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Research article

Oscillation of arbitrary-order derivatives of solutions to the higher order non-homogeneous linear differential equations taking small functions in the unit disc

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Abstract: In this article, we study the relationship between solutions and their arbitrary-order derivatives of the higher order non-homogeneous linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = F(z)$$

in the unit disc $\triangle$ with analytic or meromorphic coefficients of finite $[p, q]$-order. We obtain some oscillation theorems for $f^{(j)}(z) - \varphi(z)$, where f is a solution and $\varphi(z)$ is a small function.

Keywords: unit disc; non-homogeneous linear differential equation; arbitrary-order derivative; small function
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1. Introduction

In this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory on the complex plane $\mathbb{C}$ and in the unit disc $\triangle = \{z \in \mathbb{C} : |z| < 1\}$ (see [1–4]). In addition, we need to give some definitions and discussions. Firstly, let us give two definitions about the degree of small growth order of functions in $\triangle$ as polynomials on the complex plane $\mathbb{C}$. There are many types of definitions of small growth order of functions in $\triangle$ (see [5, 6]).

Definition 1.1. (see [5, 6]). *Let f be a meromorphic function in $\triangle$, and*

$$D(f) = \varlimsup_{r\to 1^-} \frac{T(r, f)}{\log \frac{1}{1-r}} = b.$$

If $b < \infty$, then we say that f is of finite b degree (or is non-admissible). If $b = \infty$, then we say that f is of infinite (or is admissible), both defined by characteristic function $T(r, f)$.

Definition 1.2. (see [5,6]). *Let f be an analytic function in $\triangle$, and*

$$D_M(f) = \overline{\lim_{r\to 1^-}} \frac{\log^+ M(r, f)}{\log \frac{1}{1-r}} = a \quad (or\ a = \infty).$$

Then we say that f is a function of finite a degree (or of infinite degree) defined by maximum modulus function $M(r, f) = \max_{|z|=r} | f(z) |$.

Moreover, for $F \subset [0, 1)$, the upper and lower densities of F are defined by

$$\overline{dens}_\triangle F = \overline{\lim_{r\to 1^-}} \frac{m(F \cap [0, r))}{m([0, r))}, \quad \underline{dens}_\triangle F = \underline{\lim_{r\to 1^-}} \frac{m(F \cap [0, r))}{m([0, r))}$$

respectively, where $m(G) = \int_G \frac{dt}{1-t}$ for $G \subset [0, 1)$.

Now we give the definition of iterated order and growth index to classify generally the functions of fast growth in $\triangle$ as those in $\mathbb{C}$, see [3, 7, 8]. Let us define inductively, for $r \in [0, 1)$, $\exp_1 r = e^r$ and $\exp_{p+1} r = \exp(\exp_p r)$, $p \in \mathbb{N}$. We also define for all r sufficiently large in $(0, 1)$, $\log_1 r = \log r$ and $\log_{p+1} r = \log(\log_p r)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r = r$, $\log_0 r = r$, $\exp_{-1} r = \log_1 r$, $\log_{-1} r = \exp_1 r$.

Definition 1.3. (see [9]). *The iterated p-order of a meromorphic function f in $\triangle$ is defined by*

$$\rho_p(f) = \overline{\lim_{r\to 1^-}} \frac{\log_p^+ T(r, f)}{\log \frac{1}{1-r}} \quad (p \geq 1).$$

For an analytic function f in $\triangle$, we also define

$$\rho_{M,p}(f) = \overline{\lim_{r\to 1^-}} \frac{\log_{p+1}^+ M(r, f)}{\log \frac{1}{1-r}} \quad (p \geq 1).$$

Remark 1.4. *It follows by M. Tsuji in ([4]) that if f is an analytic function in $\triangle$, then*

$$\rho_1(f) \leq \rho_{M,1}(f) \leq \rho_1(f) + 1.$$

However it follows by (Proposition 2.2.2 in [3]) that

$$\rho_{M,p}(f) = \rho_p(f) \quad (p \geq 2).$$

Definition 1.5. (see [9]). *The growth index of the iterated order of a meromorphic function f in $\triangle$ is defined by*

$$i(f) = \begin{cases} 0, & \text{if } f \text{ is non-admissible}; \\ \min\{p \in \mathbb{N}, \rho_p(f) < \infty\}, & \text{if } f \text{ is admissible}; \\ \infty, & \text{if } \rho_p(f) = \infty \text{ for all } p \in \mathbb{N}. \end{cases}$$

For an analytic function f in $\triangle$, we also define

$$i_M(f) = \begin{cases} 0, & \text{if } f \text{ is non-admissible}; \\ \min\{p \in \mathbb{N}, \rho_{M,p}(f) < \infty\}, & \text{if } f \text{ is admissible}; \\ \infty, & \text{if } \rho_{M,p}(f) = \infty \text{ for all } p \in \mathbb{N}. \end{cases}$$

Definition 1.6. (see [10, 11]). *Let f be a meromorphic function in $\triangle$. Then the iterated p-exponent of convergence of the sequence of zeros of f is defined by*

$$\lambda_p(f) = \overline{\lim_{r\to 1^-}} \frac{\log_p^+ N(r, \frac{1}{f})}{\log \frac{1}{1-r}},$$

where $N(r, \frac{1}{f})$ is the integrated counting function of zeros of $f(z)$ in $\{z \in \mathbb{C} :| z |< r\}$. Similarly, the iterated p-exponent of convergence of the sequence of distinct zeros of f is defined by

$$\overline{\lambda}_p(f) = \overline{\lim_{r\to 1^-}} \frac{\log_p^+ \overline{N}(r, \frac{1}{f})}{\log \frac{1}{1-r}},$$

where $\overline{N}(r, \frac{1}{f})$ is the integrated counting function of distinct zeros of f in $\{z \in \mathbb{C} :| z |< r\}$.

Definition 1.7. (see [12]). *Let $p \geq q \geq 1$ be integers. Let f be meromorphic function in $\triangle$, the $[p, q]$-order of f is defined by*

$$\rho_{[p,q]}(f) = \overline{\lim_{r\to 1^-}} \frac{\log_p^+ T(r, f)}{\log_q \frac{1}{1-r}}.$$

For an analytic function f in $\triangle$, we also define

$$\rho_{M,[p,q]}(f) = \overline{\lim_{r\to 1^-}} \frac{\log_{p+1}^+ M(r, f)}{\log_q \frac{1}{1-r}}.$$

Remark 1.8. *It is easy to see that $0 \leq \rho_{[p,q]}(f) \leq \infty$. If f is non-admissible, then $\rho_{[p,q]} = 0$ for any $p \geq q \geq 1$. By Definition 1.7, we have that $\rho_{[1,1]}(f) = \rho_1(f) = \rho(f)$, $\rho_{[2,1]}(f) = \rho_2(f)$ and $\rho_{[p+1,1]}(f) = \rho_{p+1}(f)$.*

Proposition 1.9. (see [12]). *Let $p \geq q \geq 1$ be integers. Let f be analytic function in $\triangle$ of $[p, q]$-order. The following two statements hold:*

(i) If $p = q$, then

$$\rho_{[p,q]}(f) \leq \rho_{M,[p,q]}(f) \leq \rho_{[p,q]}(f) + 1.$$

(ii) If $p > q$, then

$$\rho_{[p,q]}(f) = \rho_{M,[p,q]}(f).$$

Definition 1.10. (see [13]). *Let $p \geq q \geq 1$ be integers. The $[p, q]$-exponent of convergence of the zero sequence of a meromorphic function f in $\triangle$ is defined by*

$$\lambda_{[p,q]}(f) = \overline{\lim_{r\to 1^-}} \frac{\log_p^+ N(r, \frac{1}{f})}{\log_q \frac{1}{1-r}}.$$

Similarly, the $[p, q]$-exponent of convergence of the sequence of distinct zeros of f is defined by

$$\overline{\lambda}_{[p,q]}(f) = \overline{\lim_{r\to 1^-}} \frac{\log_p^+ \overline{N}(r, \frac{1}{f})}{\log_q \frac{1}{1-r}}.$$

Definition 1.11. (see [1]). *For $a \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the deficiency of f is defined by*

$$\delta(a, f) = 1 - \overline{\lim_{r \to 1^-}} \frac{N(r, \frac{1}{f-a})}{T(r, f)},$$

provided f has unbounded characteristic.

The complex oscillation theory of solutions of linear differential equations in the complex plane $\mathbb{C}$ was started by S. Bank and I. Laine in 1982. Many authors have investigated the growth and oscillation of the solutions of complex linear differential equations in $\mathbb{C}$. In 2000, J. Heittokangas first studied the growth of the solution of linear differential equations in the unit disc $\triangle$. There already exist many results (see [2, 9–13]) in $\triangle$, but the study is more difficult than that in $\mathbb{C}$, because the efficient tool, Wiman-Valiron theory, doesn't hold in $\triangle$. In 2015, author and L. P. Xiao (see [14]) studied the relationship between solutions and their derivatives of the differential equation

$$f'' + A(z)f' + B(z)f = F(z), \tag{1.1}$$

where $A(z), B(z) \not\equiv 0$ and $F(z) \not\equiv 0$ are meromorphic functions of finite iterated p-order in $\triangle$. Author obtained some oscillation theorems for $f^{(j)}(z) - \varphi(z)$, where f is a solution and $\varphi(z)$ is a small function. Before we state author's results we need to define the following:

$$A_j(z) = A_{j-1}(z) - \frac{B'_{j-1}(z)}{B_{j-1}(z)}, \quad (j = 1, 2, 3, \cdots), \tag{1.2}$$

$$B_j(z) = A'_{j-1}(z) - A_{j-1}(z)\frac{B'_{j-1}(z)}{B_{j-1}(z)} + B_{j-1}(z), \quad (j = 1, 2, 3, \cdots), \tag{1.3}$$

$$F_j(z) = F'_{j-1}(z) - F_{j-1}(z)\frac{B'_{j-1}(z)}{B_{j-1}(z)}, \quad (j = 1, 2, 3, \cdots), \tag{1.4}$$

$$D_j = F_j - (\varphi'' + A_j\varphi' + B_j\varphi), \quad (j = 1, 2, 3, \cdots), \tag{1.5}$$

where $A_0(z) = A(z), B_0(z) = B(z)$ and $F_0(z) = F(z)$. Author and L. P. Xiao obtained the following results.

Theorem 1.1. *(see [14]). Let $\varphi(z)$ be a meromorphic function in $\triangle$ with $\rho_p(\varphi) < \infty$. Let $A(z)$, $B(z) \not\equiv 0$ and $F(z) \not\equiv 0$ be meromorphic functions of finite iterated p-order in $\triangle$ such that $B_j(z) \not\equiv 0$ and $D_j(z) \not\equiv 0$ $(j = 0, 1, 2, \cdots)$.*

(i) If f is a meromorphic solution in $\triangle$ of (1.1) with $\rho_p(f) = \infty$ and $\rho_{p+1}(f) = \rho < \infty$, then f satisfies

$$\overline{\lambda}_p(f^{(j)} - \varphi) = \lambda_p(f^{(j)} - \varphi) = \rho_p(f) = \infty \quad (j = 0, 1, 2, \cdots),$$

$$\overline{\lambda}_{p+1}(f^{(j)} - \varphi) = \lambda_{p+1}(f^{(j)} - \varphi) = \rho_{p+1}(f) = \rho \quad (j = 0, 1, 2, \cdots).$$

(ii) If f is a meromorphic solution in $\triangle$ of (1.1) with

$$\max\{\rho_p(A), \rho_p(B), \rho_p(F), \rho_p(\varphi)\} < \rho_p(f) < \infty,$$

then

$$\overline{\lambda}_p(f^{(j)} - \varphi) = \lambda_p(f^{(j)} - \varphi) = \rho_p(f) \quad (j = 0, 1, 2, \cdots).$$

Theorem 1.2. *(see [14]). Let $\varphi(z)$ be an analytic function in $\triangle$ with $\rho_p(\varphi) < \infty$ and be not a solution of (1.1). Let $A(z)$, $B(z) \not\equiv 0$ and $F(z) \not\equiv 0$ be analytic functions in $\triangle$ with finite iterated p-order such that $\beta = \rho_p(B) > \max\{\rho_p(A), \rho_p(F), \rho_p(\varphi)\}$ and $\rho_{M,p}(A) \leq \rho_{M,p}(B)$. Then all nontrivial solutions of (1.1) satisfy*

$$\rho_p(B) \leq \overline{\lambda}_{p+1}(f^{(j)} - \varphi) = \lambda_{p+1}(f^{(j)} - \varphi) = \rho_{p+1}(f) \leq \rho_{M,p}(B) \quad (j = 0, 1, 2, \cdots)$$

with at most one possible exceptional solution f_0 such that

$$\rho_{p+1}(f_0) < \rho_p(B).$$

Theorem 1.3. *(see [14]). Let $\varphi(z)$ be a meromorphic function in $\triangle$ with $\rho_p(\varphi) < \infty$ and be not a solution of (1.1). Let $A(z)$, $B(z) \not\equiv 0$ and $F(z) \not\equiv 0$ be meromorphic functions in $\triangle$ with finite iterated p-order such that $\rho_p(B) > \max\{\rho_p(A), \rho_p(F), \rho_p(\varphi)\}$ and $\delta(\infty, B) > 0$. If f is a meromorphic solution in $\triangle$ of (1.1) with $\rho_p(f) = \infty$ and $\rho_{p+1}(f) = \rho$, then f satisfies*

$$\overline{\lambda}_p(f^{(j)} - \varphi) = \lambda_p(f^{(j)} - \varphi) = \rho_p(f) = \infty \quad (j = 0, 1, 2, \cdots),$$

$$\overline{\lambda}_{p+1}(f^{(j)} - \varphi) = \lambda_{p+1}(f^{(j)} - \varphi) = \rho_{p+1}(f) = \rho \quad (j = 0, 1, 2, \cdots).$$

In 2018, Z. Dahmani and M. A. Abdelaoui (see [15]) studied the higher order non-homogeneous linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = F(z), k \geq 2, \tag{1.6}$$

where $A_j(z)(j = 0, 1, \cdots, k-1)$, and $F(z) \not\equiv 0$ are meromorphic functions of finite iterated $[p, q]$-order in $\triangle$. Before we state their results we need to define the following:

$$A_j^0 = A_j, \quad (j = 0, 1, \cdots, k-1), \tag{1.7}$$

$$A_{k-1}^i = A_{k-1}^{i-1} - \frac{(A_0^{i-1})'}{A_0^{i-1}}, \quad (i = 1, 2, 3, \cdots), \tag{1.8}$$

$$A_j^i = A_j^{i-1} + A_{j+1}^{i-1}\frac{(\Psi_{j+1}^{i-1})'}{\Psi_{j+1}^{i-1}}, \quad (j = 0, 1, \cdots, k-2, i = 1, 2, 3, \cdots), \tag{1.9}$$

$$F_i = F_{i-1}' - \frac{(A_0^{i-1})'}{A_0^{i-1}}F_{i-1}, F_0 = F, \quad (i = 1, 2, 3, \cdots), \tag{1.10}$$

$$D_i = F_i - (\varphi^{(k)} + A_{k-1}^i\varphi^{(k-1)} + \cdots + A_0^i\varphi), \quad (i = 0, 1, 2, \cdots), \tag{1.11}$$

where $\Psi_{j+1}^{i-1} = \frac{A_{j+1}^{i-1}}{A_0^{i-1}}$. Z. Dahmani and M. A. Abdelaoui obtained the following results.

Theorem 1.4. *(see [15]) Let $p \geq q \geq 1$ be integers, and let $A_j(z)(j = 0, 1, \cdots, k-1), F(z) \not\equiv 0$ and $\varphi(z)$ be meromorphic functions in $\triangle$ of finite $[p, q]$-order such that $D_i(z) \not\equiv 0$ $(i = 0, 1, 2, \cdots)$. If f is a meromorphic solution of the Eq (1.6) of infinite $[p, q]$-order and $\rho_{[p+1,q]}(f) = \rho$, then f satisfies*

$$\overline{\lambda}_{[p,q]}(f^{(j)} - \varphi) = \lambda_{[p,q]}(f^{(j)} - \varphi) = \rho_{[p,q]}(f) = \infty \quad (j = 0, 1, 2, \cdots),$$

$$\overline{\lambda}_{[p+1,q]}(f^{(j)} - \varphi) = \lambda_{[p+1,q]}(f^{(j)} - \varphi) = \rho_{[p+1,q]}(f) = \rho \quad (j = 0, 1, 2, \cdots).$$

Theorem 1.5. *(see [15]). Let $p \geq q \geq 1$ be integers, and let $A_j(z)(j = 0, 1, \cdots, k-1), F(z) \not\equiv 0$ and $\varphi(z)$ be meromorphic functions in $\triangle$ of finite $[p, q]$-order such that $D_i(z) \not\equiv 0$ $(i = 0, 1, 2, \cdots)$. If f is a meromorphic solution of the Eq (1.6) with*

$$\max\{\rho_{[p,q]}(A_j)(j = 0, 1, 2, \cdots, k-1), \rho_{[p,q]}(F), \rho_{[p,q]}(\varphi)\} < \rho_{[p,q]}(f) = \rho,$$

then f satisfies

$$\overline{\lambda}_{[p,q]}(f^{(j)} - \varphi) = \lambda_{[p,q]}(f^{(j)} - \varphi) = \rho_{[p,q]}(f) = \rho \quad (j = 0, 1, 2, \cdots).$$

2. Main results

According to the proof process of Theorem 1.4 and Theorem 1.5, we know that it is necessary to increase the condition $A_0^i(z) \not\equiv 0$ and $D_i(z) \not\equiv 0$ $(i = 0, 1, 2, \cdots)$ to ensure that the Theorem 1.4 and the Theorem 1.5 are established, because we need to divide both sides of the higher order non-homogeneous linear differential equations by $A_0^i(z)$. Where $A_0^i(z)$ and $D_i(z)$ are defined in (1.7), (1.9) and (1.11). In this article, we give some sufficient conditions on the coefficients which guarantee $A_0^i(z) \not\equiv 0$ and $D_i(z) \not\equiv 0$ $(i = 0, 1, 2, \cdots)$, and we obtain:

Theorem 2.1. *Let $p \geq q \geq 1$ be integers, and let $\varphi(z)$ be an analytic function in $\triangle$ with $\rho_{[p,q]}(\varphi) < \infty$ and be not a solution of (1.6). Let $A_j(z)(j = 1, 2, \cdots, k-1)$, $A_0(z) \not\equiv 0$ and $F(z) \not\equiv 0$ be analytic functions in $\triangle$ of finite $[p, q]$-order such that $\beta = \rho_{[p,q]}(A_0) > \max\{\rho_{[p,q]}(A_j)(j = 1, 2, \cdots, k-1), \rho_{[p,q]}(F), \rho_{[p,q]}(\varphi)\}$ and $\rho_{M,[p,q]}(A_j) \leq \rho_{M,[p,q]}(A_0)$ $(j = 1, 2, \cdots, k-1)$. Then all nontrivial solutions of (1.6) satisfy*

$$\rho_{[p,q]}(A_0) \leq \overline{\lambda}_{[p+1,q]}(f^{(j)} - \varphi) = \lambda_{[p+1,q]}(f^{(j)} - \varphi) = \rho_{[p+1,q]}(f) \leq \rho_{M,[p,q]}(A_0) \quad (j = 0, 1, 2, \cdots),$$

with at most one possible exceptional solution f_0 such that

$$\rho_{[p+1,q]}(f_0) < \rho_{[p,q]}(A_0).$$

Theorem 2.2. *Let $p \geq q \geq 1$ be integers, and let $\varphi(z)$ be an meromorphic function in $\triangle$ with $\rho_{[p,q]}(\varphi) < \infty$ and be not a solution of (1.6). Let $A_j(z)(j = 1, 2, \cdots, k-1)$, $A_0(z) \not\equiv 0$ and $F(z) \not\equiv 0$ be meromorphic functions in $\triangle$ of finite $[p, q]$-order such that $\rho_{[p,q]}(A_0) > \max\{\rho_{[p,q]}(A_j)(j = 1, 2, \cdots, k-1), \rho_{[p,q]}(F), \rho_{[p,q]}(\varphi)\}$ and $\delta(\infty, A_0) > 0$. If f is a meromorphic solution in $\triangle$ of (1.6) with $\rho_{[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) = \rho$, then f satisfies*

$$\overline{\lambda}_{[p,q]}(f^{(j)} - \varphi) = \lambda_{[p,q]}(f^{(j)} - \varphi) = \rho_{[p,q]}(f) = \infty \quad (j = 0, 1, 2, \cdots),$$

$$\overline{\lambda}_{[p+1,q]}(f^{(j)} - \varphi) = \lambda_{[p+1,q]}(f^{(j)} - \varphi) = \rho_{[p+1,q]}(f) = \rho \quad (j = 0, 1, 2, \cdots).$$

3. Some lemmas

To prove our theorems, we require the following lemmas.

Lemma 3.1. *(see [13]). Let $p \geq q \geq 1$ be integers, and let $A_0, A_1, \cdots, A_{k-1}$ be analytic functions in $\triangle$ satisfying*

$$\max\{\rho_{[p,q]}(A_j) : j = 1, 2, \cdots, k-1\} < \rho_{[p,q]}(A_0).$$

If $f \not\equiv 0$ is a solution of (3.1), then $\rho_{[p,q]}(f) = \infty$ and

$$\rho_{[p,q]}(A_0) \le \rho_{[p+1,q]}(f) \le \max\{\rho_{M,[p,q]}(A_j) : j = 0, 1, \cdots, k-1\}.$$

Furthermore, if $p > q$, then

$$\rho_{[p+1,q]}(f) = \rho_{[p,q]}(A_0).$$

Lemma 3.2. *(see [15]). Let $p \ge q \ge 1$ be integers. Let $A_0, A_1, \cdots, A_{k-1}$ and $F \not\equiv 0$ be meromorphic functions in $\triangle$ and let f be a meromorphic solution of (1.6) satisfying* $\max\{\rho_{[p,q]}(A_j)(j = 0, 1, 2, \cdots, k-1), \rho_{[p,q]}(F)\} < \rho_{[p,q]}(f) \le \infty$, *then we have*

$$\overline{\lambda}_{[p,q]}(f) = \lambda_{[p,q]}(f) = \rho_{[p,q]}(f),$$

$$\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \rho_{[p+1,q]}(f).$$

Lemma 3.3. *Let $p \ge q \ge 1$ be integers, and assume that coefficients $A_0, A_1, \cdots, A_{k-1}$ and $F \not\equiv 0$ are analytic in $\triangle$ and $\rho_{[p,q]}(A_j) < \rho_{[p,q]}(A_0)$ for all $j = 1, 2, \cdots, k-1$. Let $\alpha_M = \max\{\rho_{M,[p,q]}(A_j) : j = 0, 1, \cdots, k-1\}$. If $\rho_{M,[p+1,q]}(F) < \rho_{[p,q]}(A_0)$, then all solutions f of (1.6) satisfy*

$$\rho_{[p,q]}(A_0) \le \overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) \le \alpha_M,$$

with at most one exceptional f_0 satisfying $\rho_{M,[p+1,q]}(f_0) < \rho_{[p,q]}(A_0)$.

Proof. Let $f_1, f_2, \cdots, f_k$ be a solution base of the differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0. \tag{3.1}$$

Then by the elementary theory of differential equations (see [3]), any solution of (1.6) can be represented in the form

$$f = (B_1 + C_1)f_1 + (B_2 + C_2)f_2 + \cdots + (B_k + C_k)f_k, \tag{3.2}$$

where $C_1, C_2, \cdots, C_k \in \mathbb{C}$ and $B_1, B_2, \cdots, B_k$ are analytic in $\triangle$ given by the system of equations

$$\begin{cases} B_1'f_1 + B_2'f_2 + \cdots + B_k'f_k = 0, \\ B_1'f_1' + B_2'f_2' + \cdots + B_k'f_k' = 0, \\ \cdots \\ B_1'f_1^{(k-2)} + B_2'f_2^{(k-2)} + \cdots + B_k'f_k^{(k-2)} = 0, \\ B_1'f_1^{(k-1)} + B_2'f_2^{(k-1)} + \cdots + B_k'f_k^{(k-1)} = F. \end{cases} \tag{3.3}$$

Since the Wronskian of $f_1, f_2, \cdots, f_k$ satisfies $W(f_1, f_2, \cdots, f_k) = \exp(-\int A_{k-1}dz)$, we obtain

$$B_j' = F \cdot G_j(f_1, f_2, \cdots, f_k) \cdot \exp\left(\int A_{k-1}dz\right) \quad (j = 1, 2, \cdots, k), \tag{3.4}$$

where $G_j(f_1, f_2, \cdots, f_k)$ is a differential polynomial of $f_1, f_2, \cdots, f_k$ and of their derivatives, with constant coefficients. Then by Lemma 3.1, we know that $\alpha_M \ge \rho_{M,[p+1,q]}(f_j) \ge \rho_{[p,q]}(A_0)$. By (3.2)–(3.4), we have

$$\rho_{M,[p+1,q]}(f) \le \max\{\rho_{M,[p+1,q]}(F), \alpha_M\}. \tag{3.5}$$

Since $\rho_{M,[p+1,q]}(F) < \rho_{[p,q]}(A_0) \leq \alpha_M$, it follows from (3.5) and (1.6) that all solutions f of (1.6) satisfy $\rho_{M,[p+1,q]}(f) \leq \alpha_M$.

Now we assert that all solutions f of (1.6) satisfy $\rho_{M,[p+1,q]}(f) \geq \rho_{[p,q]}(A_0)$ with at most one exception. In fact, if there exist two distinct solutions g_1, g_2 of (1.6) with $\rho_{M,[p+1,q]}(g_i) < \rho_{[p,q]}(A_0)$ $(i = 1, 2)$, then $g = g_1 - g_2$ satisfies $\rho_{M,[p+1,q]}(g) = \rho_{M,[p+1,q]}(g_1 - g_2) < \rho_{[p,q]}(A_0)$. But g is a nonzero solution of (3.1) satisfying $\rho_{M,[p+1,q]}(g) = \rho_{M,[p+1,q]}(g_1 - g_2) \geq \rho_{[p,q]}(A_0)$ by Lemma 3.1. This is a contradiction.

By Lemma 3.2, all solutions f of (1.6) satisfy $\alpha_M \geq \rho_{M,[p+1,q]}(f) = \overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) \geq \rho_{[p,q]}(A_0)$, with at most one exceptional f_0 satisfying $\rho_{M,[p+1,q]}(f_0) < \rho_{[p,q]}(A_0)$.

Lemma 3.4. *Let $p \geq q \geq 1$ be integers, φ be finite $[p,q]$-order analytic functions in $\triangle$ and assume that coefficients $A_0, A_1, \cdots, A_{k-1}$, $F \not\equiv 0$ and $F - \varphi^{(k)} - A_{k-1}\varphi^{(k-1)} - \cdots - A_1\varphi' - A_0\varphi \not\equiv 0$ are analytic in $\triangle$ and $\rho_{[p,q]}(A_j) < \rho_{[p,q]}(A_0)$ for all $j = 1, 2, \cdots, k-1$. Let $\alpha_M = \max\{\rho_{M,[p,q]}(A_j) : j = 0, 1, \cdots, k-1\}$. If $\rho_{M,[p+1,q]}(F - \varphi^{(k)} - A_{k-1}\varphi^{(k-1)} - \cdots - A_1\varphi' - A_0\varphi) < \rho_{[p,q]}(A_0)$, then all solutions f of (1.6) satisfy*

$$\rho_{[p,q]}(A_0) \leq \overline{\lambda}_{[p+1,q]}(f-\varphi) = \lambda_{[p+1,q]}(f-\varphi) = \rho_{M,[p+1,q]}(f) \leq \alpha_M,$$

with at most one exceptional f_0 satisfying $\rho_{M,[p+1,q]}(f_0) < \rho_{[p,q]}(A_0)$.

Proof. Suppose that $g = f - \varphi$, obtain $f = g + \varphi$, then from (1.6) we have $g^{(k)} + A_{k-1}g^{(k-1)} + \cdots + A_1 g' + A_0 g = F - \varphi^{(k)} - A_{k-1}\varphi^{(k-1)} - \cdots - A_1\varphi' - A_0\varphi$. By Lemma 3.3 we obtain all solutions f of (1.6) satisfy

$$\rho_{[p,q]}(A_0) \leq \overline{\lambda}_{[p+1,q]}(f-\varphi) = \lambda_{[p+1,q]}(f-\varphi) = \rho_{M,[p+1,q]}(f) \leq \alpha_M,$$

with at most one exceptional f_0 satisfying $\rho_{M,[p+1,q]}(f_0) < \rho_{[p,q]}(A_0)$.

Lemma 3.5. *(see [12]). Let $p \geq q \geq 1$ be integers. Let f be a meromorphic function in $\triangle$ such that $\rho_{[p,q]}(f) = \rho < \infty$, and let $k \geq 1$ be an integer. Then for any $\varepsilon > 0$,*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\exp_{p-1}\left\{(\rho+\varepsilon)\log_q\left(\frac{1}{1-r}\right)\right\}\right)$$

holds for all r outside a set $E_1 \subset [0,1)$ with $\int_{E_1} \frac{dr}{1-r} < \infty$.

4. Proofs of Theorems 2.1 and 2.2

4.1. The proof of Theorem 2.1

Since $F - (\varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + \cdots + A_1\varphi' + A_0\varphi) \not\equiv 0$, $\rho_{M,[p+1,q]}(F - (\varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + \cdots + A_1\varphi' + A_0\varphi)) < \rho_{[p,q]}(A_0)$. By Lemma 3.4, all nontrivial solutions of (1.6) satisfy

$$\rho_{[p,q]}(A_0) \leq \overline{\lambda}_{[p+1,q]}(f-\varphi) = \lambda_{[p+1,q]}(f-\varphi) = \rho_{[p+1,q]}(f) \leq \rho_{M,[p,q]}(A_0),$$

with at most one exceptional f_0 such that $\rho_{[p+1,q]}(f_0) < \rho_{[p,q]}(A_0)$. By using (1.9) we have

$$\begin{aligned} A_0^i &= A_1^{i-1}\left(\frac{(A_1^{i-1})'}{A_1^{i-1}} - \frac{(A_0^{i-1})'}{A_0^{i-1}}\right) + A_0^{i-1} \\ &= A_1^{i-1}\left(\frac{(A_1^{i-1})'}{A_1^{i-1}} - \frac{(A_0^{i-1})'}{A_0^{i-1}}\right) + A_1^{i-2}\left(\frac{(A_1^{i-2})'}{A_1^{i-2}} - \frac{(A_0^{i-2})'}{A_0^{i-2}}\right) + A_0^{i-2} \\ &= \sum_{k=0}^{i-1} A_1^k\left(\frac{(A_1^k)'}{A_1^k} - \frac{(A_0^k)'}{A_0^k}\right) + A_0. \end{aligned} \tag{4.1}$$

Now we prove that $A_0^i \not\equiv 0$ for all $i = 1, 2, 3, \cdots$. For that we suppose there exists $i \in \mathbb{N}$ such that $A_0^i = 0$. By (4.1) and Lemma 3.5 we have for any $\varepsilon > 0$,

$$\begin{aligned} T(r, A_0) = m(r, A_0) &\le \sum_{k=0}^{i-1} m(r, A_1^k) + O\left(\exp_{p-1}\left\{(\beta+\varepsilon)\log_q\left(\frac{1}{1-r}\right)\right\}\right) \\ &= \sum_{k=0}^{i-1} T(r, A_1^k) + O\left(\exp_{p-1}\left\{(\beta+\varepsilon)\log_q\left(\frac{1}{1-r}\right)\right\}\right), \end{aligned} \tag{4.2}$$

outside a set $E_1 \subset [0, 1)$ with $\int_{E_1} \frac{dr}{1-r} < \infty$, for all $i = 1, 2, 3, \cdots$, $\beta = \rho_{[p,q]}(A_0)$. Which implies the contradiction

$$\rho_{[p,q]}(A_0) \le \max\{\rho_{[p,q]}(A_j)(j = 1, 2, \cdots, k-1)\}.$$

Hence $A_0^i \not\equiv 0$ for all $i = 1, 2, 3, \cdots$. We prove that $D_i \not\equiv 0$ for all $i = 1, 2, 3, \cdots$. For that we suppose there exists $i \in \mathbb{N}$ such that $D_i = 0$. We have $F_i - (\varphi^{(k)} + A_{k-1}^i \varphi^{(k-1)} + \cdots + A_0^i \varphi) = 0$ from (1.11), which implies

$$\begin{aligned} F_i &= \varphi\left(\frac{\varphi^{(k)}}{\varphi} + A_{k-1}^i \frac{\varphi^{(k-1)}}{\varphi} + \cdots + A_1^i \frac{\varphi'}{\varphi} + A_0^i\right) \\ &= \varphi\left[\frac{\varphi^{(k)}}{\varphi} + A_{k-1}^i \frac{\varphi^{(k-1)}}{\varphi} + \cdots + A_1^i \frac{\varphi'}{\varphi} + \sum_{k=0}^{i-1} A_1^k\left(\frac{(A_1^k)'}{A_1^k} - \frac{(A_0^k)'}{A_0^k}\right) + A_0\right]. \end{aligned}$$

Here we suppose that $\varphi(z) \not\equiv 0$,

$$A_0 = \frac{F_i}{\varphi} - \left[\frac{\varphi^{(k)}}{\varphi} + A_{k-1}^i \frac{\varphi^{(k-1)}}{\varphi} + \cdots + A_1^i \frac{\varphi'}{\varphi} + \sum_{k=0}^{i-1} A_1^k\left(\frac{(A_1^k)'}{A_1^k} - \frac{(A_0^k)'}{A_0^k}\right)\right]. \tag{4.3}$$

On the other hand, from (1.10),

$$m(r, F_i) \le m(r, F) + O\left(\exp_{p-1}\left\{(\beta+\varepsilon)\log_q\left(\frac{1}{1-r}\right)\right\}\right). \tag{4.4}$$

By (4.3), (4.4) and Lemma 3.5 we have

$$\begin{aligned} T(r, A_0) =& m(r, A_0) \le m(r, F) + m(r, \frac{1}{\varphi}) + \sum_{k=0}^{i-1} m(r, A_1^k) \\ &+ \sum_{j=1}^{k-1} m(r, A_j^i) + O\left(\exp_{p-1}\left\{(\beta+\varepsilon)\log_q\left(\frac{1}{1-r}\right)\right\}\right), \end{aligned} \tag{4.5}$$

which implies the contradiction

$$\rho_{[p,q]}(A_0) \leq \max\{\rho_{[p,q]}(A_j)(j=1,2,\cdots,k-1), \rho_{[p,q]}(F), \rho_{[p,q]}(\varphi)\}.$$

If $\varphi(z) \equiv 0$, then from (1.10) and (1.11)

$$F'_{i-1} - \frac{(A_0^{i-1})'}{A_0^{i-1}} F_{i-1} = 0, \tag{4.6}$$

which implies $F_{i-1}(z) = cA_0^{i-1}(z)$, where c is some constant. By (4.1) and (4.6), we have

$$\frac{1}{c} F_{i-1} = \sum_{k=0}^{i-2} A_1^k \left(\frac{(A_1^k)'}{A_1^k} - \frac{(A_0^k)'}{A_0^k} \right) + A_0. \tag{4.7}$$

By (4.4), (4.7) and Lemma 3.5 we have

$$T(r, A_0) = m(r, A_0) \leq m(r, F) + \sum_{k=0}^{i-2} m(r, A_1^k) + O\left(\exp_{p-1} \left\{ (\beta + \varepsilon) \log_q \left(\frac{1}{1-r} \right) \right\} \right),$$

which implies the contradiction

$$\rho_{[p,q]}(A_0) \leq \max\{\rho_{[p,q]}(A_j)(j=1,2,\cdots,k-1), \rho_{[p,q]}(F)\}.$$

Hence $D_i \not\equiv 0$ for all $i = 1, 2, 3, \cdots$. Since $A_0^i \not\equiv 0$, $D_i \not\equiv 0$ $(i = 1, 2, 3, \cdots)$, then by Theorem 1.4 and Lemma 3.4 we have

$$\rho_{[p,q]}(A_0) \leq \overline{\lambda}_{[p+1,q]}(f^{(j)} - \varphi) = \lambda_{[p+1,q]}(f^{(j)} - \varphi) = \rho_{[p+1,q]}(f) \leq \rho_{M,[p,q]}(A_0) \quad (j = 0, 1, 2, \cdots)$$

with at most one possible exceptional solution f_0 such that

$$\rho_{[p+1,q]}(f_0) < \rho_{[p,q]}(A_0).$$

Therefore, the proof of Theorem 2.1 is completely.

4.2. The proof of Theorem 2.2

We need only to prove that $A_0^i \not\equiv 0$ and $D_i \not\equiv 0$ for all $j = 1, 2, 3, \cdots$. Then by Theorem 1.4 we can obtain Theorem 2.2. Consider the assumption $\delta(\infty, A_0) = \delta > 0$. Then for $r \to 1^-$ we have

$$T(r, A_0) \leq \frac{2}{\delta} m(r, A_0). \tag{4.8}$$

Now we prove that $A_0^i \not\equiv 0$ for all $i = 1, 2, 3, \cdots$. For that we suppose there exists $i \in \mathbb{N}$ such that $A_0^i = 0$. By (4.1) and (4.8) we obtain

$$\begin{aligned} T(r, A_0) &\leq \frac{2}{\delta} m(r, A_0) \leq \frac{2}{\delta} \sum_{k=0}^{i-1} m(r, A_1^k) + \frac{2}{\delta} O\left(\exp_{p-1} \left\{ (\beta + \varepsilon) \log_q \left(\frac{1}{1-r} \right) \right\} \right) \\ &\leq \frac{2}{\delta} \sum_{k=0}^{i-1} T(r, A_1^k) + \frac{2}{\delta} O\left(\exp_{p-1} \left\{ (\beta + \varepsilon) \log_q \left(\frac{1}{1-r} \right) \right\} \right), \end{aligned} \tag{4.9}$$

which implies the contradiction

$$\rho_{[p,q]}(A_0) \le \max\{\rho_{[p,q]}(A_j)(j=1,2,\cdots,k-1)\}.$$

Hence $A_0^i \not\equiv 0$ for all $i = 1,2,3,\cdots$. We prove that $D_i \not\equiv 0$ for all $i = 1,2,3,\cdots$. For that we suppose there exists $i \in \mathbb{N}$ such that $D_i = 0$. If $\varphi(z) \not\equiv 0$, then by (4.3), (4.4), (4.8) and Lemma 3.5 we have

$$\begin{aligned} T(r,A_0) \le \frac{2}{\delta}m(r,A_0) \le & \frac{2}{\delta}\left[m(r,F)+m(r,\frac{1}{\varphi})+\sum_{k=0}^{i-1}m(r,A_1^k)+\sum_{j=1}^{k-1}m(r,A_j^i)\right] \\ & +\frac{2}{\delta}\left[O\left(\exp_{p-1}\left\{(\beta+\varepsilon)\log_q\left(\frac{1}{1-r}\right)\right\}\right)\right], \end{aligned} \tag{4.10}$$

which implies the contradiction

$$\rho_{[p,q]}(A_0) \le \max\{\rho_{[p,q]}(A_j)(j=1,2,\cdots,k-1), \rho_{[p,q]}(F), \rho_{[p,q]}(\varphi)\}.$$

If $\varphi(z) \equiv 0$, then by (4.4), (4.7) and Lemma 3.5 we have

$$\begin{aligned} T(r,A_0) &\le \frac{2}{\delta}m(r,A_0) \\ &\le \frac{2}{\delta}m(r,F)+\frac{2}{\delta}\sum_{k=0}^{i-2}m(r,A_1^k)+O\left(\exp_{p-1}\left\{(\beta+\varepsilon)\log_q\left(\frac{1}{1-r}\right)\right\}\right) \\ &\le \frac{2}{\delta}T(r,F)+\frac{2}{\delta}\sum_{k=0}^{i-2}T(r,A_1^k)+O\left(\exp_{p-1}\left\{(\beta+\varepsilon)\log_q\left(\frac{1}{1-r}\right)\right\}\right), \end{aligned} \tag{4.11}$$

which implies the contradiction

$$\rho_{[p,q]}(A_0) \le \max\{\rho_{[p,q]}(A_j)(j=1,2,\cdots,k-1), \rho_{[p,q]}(F)\}.$$

Hence $D_i \not\equiv 0$ for all $i = 1,2,3,\cdots$. By Theorem 1.4, we have Theorem 2.2.

Therefore, this completes the proof of Theorem 2.2.

5. Conclusions

We first obtained some oscillation theorems (see [14]) which consider the distribution of meromorphic solutions and their arbitrary-order derivatives taking small function values instead of taking zeros. Moreover, Z. Dahmani and M. A. Abdelaoui (see [15]) investigated the higher order non-homogeneous linear differential equation which can be seen as an improvement of [14]. By using those theorems, we obtain some oscillation theorems for $f^{(j)}(z) - \varphi(z)$, where f is a solution and $\varphi(z)$ is a small function. We believe our results will attract the attentions of the related readers.

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Conflict of interest

The authors declare that none of the authors have any competing interests in the manuscript.

References

1. W. K. Hayman, *Meromorphic functions*, Oxford: The Clarendon Press, 1964.
2. J. Heittokangas, On complex differential equations in the unit disc, *Ann. Acad. Sci. Fenn. Math. Diss.,* **122** (2000), 1–54.
3. I. Laine, *Nevanlinna theory and complex differential equations*, Berlin: Walter de Gruyter, 1993.
4. M. Tsuji, *Potential theory in modern function theory*, New York: Chelsea, 1975.
5. Z. X. Chen, K. H. Shon, The growth of solutions of differential equations with coefficients of small growth in the disc, *J. Math. Anal. Appl.,* **297** (2004), 285–304.
6. I. E. Chyzhykov, G. G. Gundersen, J. Heittokangas, Linear differential equations and logarithmic derivative estimates, *Proc. London Math. Soc.,* **86** (2003), 735–754.
7. L. G. Bernal, On growth k-order of solutions of a complex homogeneous linear differential equation, *Proc. Am. Math. Soc.,* **101** (1987), 317–322.
8. L. Kinnunen, Linear differential equations with solutions of finite iterated order, *Southeast Asian Bull. Math.,* **22** (1998), 385–406.
9. T. B. Cao, H. Y. Yi, The growth of solutions of linear differential equations with coefficients of iterated order in the unit disc, *J. Math. Anal. Appl.,* **319** (2006), 278–294.
10. T. B. Cao, The growth, oscillation and fixed points of solutions of complex linear differential equations in the unit disc, *J. Math. Anal. Appl.,* **352** (2009), 739–748.
11. T. B. Cao, Z. S. Deng, Solutions of non-homogeneous linear differential equations in the unit disc, *Ann. Pol. Math.,* **97** (2010), 51–61.
12. B. Belaïdi, Growth of solutions to linear differential equations with analytic coefficients of $[p, q]$-order in the unit disc, *Electron. J. Differ. Equ.,* **2011** (2011), 1–11.
13. Z. Latreuch, B. Belaïdi, Linear differential equations with analytic coefficients of $[p, q]$-order in the unit disc, *Sarajevo J. Math.,* **9** (2013), 71–84.
14. P. Gong, L. P. Xiao, Oscillation of arbitrary-order derivatives of solutions to linear differential equations taking small funcions in the unit disc, *Electron. J. Differ. Equ.,* **2015** (2015), 1–12.
15. Z. Dahmani, M. A. Abdelaoui, A unit disc study for $[p, q]$-order meromorphic solutions of complex differential equations, *J. Interdiscip. Math.,* **21** (2018), 595–609.

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