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Research article

Pringsheim and statistical convergence for double sequences on $L-$fuzzy normed space

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Abstract: In this paper, we study the concept of statistical convergence for double sequences on $L-$fuzzy normed spaces. Then we give a useful characterization on the statistical convergence of double sequences with respect to their convergence in the classical sense and we illustrate that our method of convergence is weaker than the usual convergence for double sequences on $L-$fuzzy normed spaces.

Keywords: $L-$fuzzy normed space; double sequence; statistical convergence
Mathematics Subject Classification: 03E72, 40A05

1. Introduction

Space-time points are calculated in a fuzzy manner in the strong quantum gravity regime, and the sequence of these points defines a sequence of fuzzy numbers. Due to this fuzziness, the position space representation of quantum mechanics fails in such cases, necessitating the use of a more generalized structure.

We come across double sequences, i.e., matrices, in many branches of science and engineering, and there are definitely situations where either the concept of ordinary convergence does not operate or the underlying space does not serve our intent.

Other than ordinary convergence, there are many approaches for dealing with the convergence problems of sequences of real numbers and fuzzy numbers, including almost everywhere convergence and statistical convergence. The terms “almost convergence” and “statistical convergence” are well-known in the literature when it comes to probability measures [4–14].

The aim of the present paper is to investigate the statistical convergence for double sequences on $L-$fuzzy normed spaces. Then we give a useful characterization for statistically convergent double sequences on $L-$ fuzzy normed spaces. Also we display an example on that our method on convergence of double sequences is weaker than the usual convergence of double sequences on $L-$fuzzy normed spaces.

2. Preliminaries

In this section we give some preliminaries on $L-$fuzzy normed spaces.

Definition 2.1. *[15] Let* $T : [0,1] \times [0,1] \to [0,1]$ *be a function satisfying the conditions*

1). $T(x,y) = T(y,x)$

2). $T(T(x,y),z) = T(x,T(y,z))$

3). $T(x,1) = T(1,x) = x$

4). if $x \leq y, z \leq t$, *then* $T(x,z) \leq T(y,t)$

Then T is called a triangular norm (or shortly t−norm).

Example 2.2. [15] The functions T_1, T_2 and T_3 given with,

$T_1(x,y) = \min\{x,y\}$,

$T_2(x,y) = xy$,

$T_3(x,y) = \max\{x+y-1, 0\}$

are some well-known examples of $t-$norms.

Definition 2.3. *[3] Given a complete lattice* $\mathscr{L} = (L, \leq)$ *and a set X which will be called the universe. A function*

$$A : X \to L$$

is called an L−fuzzy set, or an L−set for short, on X. The family of all L−subsets on a set X is denoted by L^X.

Intersection of two $L-$sets on X is given by

$$(A \cap B)(x) := A(x) \wedge B(x)$$

for all $x \in X$. Similarly union of two $L-$sets and intersection and union of a family $\{A_i : i \in I\}$ of $L-$sets is given by

$$(A \cup B)(x) := A(x) \vee B(x)$$

$$\left(\bigcap_{i\in I} A_i\right)(x) := \bigwedge_{i\in I} A_i(x)$$

$$\left(\bigcup_{i\in I} A_i\right)(x) := \bigvee_{i\in I} A_i(x)$$

respectively.

We denote the smallest and the greatest elements of the complete lattice L by 0_L and 1_L. We also use the symbols $\succeq, \prec$ and $\succ$ given a lattice $(L, \preceq)$, in the obvious meanings.

Definition 2.4. *[15] A triangular norm (t−norm) on a complete lattice $\mathscr{L} = (L, \preceq)$ is a function $\mathscr{T} : L \times L \to L$ satisfying the following conditions for all $x, y, z, t \in L$:*

1). $\mathscr{T}(x, y) = \mathscr{T}(y, x)$

2). $\mathscr{T}(\mathscr{T}(x, y), z) = \mathscr{T}(x, \mathscr{T}(y, z))$

3). $\mathscr{T}(x, 1_L) = \mathscr{T}(1_L, x) = x$

4). if $x \preceq y$ and $z \preceq t$, then $\mathscr{T}(x, z) \preceq \mathscr{T}(y, t)$.

A t−norm $\mathscr{T}$ on a complete lattice $\mathscr{L} = (L, \preceq)$ is called continuous, if for every pair of sequences (x_n) and (y_n) on L such that $(x_n) \to x \in L$ and $(y_n) \to y \in L$, one have the property that the sequence $\mathscr{T}(x_n, y_n) \to \mathscr{T}(x, y)$ with respect to the order topology on L.

Definition 2.5. *[16] A mapping $\mathscr{N} : L \to L$ is called a negator on $\mathscr{L} = (L, \preceq)$ if,*
$N_1)$ $\mathscr{N}(0_L) = 1_L$
$N_2)$ $\mathscr{N}(1_L) = 0_L$
$N_3)$ *$x \preceq y$ implies $\mathscr{N}(y) \preceq \mathscr{N}(x)$ for all $x, y \in L$.*
In addition, if
$N_4)$ *$\mathscr{N}(\mathscr{N}(x)) = x$ for all $x \in L$,*
then the negator $\mathscr{N}$ is said to be involutive.

On the lattice $([0, 1], \leq)$ the function $\mathscr{N}_s : [0, 1] \to [0, 1]$ defined as $\mathscr{N}_s(x) = 1 - x$ is an example of an involutive negator, called standart negator on $[0, 1]$, which is used in the theory of fuzzy sets. On the other hand, given the lattice $([0, 1]^2, \preceq)$ with the order

$$(\mu_1, \nu_1) \preceq (\mu_2, \nu_2) \iff \mu_1 \leq \mu_2 \quad \text{and} \quad \nu_1 \geq \nu_2$$

for all $(\mu_i, \nu_i) \in [0, 1]^2$, $i = 1, 2$. Then the mapping $\mathscr{N}_1 : [0, 1]^2 \to [0, 1]^2$,

$$\mathscr{N}_1(\mu, \nu) = (\nu, \mu)$$

is an involutive negator used in the theory of intuitionistic fuzzy sets in the sense of Atanassov [1]. A possible candidate for a non-involutive negator on $([0, 1]^2, \preceq)$ would be given by

$$\mathscr{N}_2(\mu, \nu) = \left(\frac{1 - \mu + \nu}{2}, \frac{1 + \mu - \nu}{2} \right).$$

Remark 2.6. *In general, for any given continuous t−norm $\mathscr{T}$ and a negator $\mathscr{N}$, it is not always possible to find for each given $\varepsilon \in L - \{0_L, 1_L\}$, an element $r \in L - \{0_L, 1_L\}$ such that $\mathscr{T}(\mathscr{N}(r), \mathscr{N}(r)) \succ \mathscr{N}(\varepsilon)$. For more details and some cases where this inequality holds, see [16]. In this study, a continuous t−norm and an involutive negator $\mathscr{N}$ such that for each $\varepsilon \in L - \{0_L, 1_L\}$, there exists an $r \in L - \{0_L, 1_L\}$ satisfying $\mathscr{T}(\mathscr{N}(r), \mathscr{N}(r)) \succ \mathscr{N}(\varepsilon)$, is supposed to be given and fixed.*

Definition 2.7. *[15] Let V be a real vector space, $\mathscr{L} = (L, \preceq)$ be a complete lattice, $\mathscr{T}$ be a continuous t−norm on $\mathscr{L}$ and ρ be a $\mathscr{L}$−set on $V \times (0, \infty)$ satisfying the following:*

(a) $\rho(x, t) \succ 0_L$ *for all* $x \in V$, $t > 0$

(b) $\rho(x,t) = 1_L$ *for all* $t > 0$*, if and only if* $x = \theta$

(c) $\rho(\alpha x,t) = \rho(x, \frac{t}{|\alpha|})$ *for all* $x \in V$*,* $t > 0$ *and* $\alpha \in \mathbb{R} - \{0\}$

(d) $\mathscr{T}(\rho(x,s), \rho(y,t)) \leq \rho(x+y, s+t)$ *for all* $x, y \in V$ *and* $s, t > 0$

(e) $\lim_{t\to\infty} \rho(x,t) = 1_L$ *and* $\lim_{t\to 0} \rho(x,t) = 0_L$ *for all* $x \in V \setminus \{\theta\}$

(f) The mappings $f_x : (0,\infty) \to L$ *given by* $f(t) = \rho(x,t)$ *are continuous.*

In this case, the triple $(V, \rho\ \mathscr{T})$ *is called a* $\mathscr{L}-$*fuzzy normed space or* $\mathscr{L}-$*normed space, for short.*

Note that under the assumptions of Remark 2.6, the following definition is equivalent to the corresponding definition given in [15].

Definition 2.8. *A sequence* (x_n) *in a* $\mathscr{L}-$*fuzzy normed space* $(V, \rho, \mathscr{T})$ *is said to be convergent to* $x \in V$*, if for each* $\varepsilon \in L - \{0_L\}$ *and* $t > 0$*, there exists some* $n_0 \in \mathbb{N}$ *such that, for all* $n > n_0$

$$\rho(x_n - x, t) \succ \mathscr{N}(\varepsilon).$$

Definition 2.9. *[15] A sequence* (x_n) *in a* $\mathscr{L}-$*fuzzy normed space* $(V, \rho, \mathscr{T})$ *is said to be a Cauchy sequence, if for each* $\varepsilon \in L - \{0_L\}$ *and* $t > 0$ *there exists some* $n_0 \in \mathbb{N}$ *such that*

$$\rho(x_n - x_m, t) \succ \mathscr{N}(\varepsilon)$$

for all $m, n > n_0$*.*

3. Statistical convergence on $\mathscr{L}-$fuzzy normed space

In this section, we will look into statistical convergence on $\mathscr{L}-$fuzzy normed spaces. Before we go any further, we should review some terminology on statistical convergence. The following discussion is due to [2].

If K is a subset of $\mathbb{N}$, the set of positive integers, then its asymptotic density, denoted by $\delta\{K\}$, is

$$\delta\{K\} := \lim_n \frac{1}{n} |\{k \leq n : k \in K\}|$$

whenever the limit occurs, where $|A|$ stands for the the cardinality of a set A.

If the set $K(\varepsilon) = \{k \leq n : |x_k - l| > \varepsilon\}$ has the asymptotic density zero for a given number $\varepsilon > 0$, that is

$$\lim_n \frac{1}{n} \{k \leq n : |x_k - l| > \varepsilon\} = 0$$

then a real sequence $x = (x_k)$ is said to be statistically convergent to l. In this scenario, we will write $st - \lim x = l$.

Although every convergent sequence is statistically convergent to the same limit, the converse is not always true.

For any given $\varepsilon > 0$, if there exists an integer N such that $|\ x_{jk} - l\ | < \varepsilon$ whenever $j, k > N$, a double sequence $x = (x_{jk})$ is said to be Pringsheim's convergent or shortly $P-$ convergent. This will be written as

$$\lim_{j,k\to\infty} x_{jk} = l$$

with j and k tending to infinity independently of one another.

Let $K \subset \mathbb{N}\times\mathbb{N}$ be a two-dimensional set of positive integers, and let $K(m,n)$ be the numbers of (j,k) in K such that $j \leq m$ and $k \leq n$. Then we can define the two-dimensional analogue of natural density as follows: The lower asymptotic density of the set $K \subset \mathbb{N}\times\mathbb{N}$ is defined as

$$\delta_2(K) = \liminf_{m,n} \frac{K(m,n)}{mn}$$

and if the sequence $\left(\frac{K(m,n)}{mn}\right)$ has a limit in the sense of Pringsheim, we say it has a double natural density, and it is defined as

$$\lim_{m,n} \frac{K(m,n)}{mn} = \delta_2(K).$$

In the following, we investigate the principles of statistical convergence of double sequences in $\mathscr{L}-$fuzzy normed space.

Definition 3.1. *Let $(V,\rho,\mathscr{T})$ be a $\mathscr{L}-$fuzzy normed space. Then a double sequence $x = (x_{jk})$ is statistically convergent to $l \in V$ with respect to ρ provided that, for each $\varepsilon \in L - \{0_L\}$ and $t > 0$,*

$$\delta_2\{(j,k) \in \mathbb{N}\times\mathbb{N} : \rho(x_{jk} - l, t) \nsucc \mathscr{N}(\varepsilon)\} = 0$$

or equivalently

$$\lim_{m,n} \frac{1}{mn}\{j \leq m, k \leq n : \rho(x_{jk} - l, t) \nsucc \mathscr{N}(\varepsilon)\} = 0.$$

In this case, we write $st_{2\mathscr{L}} - \lim x = l$.

Lemma 3.2. *Let $(V,\rho,\mathscr{T})$ be a $\mathscr{L}-$fuzzy normed space. Then, the following statements are equivalent, for every $\varepsilon \in L - \{0_L\}$ and $t > 0$:*

(a) $st_{2\mathscr{L}} - \lim x = l$.

(b) $\delta_2\{(j,k) \in \mathbb{N}\times\mathbb{N} : \rho(x_{jk} - l, t) \nsucc \mathscr{N}(\varepsilon)\} = 0$.

(c) $\delta_2\{(j,k) \in \mathbb{N}\times\mathbb{N} : \rho(x_{jk} - l, t) \succ \mathscr{N}(\varepsilon)\} = 1$.

(d) $st_{2\mathscr{L}} - \lim \rho(x_{jk} - l, t) = 1_L$.

Theorem 3.3. *Let $(V,\rho,\mathscr{T})$ be a $\mathscr{L}-$fuzzy normed space. If $\lim x = l$ then $st_{2\mathscr{L}} - \lim x = l$.*

Proof. Let $\lim x = l$. Then for every $\varepsilon \in L - \{0_L\}$ and $t > 0$, there is a number $k_0 \in \mathbb{N}$ such that

$$\rho(x_{jk} - l, t) \succ \mathscr{N}(\varepsilon)$$

for all $j,k \geq k_0$. Therefore,

$$\{(j,k) \in \mathbb{N}\times\mathbb{N} : \rho(x_{jk} - l, t) \nsucc \mathscr{N}(\varepsilon)\}$$

has at most finitely many terms. We can see right away that any finite subset of the natural numbers has density zero. Hence,

$$\delta_2\{(j,k) \in \mathbb{N}\times\mathbb{N} : \rho(x_{jk} - l, t) \nsucc \mathscr{N}(\varepsilon)\} = 0$$

completes the proof. □

The converse of the Theorem 3.3 is generally not true, as would be seen in the following example.

Example 3.4. Let $V = \mathbb{R}$ and $\mathscr{L} = (\mathscr{P}(\mathbb{R}^+), \subseteq)$, the lattice of all subsets of the set of positive real numbers. Define the function $\rho : \mathbb{R} \times (0, \infty) \to \mathscr{P}(\mathbb{R}^+)$ with $\rho(x, t) = \{r \in \mathbb{R}^+ : |rx| < t\}$. Then $(\mathbb{R}, \mathscr{P}(\mathbb{R}^+), \rho)$ is a $\mathscr{L}-$normed space. On this space, consider the double sequence (a_{kl}) given by the rule $a_{kl} = \operatorname{sgn}(k \sin k + k - 1) + \operatorname{sgn}(l \cos l + l - 1)$. Then it can be conjectured that, while $st_{2\mathscr{L}} - \lim a = 2 \in \mathbb{R}$, the sequence itself is not convergent.

Theorem 3.5. *Let $(V, \rho, \mathscr{T})$ be a $\mathscr{L}-$fuzzy normed space. If a double sequence $x = (x_{jk})$ is statistically convergent with respect to the $\mathscr{L}-$fuzzy norm ρ, then this statistical limit is unique.*

Proof. Suppose that $st_{2\mathscr{L}} - \lim x = l_1$ and $st_{2\mathscr{L}} - \lim x = l_2$. For any given $\varepsilon \in L - \{0_L\}$ and $t > 0$, we can choose an $r \in L - \{0_L\}$ such that

$$\mathscr{T}(\mathscr{N}(r), \mathscr{N}(r)) \succ \mathscr{N}(\varepsilon).$$

Define the following sets:

$$K_1 = \left\{(j, k) \in \mathbb{N} \times \mathbb{N} : \rho\left(x_{jk} - l_1, \frac{t}{2}\right) \nsucc \mathscr{N}(r)\right\}$$

and

$$K_2 = \left\{(j, k) \in \mathbb{N} \times \mathbb{N} : \rho\left(x_{jk} - l_2, \frac{t}{2}\right) \nsucc \mathscr{N}(r)\right\}$$

for any $t > 0$. Then $\delta_2\{K_1\} = \delta_2\{K_2\} = 0$. Say $K = K_1 \cup K_2$. Then also $\delta_2\{K\} = 0$ so that $\delta_2\{K^c\} = 1$, and for each $(j, k) \in \mathbb{N} \times \mathbb{N}$

$$\rho(l_1 - l_2, t) \geq \mathscr{T}\left(\rho\left(x_{jk} - l_1, \frac{t}{2}\right), \rho\left(x_{jk} - l_2, \frac{t}{2}\right)\right) \succ \mathscr{T}(\mathscr{N}(r), \mathscr{N}(r)) \succ \mathscr{N}(\varepsilon).$$

Then, it is obvious that $l_1 = l_2$. □

Note that from the definition of a $\mathscr{L}-$normed space, one have $\rho(x, t) \succ 0_L$ for all $x \in V$ and $t \in (0, \infty)$. In contrast to this, one also have $\lim_{t\to\infty} \rho(x, t) = 0_L$. In particular, defining $a_n = \rho(x, \frac{1}{n})$ will give a sequence (a_n) on L such that $a_n \neq 0_L$ for all positive integer n, while $(a_n) \to 0_L$ on L. Now say $b_n := \mathscr{N}(a_n)$. Then for each n, $b_n \neq 1_L$, since otherwise one would have

$$a_n = \mathscr{N}(\mathscr{N}(a_n)) = \mathscr{N}(b_n) = \mathscr{N}(1_L) = 0_L,$$

which would be a contradiction. Being a decreasing mapping and bijective by the identity $\mathscr{N}(\mathscr{N}(x)) = x$, the involutive negator $\mathscr{N}$ is order continuous, so that $(b_n) \to \mathscr{N}(0_L) = 1_L$.

Since $(a_n) \to 0_L$, for every open basic neighborhood $A_c = \{x \in L : x < c\}$ of 0_L, where $c \in L - \{0_L\}$, there exists an $n_0 = n_0(c) \in \mathbb{N}$ such that $a_n \in A_c$ for all $n \geq n_0$. Saying $i_1 = 1$, $i_2 = n_0(a_1)+1 = n_0(a_{i_1})+1$, $i_3 = n_0(a_{i_2}) + 1$ and $i_{k+1} = n_0(a_{i_k}) + 1$ in general, we have an increasing subsequence of (a_n).

The discussion above guarantees that given any $\mathscr{L}-$ normed space, it is always possible to find a sequence (a_n) in $L - \{0_L\}$ such that $(a_n) \to 0_L$ so that $\mathscr{N}(a_n) \to 1_L$. In particular, we can always find an increasing sequence (ε_n) in $L - \{0_L\}$ such that $\mathscr{N}(\varepsilon_n) \to 1_L$.

Theorem 3.6. *Let $(V, \rho, \mathscr{T})$ be a $\mathscr{L}-$fuzzy normed space. Then, $st_{2\mathscr{L}} - \lim x = \ell$ if and only if there exists a subset $K \subseteq \mathbb{N} \times \mathbb{N}$ such that $\delta_2(K) = 1$ and $\lim_{m,n\to\infty} x_{mn} = \ell$.*

Proof. Suppose that $st_{2\mathscr{L}} - \lim x = \ell$. Let (ε_n) be an increasing sequence in $L - \{0_L\}$ such that $\mathscr{N}(\varepsilon_n) \to 1_L$ in L, and for any $t > 0$ and $j \in \mathbb{N}$, let

$$K(j) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn} - \ell, t) \succ \mathscr{N}(\varepsilon_j)\}$$

Then observe that, for any $t > 0$ and $j \in \mathbb{N}$,

$$K(j+1) \subseteq K(j).$$

Since $st_{2\mathscr{L}} - \lim x = \ell$, it is obvious that $\delta_2\{K(j)\} = 1$ for each $j \in \mathbb{N}$ and $t > 0$. Now let (p_1, q_1) be an arbitrary number pair in $K(1)$. Then there are numbers p_2 and q_2 such that $p_2 > p_1$, $q_2 > q_1$, $(p_2, q_2) \in K(2)$ and for all $n > p_2$, $m > q_2$

$$\frac{1}{mn} |\{k \le n,\ l \le m : \rho(x_{kl} - \ell, t) \succ \mathscr{N}(\varepsilon_2)\}| > \frac{1}{2}.$$

Furthermore, there is a pair $(p_3, q_3) \in K(3)$, $p_3 > p_2$, $q_3 > q_2$ such that for all $n > p_3$, $m > q_3$,

$$\frac{1}{mn} |\{k \le n,\ l \le m : \rho(x_{kl} - \ell, t) \succ \mathscr{N}(\varepsilon_3)\}| > \frac{2}{3}$$

and so on. In this way, we can construct, by induction, an index sequence $(p_j, q_k)_{j,k \in \mathbb{N}}$ of pairs of natural numbers increasing in both coordinates, such that $(p_j, q_j) \in K(j)$ and that the following statement holds for all $n > p_j$, $m > q_j$:

$$\frac{1}{mn} \left|\{k \le n,\ l \le m : \rho(x_{kl} - \ell, t) \succ \mathscr{N}(\varepsilon_j)\}\right| > \frac{j-1}{j}.$$

Now we construct an index sequence increasing in both coordinates as follows:

$$K := \{(m,n) \in \mathbb{N} \times \mathbb{N} : 1 < n < p_1,\ 1 < m < q_1\} \cup \left[\bigcup_{j \in \mathbb{N}} \{(m,n) \in K(j) : p_j \le n < p_{j+1},\ q_j \le m < q_{j+1}\}\right]$$

Hence it follows that $\delta_2(K) = 1$. Now let $\varepsilon \succ 0_L$ and choose a positive integer j such that $\varepsilon_j \prec \varepsilon$. Such a number j always exists since $(\varepsilon_n) \to 0_L$. Assume that $n \ge p_j$, $m \ge q_j$ and $m, n \in K$. Then by the definiton of K, there exists a number $k \ge j$ such that $p_k \le n < p_{k+1}$, $q_k \le m < q_{k+1}$ and $(m,n) \in K(j)$. Hence, we have, for every $\varepsilon \succ 0_L$

$$\rho(x_{mn} - \ell, t) \succ \mathscr{N}(\varepsilon_j) \succ \mathscr{N}(\varepsilon)$$

for all $n \ge p_j$, $m \ge q_j$ and $(m,n) \in K$ and this means

$$\mathscr{L} - \lim_{m,n \in K} = \ell.$$

Conversely, suppose that there exists an increasing index sequence $K = (k_{mn})_{m,n \in \mathbb{N}}$ of pairs of natural numbers such that $\delta_2\{K\} = 1$ and $\mathscr{L} - \lim_{m,n \in K} x_{mn} = \ell$. Then, for every $\varepsilon \succ 0_L$ there is a number n_0 such that for each $m, n \ge n_0$ the inequality $\rho(x_{mn} - \ell, t) \succ \mathscr{N}(\varepsilon)$ holds. Now define

$$M(\varepsilon) := \{(m,n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn} - \ell, t) \not\succ \mathscr{N}(\varepsilon)\}.$$

Then there exists an $n_0 \in \mathbb{N}$ such that

$$M(\varepsilon) \subseteq (\mathbb{N} \times \mathbb{N}) - (K - \{(k_m, k_n) : m, n \le n_0\}).$$

Since $\delta_2\{K\} = 1$ and $\{(k_m, k_n) : m, n \le n_0\}$ is finite, we get $\delta_2\{(\mathbb{N} \times \mathbb{N}) - (K - \{(k_m, k_n) : m, n \le n_0\})\} = 0$, which yields that $\delta_2\{M(\varepsilon)\} = 0$. In other words, $st_{2\mathscr{L}} - \lim x = \ell$. □

Conflict of interest

The authors declare that he has no conflict of interest.

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