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Research article

Classification of nonnegative solutions to fractional Schrödinger-Hatree-Maxwell type system

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Abstract: In this paper, we are concerned with the fractional Schrödinger-Hatree-Maxwell type system. We derive the forms of the nonnegative solution and classify nonlinearities by appling a variant (for nonlocal nonlinearity) of the direct moving spheres method for fractional Laplacians. The main ingredients are the variants (for nonlocal nonlinearity) of the maximum principles, i.e., *narrow region principle* (Theorem 2.3).

Keywords: fractional Laplacians; nonnegative solutions; nonlocal nonlinearities; direct method of moving spheres
Mathematics Subject Classification: 35B08, 35B50, 35J61, 35R11

1. Introduction

In this paper, we consider the following fractional Schrödinger-Hatree-Maxwell type system

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}}u(x) = \Big(\frac{1}{|\cdot|^{\sigma_1}} * f_1(v(x))\Big)f_2(v(x)), & x \in \mathbb{R}^n, \\ (-\Delta)^{\frac{\alpha}{2}}v(x) = \Big(\frac{1}{|\cdot|^{\sigma_2}} * g_1(u(x))\Big)g_2(u(x)), & x \in \mathbb{R}^n, \\ u(x) \geq 0, v(x) \geq 0, & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $0 < \alpha \leq 2$, $n \geq 2$, $0 < \sigma_1, \sigma_2 < n$, f_i, g_i $(i = 1, 2)$ are strictly increasing on $[0, +\infty)$, and $f_i(t) \cdot t^{-p_i}$, $g_i(t) \cdot t^{-q_i}$ $(i = 1, 2)$ are nonincreasing on $(0, +\infty)$.

We assume $u, v \in C^{1,1}_{loc} \cap \mathcal{L}_\alpha(\mathbb{R}^n)$ if $0 < \alpha < 2$, and $u, v \in C^2(\mathbb{R}^n)$ if $\alpha = 2$, where

$$\mathcal{L}_\alpha(\mathbb{R}^n) := \Big\{u : \mathbb{R}^n \to \mathbb{R} \,\Big|\, \int_{\mathbb{R}^n} \frac{|u(y)|}{1 + |y|^{n+\alpha}} dy < \infty\Big\}. \tag{1.2}$$

The nonlocal fractional Laplacians $(-\Delta)^{\frac{\alpha}{2}}$ with $0<\alpha<2$ are defined by [10, 16, 20, 51, 54]

$$(-\Delta)^{\frac{\alpha}{2}}u(x)=C_{\alpha,n}\,P.V.\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+\alpha}}dy:=C_{\alpha,n}\lim_{\epsilon\to 0}\int_{|y-x|\geq\epsilon}\frac{u(x)-u(y)}{|x-y|^{n+\alpha}}dy, \tag{1.3}$$

for functions $u,v\in C^{1,1}_{loc}\cap\mathcal{L}_{\alpha}(\mathbb{R}^n)$, where $C_{\alpha,n}=\left(\int_{\mathbb{R}^n}\frac{1-\cos(2\pi\zeta_1)}{|\zeta|^{n+\alpha}}d\zeta\right)^{-1}$ is the normalization constant. The fractional Laplacians $(-\Delta)^{\frac{\alpha}{2}}$ can also be defined equivalently [17] by Caffarelli and Silvestre's extension method [6] for $u,v\in C^{1,1}_{loc}\cap\mathcal{L}_{\alpha}(\mathbb{R}^n)$.

We should mention that the fractional Laplacian is different from the fractional differentiation. Both are defined through a singular convolution integral, but the former is guaranteed to be the positive definition via the Riesz potential as the standard Laplace operator, while the latter via the Riemann-Liouville integral is not. It is noted that the fractional Laplacian can not be interpreted by the fractional differentiation in the sense of either Riemann-Liouville or Caputo. Both the fractional Laplacian and the fractional differentiation have found applications in many complicated engineering problems. In particular, the fractional Laplacian attracts new attentions in recent years owing to its unique capability describing anomalous diffusion problems [34].

The fractional Laplacian can be regarded as an infinitesimal generator of a stable Lévy process, which has many applications in probability, optimization and finance [1, 4]. It is also widely used to simulate various physical phenomena, such as anomalous diffusion and quasi-geostrophic flows, turbulence and water waves, molecular dynamics and relativistic quantum mechanics of stars (see [2, 22, 23, 36] and the references therein). But, it is difficult to study the fractional Laplacians because of its non-local feature. In order to overcome this difficulty, Chen, Li and Ou [18] put forward the method of moving planes in integral forms. Subsequently, Caffarelli and Silvestre [6] introduced an extension method to conquer this difficulty, which transformed this nonlocal problem into a local one in higher dimensions. This extension method provides a powerful tool and leads to a very active study in equations containing the fractional Laplacians, and obtains a series of fruitful results (see [3, 21] and the references therein).

Chen, Li and Li developed a direct method of moving planes for the fractional Laplacians in [16, 25]. Rather than use the extension method of Caffarelli and Silvestre [6], they worked directly on the non-local operator to establish strong maximum principles for anti-symmetric functions and narrow region principles, and then for nonnegative solutions they obtained classification and Liouville type results. The direct method of moving planes introduced in [16] has been used to study more general nonlocal operators with general nonlinearities [15, 25]. In the early 1950s, the method of moving planes was originally invented by Alexanderoff. Later, it was further developed by Serrin [51], Gidas, Ni and Nirenberg [33], Caffarelli, Gidas and Spruck [5], Chen and Li [11], Li and Zhu [40], Lin [43], Chen, Li and Ou [18], Chen, Li and Li [16], and so on. For more literatures on the classification of solutions and Liouville type theorems for various PDE and IE problems through the methods of moving planes or spheres, please refer to [7, 9, 10, 14, 20, 24, 27, 29, 30, 44–47, 52] and the references therein.

In [20], Chen, Li and Zhang introduced another direct method—-the method of moving spheres on the fractional Laplacians, which is more convenient than the method of moving planes. The method of moving spheres was initially used by Padilla [49], Chen and Li [12] and Li and Zhu [40]. It can be applied to obtain the explicit form of solutions directly instead of deriving radial symmetry of solutions and then classifying radial solutions.

There are many literatures on the qualitative properties of solutions to Hartree and Choquard equations of fractional or higher order, please see e.g., Cao and Dai [7], Chen and Li [13], Dai, Fang, et al., [24], Dai and Qin [29], Dai and Liu [26], Lei [38], Liu [44], Le [37], Ma and Zhao [48], Xu and Lei [53] and the references therein.

System (1.1) is closely related to the following integral system

$$\begin{cases} u(y) = \int_{\mathbb{R}^n} \frac{R_{\alpha,n}}{|y-z|^{n-\alpha}} \Big(\int_{\mathbb{R}^n} \frac{f_1(v(\xi))}{|z-\xi|^{\sigma_1}} d\xi \Big) f_2(v(z)) dz, \\ v(y) = \int_{\mathbb{R}^n} \frac{R_{\alpha,n}}{|y-z|^{n-\alpha}} \Big(\int_{\mathbb{R}^n} \frac{g_1(u(\zeta))}{|z-\zeta|^{\sigma_2}} d\zeta \Big) g_2(u(z)) dz, \end{cases} \tag{1.4}$$

where the Riesz potential's constants $R_{\alpha,n} := \frac{\Gamma(\frac{n-\alpha}{2})}{\pi^{\frac{n}{2}} 2^{\alpha} \Gamma(\frac{\alpha}{2})}$ [50].

In the special case $f_i(t) = t^{p_i}$, $g_i(t) = t^{q_i}$ $(i = 1, 2)$, system (1.4) turns into the following Schrödinger-Hatree-Maxwell type system

$$\begin{cases} u(y) = \int_{\mathbb{R}^n} \frac{R_{\alpha,n}}{|y-z|^{n-\alpha}} \Big(\int_{\mathbb{R}^n} \frac{v^{p_1}(x)}{|z-\xi|^{\sigma_1}} d\xi \Big) v^{p_2}(x) dz, \\ v(y) = \int_{\mathbb{R}^n} \frac{R_{\alpha,n}}{|y-z|^{n-\alpha}} \Big(\int_{\mathbb{R}^n} \frac{v^{q_1}(x)}{|z-\zeta|^{\sigma_2}} d\zeta \Big) u^{q_2}(x) dz. \end{cases} \tag{1.5}$$

When $p_1 = q_1 = 2$ and $p_2 = q_2 = 1$, PDEs of type (1.1) arise in the Hartree-Fock theory of the nonlinear Schrödinger equations [41]. When $f_1(v(x)) = v^2(x)$, $f_2(v(x)) = v(x)$, $g_1(u(x)) = u^2(x)$, $g_2(u(x)) = u(x)$, and $\sigma_1 = \sigma_2 = 2\alpha$, the solution (u, v) to problem (1.1) is also a ground state or a stationary solution to the following $\dot{H}^{\frac{\alpha}{2}}$-critical focusing dynamic Schrödinger-Hartree system

$$\begin{cases} i\partial_t u + (-\Delta)^{\frac{\alpha}{2}} u = \Big(\frac{1}{|x|^{2\alpha}} * |v|^2 \Big) v, & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\ i\partial_t v + (-\Delta)^{\frac{\alpha}{2}} v = \Big(\frac{1}{|x|^{2\alpha}} * |u|^2 \Big) u, & (t, x) \in \mathbb{R} \times \mathbb{R}^n. \end{cases} \tag{1.6}$$

In the special case $\alpha = 2$, the above Schrödinger-Hartree equations have many interesting applications in the quantum theory of large systems of non-relativistic bosonic atoms and molecules [32].

When $\sigma_1, \sigma_2 = 2\alpha$, $\alpha \in (0, \frac{n}{2})$, $p_1 = q_1 = 2$, $p_2 = q_2 = 1$, Dai, Fang, et al., [24] classified all the positive $H^{\frac{\alpha}{2}}(\mathbb{R}^n)$ weak solutions to (1.5) by using the method of moving planes in integral forms for the equivalent integral equation system (1.4) due to Chen, Li and Ou [18, 19]. They also classified all the $L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)$ integrable solutions to the equivalent integral equations. For $0 < \alpha < \min\{2, \frac{n}{2}\}$, Dai, Fang and Qin [25] classified all the $C^{1,1}_{loc} \cap \mathcal{L}_\alpha$ solutions to (1.5) with $\sigma_1, \sigma_2 = 2\alpha$, $p_1 = q_1 = 2$, $p_2 = q_2 = 1$ by applying a variant (for nonlocal nonlinearity) of the direct method of moving planes for fractional Laplacians. The qualitative properties of solutions to general fractional order or higher order elliptic equations have also been extensively studied, for instance, see Chen, Fang and Yang [10], Chen, Li and Li [16], Chen, Li and Ou [18], Caffarelli and Silvestre [6], Chang and Yang [9], Dai and Qin [29], Cao, Dai and Qin [8], Dai, Liu and Qin [28], Fang and Chen [30], Lin [43], Wei and Xu [52] and the references therein.

Our main theorem is the following classification theorem for PDEs system (1.1).

Theorem 1.1. *Let* $(u, v) \in C^0(\mathbb{R}^n) \times C^0(\mathbb{R}^n)$ *be a pair of nonnegative solution to the problem* (1.4). *Assume that* $f_1, f_2, g_1, g_2 : [0, +\infty) \to \mathbb{R}_+$ *satisfy the following conditions:*

(i) $f_1(t)$, $f_2(t)$, $g_1(t)$ *and* $g_2(t)$ *are strictly increasing on* $[0,+\infty)$*;*

(ii) $F_1(t) = f_1(t)\cdot t^{-p_1}$, $F_2(t) = f_2(t)\cdot t^{-p_2}$, $G_1(t) = g_1(t)\cdot t^{-q_1}$ *and* $G_2(t) = g_2(t)\cdot t^{-q_2}$ *(*$p_1 = \frac{2n-\sigma_1}{n-\alpha}$, $p_2 = \frac{n+\alpha-\sigma_1}{n-\alpha}$, $q_1 = \frac{2n-\sigma_2}{n-\alpha}$, $q_2 = \frac{n+\alpha-\sigma_2}{n-\alpha}$*) are nonincreasing on* $(0,+\infty)$.

Then for some $z_0 \in \mathbb{R}^n$*, for any* $y \in \mathbb{R}^n$*,* u*,* v *must take the following form as*

$$u(y) = \frac{c_1}{(d^2+|y-z_0|^2)^{\frac{n-\alpha}{2}}}, \quad v(y) = \frac{c_2}{(d^2+|y-z_0|^2)^{\frac{n-\alpha}{2}}},$$

for some $c_1 > 0$*,* $c_2 > 0$ *and* $d > 0$.

Furthermore, $f_1(t)$*,* $f_2(t)$*,* $g_1(t)$ *and* $g_2(t)$ *must be the form of*

$$f_1(t) = C_1 t^{p_1}, f_2(t) = C_2 t^{p_2},\ t \in (0, \max_{x\in\mathbb{R}^n} v(x)],$$

$$g_1(t) = C_3 t^{q_1}, g_2(t) = C_4 t^{q_2},\ t \in (0, \max_{x\in\mathbb{R}^n} u(x)],$$

where C_1*,* C_2*,* C_3 *and* C_4 *are some positive constants.*

Remark 1.2. In the conformal invariant case ($p_1 = \frac{2n-\sigma_1}{n-\alpha}$, $p_2 = \frac{n+\alpha-\sigma_1}{n-\alpha}$, $q_1 = \frac{2n-\sigma_2}{n-\alpha}$, $q_2 = \frac{n+\alpha-\sigma_2}{n-\alpha}$), and $u = v$, $\sigma_1 = \sigma_2$, the classification of nonnegative solutions to system (1.1) would provide the best constants and extremal functions for the corresponding Hardy-Littlewood-Sobolev inequality [28, 31, 42].

In the following, we introduce some notation, we define the Kelvin transforms. Take arbitrary $x \in \mathbb{R}^n$ and $\lambda > 0$,

$$u_{x,\lambda}(y) := \left(\frac{\lambda}{|y-x|}\right)^{n-\alpha} u(y^{x,\lambda}), \qquad \forall\ y \in \mathbb{R}^n \setminus \{x\},$$

$$v_{x,\lambda}(y) := \left(\frac{\lambda}{|y-x|}\right)^{n-\alpha} v(y^{x,\lambda}), \qquad \forall\ y \in \mathbb{R}^n \setminus \{x\},$$

where

$$y^{x,\lambda} = \frac{\lambda^2(y-x)}{|y-x|^2} + x.$$

For any $\lambda > 0$, we denote

$$B_\lambda(x) := \{y \in \mathbb{R}^n \,|\, |y-x| < \lambda\},$$

and $U_{x,\lambda}(y) = u_{x,\lambda}(y) - u(y)$, $V_{x,\lambda}(y) = v_{x,\lambda}(y) - v(y)$ for any $y \in B_\lambda(x) \setminus \{x\}$.

The rest of our paper is organized as follows. In Section 2, we carry out our proof of Theorem 1.1. In order to prove Theorem 1.1, we divide the proof into two parts. In the first part, we will apply a variant (for nonlocal nonlinearity) of the direct method of moving spheres for fractional Laplacians developed by Chen, Li and Zhang [20] and Dai and Liu [26] to derive the forms for nonnegative solution (u, v). In the second part, we classify nonlinearities f_1, f_2, g_1 and g_2 due to Hu and Liu [35].

In the following, we will use C to denote general positive constants that may depend on n, α, p_1, p_2, q_1, q_2, σ_1, σ_2, u and v, and whose values may differ from line to line.

2. Preparation of Theorem 1.1

In this section, we will use a direct method of moving spheres for nonlocal nonlinearity with the help of narrow region principle to classify the nonnegative solutions of PDEs system (1.1).

2.1. The direct method of moving spheres for nonlocal nonlinearity

Assume that, $n \geq 2$, $0 < \sigma_1, \sigma_2 < n$, $0 < \alpha \leq 2$, f_1, f_2, g_1 and g_2 are strictly increasing on $[0, +\infty)$, furthermore, $F_1(t) = f_1(t) \cdot t^{-p_1}$, $F_2(t) = f_2(t) \cdot t^{-p_2}$, $G_1(t) = g_1(t) \cdot t^{-q_1}$ and $G_2(t) = g_2(t) \cdot t^{-q_2}$ are nonincreasing on $(0, +\infty)$£ with $p_1 = \frac{2n-\sigma_1}{n-\alpha}$, $p_2 = \frac{n+\alpha-\sigma_1}{n-\alpha}$, $q_1 = \frac{2n-\sigma_2}{n-\alpha}$ and $q_2 = \frac{n+\alpha-\sigma_2}{n-\alpha}$. Suppose (u, v) is a pair of nonnegative classical solution of (1.1) which is not identically zero. Then we can derive that $u, v > 0$ in $\mathbb{R}^n$ and $\int_{\mathbb{R}^n} \frac{f_1(v(x))}{|x|^{\sigma_1}} dx < +\infty$, $\int_{\mathbb{R}^n} \frac{g_1(u(x))}{|x|^{\sigma_2}} dx < +\infty$. Thus we assume (u, v) is actually a positive solution from now on. For convenience, we define the following conformal transforms. Take arbitrary $x \in \mathbb{R}^n$ and $\lambda > 0$,

$$u_{x,\lambda}(y) := \left(\frac{\lambda}{|y-x|}\right)^{n-\alpha} u(y^{x,\lambda}), \qquad \forall\ y \in \mathbb{R}^n \setminus \{x\},$$

$$v_{x,\lambda}(y) := \left(\frac{\lambda}{|y-x|}\right)^{n-\alpha} v(y^{x,\lambda}), \qquad \forall\ y \in \mathbb{R}^n \setminus \{x\},$$

where

$$y^{x,\lambda} = \frac{\lambda^2(y-x)}{|y-x|^2} + x.$$

Then, because (u, v) is a pair of positive classical solution of (1.1), we are able to verify that $u_{x,\lambda}, v_{x,\lambda} \in \mathcal{L}_\alpha(\mathbb{R}^n) \cap C^{1,1}_{loc}(\mathbb{R}^n \setminus \{x\})$ if $0 < \alpha < 2$ ($u_{x,\lambda}, v_{x,\lambda} \in C^2(\mathbb{R}^n \setminus \{x\})$ if $\alpha = 2$) and satisfies the integral property

$$\int_{\mathbb{R}^n} \frac{f_1(v_{x,\lambda}(y))}{\lambda^{\sigma_1}} dy = \int_{\mathbb{R}^n} \frac{f_1(v(x))}{|x|^{\sigma_1}} dx < +\infty,$$
$$\int_{\mathbb{R}^n} \frac{g_1(u_{x,\lambda}(y))}{\lambda^{\sigma_2}} dy = \int_{\mathbb{R}^n} \frac{g_1(u(x))}{|x|^{\sigma_2}} dx < +\infty,$$

and a similar equation as u, v for any $x \in \mathbb{R}^n$ and $\lambda > 0$. Actually, without loss of generality, we may assume $x = 0$ for simplicity and get, for $0 < \alpha < 2$ ($\alpha = 2$ is similar),

$$\begin{aligned}
(-\Delta)^{\frac{\alpha}{2}} u_{0,\lambda}(y) &= C_{\alpha,n} P.V. \int_{\mathbb{R}^n} \frac{\Big((\frac{\lambda}{|y|})^{n-\alpha} - (\frac{\lambda}{|z|})^{n-\alpha}\Big) u(\frac{\lambda^2 y}{|y|^2}) + (\frac{\lambda}{|z|})^{n-\alpha}\Big(u(\frac{\lambda^2 y}{|y|^2}) - u(\frac{\lambda^2 z}{|z|^2})\Big)}{|y-z|^{n+\alpha}} dz \\
&= u\Big(\frac{\lambda^2 y}{|y|^2}\Big)(-\Delta)^{\frac{\alpha}{2}} \left[\left(\frac{\lambda}{|y|}\right)^{n-\alpha}\right] + C_{\alpha,n} P.V. \int_{\mathbb{R}^n} \frac{u(\frac{\lambda^2 y}{|y|^2}) - u(z)}{\Big|y - \frac{\lambda^2 z}{|z|^2}\Big|^{n+\alpha}} \cdot \frac{\lambda^{n+\alpha}}{|z|^{n+\alpha}} dz \\
&= \frac{\lambda^{n+\alpha}}{|y|^{n+\alpha}} (-\Delta)^{\frac{\alpha}{2}} u\Big(\frac{\lambda^2 y}{|y|^2}\Big) \\
&= \frac{\lambda^{n+\alpha}}{|y|^{n+\alpha}} \int_{\mathbb{R}^n} \frac{f_1(v(z))}{\Big|\frac{\lambda^2 y}{|y|^2} - z\Big|^{\sigma_1}} dz \cdot f_2\Big(v\Big(\frac{\lambda^2 y}{|y|^2}\Big)\Big) \\
&= \frac{\lambda^{n+\alpha}}{|y|^{n+\alpha}} \int_{\mathbb{R}^n} \frac{\lambda^{2n}|z|^{-2n}}{\Big|\frac{\lambda^2 y}{|y|^2} - \frac{\lambda^2 z}{|z|^2}\Big|^{\sigma_1}} f_1\Big(v\Big(\frac{\lambda^2 z}{|z|^2}\Big)\Big) dz \cdot f_2\Big(v\Big(\frac{\lambda^2 y}{|y|^2}\Big)\Big) \\
&= \int_{\mathbb{R}^n} \frac{v_{0,\lambda}^{p_1}(z)}{|y-z|^{\sigma_1}} F_1(v(z^{0,\lambda})) dz \cdot v_{0,\lambda}^{p_2}(y) F_2(v(y^{0,\lambda})),
\end{aligned}$$

$$
\begin{aligned}
(-\Delta)^{\frac{\alpha}{2}} v_{0,\lambda}(y) &= C_{\alpha,n} P.V. \int_{\mathbb{R}^n} \frac{\Big((\frac{\lambda}{|y|})^{n-\alpha} - (\frac{\lambda}{|z|})^{n-\alpha} \Big) v(\frac{\lambda^2 y}{|y|^2}) + (\frac{\lambda}{|z|})^{n-\alpha} \Big(v(\frac{\lambda^2 y}{|y|^2}) - v(\frac{\lambda^2 z}{|z|^2}) \Big)}{|y-z|^{n+\alpha}} dz \\
&= v\Big(\frac{\lambda^2 y}{|y|^2}\Big)(-\Delta)^{\frac{\alpha}{2}} \left[\left(\frac{\lambda}{|y|} \right)^{n-\alpha} \right] + C_{\alpha,n} P.V. \int_{\mathbb{R}^n} \frac{v(\frac{\lambda^2 y}{|y|^2}) - v(z)}{\left| y - \frac{\lambda^2 z}{|z|^2} \right|^{n+\alpha}} \cdot \frac{\lambda^{n+\alpha}}{|z|^{n+\alpha}} dz \\
&= \frac{\lambda^{n+\alpha}}{|y|^{n+\alpha}} (-\Delta)^{\frac{\alpha}{2}} v\Big(\frac{\lambda^2 y}{|y|^2}\Big) \\
&= \frac{\lambda^{n+\alpha}}{|y|^{n+\alpha}} \int_{\mathbb{R}^n} \frac{g_1(u(z))}{\left| \frac{\lambda^2 y}{|y|^2} - z \right|^{\sigma_2}} dz \cdot g_2\Big(u\Big(\frac{\lambda^2 y}{|y|^2}\Big) \Big) \\
&= \frac{\lambda^{n+\alpha}}{|y|^{n+\alpha}} \int_{\mathbb{R}^n} \frac{\lambda^{2n} |z|^{-2n}}{\left| \frac{\lambda^2 y}{|y|^2} - \frac{\lambda^2 z}{|z|^2} \right|^{\sigma_2}} g_1\Big(u\Big(\frac{\lambda^2 z}{|z|^2}\Big) \Big) dz \cdot g_2\Big(u\Big(\frac{\lambda^2 y}{|y|^2}\Big) \Big) \\
&= \int_{\mathbb{R}^n} \frac{u_{0,\lambda}^{q_1}(z)}{|y-z|^{\sigma_2}} G_1(u(z^{0,\lambda})) dz \cdot u_{0,\lambda}^{q_2}(y) G_2(u(y^{0,\lambda})),
\end{aligned}
$$

this means, the conformal transforms $u_{x,\lambda}, v_{x,\lambda} \in \mathcal{L}_\alpha(\mathbb{R}^n) \cap C^{1,1}_{loc}(\mathbb{R}^n \setminus \{x\})$ if $0 < \alpha < 2$ ($u_{x,\lambda}, v_{x,\lambda} \in C^2(\mathbb{R}^n \setminus \{x\})$ if $\alpha = 2$) satisfies

$$
\begin{cases}
(-\Delta)^{\frac{\alpha}{2}} u_{x,\lambda}(y) = \displaystyle\int_{\mathbb{R}^n} \frac{v_{x,\lambda}^{p_1}(z)}{|y-z|^{\sigma_1}} F_1(v(z^{x,\lambda})) dz \cdot v_{x,\lambda}^{p_2}(y) F_2(v(y^{x,\lambda})), \\
(-\Delta)^{\frac{\alpha}{2}} v_{x,\lambda}(y) = \displaystyle\int_{\mathbb{R}^n} \frac{u_{x,\lambda}^{q_1}(z)}{|y-z|^{\sigma_2}} G_1(u(z^{x,\lambda})) dz \cdot u_{x,\lambda}^{q_2}(y) G_2(u(y^{x,\lambda})),
\end{cases} \tag{2.1}
$$

for every $y \in \mathbb{R}^n \setminus \{x\}$, where $p_1 = \frac{2n-\sigma_1}{n-\alpha}$, $p_2 = \frac{n+\alpha-\sigma_1}{n-\alpha}$, $q_1 = \frac{2n-\sigma_2}{n-\alpha}$ and $q_2 = \frac{n+\alpha-\sigma_2}{n-\alpha}$. For any $\lambda > 0$, we denote

$$
B_\lambda(x) := \{y \in \mathbb{R}^n \,|\, |y-x| < \lambda\},
$$

and define

$$
P(y) := \Big(\frac{1}{|\cdot|^{\sigma_1}} * f_1 \Big)(y), \qquad \widetilde{P}_{x,\lambda}(y) := \int_{B_\lambda(x)} \frac{v^{p_1-1}(z)}{|y-z|^{\sigma_1}} F_1(v(z)) dz,
$$

$$
Q(y) := \Big(\frac{1}{|\cdot|^{\sigma_2}} * g_1 \Big)(y), \qquad \widetilde{Q}_{x,\lambda}(y) := \int_{B_\lambda(x)} \frac{u^{q_1-1}(z)}{|y-z|^{\sigma_2}} G_1(u(z)) dz.
$$

Let $U_{x,\lambda}(y) = u_{x,\lambda}(y) - u(y)$, $V_{x,\lambda}(y) = v_{x,\lambda}(y) - v(y)$ for any $y \in B_\lambda(x) \setminus \{x\}$. By the definitions of $u_{x,\lambda}$, $v_{x,\lambda}$ and $U_{x,\lambda}$, $V_{x,\lambda}$, we have

$$
\begin{aligned}
U_{x,\lambda}(y) &= u_{x,\lambda}(y) - u(y) = \left(\frac{\lambda}{|y-x|} \right)^{n-\alpha} u(y^{x,\lambda}) - u(y) \\
&= \left(\frac{\lambda}{|y-x|} \right)^{n-\alpha} \left(u(y^{x,\lambda}) - \left(\frac{\lambda}{|y^{x,\lambda}-x|} \right)^{n-\alpha} u((y^{x,\lambda})^{x,\lambda}) \right) \\
&= -\left(\frac{\lambda}{|y-x|} \right)^{n-\alpha} U_{x,\lambda}(y^{x,\lambda}) = -(U_{x,\lambda})_{x,\lambda}(y),
\end{aligned} \tag{2.2}
$$

$$
\begin{aligned}
V_{x,\lambda}(y) &= v_{x,\lambda}(y) - v(y) = \left(\frac{\lambda}{|y-x|}\right)^{n-\alpha} v(y^{x,\lambda}) - v(y) \\
&= \left(\frac{\lambda}{|y-x|}\right)^{n-\alpha} \left(v(y^{x,\lambda}) - \left(\frac{\lambda}{|y^{x,\lambda}-x|}\right)^{n-\alpha} v((y^{x,\lambda})^{x,\lambda})\right) \\
&= -\left(\frac{\lambda}{|y-x|}\right)^{n-\alpha} V_{x,\lambda}(y^{x,\lambda}) = -(V_{x,\lambda})_{x,\lambda}(y),
\end{aligned}
\tag{2.3}
$$

for every $y \in B_\lambda(x) \setminus \{x\}$.

We will first show that there exists a $\epsilon_0 > 0$ (depending on x) sufficiently small, such that, for any $0 < \lambda \leq \epsilon_0$, it holds that $U_{x,\lambda}(y) \geq 0$, $V_{x,\lambda}(y) \geq 0$, for every $y \in B_\lambda(x) \setminus \{x\}$.

We first need to show that the nonnegative solution (u, v) to PDEs system (1.1) also satisfies the equivalent integral system (1.4).

Lemma 2.1. *Suppose* (u, v) *is a nonnegative solution to* (1.1)*, then* (u, v) *also satisfies the equivalent integral system* (1.4)*, and vice versa.*

The proof of Lemma 2.1 is similar to [25, 26], so we omit the details here.

Based on Lemma 2.1, we can prove that $U_{x,\lambda}$, $V_{x,\lambda}$ has a strictly positive lower bound in a small neighborhood of x.

Lemma 2.2. *For each fixed* $x \in \mathbb{R}^n$*, there exists a* $\eta_0 > 0$ *(depending on x) sufficiently small such that, if* $0 < \lambda \leq \eta_0$*, then*

$$
U_{x,\lambda}(y) \geq 1, V_{x,\lambda}(y) \geq 1, \quad y \in \overline{B_{\lambda^2}(x)} \setminus \{x\}.
$$

Proof. According to the idea of [20], we will prove Lemma 2.2 as follows. Define

$$
f(v(y)) := f_2(v(y)) \int_{\mathbb{R}^n} \frac{f_1(v(\xi))}{|y-\xi|^{\sigma_1}} d\xi,
$$

$$
g(u(y)) := g_2(u(y)) \int_{\mathbb{R}^n} \frac{g_1(u(\zeta))}{|y-\zeta|^{\sigma_2}} d\zeta.
$$

For any $|y| \geq 1$, since (u, v) satisfy the integral system (1.4), f_i, g_i $(i = 1, 2)$ are positive on $[0, +\infty)$, so $f(v(z))$ and $g(u(z))$ have lower bounds on compact sets, we can derive that

$$
\begin{aligned}
u(y) &= R_{\alpha,n} \int_{\mathbb{R}^n} \frac{f(v(z))}{|y-z|^{n-\alpha}} dz \\
&\geq R_{\alpha,n} \int_{B_{\frac{1}{2}}(0)} \frac{f(v(z))}{|y-z|^{n-\alpha}} dz \\
&\geq \frac{b_1}{|y|^{n-\alpha}} \int_{B_{\frac{1}{2}}(0)} f(v(z)) dz \\
&\geq \frac{\widetilde{b}_1}{|y|^{n-\alpha}},
\end{aligned}
\tag{2.4}
$$

$$
\begin{aligned}
v(y) &= R_{\alpha,n}\int_{\mathbb{R}^n}\frac{g(u(z))}{|y-z|^{n-\alpha}}dz\\
&\geq R_{\alpha,n}\int_{B_{\frac{1}{2}}(0)}\frac{g(u(z))}{|y-z|^{n-\alpha}}dz\\
&\geq \frac{b_2}{|y|^{n-\alpha}}\int_{B_{\frac{1}{2}}(0)}g(u(z))dz\\
&\geq \frac{\widetilde{b_2}}{|y|^{n-\alpha}},
\end{aligned}
\tag{2.5}
$$

where $\widetilde{b_1} := b_1\cdot\int_{B_{\frac{1}{2}}(0)}f(v(z))dz$ and $\widetilde{b_2} := b_2\cdot\int_{B_{\frac{1}{2}}(0)}g(u(z))dz$.

It follows immediately that

$$
u_{x,\lambda}(y) = \left(\frac{\lambda}{|y-x|}\right)^{n-\alpha}u(y^{x,\lambda}) \geq \left(\frac{\lambda}{|y-x|}\right)^{n-\alpha}\frac{\widetilde{b_1}}{|y^{x,\lambda}|^{n-\alpha}} = \frac{\widetilde{b_1}}{\lambda^{n-\alpha}},
$$

$$
v_{x,\lambda}(y) = \left(\frac{\lambda}{|y-x|}\right)^{n-\alpha}v(y^{x,\lambda}) \geq \left(\frac{\lambda}{|y-x|}\right)^{n-\alpha}\frac{\widetilde{b_2}}{|y^{x,\lambda}|^{n-\alpha}} = \frac{\widetilde{b_2}}{\lambda^{n-\alpha}},
$$

for all $y\in\overline{B_{\lambda^2}(x)}\setminus\{x\}$. Therefore, we have if $0<\lambda\leq\eta_0$ for some $\eta_0(x)>0$ small enough, then

$$
U_{x,\lambda}(y) = u_{x,\lambda}(y)-u(y) \geq \frac{\widetilde{b_1}}{\lambda^{n-\alpha}} - \max_{|y-x|\leq\lambda^2}u(y) \geq 1,
$$

$$
V_{x,\lambda}(y) = v_{x,\lambda}(y)-v(y) \geq \frac{\widetilde{b_2}}{\lambda^{n-\alpha}} - \max_{|y-x|\leq\lambda^2}v(y) \geq 1,
$$

for any $y\in\overline{B_{\lambda^2}(x)}\setminus\{x\}$, this finishes the proof of Lemma 2.2. □

For every fixed $x\in\mathbb{R}^n$, define

$$
B_\lambda^- = \{y\in B_\lambda(x)\setminus\{x\}\,|\,U_{x,\lambda}(y)<0, V_{x,\lambda}(y)<0\}.
$$

Now we need the main theorem, which is a variant (for nonlocal nonlinearity) of the *Narrow region principle* (Theorem 2.2 in [20]).

Theorem 2.3. *(Narrow region principle) Assume $x\in\mathbb{R}^n$ is arbitrarily fixed. Let Ω be a narrow region in $B_\lambda(x)\setminus\{x\}$ with small thickness $0<l<\lambda$ such that $\Omega\subseteq A_{\lambda,l}(x) := \{y\in\mathbb{R}^n|\,\lambda-l<|y-x|<\lambda\}$. Suppose $U_{x,\lambda}$, $V_{x,\lambda}\in\mathcal{L}_\alpha(\mathbb{R}^n)\cap C^{1,1}_{loc}(\Omega)$ if $0<\alpha<2$ ($U_{x,\lambda}$, $V_{x,\lambda}\in C^2(\Omega)$ if $\alpha=2$) and satisfies*

$$
\begin{cases}
(-\Delta)^{\frac{\alpha}{2}}U_{x,\lambda}(y)-\mathcal{L}_1(y)V_{x,\lambda}(y)-p_1\left(\int_{B_\lambda^-}\frac{v^{p_1-1}(z)V_{x,\lambda}(z)}{|y-z|^{\sigma_1}}F_1(v(z))dz\right)v^{p_2}(y)F_2(v(y))\geq 0 \quad \textit{in } \Omega\cap B_\lambda^-,\\
(-\Delta)^{\frac{\alpha}{2}}V_{x,\lambda}(y)-\mathcal{L}_2(y)U_{x,\lambda}(y)-q_1\left(\int_{B_\lambda^-}\frac{u^{q_1-1}(z)U_{x,\lambda}(z)}{|y-z|^{\sigma_2}}G_1(u(z))dz\right)u^{q_2}(y)G_2(u(y))\geq 0 \quad \textit{in } \Omega\cap B_\lambda^-,\\
\textit{negative minimum of } U_{x,\lambda},V_{x,\lambda} \textit{ is attained in the interior of } B_\lambda(x)\setminus\{x\} \textit{ if } B_\lambda^-\neq\emptyset,\\
\textit{negative minimum of } U_{x,\lambda},V_{x,\lambda} \textit{ cannot be attained in } (B_\lambda(x)\setminus\{x\})\setminus\Omega,
\end{cases}
\tag{2.6}
$$

where $\mathcal{L}_1(y) := p_2 v_{x,\lambda}^{p_2-1}(y)P(y)F_2(v(y))$, $\mathcal{L}_2(y) := q_2 u_{x,\lambda}^{q_2-1}(y)Q(y)G_2(u(y))$. *Then, we have*
(i) there exists a sufficiently small constant $\gamma_0(x) > 0$*, such that, for all* $0 < \lambda \leq \gamma_0$,

$$U_{x,\lambda}(y) \geq 0, V_{x,\lambda}(y) \geq 0, \quad \forall\, y \in \Omega, \tag{2.7}$$

(ii) there exists a sufficiently small $l_0(x,\lambda) > 0$ *depending on* λ *continuously, such that, for all* $0 < l \leq l_0$,

$$U_{x,\lambda}(y) \geq 0, V_{x,\lambda}(y) \geq 0, \quad \forall\, y \in \Omega. \tag{2.8}$$

Proof. Without loss of generality, we may assume $x = 0$ here for convenience. Suppose (2.7) and (2.8) do not hold, we will obtain a contradiction for any $0 < \lambda \leq \gamma_0$ with constant γ_0 small enough and any $0 < l \leq l_0(\lambda)$ with $l_0(\lambda)$ sufficiently small respectively. We divide the proof into two parts.

Part 1: For $0 < \alpha < 2$ and $\alpha = 2$, if (2.7) fails, we show that there exists some $\hat{y} \in \Omega \cap B_\lambda^-$ such that

$$(-\Delta)^{\frac{\alpha}{2}} U_{0,\lambda}(\hat{y}) \leq \frac{C}{l^\alpha} U_{0,\lambda}(\hat{y}) < 0. \tag{2.9}$$

Recall (2.6) and our hypothesis, there exists $\tilde{y} \in (\Omega \cap B_\lambda^-) \subseteq A_{\lambda,l}(0) := \{y \in \mathbb{R}^n | \, \lambda - l < |y| < \lambda\}$ such that

$$U_{0,\lambda}(\tilde{y}) = \min_{B_\lambda(0)\setminus\{0\}} U_{0,\lambda}(y) < 0. \tag{2.10}$$

We first consider the cases $0 < \alpha < 2$. Let $\tilde{U}_{0,\lambda}(y) = U_{0,\lambda}(y) - U_{0,\lambda}(\tilde{y})$, then $\tilde{U}_{0,\lambda}(\tilde{y}) = 0$ and

$$(-\Delta)^{\alpha/2}\tilde{U}_{0,\lambda}(y) = (-\Delta)^{\alpha/2} U_{0,\lambda}(y).$$

Using the anti-symmetry property $U_{x,\lambda}(y) = -(U_{x,\lambda})_{x,\lambda}(y)$, it holds

$$\begin{aligned}
\left(\frac{\lambda}{|y|}\right)^{n-\alpha} \tilde{U}_{0,\lambda}(y^{0,\lambda}) &= \left(\frac{\lambda}{|y|}\right)^{n-\alpha} U_{0,\lambda}(y^{0,\lambda}) - \left(\frac{\lambda}{|y|}\right)^{n-\alpha} U_{0,\lambda}(\tilde{y}) \\
&= -U_{0,\lambda}(y) + U_{0,\lambda}(\tilde{y}) - \left(1 + \left(\frac{\lambda}{|y|}\right)^{n-\alpha}\right) U_{0,\lambda}(\tilde{y}) \\
&= -\tilde{U}_{0,\lambda}(y) - \left(1 + \left(\frac{\lambda}{|y|}\right)^{n-\alpha}\right) U_{0,\lambda}(\tilde{y}).
\end{aligned}$$

Consequently, it follows that

$$\begin{aligned}
(-\Delta)^{\alpha/2}\tilde{U}_{0,\lambda}(\tilde{y}) &= C_{\alpha,n}\, P.V. \int_{\mathbb{R}^n} \frac{\tilde{U}_{0,\lambda}(\tilde{y}) - \tilde{U}_{0,\lambda}(z)}{|\tilde{y} - z|^{n+\alpha}} dz \\
&= C_{\alpha,n}\, P.V. \left(\int_{B_\lambda(0)} \frac{-\tilde{U}_{0,\lambda}(z)}{|\tilde{y} - z|^{n+\alpha}} dz + \int_{\mathbb{R}^n \setminus B_\lambda(0)} \frac{-\tilde{U}_{0,\lambda}(z)}{|\tilde{y} - z|^{n+\alpha}} dz \right) \\
&= C_{\alpha,n}\, P.V. \Bigg(\int_{B_\lambda(0)} \frac{-\tilde{U}_{0,\lambda}(z)}{|\tilde{y} - z|^{n+\alpha}} dz + \int_{\mathbb{R}^n \setminus B_\lambda(0)} \frac{\left(\frac{\lambda}{|z|}\right)^{n-\alpha} \tilde{U}_{0,\lambda}(z^{0,\lambda})}{|\tilde{y} - z|^{n+\alpha}} dz \\
&\quad + \int_{\mathbb{R}^n \setminus B_\lambda(0)} \frac{\left(1 + (\frac{\lambda}{|z|})^{n-\alpha}\right) U_{0,\lambda}(\tilde{z})}{|\tilde{y} - z|^{n+\alpha}} dz \Bigg) \\
&= C_{\alpha,n}\, P.V. \Bigg(\int_{B_\lambda(0)} \frac{-\tilde{U}_{0,\lambda}(z)}{|\tilde{y} - z|^{n+\alpha}} dz + \int_{B_\lambda(0)} \frac{\tilde{U}_{0,\lambda}(z)}{\left|\frac{|z|\tilde{y}}{\lambda} - \frac{\lambda z}{|z|}\right|^{n+\alpha}} dz \\
&\quad + \int_{\mathbb{R}^n \setminus B_\lambda(0)} \frac{\left(1 + (\frac{\lambda}{|z|})^{n-\alpha}\right) U_{0,\lambda}(\tilde{z})}{|\tilde{y} - z|^{n+\alpha}} dz \Bigg).
\end{aligned}$$

Notice that, for any $z \in B_\lambda(0) \setminus \{0\}$,

$$\left|\frac{|z|\tilde{y}}{\lambda} - \frac{\lambda z}{|z|}\right|^2 - |\tilde{y} - z|^2 = \frac{(|\tilde{y}|^2 - \lambda^2)(|z|^2 - \lambda^2)}{\lambda^2} > 0.$$

Then combining this with $U_{0,\lambda}(\tilde{y}) < 0$, we get

$$\begin{aligned}(-\Delta)^{\alpha/2} U_{0,\lambda}(\tilde{y}) &\leq C_{\alpha,n} U_{0,\lambda}(\tilde{y}) \int_{\mathbb{R}^n \setminus B_\lambda(0)} \frac{1}{|\tilde{y} - z|^{n+\alpha}} dz \\ &\leq C_{\alpha,n} U_{0,\lambda}(\tilde{y}) \int_{(\mathbb{R}^n \setminus B_\lambda(0)) \cap (B_{4l}(\tilde{y}) \setminus B_l(\tilde{y}))} \frac{1}{|\tilde{y} - z|^{n+\alpha}} dz \\ &\leq \frac{C}{l^\alpha} U_{0,\lambda}(\tilde{y}) < 0.\end{aligned} \tag{2.11}$$

Now we consider $\alpha = 2$, we can also derive the same estimate as (2.11) at some point $y_0 \in \Omega \cap B_\lambda^-$. The last, we define

$$\phi(y) := \cos \frac{|y| - \lambda + l}{l}, \tag{2.12}$$

then it is obvious that $\phi(y) \in [\cos 1, 1]$ for any $y \in \overline{A_{\lambda,l}(0)} = \{y \in \mathbb{R}^n \,|\, \lambda - l \leq |y| \leq \lambda\}$ and $-\frac{\Delta\phi(y)}{\phi(y)} \geq \frac{1}{l^2}$. Define

$$\overline{U}_{0,\lambda}(y) := \frac{U_{0,\lambda}(y)}{\phi(y)}, \tag{2.13}$$

for $y \in \overline{A_{\lambda,l}(0)}$. Then there exists a $y_0 \in \Omega \cap B_\lambda^-$ such that

$$\overline{U}_{0,\lambda}(y_0) = \min_{\overline{A_{\lambda,l}(0)}} \overline{U}_{0,\lambda}(y) < 0. \tag{2.14}$$

Since

$$-\Delta U_{0,\lambda}(y_0) = -\Delta \overline{U}_{0,\lambda}(y_0) \cdot \phi(y_0) - 2\nabla \overline{U}_{0,\lambda}(y_0) \cdot \nabla\phi(y_0) - \overline{U}_{0,\lambda}(y_0) \cdot \Delta\phi(y_0), \tag{2.15}$$

it can be deduced immediately that

$$-\Delta U_{0,\lambda}(y_0) \leq \frac{1}{l^2} U_{0,\lambda}(y_0). \tag{2.16}$$

In summary, we have proved that for both $0 < \alpha < 2$ and $\alpha = 2$, there exists some $\hat{y} \in \Omega \cap B_\lambda^-$ such that

$$(-\Delta)^{\frac{\alpha}{2}} U_{0,\lambda}(\hat{y}) \leq \frac{C}{l^\alpha} U_{0,\lambda}(\hat{y}) < 0. \tag{2.17}$$

Similar, we have

$$V_{0,\lambda}(\hat{y}) < 0,$$

then we know that exists a $\bar{y}$ such that

$$V_{0,\lambda}(\bar{y}) = \min_{B_\lambda(0) \setminus (0)} V_{0,\lambda}(y) < 0,$$

and we can derive that

$$(-\Delta)^{\frac{\alpha}{2}} V_{0,\lambda}(\bar{y}) \leq \frac{C}{l^\alpha} V_{0,\lambda}(\bar{y}) < 0. \tag{2.18}$$

Part 2: We obtain a contradiction for any $0 < \lambda \le \gamma_0$ with constant γ_0 small enough and any $0 < l \le l_0(\lambda)$ with $l_0(\lambda)$ sufficiently small respectively.

By (2.6), we have at the point $\hat{y}$,

$$\begin{aligned}
0 &\le (-\Delta)^{\frac{\alpha}{2}} U_{0,\lambda}(\hat{y}) - \mathcal{L}_1(\hat{y}) V_{0,\lambda}(\hat{y}) \qquad (2.19)\\
&\quad - p_1\Big(\int_{B_\lambda^-} \frac{v^{p_1-1}(z) V_{0,\lambda}(z)}{|\hat{y}-z|^{\sigma_1}} F_1(v(z))dz\Big) v^{p_2}(\hat{y}) F_2(v(\hat{y}))\\
&\le (-\Delta)^{\frac{\alpha}{2}} U_{0,\lambda}(\hat{y}) - \mathcal{L}_1(\hat{y}) V_{0,\lambda}(\bar{y}) \qquad (2.20)\\
&\quad - \Big(p_1 \int_{B_\lambda^-} \frac{v^{p_1-1}(z)}{|\hat{y}-z|^{\sigma_1}} F_1(v(z))dz \cdot v^{p_2}(\hat{y}) F_2(v(\hat{y}))\Big) V_{0,\lambda}(\bar{y})\\
&\le (-\Delta)^{\frac{\alpha}{2}} U_{0,\lambda}(\hat{y}) - c'_{0,\lambda}(\hat{y}) V_{0,\lambda}(\bar{y}),
\end{aligned}$$

where

$$\begin{aligned}
c'_{x,\lambda}(y) &:= \mathcal{L}_1(y) + p_1 \widetilde{P}_{x,\lambda}(y) v^{p_2}(y) F_2(v(y))\\
&= p_2 P(y) v_{x,\lambda}^{p_2-1}(y) F_2(v(y)) + p_1 \widetilde{P}_{x,\lambda}(y) v^{p_2}(y) F_2(v(y)) > 0.
\end{aligned}$$

Since $\lambda - l < |y| < \lambda$, we have

$$\begin{aligned}
P(y) &\le \left\{\int_{|y-z|<\frac{|z|}{2}} + \int_{|y-z|\ge\frac{|z|}{2}}\right\} \frac{f_1(v(z))}{|y-z|^{\sigma_1}} dz \qquad (2.21)\\
&\le \Big[\max_{|y|\le 2\lambda} f_1(v(y))\Big] \int_{|y-z|<\lambda} \frac{1}{|y-z|^{\sigma_1}} dz + 2^{\sigma_1} \int_{\mathbb{R}^n} \frac{f_1(v(z))}{|z|^{\sigma_1}} dz\\
&\le C\lambda^{n-\sigma_1} \Big[\max_{|y|\le 2\lambda} f_1(v(y))\Big] + 2^{\sigma_1} \int_{\mathbb{R}^n} \frac{f_1(v(x))}{|x|^{\sigma_1}} dx =: C'_{1,\lambda},
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{P}_{0,\lambda}(y) &\le \int_{|y-z|<2\lambda} \frac{1}{|y-z|^{\sigma_1}} v^{p_1-1}(z) F_1(v(z)) dz \qquad (2.22)\\
&\le C\lambda^{n-\sigma_1} \Big[\max_{|y|\le 4\lambda} \frac{f_1(v(y))}{v(y)}\Big] =: C''_{1,\lambda}.
\end{aligned}$$

It is obvious that $C'_{1,\lambda}$ and $C''_{1,\lambda}$ depend on λ continuously and monotone increasing with respect to $\lambda > 0$.

Therefore, we can derive from (2.21) and (2.22) that, for any $\lambda - l \le |y| \le \lambda$,

$$\begin{aligned}
0 &< c'_{0,\lambda}(y) = p_2 P(y) v_{0,\lambda}^{p_2-1}(y) F_2(v(y)) + p_1 \widetilde{P}_{0,\lambda}(y) v^{p_2}(y) F_2(v(y)) \qquad (2.23)\\
&\le p_2 C'_{1,\lambda} \Big[\min_{|y|\le\lambda} v_{0,\lambda}(y)\Big]^{p_2-1} \Big[\max_{|y|\le\lambda} F_2(v(y))\Big] + p_1 C''_{1,\lambda} \Big[\max_{|y|\le\lambda} f_2(v(y))\Big] =: C_{1,\lambda},
\end{aligned}$$

where $C_{1,\lambda}$ depends continuously on λ and monotone increasing with respect to $\lambda > 0$.

From (2.17) and (2.19) we have

$$U_{0,\lambda}(\hat{y}) \ge c'_{0,\lambda}(\hat{y}) l^\alpha V_{0,\lambda}(\bar{y}). \qquad (2.24)$$

According to (2.6), we also have at the point $\bar{y}$

$$\begin{aligned} 0 &\leq (-\Delta)^{\frac{\alpha}{2}} V_{0,\lambda}(\bar{y}) - \mathcal{L}_2(\bar{y}) U_{0,\lambda}(\bar{y}) \qquad (2.25)\\ &\quad - q_1\Big(\int_{B_\lambda^-} \frac{u^{q_1-1}(z)U_{0,\lambda}(z)}{|\bar{y}-z|^{\sigma_2}} G_1(u(z))dz\Big) u^{q_2}(\bar{y}) G_2(u(\bar{y}))\\ &\leq (-\Delta)^{\frac{\alpha}{2}} V_{0,\lambda}(\bar{y}) - \mathcal{L}_2(\bar{y}) U_{0,\lambda}(\hat{y}) \qquad (2.26)\\ &\quad - \Big(q_1 \int_{B_\lambda^-} \frac{u^{q_1-1}(z)}{|\bar{y}-z|^{\sigma_2}} G_1(u(z))dz \cdot u^{q_2}(\bar{y}) G_2(u(\bar{y}))\Big) U_{0,\lambda}(\hat{y})\\ &\leq (-\Delta)^{\frac{\alpha}{2}} V_{0,\lambda}(\bar{y}) - c''_{0,\lambda}(\bar{y}) U_{0,\lambda}(\hat{y}), \end{aligned}$$

where

$$\begin{aligned} c''_{x,\lambda}(y) &:= \mathcal{L}_2(y) + q_1 \widetilde{Q}_{x,\lambda}(y) u^{q_2}(y) G_2(u(y))\\ &= q_2 Q(y) u_{x,\lambda}^{q_2-1}(y) G_2(u(y)) + q_1 \widetilde{Q}_{x,\lambda}(y) u^{q_2}(y) G_2(u(y)) > 0. \end{aligned}$$

Since $\lambda - l < |y| < \lambda$, we have

$$\begin{aligned} Q(y) &\leq \left\{\int_{|y-z|<\frac{|z|}{2}} + \int_{|y-z|\geq\frac{|z|}{2}}\right\} \frac{g_1(u(z))}{|y-z|^{\sigma_2}} dz \qquad (2.27)\\ &\leq \Big[\max_{|y|\leq 2\lambda} g_1(u(y))\Big] \int_{|y-z|<\lambda} \frac{1}{|y-z|^{\sigma_2}} dz + 2^{\sigma_2} \int_{\mathbb{R}^n} \frac{g_1(u(z))}{|z|^{\sigma_2}} dz\\ &\leq C\lambda^{n-\sigma_2}\Big[\max_{|y|\leq 2\lambda} g_1(u(y))\Big] + 2^{\sigma_2} \int_{\mathbb{R}^n} \frac{g_1(u(x))}{|x|^{\sigma_2}} dx =: C'_{2,\lambda}, \end{aligned}$$

and

$$\begin{aligned} \widetilde{Q}_{0,\lambda}(y) &\leq \int_{|y-z|<2\lambda} \frac{1}{|y-z|^{\sigma_2}} u^{q_1-1}(z) G_1(u(z)) dz \qquad (2.28)\\ &\leq C\lambda^{n-\sigma_2}\Big[\max_{|y|\leq 4\lambda} \frac{g_1(u(y))}{u(y)}\Big] =: C''_{2,\lambda}. \end{aligned}$$

It is obvious that $C'_{2,\lambda}$ and $C''_{2,\lambda}$ depend on λ continuously and monotone increasing with respect to $\lambda > 0$.

As a result, we obtain from (2.27) and (2.28) that, for any $\lambda - l \leq |y| \leq \lambda$

$$\begin{aligned} 0 &< c''_{0,\lambda}(y) = q_2 Q(y) u_{0,\lambda}^{q_2-1}(y) G_2(u(y)) + q_1 \widetilde{Q}_{0,\lambda}(y) u^{q_2}(y) G_2(u(y)) \qquad (2.29)\\ &\leq q_2 C'_{2,\lambda}\Big[\min_{|y|\leq\lambda} u_{0,\lambda}(y)\Big]\Big[\max_{|y|\leq\lambda} G_2(u(y))\Big] + q_1 C''_{2,\lambda}\Big[\max_{|y|\leq\lambda} g_2(u(y))\Big] =: C_{2,\lambda}, \end{aligned}$$

where $C_{2,\lambda}$ depends continuously on λ and monotone increasing with respect to $\lambda > 0$.

From (2.18) and (2.25), we have

$$V_{0,\lambda}(\bar{y}) \geq c''_{0,\lambda}(\bar{y}) l^{\alpha} U_{0,\lambda}(\hat{y}). \qquad (2.30)$$

As a consequence, it follows from (2.17), (2.19), (2.23), (2.29) and (2.30) that

$$0 \leq (-\Delta)^{\frac{\alpha}{2}} U_{0,\lambda}(\hat{y}) - c'_{0,\lambda}(\hat{y}) V_{0,\lambda}(\bar{y})$$

$$
\begin{aligned}
&\le \frac{C}{l^\alpha}U_{0,\lambda}(\hat{y}) - c'_{0,\lambda}(\hat{y})c''_{0,\lambda}(\bar{y})l^\alpha U_{0,\lambda}(\hat{y})\\
&\le \frac{C}{l^\alpha}U_{0,\lambda}(\hat{y}) - C_{1,\lambda}C_{2,\lambda}l^\alpha U_{0,\lambda}(\hat{y})\\
&= (\frac{C}{l^\alpha} - C_\lambda l^\alpha)U_{0,\lambda}(\hat{y}),
\end{aligned}
$$

that is,

$$
\frac{C}{\lambda^\alpha} \le \frac{C}{l^\alpha} \le C_\lambda l^\alpha. \tag{2.31}
$$

We can draw a contradiction from (2.31) directly if $0 < \lambda \le \gamma_0$ for some constants γ_0 small enough, or if $0 < l \le l_0$ for some sufficiently small l_0 depending on λ continuously. This shows that (2.7) and (2.8) must hold. Moreover, by (2.6), we can actually deduce from $U_{x,\lambda}(y) \ge 0$, $V_{x,\lambda} \ge 0$ in Ω that

$$
U_{x,\lambda}(y) \ge 0, V_{x,\lambda}(y) \ge 0, \quad \forall\ y \in B_\lambda(x) \setminus \{x\}. \tag{2.32}
$$

This completes the proof of Theorem 2.3. □

The following lemma provides a start point for us to move the spheres.

Lemma 2.4. *For every $x \in \mathbb{R}^n$, there exists $\epsilon_0(x) > 0$ such that, $u_{x,\lambda}(y) \ge u(y)$ and $v_{x,\lambda}(y) \ge v(y)$ for all $\lambda \in (0, \epsilon_0(x)]$ and $y \in B_\lambda(x) \setminus \{x\}$.*

Proof. For every $x \in \mathbb{R}^n$, recalling that

$$
B_\lambda^- = \{y \in B_\lambda(x) \setminus \{x\} \,|\, U_{x,\lambda}(y) < 0, V_{x,\lambda}(y) < 0\}.
$$

Define $\epsilon_0(x) := \min\{\eta_0(x), \gamma_0(x)\}$, where $\eta_0(x)$ and $\gamma_0(x)$ are defined the same as in Lemma 2.2 and Theorem 2.3. We will show via contradiction arguments that, for any $0 < \lambda \le \epsilon_0$,

$$
B_\lambda^- = \emptyset. \tag{2.33}
$$

Suppose (2.33) does not hold, that is, $B_\lambda^- \neq \emptyset$ and hence $U_{x,\lambda}$, $V_{x,\lambda}$ is negative somewhere in $B_\lambda(x) \setminus \{x\}$. For arbitrary $y \in B_\lambda^-$, we deduce from (1.1) and (2.1) that

$$
\begin{aligned}
(-\Delta)^{\frac{\alpha}{2}}U_{x,\lambda}(y) = &\int_{\mathbb{R}^n} \frac{v_{x,\lambda}^{p_1}(z)}{|y-z|^{\sigma_1}}F_1\left(\left(\frac{\lambda}{|z-x|}\right)^{\alpha-n} v_{x,\lambda}(z)\right)dz \cdot v_{x,\lambda}^{p_2}(y)F_2\left(\left(\frac{\lambda}{|y-x|}\right)^{\alpha-n} v_{x,\lambda}(y)\right)\\
&- \int_{\mathbb{R}^n} \frac{v^{p_1}(z)}{|y-z|^{\sigma_1}}F_1(v(z))dz \cdot v^{p_2}(y)F_2(v(y))
\end{aligned} \tag{2.34}
$$

$$
\begin{aligned}
&\geq \int_{\mathbb{R}^n} \frac{v_{x,\lambda}^{p_1}(z)}{|y-z|^{\sigma_1}} F_1(v_{x,\lambda}(z))dz \cdot v_{x,\lambda}^{p_2}(y) F_2(v(y)) - \int_{\mathbb{R}^n} \frac{v^{p_1}(z)}{|y-z|^{\sigma_1}} F_1(v(z))dz \cdot v^{p_2}(y) F_2(v(y)) \\
&\geq p_2 \int_{\mathbb{R}^n} \frac{v^{p_1}(z)}{|y-z|^{\sigma_1}} F_1(v(z))dz \cdot v_{x,\lambda}^{p_2-1}(y) F_2(v(y)) V_{x,\lambda}(y) \\
&\quad + \int_{\mathbb{R}^n} \frac{v_{x,\lambda}^{p_1}(z) F_1(v_{x,\lambda}(z)) - v^{p_1}(z) F_1(v(z))}{|y-z|^{\sigma_1}} dz \cdot v_{x,\lambda}^{p_2}(y) F_2(v(y)) \\
&\geq \mathcal{L}_1(y) V_{x,\lambda}(y) + v_{x,\lambda}^{p_2}(y) F_2(v(y)) \int_{B_\lambda(x)} \left(\frac{1}{\left| \frac{(y-x)|z-x|}{\lambda} - \frac{\lambda(z-x)}{|z-x|} \right|^{\sigma_1}} - \frac{1}{|y-z|^{\sigma_1}} \right) \\
&\quad \left(v^{p_1}(z) F_1(v(z)) - v_{x,\lambda}^{p_1}(z) F_1(v_{x,\lambda}(z)) \right) dz \\
&\geq \mathcal{L}_1(y) V_{x,\lambda}(y) + v^{p_2}(y) F_2(v(y)) \int_{B_\lambda^-(x)} \frac{1}{|y-z|^{\sigma_1}} (v_{x,\lambda}^{p_1}(z) F_1(v_{x,\lambda}(z)) - v^{p_1}(z) F_1(v(z))) dz \\
&\geq \mathcal{L}_1(y) V_{x,\lambda}(y) + v^{p_2}(y) F_2(v(y)) \int_{B_\lambda^-(x)} \frac{1}{|y-z|^{\sigma_1}} (v_{x,\lambda}^{p_1}(z) - v^{p_1}(z)) F_1(v(z)) dz \\
&\geq \mathcal{L}_1(y) V_{x,\lambda}(y) + p_1 \left(\int_{B_\lambda^-} \frac{v^{p_1-1}(z) V_{x,\lambda}(z)}{|y-z|^{\sigma_1}} F_1(v(z)) dz \right) v^{p_2}(y) F_2(v(y)),
\end{aligned} \tag{2.35}
$$

where $\mathcal{L}_1 = p_2 v_{x,\lambda}^{p_2-1}(y) P(y) F_2(v(y))$.

Similarly, one can calculate

$$
\begin{aligned}
&(-\Delta)^{\frac{\alpha}{2}} V_{x,\lambda}(y) \\
&\geq \mathcal{L}_2(y) U_{x,\lambda}(y) + q_1 \left(\int_{B_\lambda^-} \frac{u^{q_1-1}(z) U_{x,\lambda}(z)}{|y-z|^{\sigma_2}} G_1(u(z)) dz \right) u^{q_2}(y) G_2(u(y)),
\end{aligned} \tag{2.36}
$$

where $\mathcal{L}_2 = q_2 u_{x,\lambda}^{q_2-1}(y) Q(y) G_2(u(y))$.

Then (2.35) and (2.36) imply that, for all $y \in B_\lambda^-$,

$$
(-\Delta)^{\frac{\alpha}{2}} U_{x,\lambda}(y) - \mathcal{L}_1(y) V_{x,\lambda}(y) - p_1 \left(\int_{B_\lambda^-} \frac{v^{p_1-1}(z) V_{x,\lambda}(z)}{|y-z|^{\sigma_1}} F_1(v(z)) dz \right) v^{p_2}(y) F_2(v(y)) \geq 0, \tag{2.37}
$$

$$
(-\Delta)^{\frac{\alpha}{2}} V_{x,\lambda}(y) - \mathcal{L}_2(y) U_{x,\lambda}(y) - q_1 \left(\int_{B_\lambda^-} \frac{u^{q_1-1}(z) U_{x,\lambda}(z)}{|y-z|^{\sigma_2}} G_1(u(z)) dz \right) u^{q_2}(y) G_2(u(y)) \geq 0. \tag{2.38}
$$

Due to $\epsilon_0(x) := \min\{\eta_0(x), \gamma_0(x)\}$, by Lemma 2.2, we have, for any $0 < \lambda \leq \epsilon_0$,

$$
U_{x,\lambda}(y) \geq 1, V_{x,\lambda}(y) \geq 1, \qquad \forall\ y \in \overline{B_{\lambda^2}(x)} \setminus \{x\}. \tag{2.39}
$$

Thus, by taking $l = \lambda - \lambda^2$ and $\Omega = A_{\lambda,l}(x)$, then it follows from (2.37)–(2.39) that all the conditions in (2.6) in Theorem 2.3 are satisfied, we can derive from (i) in Theorem 2.3 that $U_{x,\lambda} \geq 0$, $V_{x,\lambda} \geq 0$ in $\Omega = A_{\lambda,l}(x)$ for any $0 < \lambda \leq \epsilon_0(x)$. In other words, there exists a $\epsilon_0(x) > 0$ such that, for all $\lambda \in (0, \epsilon_0(x)]$,

$$
U_{x,\lambda}(y) \geq 0, V_{x,\lambda}(y) \geq 0, \qquad \forall\ y \in B_\lambda(x) \setminus \{x\}.
$$

This completes the proof of Lemma 2.4. □

For each fixed $x \in \mathbb{R}^n$, we define

$$\bar{\lambda}(x) = \sup\{\lambda > 0 \,|\, u_{x,\mu} \geq u, v_{x,\mu} \geq v \text{ in } B_\mu(x) \setminus \{x\}, \ \forall\ 0 < \mu \leq \lambda\}. \tag{2.40}$$

By Lemma 2.4, $\bar{\lambda}(x)$ is well-defined and $0 < \bar{\lambda}(x) \leq +\infty$ for any $x \in \mathbb{R}^n$.

We need the following lemma, which is crucial in our proof.

Lemma 2.5. *If $\bar{\lambda}(\bar{x}) < +\infty$ for some $\bar{x} \in \mathbb{R}^n$, then*

$$u_{\bar{x},\bar{\lambda}(\bar{x})}(y) = u(y), v_{\bar{x},\bar{\lambda}(\bar{x})}(y) = v(y), \qquad \forall\ y \in B_{\bar{\lambda}}(\bar{x}) \setminus \{\bar{x}\}.$$

Proof. Without loss of generality, we may assume $x = 0$ for convenience. Since (u, v) is a pair of positive solution to integral system (1.4), one can verify that $u_{0,\lambda}, v_{0,\lambda}$ also satisfies a similar integral system as (1.4) in $\mathbb{R}^n \setminus \{0\}$. In fact, by (1.4) and direct calculations, we have, for any $y \in \mathbb{R}^n \setminus \{0\}$,

$$\begin{aligned}
u_{0,\lambda}(y) &= \left(\frac{\lambda}{|y|}\right)^{n-\alpha} u\left(\frac{\lambda^2 y}{|y^2|}\right) \\
&= \frac{\lambda^{n-\alpha}}{|y|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{R_{\alpha,n}}{\left|\frac{\lambda^2 y}{|y|^2} - z\right|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{f_1(v(\xi))}{|z-\xi|^{\sigma_1}} d\xi \cdot f_2(v(z)) dz \\
&= \frac{\lambda^{n-\alpha}}{|y|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{R_{\alpha,n}}{\left|\frac{\lambda^2 y}{|y|^2} - \frac{\lambda^2 z}{|z|^2}\right|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{f_1\left(v\left(\frac{\lambda^2 \xi}{|\xi|^2}\right)\right)}{\left|\frac{\lambda^2 z}{|z|^2} - \frac{\lambda^2 \xi}{|\xi|^2}\right|^{\sigma_1}} \frac{\lambda^{2n}}{|\xi|^{2n}} d\xi \cdot f_2\left(v\left(\frac{\lambda^2 z}{|z|^2}\right)\right) \frac{\lambda^{2n}}{|z|^{2n}} dz \\
&= \int_{\mathbb{R}^n} \frac{R_{\alpha,n}}{|y-z|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{v_{0,\lambda}^{p_1}(\xi)}{|z-\xi|^{\sigma_1}} F_1\left(v(\xi^{0,\lambda})\right) d\xi \cdot v_{0,\lambda}^{p_2}(z) F_2\left(v(z^{0,\lambda})\right) dz,
\end{aligned}$$

$$\begin{aligned}
v_{0,\lambda}(y) &= \left(\frac{\lambda}{|y|}\right)^{n-\alpha} v\left(\frac{\lambda^2 y}{|y^2|}\right) \\
&= \frac{\lambda^{n-\alpha}}{|y|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{R_{\alpha,n}}{\left|\frac{\lambda^2 y}{|y|^2} - z\right|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{g_1(u(\zeta))}{|z-\zeta|^{\sigma_2}} d\zeta \cdot g_2(u(z)) dz \\
&= \frac{\lambda^{n-\alpha}}{|y|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{R_{\alpha,n}}{\left|\frac{\lambda^2 y}{|y|^2} - \frac{\lambda^2 z}{|z|^2}\right|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{g_1\left(u\left(\frac{\lambda^2 \zeta}{|\zeta|^2}\right)\right)}{\left|\frac{\lambda^2 z}{|z|^2} - \frac{\lambda^2 \zeta}{|\zeta|^2}\right|^{\sigma_2}} \frac{\lambda^{2n}}{|\zeta|^{2n}} d\zeta \cdot g_2\left(u\left(\frac{\lambda^2 z}{|z|^2}\right)\right) \frac{\lambda^{2n}}{|z|^{2n}} dz \\
&= \int_{\mathbb{R}^n} \frac{R_{\alpha,n}}{|y-z|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{u_{0,\lambda}^{q_1}(\zeta)}{|z-\zeta|^{\sigma_2}} G_1\left(u(\zeta^{0,\lambda})\right) d\zeta \cdot u_{0,\lambda}^{q_2}(z) G_2\left(u(z^{0,\lambda})\right) dz.
\end{aligned}$$

Suppose on the contrary that $U_{0,\bar{\lambda}} \geq 0$ but $U_{0,\bar{\lambda}}$ is not identically zero in $B_{\bar{\lambda}}(0) \setminus \{0\}$, then we will get a contradiction with the definition (2.40) of $\bar{\lambda}$. Now, we divide the proof into two parts.

Part 1: We prove that

$$U_{0,\bar{\lambda}}(y) > 0, V_{0,\bar{\lambda}}(y) > 0, \quad \forall\, y \in B_{\bar{\lambda}}(0) \setminus \{0\}. \tag{2.41}$$

Actually, if there exists a point $y^0 \in B_{\bar{\lambda}}(0) \setminus \{0\}$ such that $U_{0,\bar{\lambda}}(y^0) > 0$, by continuity, there exists a small $\gamma > 0$ and a constant $c_0 > 0$ such that

$$B_\gamma(y^0) \subset B_{\bar{\lambda}}(0) \setminus \{0\} \quad \text{and} \quad U_{0,\bar{\lambda}}(y) \geq c_0 > 0, \quad \forall\, y \in B_\gamma(y^0).$$

For any $y \in B_{\bar{\lambda}}(0) \setminus \{0\}$, one can deduce that

$$
\begin{aligned}
u(y) &= \int_{\mathbb{R}^n} \frac{R_{\alpha,n}}{|y-z|^{n-\alpha}} P(z) f_2(v(z)) dz \\
&= \int_{B_{\bar{\lambda}}(0)} \frac{R_{\alpha,n}}{|y-z|^{n-\alpha}} P(z) f_2(v(z)) dz + \int_{\mathbb{R}^n \setminus B_{\bar{\lambda}}(0)} \frac{R_{\alpha,n}}{|y-z|^{n-\alpha}} P(z) f_2(v(z)) dz \\
&= \int_{B_{\bar{\lambda}}(0)} \frac{R_{\alpha,n}}{|y-z|^{n-\alpha}} P(z) f_2(v(z)) dz \\
&\quad + \int_{B_{\bar{\lambda}}(0)} \frac{R_{\alpha,n}}{|\frac{y|z|}{\bar{\lambda}} - \frac{\bar{\lambda} z}{|z|}|^{n-\alpha}} P(z^{\bar{\lambda}}) \left(\frac{\bar{\lambda}}{|z|}\right)^{\sigma_1} v_{0,\bar{\lambda}}^{p_2}(z) F_2\left(\left(\frac{\bar{\lambda}}{|z|}\right)^{\alpha-n} v_{0,\bar{\lambda}}(z)\right) dz,
\end{aligned}
$$

and

$$
\begin{aligned}
u_{0,\bar{\lambda}}(y) &= \int_{\mathbb{R}^n} \frac{R_{\alpha,n}}{|y-z|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{v_{0,\bar{\lambda}}^{p_1}(\xi)}{|z-\xi|^{\sigma_1}} F_1\left(v(\xi^{0,\bar{\lambda}})\right) d\xi \cdot v_{0,\bar{\lambda}}^{p_2}(z) F_2\left(v(z^{0,\bar{\lambda}})\right) dz \\
&= \int_{B_{\bar{\lambda}}(0)} \frac{R_{\alpha,n}}{|y-z|^{n-\alpha}} \overline{P}_{0,\bar{\lambda}}(z) v_{0,\bar{\lambda}}^{p_2}(z) F_2\left(\left(\frac{\bar{\lambda}}{|z|}\right)^{\alpha-n} v_{0,\bar{\lambda}}(z)\right) dz \\
&\quad + \int_{B_{\bar{\lambda}}(0)} \frac{R_{\alpha,n}}{|\frac{y|z|}{\bar{\lambda}} - \frac{\bar{\lambda} z}{|z|}|^{n-\alpha}} \overline{P}_{0,\bar{\lambda}}(z^{\bar{\lambda}}) \left(\frac{\bar{\lambda}}{|z|}\right)^{\sigma_1} v^{p_2}(z) F_2(v(z)) dz,
\end{aligned}
$$

where

$$
\overline{P}_{x,\lambda}(y) := \int_{\mathbb{R}^n} \frac{v_{x,\lambda}^{p_1}(\xi)}{|y-\xi|^{\sigma_1}} F_1\left(\left(\frac{\lambda}{|x-\xi|}\right)^{\alpha-n} v_{x,\lambda}(\xi)\right) d\xi.
$$

Let us define

$$
K_{1,\bar{\lambda}}(y,z) = R_{\alpha,n}\left(\frac{1}{\left|y-z\right|^{n-\alpha}} - \frac{1}{\left|\frac{y|z|}{\bar{\lambda}} - \frac{\bar{\lambda} z}{|z|}\right|^{n-\alpha}}\right),
$$

$$
K_{2,\bar{\lambda}}(y,z) = R_{\alpha,n}\left(\frac{1}{\left|y-z\right|^{\sigma_1}} - \frac{1}{\left|\frac{y|z|}{\bar{\lambda}} - \frac{\bar{\lambda} z}{|z|}\right|^{\sigma_1}}\right).
$$

It is easy to verify that $K_{1,\bar{\lambda}}(y,z) > 0$, $K_{2,\bar{\lambda}}(y,z) > 0$, and

$$
\overline{P}_{0,\bar{\lambda}}(z) = P(z^{\bar{\lambda}}) \left(\frac{\bar{\lambda}}{|z|}\right)^{\sigma_1}, \qquad P(z) = \overline{P}_{0,\bar{\lambda}}(z^{\bar{\lambda}}) \left(\frac{\bar{\lambda}}{|z|}\right)^{\sigma_1},
$$

and moreover,

$$
\overline{P}_{0,\bar{\lambda}}(z) - P(z) = \int_{B_{\bar{\lambda}}(0)} K_{2,\bar{\lambda}}(z,\xi) \left(v_{0,\bar{\lambda}}^{p_1}(\xi) F_1\left(\left(\frac{\bar{\lambda}}{|\xi|}\right)^{\alpha-n} v_{0,\bar{\lambda}}(\xi)\right) - v^{p_1}(\xi) F_1(v(\xi))\right) d\xi > 0.
$$

As a result, it follows immediately that, for any $y \in B_{\bar{\lambda}}(0) \setminus \{0\}$,

$$
\begin{aligned}
U_{0,\bar{\lambda}}(y) &= \int_{B_{\bar{\lambda}}(0)} K_{1,\bar{\lambda}}(y,z)\overline{P}_{0,\bar{\lambda}}(z)v_{0,\bar{\lambda}}^{p_2}(z)F_2\left(\left(\frac{\bar{\lambda}}{|z|}\right)^{\alpha-n} v_{0,\bar{\lambda}}(z)\right)dz \\
&\quad - \int_{B_{\bar{\lambda}}(0)} K_{1,\bar{\lambda}}(y,z)P(z)v^{p_2}(z)F_2(v(z))dz \\
&= \int_{B_{\bar{\lambda}}(0)} K_{1,\bar{\lambda}}(y,z)\left[\overline{P}_{0,\bar{\lambda}}(z)v_{0,\bar{\lambda}}^{p_2}(z)F_2\left(\left(\frac{\bar{\lambda}}{|z|}\right)^{\alpha-n} v_{0,\bar{\lambda}}(z)\right) - P(z)v^{p_2}(z)F_2(v(z))\right]dz \\
&\geq \int_{B_{\bar{\lambda}}(0)} K_{1,\bar{\lambda}}(y,z)P(z)\left[v_{0,\bar{\lambda}}^{p_2}(z)F_2\left(\left(\frac{\bar{\lambda}}{|z|}\right)^{\alpha-n} v_{0,\bar{\lambda}}(z)\right) - v^{p_2}(z)F_2(v(z))\right]dz \\
&\geq \int_{B_{\bar{\lambda}}(0)} K_{1,\bar{\lambda}}(y,z)P(z)\left[v_{0,\bar{\lambda}}^{p_2}(z)F_2\left(v_{0,\bar{\lambda}}(z)\right) - v^{p_2}(z)F_2(v(z))\right]dz \\
&\geq \int_{B_{\gamma}(y^0)} K_{1,\bar{\lambda}}(y,z)P(z)(f_2(v_{0,\bar{\lambda}}(z)) - f_2(v(z)))dz > 0,
\end{aligned}
\tag{2.42}
$$

therefore we arrive at (2.41).

Part 2: We draw a contradiction with the definition (2.40) of $\bar{\lambda}(0)$.

Furthermore, (2.42) also shows that there exists a $0 < \eta < \bar{\lambda}$ small enough such that, for any $y \in \overline{B_{\eta}(0)} \setminus \{0\}$,

$$
U_{0,\bar{\lambda}}(y) \geq \int_{B_{\frac{\gamma}{2}}(y^0)} c_8 c_7 c_0 \, dz =: \widetilde{c}_0 > 0. \tag{2.43}
$$

Now we define

$$
\tilde{l}_0 := \min_{\lambda \in [\bar{\lambda}, 2\bar{\lambda}]} l_0(0,\lambda) > 0, \tag{2.44}
$$

where $l_0(0,\lambda)$ is given by Theorem 2.3. For a fixed small $0 < r_0 < \frac{1}{2}\min\{\tilde{l}_0, \bar{\lambda}\}$, by (2.41) and (2.43), we can define

$$
m_1 := \inf_{y \in \overline{B_{\bar{\lambda}-r_0}(0)} \setminus \{0\}} U_{0,\bar{\lambda}}(y) > 0. \tag{2.45}
$$

Similarly, we can define

$$
m_2 := \inf_{y \in \overline{B_{\bar{\lambda}-r_0}(0)} \setminus \{0\}} V_{0,\bar{\lambda}}(y) > 0. \tag{2.46}
$$

Since u is uniformly continuous on arbitrary compact set $K \subset \mathbb{R}^n$ (say, $K = \overline{B_{4\bar{\lambda}}(0)}$), we can deduce from (2.45) that, there exists a $0 < \varepsilon_0 < \frac{1}{2}\min\{\tilde{l}_0, \bar{\lambda}\}$ sufficiently small, such that, for any $\lambda \in [\bar{\lambda}, \bar{\lambda} + \varepsilon_0]$,

$$
U_{0,\lambda}(y) \geq \frac{m_1}{2} > 0, \quad \forall\, y \in \overline{B_{\bar{\lambda}-r_0}(0)} \setminus \{0\}. \tag{2.47}
$$

In order to prove (2.47), one should observe that (2.45) is equivalent to

$$
|y|^{n-\alpha}u(y) - \bar{\lambda}^{n-\alpha}u(y^{0,\bar{\lambda}}) \geq m_1\bar{\lambda}^{n-\alpha}, \qquad \forall\, |y| \geq \frac{\bar{\lambda}^2}{\bar{\lambda} - r_0}. \tag{2.48}
$$

Since u is uniformly continuous on $\overline{B_{4\bar{\lambda}}(0)}$, we infer from (2.48) that there exists a $0 < \varepsilon_0 < \frac{1}{2}\min\{\tilde{l}_0, \bar{\lambda}\}$ sufficiently small, such that, for any $\lambda \in [\bar{\lambda}, \bar{\lambda} + \varepsilon_0]$,

$$|y|^{n-\alpha}u(y) - \lambda^{n-\alpha}u(y^{0,\lambda}) \geq \frac{m_1}{2}\lambda^{n-\alpha}, \qquad \forall\, |y| \geq \frac{\lambda^2}{\lambda - r_0}, \tag{2.49}$$

which is equivalent to (2.47), hence we have proved (2.47).

Similarly, we can prove

$$V_{0,\lambda}(y) \geq \frac{m_2}{2} > 0, \quad \forall\, y \in \overline{B_{\bar{\lambda}-r_0}(0)} \setminus \{0\}. \tag{2.50}$$

For any $\lambda \in [\bar{\lambda}, \bar{\lambda} + \varepsilon_0]$, let $l := \lambda - \bar{\lambda} + r_0 \in (0, \tilde{l}_0)$ and $\Omega := A_{\lambda,l}(0)$, then it follows from (2.37), (2.38) and (2.47) that all the conditions (2.6) in Theorem 2.3 are satisfied, hence we can derive from (ii) in Theorem 2.3 that

$$U_{0,\lambda}(y) \geq 0, V_{0,\lambda}(y) \geq 0, \qquad \forall\; y \in \Omega = A_{\lambda,l}(0). \tag{2.51}$$

Thus, we get from (2.47) and (2.51) that, $B^-_\lambda = \emptyset$ for all $\lambda \in [\bar{\lambda}, \bar{\lambda} + \varepsilon_0]$, that is,

$$U_{0,\lambda}(y) \geq 0, V_{0,\lambda}(y) \geq 0, \quad \forall\; y \in B_\lambda(0) \setminus \{0\}, \tag{2.52}$$

which contradicts with the definition (2.40) of $\bar{\lambda}(0)$. As a result, in the case $0 < \bar{\lambda}(0) < +\infty$, we must have $U_{0,\bar{\lambda}} \equiv 0$, $V_{0,\bar{\lambda}} \equiv 0$ in $B_{\bar{\lambda}}(0) \setminus \{0\}$, that is,

$$u_{0,\bar{\lambda}(0)}(y) \equiv u(y), v_{0,\bar{\lambda}(0)}(y) \equiv v(y), \qquad \forall\; y \in B_{\bar{\lambda}}(0) \setminus \{0\}. \tag{2.53}$$

This finishes our proof of Lemma 2.5. □

We also need the following property about the limiting radius $\bar{\lambda}(x)$.

Lemma 2.6. *If $\bar{\lambda}(\bar{x}) = +\infty$ for some $\bar{x} \in \mathbb{R}^n$, then $\bar{\lambda}(x) = +\infty$ for all $x \in \mathbb{R}^n$.*

Proof. Since $\bar{\lambda}(\bar{x}) = +\infty$, recalling the definition of $\bar{\lambda}$, we get

$$u_{\bar{x},\lambda}(y) \geq u(y), v_{\bar{x},\lambda}(y) \geq v(y), \quad \forall\; y \in B_\lambda(\bar{x}) \setminus \{\bar{x}\}, \qquad \forall\; 0 < \lambda < +\infty.$$

That is,

$$u(y) \geq u_{\bar{x},\lambda}(y), v(y) \geq v_{\bar{x},\lambda}(y), \qquad \forall\; |y - \bar{x}| \geq \lambda, \quad \forall\; 0 < \lambda < +\infty.$$

It follows immediately that

$$\lim_{|y|\to\infty} |y|^{n-\alpha}u(y) = +\infty, \lim_{|y|\to\infty} |y|^{n-\alpha}v(y) = +\infty. \tag{2.54}$$

On the other hand, if we assume $\bar{\lambda}(x) < +\infty$ for some $x \in \mathbb{R}^n$, then by Lemma 2.5, one gets that

$$\lim_{|y|\to\infty} |y|^{n-\alpha}u(y) = \lim_{|y|\to\infty} |y|^{n-\alpha}u_{x,\bar{\lambda}(x)}(y) = (\bar{\lambda}(x))^{n-\alpha}u(x) < +\infty,$$

$$\lim_{|y|\to\infty} |y|^{n-\alpha}v(y) = \lim_{|y|\to\infty} |y|^{n-\alpha}v_{x,\bar{\lambda}(x)}(y) = (\bar{\lambda}(x))^{n-\alpha}v(x) < +\infty,$$

which contradicts with (2.54).

This finishes the proof of Lemma 2.6. □

We are now ready to give a proof of Theorem 1.1.

3. Proof of Theorem 1.1

We derive the forms for nonnegative solution (u, v) by discussing two different possible cases.
Case (i). $\bar{\lambda}(\bar{z}) = +\infty$ for some $\bar{z} \in \mathbb{R}^n$.

By Lemma 2.6, one can derive that

$$\bar{\lambda}(z) = +\infty, \quad \forall z \in \mathbb{R}^n.$$

Thus, for all $z \in \mathbb{R}^n$ and $0 < \lambda < +\infty$, we have

$$u_{z,\lambda}(y) \geq u(y), \quad \forall\ y \in B_\lambda(z) \setminus \{z\}, \quad \forall\ 0 < \lambda < +\infty,$$

$$v_{z,\lambda}(y) \geq v(y), \quad \forall\ y \in B_\lambda(z) \setminus \{z\}, \quad \forall\ 0 < \lambda < +\infty.$$

By a calculus lemma (Lemma 11.2 in Li [39]), we conclude that $u = b_1, v = b_2$ is a constant.

This is a contradiction, therefore *Case (i)* is impossible.
Case (ii). $\bar{\lambda}(z) < \infty$ for all $z \in \mathbb{R}^n$.

By Lemma 2.5, we infer that

$$u_{z,\bar{\lambda}(z)}(y) = u(y), \ \ v_{z,\bar{\lambda}(z)}(y) = v(y), \qquad \forall\ y \in B_{\bar{\lambda}(z)}(z) \setminus \{z\}. \tag{3.1}$$

From a calculus lemma (Lemma 11.1 in Li [39]) and (3.1), we derive that for any $y \in \mathbb{R}^n$,

$$u(y) = \frac{c_1}{(d^2 + |y - z_0|^2)^{\frac{n-\alpha}{2}}},$$

for some $c_1 > 0$, $d > 0$, $z_0 \in \mathbb{R}^n$. In a similar way, for any $y \in \mathbb{R}^n$, we have

$$v(y) = \frac{c_2}{(d^2 + |y - z_0|^2)^{\frac{n-\alpha}{2}}},$$

for some $c_2 > 0$, $d > 0$, $z_0 \in \mathbb{R}^n$.

Now we verify that F_1, F_2, G_1 and G_2 must be constants. It follows from Lemma 2.5 and (2.42) that

$$\begin{aligned}
0 &= u_{x,\bar{\lambda}}(y) - u(y) \\
&= \int_{B_{\bar{\lambda}}(x)} K_{1,\bar{\lambda}}(y,z) \left[\overline{P}_{x,\bar{\lambda}}(z) v_{x,\bar{\lambda}}^{p_2}(z) F_2\left(\left(\frac{\bar{\lambda}}{|z-x|} \right)^{\alpha-n} v_{x,\bar{\lambda}}(z) \right) - P(z) v^{p_2}(z) F_2(v(z)) \right] dz \\
&= \int_{B_{\bar{\lambda}}(x)} K_{1,\bar{\lambda}}(y,z) \left[\overline{P}_{x,\bar{\lambda}}(z) \frac{f_2\left(\left(\frac{\bar{\lambda}}{|z-x|} \right)^{\alpha-n} v_{x,\bar{\lambda}}(z) \right)}{\left(\left(\frac{\bar{\lambda}}{|z-x|} \right)^{\alpha-n} v_{x,\bar{\lambda}}(z) \right)^{p_2}} - P(z) \frac{f_2(v(z))}{v^{p_2}(z)} \right] v^{p_2}(z) dz \\
&\geq \int_{B_{\bar{\lambda}}(x)} K_{1,\bar{\lambda}}(y,z) P(z) v^{p_2}(z) \left(\frac{f_2\left(\left(\frac{\bar{\lambda}}{|z-x|} \right)^{\alpha-n} v_{x,\bar{\lambda}}(z) \right)}{\left(\left(\frac{\bar{\lambda}}{|z-x|} \right)^{\alpha-n} v_{x,\bar{\lambda}}(z) \right)^{p_2}} - \frac{f_2(v(z))}{v^{p_2}(z)} \right) dz \\
&\geq 0,
\end{aligned}$$

where

$$
\begin{aligned}
\overline{P}_{x,\bar{\lambda}}(z) - P(z) &= \int_{B_{\bar{\lambda}}(x)} K_{2,\bar{\lambda}}(z,\xi)\left(v_{x,\bar{\lambda}}^{p_1}(\xi)F_1\left(\left(\frac{\bar{\lambda}}{|x-\xi|}\right)^{\alpha-n} v_{x,\bar{\lambda}}(\xi)\right) - v^{p_1}(\xi)F_1(v(\xi))\right)d\xi \\
&= \int_{B_{\bar{\lambda}}(x)} K_{2,\bar{\lambda}}(z,\xi)v^{p_1}(\xi)\left(\frac{f_1\left(\left(\frac{\bar{\lambda}}{|x-\xi|}\right)^{\alpha-n} v_{x,\bar{\lambda}}(\xi)\right)}{\left(\left(\frac{\bar{\lambda}}{|x-\xi|}\right)^{\alpha-n} v_{x,\bar{\lambda}}(\xi)\right)^{p_1}} - \frac{f_1(v(\xi))}{v^{p_1}(\xi)}\right)d\xi \\
&\geq 0,
\end{aligned}
\tag{3.2}
$$

that is,

$$
\overline{P}_{x,\bar{\lambda}}(z)\frac{f_2\left(\left(\frac{\bar{\lambda}}{|z-x|}\right)^{\alpha-n} v_{x,\bar{\lambda}}(z)\right)}{\left(\left(\frac{\bar{\lambda}}{|z-x|}\right)^{\alpha-n} v_{x,\bar{\lambda}}(z)\right)^{p_2}} = P(z)\frac{f_2(v(z))}{v^{p_2}(z)}.
$$

Consequently, it follows that for any $z \in B_{\bar{\lambda}}(x)$, we have

$$
\frac{f_1\left(\left(\frac{\bar{\lambda}}{|z-x|}\right)^{\alpha-n} v_{x,\bar{\lambda}}(z)\right)}{\left(\left(\frac{\bar{\lambda}}{|z-x|}\right)^{\alpha-n} v_{x,\bar{\lambda}}(z)\right)^{p_1}} \equiv \frac{f_1(v(\xi))}{v^{p_1}(\xi)},
$$

and

$$
\frac{f_2\left(\left(\frac{\bar{\lambda}}{|z-x|}\right)^{\alpha-n} v_{x,\bar{\lambda}}(z)\right)}{\left(\left(\frac{\bar{\lambda}}{|z-x|}\right)^{\alpha-n} v_{x,\bar{\lambda}}(z)\right)^{p_2}} \equiv \frac{f_2(v(\xi))}{v^{p_2}(\xi)}.
$$

Thus, for some positive constant C_1 and C_2, we have

$$
F_1(t) = \frac{f_1(t)}{t^{p_1}} = C_1, \quad t \in (0, \max_{x\in\mathbb{R}^n} v(x)],
$$

$$
F_2(t) = \frac{f_2(t)}{t^{p_2}} = C_2, \quad t \in (0, \max_{x\in\mathbb{R}^n} v(x)].
$$

Similarly, we deduce that for some positive constant C_3 and C_4,

$$
G_1(t) = \frac{g_1(t)}{t^{q_1}} = C_3, \quad t \in (0, \max_{x\in\mathbb{R}^n} u(x)],
$$

$$
G_2(t) = \frac{g_2(t)}{t^{q_2}} = C_4, \quad t \in (0, \max_{x\in\mathbb{R}^n} u(x)].
$$

This completes our proof of Theorem 1.1.

4. Conclusions

In this paper, we obtain the forms of the nonnegative solution and classify nonlinearities by appling a variant (for nonlocal nonlinearity) of the direct moving spheres method for fractional Laplacians.

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Conflict of interest

The authors declare that they have no competing interests in this paper.

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