounds for blow-up time in a nonlinear parabolic problem, *J. Math. Anal. Appl.,* **354** (2009), 394–396.

23. H. X. Li, W. J. Gao, Y. Z. Han, Lower bounds for the blowup time of solutions to a nonlinear parabolic problem, *Electron. J. Differ. Equ.,* **2014** (2014), 1–6.

24. Y. Liu, S. G. Luo, Y. H. Ye, Blow-up phenomena for a parabolic problem with a gradient nonlinearity under nonlinear boundary conditions, *Comput. Math. Appl.,* **65** (2013), 1194–1199.

25. M. Marras, S. V. Piro, G. Viglialoro, Lower bounds for blow-up time in a parabolic problem with a gradient term under various boundary conditions, *Kodai Math. J.,* **37** (2014), 532–543.

26. Q. Y. Zhang, Z. X. Jiang, S. N. Zheng, Blow-up time estimate for a degenerate diffusion equation with gradient absorption, *Appl. Math. Comput.,* **251** (2015), 331–335.

27. G. S. Tang, Blow-up phenomena for a parabolic system with gradient nonlinearity under nonlinear boundary conditions, *Comput. Math. Appl.,* **74** (2017), 360–368.

28. L. W. Ma, Z. B. Fang, Bounds for blow-up time of a reaction-diffusion equation with weighted gradient nonlinearity, *Comput. Math. Appl.,* **76** (2018), 508–519.

29. H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, New York: Springer, 2011.